Research Article

Maximum Norm Estimates of the Solution of the Navier-Stokes Equations in the Halfspace with Bounded Initial Data

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In this paper, I consider the Cauchy problem for the incompressible Navier-Stokes equations in $\mathbb{R}_+^n$ for $n \geq 3$ with bounded initial data and derive a priori estimates of the maximum norm of all derivatives of the solution in terms of the maximum norm of the initial data. This paper is a continuation of my work in my previous papers, where the initial data are considered in $\mathbb{T}^n$ and $\mathbb{R}^n$ respectively. In this paper, because of the nonempty boundary in our domain of interest, the details in obtaining the desired result are significantly different and more challenging than the work of my previous papers. This challenges arise due to the possible noncommutativity nature of the Leray projector with the derivatives in the direction of normal to the boundary of the domain of interest. Therefore, we only consider one derivative of the velocity field in that direction.

1. Introduction

We consider the Cauchy problem of the incompressible Navier-Stokes equations in $\mathbb{R}_+^n$, $n \geq 3$:

$$\left. \begin{aligned} u_t + u \cdot \nabla u + \nabla p &= \Delta u & \text{for } x \in \mathbb{R}_+^n, t > 0, \\ \nabla \cdot u &= 0 & \text{for } x \in \mathbb{R}_+^n, t > 0, \\ u|_{t=0} &= f & \text{for } x \in \mathbb{R}_+^n, \\ u|_{x_n=0} &= 0 & \text{for } t > 0, \end{aligned} \right\} \quad (1)$$

where $u = u(x, t) = (u_1(x, t), \dots, u_n(x, t))$ and $p = p(x, t)$ stand for the unknown velocity vector field of the fluid and its pressure, while $f = f(x) = (f_1(x), \dots, f_n(x))$ is the given initial velocity vector field, with $\nabla \cdot f = 0$ and $f|_{x_n=0} = 0$. In what follows, we will use the same notations for the space of vector-valued and scalar functions for convenience in writing.

There is a large literature on the existence and uniqueness of solution of the Navier-Stokes equations in $\mathbb{R}^n$. For the given initial data, solutions of (1) have been constructed in various function spaces. For example, if $f \in L^r$ for some r with $3 \leq r < \infty$, then it is well known that there is a unique

classical solution in some maximum interval of time: $0 \leq t < T_f$, where $0 < T_f \leq \infty$. But, for the uniqueness of the pressure, one requires $|p(x, t)| \rightarrow 0$ as $|x| \rightarrow \infty$. See [1] and [2] for $r = 3$ and [3] for $3 < r < \infty$. The solution is C^∞ for $0 < T_f < \infty$.

It is well known that for $f \in L^\infty(\mathbb{R}^n)$, there is a unique, smooth, and local-in-time solution u for the Navier-Stokes equations with

$$p = \sum_{i,j} R_i R_j u_i u_j, \quad (2)$$

where $R_i = (-\Delta)^{-1/2} \partial_{x_i}$ is the i th Riesz operator. It is known that in $\mathbb{R}^2$, this solution can be extended globally in time. For $f \in L^\infty(\mathbb{R}^n)$, where $n \geq 3$, the existence of a regular solution follows from [4]. The solution is only unique if one puts some growth restrictions on the pressure as $|x| \rightarrow \infty$. A simple example of nonuniqueness is demonstrated in [5], where the velocity u is bounded, but $|p(x, t)| \leq C|x|$. In addition, an estimate $|p(x, t)| \leq C(1 + |x|^\sigma)$ with $\sigma < 1$ (see [6]) implies uniqueness. Also, the assumption $p \in L_{loc}^1(0, T; \text{BMO})$ (see [7]) implies uniqueness.

For $f \in L^\infty(\mathbb{R}_+^n)$, where $n \geq 3$, the existence of a local mild solution is proved by Bae and Jin in [8]. In the same paper, it is also proved that such mild solution is indeed a strong solution of the Navier-Stokes equations (1). Before the result of Bae and Jin, the local-in-time existence of mild (strong) solution of the halfspace problem was provided in [9] by Solonikov for continuous bounded initial data in $\mathbb{R}_+^n$.

In this paper, I am interested in obtaining estimates of the maximum norm of the derivatives of u in terms of the maximum norm of the initial function f , assuming that the solution exists, and it is $C^\infty(\mathbb{R}_+^n)$ for $0 < t < T_f$. The work of this paper is a continuation of the work of my papers [10] and [11] to the halfspace case for nondecaying initial data. Non-empty boundary in the domain in this paper makes this work different, in some aspects, and significantly more challenging in proving the key lemmas than the work in my previous works where the initial functions are in $\mathbb{R}^n$ or $\mathbb{T}^n$.

We begin by transforming the momentum equations of (1) into the abstract ordinary differential equations:

$$u_t + Au = -\mathbb{P}(u \cdot \nabla u), \quad (3)$$

where $A = -\mathbb{P}\Delta$ is the Stokes operator and $\mathbb{P}$ is the Leray projector, which is given by

$$\mathbb{P}f(x) \equiv f(x) + \nabla_x \int_{\mathbb{R}_+^n} \nabla_y \mathcal{G}(x, y) \cdot f(y) dy, \quad (4)$$

where $f_n|_{x_n=0} = 0$. Note that

$$\mathcal{G}(x, y) \equiv N(x - y) + N(x - y^*), \quad (5)$$

where $y^* = (y_1, \dots, y_{n-1}, -y_n)$, $N(x) = (1/(2-n)\omega_n)|x|^{2-n}$, if $n \geq 3$, and ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$ which is given by $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$.

The solution of (3) is formally expressed in the integral form:

$$u(t) = e^{-At}f - \int_0^t e^{-A(t-s)}\mathbb{P}(u \cdot \nabla u)(s)ds. \quad (6)$$

Solonikov [9] has expressed the solution operator of the Stokes equations in $\mathbb{R}_+^n$ in the integral form

$$e^{-At}f \equiv \int_{\mathbb{R}_+^n} G(x, y, t) \cdot f(y) dy, \quad (7)$$

where $G = (G_{ij})_{i,j=1,\dots,n}$ is given by

$$\begin{aligned} G_{ij}(x, y, t) &= \delta_{ij}(\Gamma(x - y, t) - \Gamma(x - y^*, t)) \\ &+ 4(1 - \delta_{jn}) \frac{\partial}{\partial x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_i} N(x - z) \\ &\cdot \Gamma(z - y^*, t) dz. \end{aligned} \quad (8)$$

The function $\Gamma(x, t)$ is the n -dimensional Gaussian kernel defined by $\Gamma(x, t) \equiv \Gamma_t(x) \equiv (1/(4\pi t)^{n/2})e^{-|x|^2/4t}$.

A solution formula of the Stokes equations (3) in $\mathbb{R}_+^n$ has also been provided by Ukai in [12]. Such solution formula has been used in the L^q setting, particularly for $1 < q < \infty$ (see [13, 14]). For L^1 and L^∞ estimates of the Stokes flow or its gradient, see [15, 16]. The solution formula provided by Solonikov [9] has mainly been used for L^∞ framework (see [14, 17]).

To formulate the main result of this paper, we first introduce some notations as follows:

$$|f|_\infty = \sup_x |f(x)| \text{ with } |f(x)|^2 = \sum_i f_i^2(x), \quad (9)$$

and $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, $\partial x_i = D_i = (\partial/\partial x_i)$ for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$. In what follows, if $|\alpha| = j$, for any $j = 0, 1, \dots$, then we will denote $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ by D^j . We also set

$$|\mathcal{D}^j u(t)|_\infty := |\mathcal{D}^j u(\cdot, t)|_\infty = \max_{|\alpha|=j} |D^\alpha u(\cdot, t)|_\infty. \quad (10)$$

Clearly, $|\mathcal{D}^j u(t)|_\infty$ measures all space derivatives of order j in maximum norm. For later purposes, let us also introduce a few other notations:

$$\begin{aligned} \Gamma(x, t) &= \Gamma_t(x) = \bar{\Gamma}_t(\bar{x})\Gamma_t^n(x_n) \\ &= \left(\frac{1}{(4\pi t)^{(n-1)/2}} e^{-|\bar{x}|^2/4t} \right) \left(\frac{1}{(4\pi t)^{1/2}} e^{-x_n^2/4t} \right), \end{aligned}$$

$$D^\beta = (D_1^{\beta_1} \dots D_{n-1}^{\beta_{n-1}}), D^\alpha = D^\beta D_n, \quad \alpha = (\beta, 1), \quad \nabla \cdot = \text{div}. \quad (11)$$

Throughout this paper, D^j will be understood as the derivative of order $j = |\alpha| = |\beta| + 1$. In addition, $1_{\mathbb{R}_+^n}$ denotes the characteristic function which is 1 on $\mathbb{R}_+^n$ and 0 otherwise. τ_a is a translation operator defined by $\tau_a f(x) := f(a - x)$.

The goal of this paper is to prove the following theorem.

Theorem 1. Consider the Cauchy problem for the Navier-Stokes equations ((1)) where $f \in L^\infty(\mathbb{R}_+^n)$ and $\nabla \cdot f = 0$ is understood in the sense of distribution. There is a constant $c_0 > 0$, and for any $\alpha = (\alpha_1, \dots, \alpha_{n-1}, 1)$ with $|\alpha| = j$ where $j = 0, 1, \dots$ there is a constant K_j so that

$$t^{j/2} |\mathcal{D}^j u(t)|_\infty \leq K_j |f|_\infty \text{ for } 0 < t \leq \frac{c_0}{|f|_\infty^2}. \quad (12)$$

The constants c_0 and K_j are independent of t and f .

One of the important tools in the proof of Theorem 1 is the uniform estimates of the composite operator $D^j e^{-At} \mathbb{P} \nabla \text{div}$. But, obtaining such uniform estimates is complicated because of the possible noncommutativity nature of the Leray projector with the derivatives in the direction of normal to the boundary of the domain; hence, D^j and $e^{-At} \mathbb{P}$ may not be commutative.

To overcome this difficulty, we will generalize the techniques of obtaining the uniform estimates on $\nabla e^{-At} \mathbb{P} \nabla \text{div}$ of the paper [8] by Bae and Jin to obtain our desired uniform estimates on $D^j e^{-At} \mathbb{P} \nabla \text{div}$. In their paper, they require the uniform estimates to prove the existence of the local solution of the Navier-Stokes equations in halfspace for bounded initial data.

This paper is organized in the following ways. In “Some Auxiliary Results,” we introduce some auxiliary results which will be labelled as propositions. In “Estimate of $D^j e^{-At} \mathbb{P} \nabla \cdot g$,” we derive an important estimate on the composite operator $D^j e^{-At} \mathbb{P} \nabla \text{div}$. In “Estimates for the Navier-Stokes Equations,” we establish some estimates on the solution of the Navier-Stokes equations. In “Estimates for the Navier-Stokes Equations,” a proof of Theorem 1 will be provided. Finally, Appendices A, B, and C contain proofs of the propositions which are introduced in “Some Auxiliary Results.”

2. Some Auxiliary Results

Let us consider the Stokes problem in $\mathbb{R}_+^n, n \geq 3$:

$$\left. \begin{aligned} u_t - \Delta u + \nabla p &= -\nabla \cdot g(u) \quad \text{for } x \in \mathbb{R}_+^n, t > 0, \\ \nabla \cdot u &= 0 \quad \text{for } x \in \mathbb{R}_+^n, t > 0, \\ u|_{t=0} &= f \quad \text{for } x \in \mathbb{R}_+^n, \\ u|_{x_n=0} &= 0 \quad \text{for } t > 0, \end{aligned} \right\} \quad (13)$$

where $g = u \otimes u = (g_{ij})_{1 \leq i, j \leq n}$. Here, we note that each g_{ij} is quadratic in components of u .

Solonikov in [9] has obtained the solution of (13) which is given by

$$\begin{aligned} u(x, t) &= \int_{\mathbb{R}_+^n} G(x, y, t) \cdot f(y) dy \\ &\quad - \int_0^t \int_{\mathbb{R}_+^n} G(x, y, t-s) \cdot (\mathbb{P} \nabla \cdot g)(y, s) dy ds. \end{aligned} \quad (14)$$

Next, we state the following proposition.

Proposition 2. *If $k = 1, \dots, n-1$, then we have*

$$\frac{\partial}{\partial x_k} G_{ij}(x, y, t) = -\frac{\partial}{\partial y_k} G_{ij}(x, y, t) \quad (15)$$

and

$$\begin{aligned} \frac{\partial}{\partial x_n} G_{ij}(x, y, t) &= \frac{\partial}{\partial y_n} G_{ij}(x, y, t) - 2\delta_{ij} \frac{\partial}{\partial y_n} \Gamma(x-y, t) \\ &\quad + 4(1-\delta_{jn}) \frac{\partial}{\partial x_j} \int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_i} N(\bar{x}-\bar{z}, x_n) \\ &\quad \cdot \Gamma(\bar{z}-\bar{y}, y_n, t) d\bar{z}. \end{aligned} \quad (16)$$

Proof. The proof is given in Appendix A.

Proposition 3. *Let $x \in \mathbb{R}_+^n$ and f be any Hölder continuous function with the exponent $0 < \alpha < 1$:*

$$[f]_\alpha = \sup_{x, z} \frac{|f(x) - f(z)|}{|x - z|^\alpha} < \infty. \quad (17)$$

Then, for $i, j \neq n$ or $i, j = n$, we have

$$\begin{aligned} \partial_{x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x-z) f(z) dz &= -\frac{\delta_{ij}}{2n} f(x) \\ &\quad + \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i x_j}^2 N(x-z) f(z) dz \\ &\quad + \delta_{jn} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(\bar{x}-\bar{z}) f(\bar{z}, x_n) d\bar{z}. \end{aligned} \quad (18)$$

For $i = n, j \neq n$ or $j = n, i \neq n$, we have

$$\partial_{x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x-z) f(z) dz = \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i x_j}^2 N(x-z) f(z) dz. \quad (19)$$

Proof. The proof is given in Appendix B.

Next, we define the Hardy space $\mathcal{H}^1$. Let $\mathcal{N}h(x) := \sup_{s>0} |h * \Gamma_s(x)|$. Let $\mathcal{H}^1(\mathbb{R}^n)$ be the space of functions h so that $\mathcal{N}h \in L^1(\mathbb{R}^n)$ with the norm $\|h\|_{\mathcal{H}^1(\mathbb{R}^n)} := \|\mathcal{N}h\|_{L^1(\mathbb{R}^n)}$. Let $\mathcal{H}^1(\mathbb{R}_+^n)$ be the space of functions h so that there is $\tilde{h} \in \mathcal{H}^1(\mathbb{R}^n)$ with $\tilde{h}|_{\mathbb{R}_+^n} = h$ with the norm $\|h\|_{\mathcal{H}^1(\mathbb{R}_+^n)} = \inf \{ \|\tilde{h}\|_{\mathcal{H}^1(\mathbb{R}^n)} : \tilde{h}|_{\mathbb{R}_+^n} = h \}$.

Next, we state a few well-known results related to the Hardy-norm estimates of the Gaussian kernel Γ_t .

Proposition 4. *Fix $a \in \mathbb{R}^n$. Then $I_{\mathbb{R}_+^n} D^j(\tau_a \Gamma_t) \in \mathcal{H}^1(\mathbb{R}^n)$ with*

$$\|I_{\mathbb{R}_+^n} D^j(\tau_a \Gamma_t)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{-j/2} \quad (20)$$

and

$$\|I_{\mathbb{R}_+^n} D^j(\tau_a * \Gamma_t - \tau_b * \Gamma_t)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{-(j+\alpha)/2} e^{-a_n^2/4t} |\bar{a} - \bar{b}|^\alpha \quad (21)$$

for $0 < \alpha < 1$.

We omit the proofs of well-known results of Proposition 4.

Proposition 5. Let $j = 1 \dots, n-1$ and $i = 1, \dots, n$. Then we have

$$\begin{aligned} & \int_{\mathbb{R}_+^n} \left| \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial^2}{\partial z_i \partial z_j} N(x-z) f(z, y) dz dy \right| \\ & \leq C \sup_{x \in \mathbb{R}_+^n} \int_{\mathbb{R}_+^n} |f(x, y)| dy \\ & \quad + C x_n^\alpha \sup_{x_n/2 < z_n < x_n, \bar{z} \in \mathbb{R}^{n-1}} \frac{\int_{\mathbb{R}_+^n} |f(\bar{z}, z_n, y) - f(\bar{x}, z_n, y)| dy}{|\bar{x} - \bar{z}|^\alpha}. \end{aligned} \quad (22)$$

Proof. The proof is given in Appendix C.

3. Estimate of $D^j e^{-At} \mathbb{P} \nabla \cdot g$

Solonikov in [9] and Shimizu in [16] provide the following estimates:

$$|\mathcal{D}^j e^{-At} f|_\infty \leq C t^{-j/2} |f|_\infty, \quad (23)$$

$$|e^{-At} \mathbb{P} \nabla \cdot g|_\infty \leq C t^{-1/2} |g|_\infty, \quad (24)$$

where $f \in L^\infty(\mathbb{R}_+^n)$, $g = (g_{ij})$, $1 \leq i, j \leq n$, and $g_{ij} \in L^\infty(\mathbb{R}_+^n)$ for each i, j . Also, f and g vanish on the boundary. In addition, in paper [8] by Bae and Jin, they prove

$$|\nabla e^{-At} \mathbb{P} \nabla \cdot g|_\infty \leq C t^{-1/2} |\nabla g|_\infty \quad (25)$$

as a critical estimate to prove their desired result.

With all the above estimates in hand, we begin to obtain the uniform estimate on the composite operator $D^j e^{-At} \mathbb{P} \nabla \cdot g$. For that purpose, recall

$$e^{-At} f = \int_{\mathbb{R}_+^n} G(x, y, t) \cdot f(y) dy, \quad (26)$$

where $G(x, y, t)$ is defined by (8). In the following, consider $i \neq n$, and denote by $H = (H_{ijk})_{i,j,k=1}^n$ the kernel tensor of the operator $e^{-At} \mathbb{P} \nabla \text{div}$. For simplicity in computational purpose, we consider g_{ij} as a Schwartz class function in $\mathbb{R}_+^n$ vanishing on the boundary for each i, j . Thus, we begin by writing

$$\left[\partial_{x_i} e^{-At} \mathbb{P} D_x^\beta \nabla \cdot g \right]_i(x) = \sum_{j,k=1}^n \int_{\mathbb{R}_+^n} \partial_{x_i} H_{ijk}(x, y, t) D_y^\beta g_{jk}(y) dy, \quad (27)$$

where

$$\begin{aligned} H_{ijk}(x, y, t) &= -\partial_{y_k} G_{ij}(x, y, t) \\ &+ \sum_{l=1}^n \partial_{y_j} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \partial_{z_l} G_{il}(x, z, t) dz. \end{aligned} \quad (28)$$

With integration by parts, we obtain

$$\left[\partial_{x_i} e^{-At} \mathbb{P} D_x^\beta \nabla \cdot g \right]_i(x) = \sum_{j,k=1}^n (-1)^{|\beta|} \int_{\mathbb{R}_+^n} \partial_{x_i} D_y^\beta H_{ijk}(x, y, t) g_{jk}(y) dy. \quad (29)$$

Use $G_{in} = 0$ for $i \neq n$ to write the following:

$$\begin{aligned} \partial_{x_i} D_y^\beta H_{ijk}(x, y, t) &= -\partial_{x_i} \partial_{y_k} D_y^\beta G_{ij}(x, y, t) \\ &+ \sum_{l=1}^{n-1} \partial_{x_i} \partial_{y_j} D_y^\beta \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \partial_{z_l} G_{il}(x, z, t) dz. \end{aligned} \quad (30)$$

Use $\partial_{x_i} G_{ij}(x, y, t) = -\partial_{y_i} G_{ij}(x, y, t)$ for $i \neq n$ and Proposition 5 to justify the following expression:

$$\begin{aligned} \partial_{x_i} D_y^\beta H_{ijk}(x, y, t) &= \partial_{y_k} D_y^{\beta+1} G_{ij}(x, y, t) \\ &+ \sum_{l=1}^{n-1} \partial_{x_i} \partial_{y_j} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \partial_{z_l} D_z^\beta G_{il}(x, z, t) \\ &\cdot dz = \partial_{y_k} D_y^{\beta+1} G_{ij}(x, y, t) \\ &+ \sum_{l=1}^{n-1} \partial_{y_j} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \partial_{x_i} D_z^{\beta+1} G_{il}(x, z, t) dz. \end{aligned} \quad (31)$$

Therefore, (29) can be rewritten as

$$\begin{aligned} \left[\partial_{x_i} e^{-At} \mathbb{P} D_x^\beta \nabla \cdot g \right]_i(x) &= \sum_{j,k=1}^n (-1)^{|\beta|} \int_{\mathbb{R}_+^n} \\ &\cdot \left[\partial_{y_k} D_y^{\beta+1} G_{ij}(x, y, t) + \sum_{l=1}^{n-1} \partial_{y_j} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \partial_{x_i} D_z^{\beta+1} G_{il}(x, z, t) dz \right] \\ &\cdot g_{jk}(y) dy = I_1(x, t) + I_2(x, t), \end{aligned} \quad (32)$$

where

$$\begin{aligned} I_1(x, t) &:= (-1)^{|\beta|} \sum_{j,k=1}^n \int_{\mathbb{R}_+^n} \partial_{y_k} D_y^{\beta+1} G_{ij}(x, y, t) g_{jk}(y) dy, \\ I_2(x, t) &:= (-1)^{|\beta|} \sum_{j,k=1}^n \int_{\mathbb{R}_+^n} \sum_{l=1}^{n-1} \partial_{y_j} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \partial_{x_i} \\ &\cdot D_z^{\beta+1} G_{il}(x, z, t) dz g_{jk}(y) dy. \end{aligned} \quad (33)$$

First, we estimate I_1 for $k \neq n$. For that purpose, recall $\partial_{y_k} G_{ij}(x, y, t) = -\partial_{x_k} G_{ij}(x, y, t)$. Therefore,

$$I_1(x, t) = \sum_{j,k=1, k \neq n}^n \int_{\mathbb{R}_+^n} D_x^{\beta+2} G_{ij}(x, y, t) g_{jk}(y) dy. \quad (34)$$

Clearly, $I_1(x, t)$ is the derivative of the i th component of the solution of the Stokes equations. So, using estimate

(23), we get the desired estimate on I_1 as below

$$|I_1(x, t)| \leq Ct^{-(j+1)/2} |g|_\infty \quad \text{for } k \neq n. \quad (35)$$

Next, we estimate I_1 for $k = n$. In [9], G_{ij} is given as

$$G_{ij}(x, y, t) = \delta_{ij} \Gamma(x - y, t) + G_{ij}^*(x, y, t), \quad (36)$$

where

$$\begin{aligned} G_{ij}^*(x, y, t) &= -\delta_{ij} \Gamma(x - y^*, t) \\ &+ 4(1 - \delta_{jn}) \partial_{x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x - z) \Gamma(z - y^*, t) dz. \end{aligned} \quad (37)$$

Next, we use the estimate

$$\begin{aligned} |D_t^s D_x^k D_y^m G_{ij}^*(x, y, t)| &\leq Ct^{-s-m_n/2} (t + x_n^2)^{-k_n/2} \\ &\cdot \left(|x - y^*|^2 + t \right)^{\frac{n+k'|+m'|}{2}} e^{-\frac{C y_n^2}{t}}, \end{aligned} \quad (38)$$

where $m' = (m_1, m_2, \dots, m_{n-1})$, $k' = (k_1, k_2, \dots, k_{n-1})$, provided in [18], to obtain the desired estimate on I_1 for $k = n$.

Therefore, we use modified G_{ij} for $k = n$ and rewrite I_1 as

$$\begin{aligned} I_1(x, t) &= (-1)^{|\beta|} \sum_{j,k=1}^n \int_{\mathbb{R}_+^n} \partial_{y_n} D_y^{\beta+1} [\delta_{ij} \Gamma(x - y, t) \\ &+ G_{ij}^*(x, y, t)] g_{jk}(y) dy \\ &= (-1)^{2|\beta|+1} \sum_{j,k=1}^n \int_{\mathbb{R}_+^n} \partial_{y_n} D_x^{\beta+1} \delta_{ij} \Gamma(x - y, t) g_{jk} \\ &\cdot (y) dy + (-1)^{|\beta|} \sum_{j,k=1}^n \int_{\mathbb{R}_+^n} \partial_{y_n} D_y^{\beta+1} G_{ij}^*(x, y, t) \\ &\cdot g_{jk}(y) dy := I_1^* + I_1^{**}. \end{aligned} \quad (39)$$

The estimate for I_1^* follows as

$$\begin{aligned} |I_1^*(x, t)| &\leq C |g|_\infty \int_{\mathbb{R}^n} |1_{\mathbb{R}_+^n} \partial_{y_n} D_x^{\beta+1} \delta_{ij} \Gamma(x - y, t)| \\ &\cdot dy \leq C |g|_\infty \left\| 1_{\mathbb{R}_+^n} D_x^{\beta+1} \partial_{x_n} \Gamma(x, -t) \right\|_{\mathcal{S}'^1(\mathbb{R}^n)}. \end{aligned} \quad (40)$$

Applying Proposition 4, we obtain

$$|I_1^*(x, t)| \leq C |g|_\infty t^{-(j+1)/2}. \quad (41)$$

To estimate I_1^{**} , we use the estimate of (38) and obtain

$$\begin{aligned} |I_1^{**}(x, t)| &\leq C |g|_\infty t^{-1/2} \int_{\mathbb{R}_+^n} \frac{e^{-C y_n^2/t}}{(|x - y^*|^2 + t)^{n+j/2}} dy \leq C |g|_\infty t^{-(j+1)/2} \\ &\cdot \int_{\mathbb{R}_+^n} \frac{e^{-C y_n^2/t}}{(|x - y^*|^2 + t)^{n/2}} dy \leq C |g|_\infty t^{-(j+1)/2} t^{1/2} (t + x_n^2)^{-1/2}. \end{aligned} \quad (42)$$

Finally, we get

$$|I_1^{**}(x, t)| \leq Ct^{-(j+1)/2} |g|_\infty. \quad (43)$$

We obtain

$$|I_1|_\infty \leq |I_1^*|_\infty + |I_1^{**}|_\infty \leq Ct^{-(j+1)/2} |g|_\infty \quad \text{for } k = n. \quad (44)$$

Therefore, from (35) and (44), we obtain

$$|I_1|_\infty \leq Ct^{-(j+1)/2} |g|_\infty \quad \text{for all } k. \quad (45)$$

Next, we estimate I_2 : For that, let us begin by rewriting I_2 after dropping the summation notations and negative signs for convenience in writing.

$$I_2(x, t) = \int_{\mathbb{R}_+^n} \partial_{y_j} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \partial_{x_i} D_z^{\beta+1} G_{il}(x, z, t) dz g_{jk}(y) dy. \quad (46)$$

Equivalently, we write

$$I_2(x, t) = \int_{\mathbb{R}_+^n} T(x, y, t) g_{jk}(y) dy, \quad (47)$$

where

$$T(x, y, t) := \partial_{y_j} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) D_z^{\beta+2} G_{il}(x, z, t) dz. \quad (48)$$

Using expression for G_{ij} from (8) for $i, l \neq n$, we obtain

$$\begin{aligned} T(x, y, t) &= \partial_{y_j} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \\ &\cdot D_z^{\beta+2} [\delta_{il} \Gamma(x - z, t) - \Gamma(x - z^*, t)] \\ &+ 4 \partial_{x_l} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x - w) \Gamma(w - z^*, t) dw \\ &\cdot dz = T_1 + T_2 + T_3, \end{aligned} \quad (49)$$

where

$$\begin{aligned} T_1(x, y, t) &:= \partial_{y_j} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) D_z^{\beta+2} \delta_{il} \Gamma(x - z, t) dz, \\ T_2(x, y, t) &:= -\partial_{y_j} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) D_z^{\beta+2} \delta_{il} \Gamma(x - z^*, t) dz, \\ T_3(x, y, t) &:= \partial_{y_j} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \\ &\cdot D_z^{\beta+2} \left[4 \partial_{x_l} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x - w) \Gamma(w - z^*, t) dw \right] dz. \end{aligned} \quad (50)$$

To estimate T_1 , let us proceed by writing

$$\begin{aligned} T_1(x, y, t) &= \partial_{y_j} \int_{\mathbb{R}_+^n} \partial_{y_k} [N(z-y) + N(z-y^*)] \\ &\quad \cdot D_{\bar{z}}^{\beta+2} \delta_{il} \Gamma(x-z, t) dz = \partial_{y_j} \int_{\mathbb{R}^n} \partial_{y_k} N(z-y) 1_{\{z_n > 0\}} \\ &\quad \cdot D_{\bar{z}}^{\beta+2} \delta_{il} \Gamma(x-z, t) dz + \partial_{y_j} \int_{\mathbb{R}^n} \partial_{y_k} N(z-y) 1_{\{z_n < 0\}} \\ &\quad \cdot D_{\bar{z}}^{\beta+2} \delta_{il} \Gamma(x-z^*, t) dz. \end{aligned} \quad (51)$$

It is well known that $1_{\mathbb{R}_+^n} D_{\bar{z}}^{\beta+2} \Gamma(x-z, t)$ and $1_{\mathbb{R}_-^n} D_{\bar{z}}^{\beta+2} \Gamma(x-z^*, t)$ are in Hardy space $\mathcal{H}^1(\mathbb{R}^n)$, for any fixed $z \in \mathbb{R}^n$. Since the Calderon-Zygmund type transforms are bounded in Hardy space, we obtain that

$$\begin{aligned} \int_{\mathbb{R}_+^n} |T_1(x, y, t)| dy &\leq C \left\| 1_{\mathbb{R}_+^n} D_{\bar{z}}^{\beta+2} \Gamma_t(z-\cdot) \right\|_{\mathcal{H}^1(\mathbb{R}^n)} \\ &\quad + C \left\| 1_{\mathbb{R}_-^n} D_{\bar{z}}^{\beta+2} \Gamma_t(z-\cdot) \right\|_{\mathcal{H}^1(\mathbb{R}^n)}. \end{aligned} \quad (52)$$

Using the estimates of Proposition 4, we arrive at

$$\int_{\mathbb{R}_+^n} |T_1(x, y, t)| dy \leq C t^{-(j+1)/2}. \quad (53)$$

With exactly the same argument as for T_1 , we also obtain

$$\int_{\mathbb{R}_+^n} |T_2(x, y, t)| dy \leq C t^{-(j+1)/2}. \quad (54)$$

It remains to obtain an estimate for T_3 . We use Proposition 3 for $i, l \neq n$ by replacing f by $\Gamma(w, \cdot)$ and also use $G_{ln} = 0$ to rewrite T_3 as

$$\begin{aligned} T_3(x, y, t) &= 4\partial_{y_j} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \\ &\quad \cdot D_{\bar{z}}^{\beta+2} \left[-\frac{\delta_{il}}{2n} \Gamma(x-z^*, t) + \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i x_l}^2 N(x-w) \right. \\ &\quad \cdot \Gamma(w-z^*, t) dw \Big] dz = -\frac{\delta_{il}}{2n} \partial_{y_j} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \\ &\quad \cdot D_{\bar{z}}^{\beta+2} \Gamma(x-z^*, t) dz + \partial_{y_j} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \\ &\quad \cdot \partial_{x_i x_l}^2 N(x-w) D_{\bar{z}}^{\beta+2} \Gamma(w-z^*, t) dw dz := T_3^* + T_3^{**}. \end{aligned} \quad (55)$$

By the same argument as for T_1 , we can obtain

$$\int_{\mathbb{R}_+^n} |T_3^*(x, y, t)| dy \leq C t^{-(j+1)/2}. \quad (56)$$

Let us rewrite T_3^{**} as

$$\begin{aligned} T_3^{**}(x, y, t) &= \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i x_l}^2 N(x-w) \\ &\quad \cdot \left[\partial_{y_j} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) D_{\bar{z}}^{\beta+2} \Gamma(w-z^*, t) dz \right] dw. \end{aligned} \quad (57)$$

Set

$$T_{jk}(w, y, t) := \partial_{y_j} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) D_{\bar{z}}^{\beta+2} \Gamma(w-z^*, t) dz. \quad (58)$$

By Proposition 5,

$$\begin{aligned} \int_{\mathbb{R}_+^n} |T_3^{**}(x, y, t)| dy &\leq C \sup_{w \in \mathbb{R}_+^n} \int_{\mathbb{R}_+^n} |T_{jk}(w, y, t)| dy \\ &\quad + C x_n^\alpha \sup_{x_n/2 < w_n < x_n} \sup_{\bar{w} \in \mathbb{R}^{n-1}} \frac{\int_{\mathbb{R}_+^n} |T_{jk}((\bar{w}, w_n), y, t) - T_{jk}((\bar{x}, w_n), y, t)| dy}{|\bar{x} - \bar{w}|^\alpha}. \end{aligned} \quad (59)$$

Notice that

$$\begin{aligned} T_{jk}(w, y, t) &= \partial_{y_j} \int_{\mathbb{R}_+^n} \partial_{y_k} [N(z-y) + N(z-y^*)] \\ &\quad \cdot D_{\bar{z}}^{\beta+2} \Gamma(w-z^*, t) dz = \partial_{y_j} \int_{\mathbb{R}^n} \partial_{y_k} N(z-y) 1_{\{z_n > 0\}} \\ &\quad \cdot D_{\bar{z}}^{\beta+2} \Gamma(w-z^*, t) dz + \partial_{y_j} \int_{\mathbb{R}^n} \partial_{y_k} N(z-y) 1_{\{z_n < 0\}} \\ &\quad \cdot D_{\bar{z}}^{\beta+2} \Gamma(w-z, t) dz. \end{aligned} \quad (60)$$

We also recall that $1_{\mathbb{R}_+^n} D_{\bar{w}}^{\beta+2} \Gamma(w-z^*, t)$ and $1_{\mathbb{R}_-^n} D_{\bar{w}}^{\beta+2} \Gamma(w-z, t)$ are in the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$, for any fixed $w \in \mathbb{R}^n$. Since the Calderon-Zygmund type transforms are bounded in Hardy space, after using $D_{\bar{w}}^{\beta+2} \Gamma(w-z, t) = (-1)^{|\beta|+2} D_{\bar{z}}^{\beta+2} \Gamma(w-z, t)$, we arrive at

$$\begin{aligned} \int_{\mathbb{R}_+^n} |T_{jk}(w, y, t)| dy &\leq C \left\| 1_{\mathbb{R}_+^n} D_{\bar{w}}^{\beta+2} \Gamma(w-\cdot, t) \right\|_{\mathcal{H}^1(\mathbb{R}^n)} \\ &\quad + \left\| 1_{\mathbb{R}_-^n} D_{\bar{w}}^{\beta+2} \Gamma(w-\cdot, t) \right\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C t^{-(j+1)/2}. \end{aligned} \quad (61)$$

Let us recall a result of Proposition 4:

$$1_{\mathbb{R}_+^n} D_{\bar{w}}^{\beta+2} [(\tau_a * \Gamma_t) - (\tau_b * \Gamma_t)] \in \mathcal{H}^1(\mathbb{R}^n), \quad (62)$$

for any $a, b \in \mathbb{R}^n$, and is bounded by $C t^{-(|\beta|+2+\gamma)/2} e^{-w_n^2/4t} |\bar{w} - \bar{x}|^\gamma$ for $0 < \gamma < 1$. Hence, in similar way as for P_{jk} , we obtain

$$\int_{\mathbb{R}_+^n} |P_{jk}((\bar{w}, w_n), y, t) - P_{jk}((\bar{x}, w_n), y, t)| dy \leq C t^{-(|\beta|+2+\gamma)/2} e^{-\frac{w_n^2}{4t}} |\bar{w} - \bar{x}|^\gamma. \quad (63)$$

Therefore,

$$\int_{\mathbb{R}_+^n} |T_3^{**}(x, y, t)| dy \leq Ct^{-(|\beta|+2)/2} + Ct^{-(|\beta|+2+\gamma)/2} x_n^\gamma e^{-\frac{\gamma^2}{4t}} \leq Ct^{-(j+1)/2}. \quad (64)$$

Using (56) and (64) leads us to obtain

$$\int_{\mathbb{R}_+^n} |T_3(x, y, t)| dy \leq Ct^{-(j+1)/2}. \quad (65)$$

Since $T = T_1 + T_2 + T_3$, with the use of (53), (54), and (65), we obtain

$$|I_2(x, t)| \leq |g|_\infty \int_{\mathbb{R}_+^n} |T(x, y, t)| dy \leq Ct^{-(j+1)/2} |g|_\infty. \quad (66)$$

Finally, using (45) and (66) with fact that e^{-At} commutes with D_x^β , we have proved the following important lemma.

Lemma 6. For any $g = (g)_{ij}$, $1 \leq i, j \leq n$ with $g_{ij} \in L^\infty(\mathbb{R}_+^n)$, and $g_{ij}(\bar{x}, 0) = 0$, there exists a constant C independent of t and g such that

$$|D^j e^{-At} \mathbb{P}(\nabla \cdot g)|_\infty \leq Ct^{-(j+1)/2} |g|_\infty, \quad (67)$$

for $0 < t < T$, for some $T > 0$.

Corollary 7. Let g be as in the previous lemma, then the solution of

$$u_t + Au = \mathbb{P}\nabla \cdot g, \quad u|_{t=0} = 0, \quad u|_{x_n=0} = 0 \quad (68)$$

satisfies

$$|u(t)|_\infty \leq Ct^{1/2} \max_{0 \leq s \leq t} |g(s)|_\infty, \quad 0 < t < T, \quad (69)$$

for some $T > 0$.

Proof. The solution of (38) is given by

$$u(t) = \int_0^t e^{-A(t-s)} \mathbb{P}\nabla \cdot g(s) ds, \quad 0 < t < T \quad (70)$$

and

$$|u(t)|_\infty \leq \int_0^t |e^{-A(t-s)} \mathbb{P}\nabla \cdot g(s)|_\infty ds. \quad (71)$$

Applying the estimate (24), we obtain

$$|u(t)|_\infty \leq \max_{0 \leq s \leq t} |g(s)|_\infty \int_0^t (t-s)^{-1/2} ds. \quad (72)$$

Hence, we obtain

$$|u(t)|_\infty \leq Ct^{1/2} \max_{0 \leq s \leq t} |g(s)|_\infty. \quad (73)$$

4. Estimates for the Navier-Stokes Equations

Recall the transformed abstract ordinary differential equation (3):

$$u_t + Au = -\mathbb{P}(u \cdot \nabla u). \quad (74)$$

Solution of (74) with given initial and boundary condition as in (1) is given by

$$u(t) = e^{-At} f - \int_0^t e^{-A(t-s)} \mathbb{P}(u \cdot \nabla u)(s) ds. \quad (75)$$

Using the solution (75) along with the use of estimates (23), (24), and (25), we prove the following important lemma.

Lemma 8. Set

$$V(t) = |u(t)|_\infty + t^{1/2} |\mathcal{D}u(t)|_\infty, \quad 0 < t < T. \quad (76)$$

There is a constant $C > 0$, independent of t and f , so that

$$V(t) \leq C|f|_\infty + Ct^{1/2} \max_{0 \leq s \leq t} V^2(s), \quad 0 < t < T. \quad (77)$$

Proof. Using estimate (23) for the solution of the Stokes equations in (75), we obtain

$$|u(t)|_\infty \leq |f|_\infty + \left| \int_0^t e^{-A(t-s)} (\mathbb{P}\nabla \cdot g)(s) ds \right|_\infty. \quad (78)$$

From (74), after using estimate (24), with the fact that g is quadratic in u gives us

$$\begin{aligned} |u(t)|_\infty &\leq |f|_\infty + C \int_0^t (t-s)^{-1/2} |u(s)|_\infty^2 ds = |f|_\infty \\ &+ C \int_0^t (t-s)^{-1/2} s^{-1/2} s^{1/2} |u(s)|_\infty^2 ds \leq |f|_\infty \\ &+ C \max_{0 \leq s \leq t} \{s^{1/2} |u(s)|_\infty^2\} \int_0^t (t-s)^{-1/2} s^{-1/2} ds. \end{aligned} \quad (79)$$

Since $\int_0^t (t-s)^{-1/2} s^{-1/2} ds = C > 0$, which is independent of t , we arrive at the following estimate

$$|u(t)|_\infty \leq |f|_\infty + C \max_{0 \leq s \leq t} \{s^{1/2} |u(s)|_\infty^2\}, \quad (80)$$

$$|u(t)|_\infty \leq |f|_\infty + Ct^{1/2} \max_{0 \leq s \leq t} V^2(s). \quad (81)$$

Next, apply D_i to $u(t)$ in the integral form to obtain and

estimate for $D_i u =: v$:

$$\begin{aligned} v(t) &= D_i e^{-At} f - D_i \int_0^t e^{-A(t-s)} \mathbb{P} \nabla \cdot g(u(s)) ds = D_i e^{-At} f \\ &\quad - \int_0^t D_i e^{-A(t-s)} \mathbb{P} \nabla \cdot g(u(s)) ds. \end{aligned} \quad (82)$$

Let us estimate the integral in the above expression as below.

$$\left| \int_0^t D_i e^{-A(t-s)} (\mathbb{P} \nabla \cdot g)(s) ds \right| \leq \int_0^t |D_i e^{-A(t-s)} (\mathbb{P} \nabla \cdot g)(s)| ds. \quad (83)$$

We use the estimate (25) again with the fact that g is quadratic in u to obtain

$$\begin{aligned} \left| \int_0^t D_i e^{-A(t-s)} (\mathbb{P} \nabla \cdot g)(s) ds \right| &\leq C \int_0^t (t-s)^{-1/2} |u(s)|_\infty |\mathcal{D}u(s)|_\infty \\ &\quad \cdot ds = C \int_0^t (t-s)^{-1/2} s^{-1/2} s^{1/2} |u(s)|_\infty |\mathcal{D}u(s)|_\infty \\ &\quad \cdot ds \leq C \max_{0 \leq s \leq t} \{s^{1/2} |u(s)|_\infty |\mathcal{D}u(s)|_\infty\} \int_0^t (t-s)^{-1/2} s^{-1/2} \\ &\quad \cdot ds \leq C \max_{0 \leq s \leq t} \{ |u(s)|_\infty^2 + s |\mathcal{D}u(s)|_\infty^2 \}. \end{aligned} \quad (84)$$

Therefore, we have the following estimate for $v = D_i u$

$$|v(t)|_\infty \leq C t^{-1/2} |f|_\infty + C \max_{0 \leq s \leq t} \{ |u(s)|_\infty^2 + s |\mathcal{D}u(s)|_\infty^2 \}, \quad (85)$$

$$t^{1/2} |\mathcal{D}u(t)|_\infty \leq C |f|_\infty + C t^{1/2} \max_{0 \leq s \leq t} V^2(t). \quad (86)$$

The combination of (80) and (85), proves Lemma 8.

Lemma 9. Let C and $u \in L^\infty(\mathbb{R}_+^n \times (0, T))$ be same as in Lemma 8 for some $T > 0$. Set

$$c_0 = \frac{1}{16C^4}. \quad (87)$$

Then $T > c_0/|f|_\infty^2$ and

$$|u(t)|_\infty + t^{1/2} |\mathcal{D}u(t)|_\infty < 2C |f|_\infty \quad \text{for } 0 \leq t < \frac{c_0}{|f|_\infty^2}. \quad (88)$$

Proof. We prove this lemma by contradiction after recalling the definition of $V(t)$ in (76). Suppose that (88) does not hold, then denote by t_0 the smallest time with $V(t_0) = 2C |f|_\infty$. Use (77) to obtain

$$2C |f|_\infty = V(t_0) \leq C |f|_\infty + C t_0^{1/2} 4C^2 |f|_\infty^2. \quad (89)$$

Thus

$$1 \leq 4C^2 t_0^{1/2} |f|_\infty^2. \quad (90)$$

Therefore, $t_0 \geq c_0/|f|_\infty^2$. This contradiction proves (88) and $T > c_0/|f|_\infty^2$.

5. Proof of Theorem 1

Lemma 9 proves Theorem 1 for $j = 0, 1$ for $0 < t < c_0/|f|_\infty^2$. Now, we apply induction on j to prove Theorem 1. Suppose $j \geq 1$ and assume

$$t^{k/2} |\mathcal{D}^k u(t)|_\infty \leq K_k |f|_\infty, \quad \text{for } 0 \leq t \leq \frac{c_0}{|f|_\infty^2}, \quad 0 \leq k \leq j-1. \quad (91)$$

Apply $D^\beta := D_x^\beta$ to $u_t + Au = -\mathbb{P} \nabla \cdot g$ with the fact that $\mathbb{P}$ commutes with D^β . Also let $D^\beta u =: v$ to obtain

$$v_t + Av = -D^\beta \mathbb{P} \nabla \cdot g \quad \text{with} \quad v|_{\partial \mathbb{R}_+^n} = 0, \quad \nabla \cdot v = 0. \quad (92)$$

The solution of above system can be written as

$$v(t) = D^\beta e^{-At} f - \int_0^t e^{-A(t-s)} D^\beta \mathbb{P} \nabla \cdot g(u)(s) ds. \quad (93)$$

Since $\nabla \cdot v = 0$, we can write

$$\partial_{x_n} v(t) = - \sum_{i=1}^{n-1} \partial_{x_i} v(t). \quad (94)$$

Using integral form of $v(t)$ from above, we can write

$$\partial_{x_n} v(t) = - \sum_{i=1}^{n-1} \partial_{x_i} \left[D^\beta e^{-At} f - \int_0^t e^{-A(t-s)} D^\beta \mathbb{P} \nabla \cdot g(u)(s) ds \right]. \quad (95)$$

Our goal is to prove $|\partial_{x_n} v(t)|_\infty \leq C t^{-j/2} |f|_\infty$. For that, let us start with the following where $i \neq n$.

$$|\partial_{x_n} v(t)|_\infty \leq C |D^{\beta+1} e^{-At} f|_\infty + C \left| \partial_{x_i} \int_0^t e^{-A(t-s)} D^\beta \mathbb{P} \nabla \cdot g(u)(s) ds \right|_\infty. \quad (96)$$

Using the estimate (23) in the first term of the above expression, we obtain

$$\begin{aligned} |\partial_{x_n} v(t)|_\infty &\leq C t^{-j/2} \\ &\quad \cdot |f|_\infty + C \left| \partial_{x_i} \int_0^{t/2} e^{-A(t-s)} D^\beta \mathbb{P} \nabla \cdot g(u)(s) ds \right|_\infty + C \left| \partial_{x_i} \int_{t/2}^t e^{-A(t-s)} \right. \\ &\quad \cdot D^\beta \mathbb{P} \nabla \cdot g(u)(s) ds \left. \right|_\infty \end{aligned} \quad (97)$$

$$\leq Ct^{-j/2} \|f\|_\infty + I_1 + I_2, \quad (98)$$

where

$$\begin{aligned} I_1(x, t) &:= C \left| \partial_{x_i} \int_0^{t/2} e^{-A(t-s)} D^\beta \mathbb{P} \nabla \cdot g(u)(s) ds \right|_\infty, \\ I_2(x, t) &:= C \left| \partial_{x_i} \int_{t/2}^t e^{-A(t-s)} D^\beta \mathbb{P} \nabla \cdot g(u)(s) ds \right|_\infty. \end{aligned} \quad (99)$$

To estimate I_1 uniformly, we proceed as

$$\left| I_1 \right|_\infty \leq C \int_0^{t/2} \left| \partial_{x_i} e^{-A(t-s)} D^\beta \mathbb{P} \nabla \cdot g(u)(s) \right|_\infty ds. \quad (100)$$

Using Lemma 6, we obtain

$$\left| I_1 \right|_\infty \leq C \int_0^{t/2} (t-s)^{-(j+1)/2} \|g(u(s))\|_\infty ds. \quad (101)$$

We use simple integration, and the fact that g is quadratic in u to arrive at

$$\left| I_1 \right|_\infty \leq Ct^{(1-j)/2} \|f\|_\infty^2. \quad (102)$$

Next, we estimate I_2 . For that, we proceed in the following way:

$$\left| I_2 \right|_\infty \leq C \int_{t/2}^t \left| \partial_{x_i} e^{-A(t-s)} D^\beta \mathbb{P} \nabla \cdot g(u)(s) \right|_\infty ds. \quad (103)$$

Since the order of the derivatives of $|D^\beta \nabla \cdot g|_\infty$ is $|\beta| + 1$, for convenience in writing, we use $|D^j g|_\infty$ to estimate $|D^\beta \nabla \cdot g|_\infty$. Since $g(u)$ is quadratic in u ; therefore

$$\left| D^j g(u) \right|_\infty \leq C \|u\|_\infty \left| \mathcal{D}^j u \right|_\infty + \sum_{k=1}^{j-1} \left| \mathcal{D}^k u \right|_\infty \left| \mathcal{D}^{j-k} u \right|_\infty. \quad (104)$$

By induction hypothesis (91) we obtain

$$\sum_{k=1}^{j-1} \left| \mathcal{D}^k u(s) \right|_\infty \left| \mathcal{D}^{j-k} u(s) \right|_\infty \leq Cs^{-j/2} \|f\|_\infty^2. \quad (105)$$

Apply estimate (25) to the integral (103) with the use of (104) to obtain

$$\begin{aligned} \left| I_2(t) \right|_\infty &\leq C \int_{t/2}^t (t-s)^{-1/2} \\ &\cdot \left(C \|u(s)\|_\infty \left| \mathcal{D}^j u(s) \right|_\infty + \sum_{k=1}^{j-1} \left| \mathcal{D}^k u(s) \right|_\infty \left| \mathcal{D}^{j-k} u(s) \right|_\infty \right) ds = I_1 + I_2, \end{aligned} \quad (106)$$

where

$$\begin{aligned} J_1(x, t) &:= C \int_{t/2}^t (t-s)^{-1/2} \|u(s)\|_\infty \left| \mathcal{D}^j u(s) \right|_\infty ds, \\ J_2(x, t) &:= C \sum_{k=1}^{j-1} \int_{t/2}^t (t-s)^{-1/2} \left| \mathcal{D}^k u(s) \right|_\infty \left| \mathcal{D}^{j-k} u(s) \right|_\infty ds. \end{aligned} \quad (107)$$

Since $\int_{t/2}^t (t-s)^{-1/2} s^{-j/2} ds = Ct^{(1-j)/2}$, where C is independent of t , and the using the estimate of (105), we obtain $|J_2(t)|_\infty \leq C \|f\|_\infty^2 t^{(1-j)/2}$. For J_1 , let us begin as below.

$$\begin{aligned} \left| J_1(t) \right|_\infty &\leq C \int_{t/2}^t (t-s)^{-1/2} \|u(s)\|_\infty \left| \mathcal{D}^j u(s) \right|_\infty \\ &\cdot ds \leq C \|f\|_\infty \int_{t/2}^t (t-s)^{-1/2} s^{-j/2} s^{j/2} \left| \mathcal{D}^j u(s) \right|_\infty \\ &\cdot ds \leq C \|f\|_\infty t^{(1-j)/2} \max_{0 \leq s \leq t} \left\{ s^{j/2} \left| \mathcal{D}^j u(s) \right|_\infty \right\}. \end{aligned} \quad (108)$$

Therefore

$$\begin{aligned} \left| I_2(t) \right|_\infty &\leq \left| J_1(t) \right|_\infty + \left| J_2(t) \right|_\infty, \\ \left| I_2(t) \right|_\infty &\leq Ct^{(1-j)/2} \|f\|_\infty^2 + C \|f\|_\infty t^{(1-j)/2} \max_{0 \leq s \leq t} \left\{ s^{j/2} \left| \mathcal{D}^j u(s) \right|_\infty \right\}. \end{aligned} \quad (109)$$

We use these bounds to bind the integral in (97). We have $D^j u = \partial_{x_n} D^\beta u$. Then, maximizing the resulting estimate for $t^{j/2} |D^j u(t)|_\infty$ over all derivatives D^j of order j and setting

$$\phi(t) := t^{j/2} \left| \mathcal{D}^j u(t) \right|_\infty, \quad (110)$$

and from (98), we obtain the following estimate:

$$\phi(t) \leq C \|f\|_\infty + Ct^{1/2} \|f\|_\infty^2 + C \|f\|_\infty t^{1/2} \max_{0 \leq s \leq t} \phi(s) \quad \text{for } 0 \leq t \leq \frac{c_0}{\|f\|_\infty^2}. \quad (111)$$

Since $t^{1/2} \|f\|_\infty \leq \sqrt{c_0}$, then $Ct^{1/2} \|f\|_\infty^2 \leq C\sqrt{c_0} \|f\|_\infty$. Therefore

$$\phi(t) \leq C_j \|f\|_\infty + C_j \|f\|_\infty t^{1/2} \max_{0 \leq s \leq t} \phi(s) \quad \text{for } 0 \leq t \leq c_0 / \|f\|_\infty^2. \quad (112)$$

Let us fix C_j so that the above estimate holds and set

$$c_j = \min \left\{ c_0, \frac{1}{4C_j^2} \right\}. \quad (113)$$

First, let us prove the following:

$$\phi(t) < 2C_j|f|_\infty \quad \text{for} \quad 0 \leq t < \frac{c_j}{|f|_\infty^2}. \quad (114)$$

Suppose there is a smallest time t_0 such that $0 < t_0 < c_j/|f|_\infty^2$ with $\phi(t_0) = 2C_j|f|_\infty$. Then, using (88), we obtain

$$2C_j|f|_\infty = \phi(t_0) \leq C_j|f|_\infty + 2C_j^2|f|_\infty^2 t_0^{1/2}. \quad (115)$$

Thus

$$1 \leq 2C_j|f|_\infty t_0^{1/2} \quad \text{gives} \quad t_0 \geq c_j/|f|_\infty^2, \quad (116)$$

which contradicts the assertion. Therefore, we proved the estimate

$$t^{j/2} |\mathcal{D}^j u(t)|_\infty \leq 2C_j|f|_\infty \quad \text{for} \quad 0 \leq t \leq c_j/|f|_\infty^2. \quad (117)$$

If

$$T_j := \frac{c_j}{|f|_\infty^2} < t \leq \frac{c_0}{|f|_\infty^2} =: T_0, \quad (118)$$

then we start the corresponding estimate at $t - T_j$. Using Lemma 9, we have $|u(t - T_j)|_\infty \leq 2|f|_\infty$ and obtain

$$T_j^{j/2} |\mathcal{D}^j u(t)|_\infty \leq 4C_j|f|_\infty. \quad (119)$$

Finally, for any t satisfying (118)

$$t^{j/2} \leq T_0^{j/2} = \left(\frac{c_0}{c_j}\right)^{j/2} T_j^{j/2}, \quad (120)$$

and (119) yield

$$t^{j/2} |\mathcal{D}^j u(t)|_\infty \leq 4C_j \left(\frac{c_0}{c_j}\right)^{j/2} |f|_\infty. \quad (121)$$

This completes the proof of Theorem 1.

In the following appendices, we provide proofs of the propositions that are introduced in ‘‘Some Auxiliary Results.’’ However, these proofs have also been provided in [8]. For the reader’s convenience, we provide them with more details in this paper as well.

Appendix

A. Proof of Proposition 2

We first let the case $k \neq n$. Differentiate G_{ij} with respect to x_k to obtain

$$\begin{aligned} \partial_{x_k} G_{ij}(x, y, t) &= \delta_{ij} \partial_{x_k} [\Gamma_t(x - y) - \Gamma_t(x - y^*)] \\ &\quad + 4(1 - \delta_{jn}) \partial_{x_k} \partial_{x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x - z) \Gamma_t(z - y^*) \\ &\quad \cdot dz = \delta_{ij} \partial_{x_k} [\Gamma_t(x - y) - \Gamma_t(x - y^*)] + 4(1 - \delta_{jn}) \\ &\quad \cdot \partial_{x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x - z) \partial_{z_k} \Gamma_t(z - y^*) dz. \end{aligned} \quad (A.1)$$

Observe that for $k \neq n$

$$\begin{aligned} \partial_{y_k} \Gamma_t(x - y) &= -\partial_{x_k} \Gamma_t(x - y), \\ \partial_{y_k} \Gamma_t(x - y^*) &= -\partial_{x_k} \Gamma_t(x - y^*). \end{aligned} \quad (A.2)$$

This proves the desired result of Proposition 2 for $k \neq n$. For the case $k = n$. We start with the expression for some appropriately chosen function g :

$$\begin{aligned} \partial_{x_n} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x - z) g(z) dz &= \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} \partial_{x_n} N(x - z) g(z) \\ &\quad \cdot dz + \frac{\delta_{in}}{2n} g(x) + \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(\bar{x} - \bar{z}, 0) g(\bar{z}, x_n) d\bar{z}, \end{aligned} \quad (A.3)$$

and

$$\begin{aligned} -\int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} \partial_{x_n} N(x - z) g(z) dz &= \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} \partial_{z_n} N(x - z) \\ &\quad \cdot g(z) dz = -\int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x - z) \partial_{z_n} g(z) dz + \frac{\delta_{in}}{2n} g(x) \\ &\quad + \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(\bar{x} - \bar{z}, 0) g(\bar{z}, x_n) \\ &\quad \cdot d\bar{z} - \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(\bar{x} - \bar{z}, z_n) g(\bar{z}, 0) d\bar{z}. \end{aligned} \quad (A.4)$$

Therefore, we arrive at

$$\begin{aligned} \partial_{x_n} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x - z) \Gamma(z - y^*, t) \\ \cdot dz &= \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x - z) \partial_{z_n} \Gamma(z - y^*, t) \\ \cdot dz &+ \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(\bar{x} - \bar{z}, x_n) \Gamma(\bar{z} - \bar{y}, y_n, t) d\bar{z}. \end{aligned} \quad (A.5)$$

Since

$$\begin{aligned}\partial_{y_n} \Gamma_t(x-y) &= -\partial_{x_n} \Gamma_t(x-y), \\ \partial_{y_n} \Gamma_t(x-y^*) &= \partial_{x_n} \Gamma_t(x-y^*),\end{aligned}\quad (\text{A.6})$$

therefore, after differentiating G_{ij} with respect to x_n variable, we have

$$\begin{aligned}\partial_{x_n} G_{ij}(x, y, t) &= -\delta_{ij} \partial_{y_n} [\Gamma(x-y, t) + \Gamma(x-y^*, t)] \\ &\quad + 4(1-\delta_{jn}) \partial_{y_n} \partial_{x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x-z) \\ &\quad \cdot \Gamma(z-y^*, t) dz + 4(1-\delta_{jn}) \\ &\quad \cdot \partial_{x_j} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(\bar{x}-\bar{z}, x_n) \Gamma(\bar{z}-\bar{y}, y_n, t) \\ &\quad \cdot d\bar{z} = \partial_{y_n} G_{ij}(x, y, t) - 2\delta_{ij} \partial_{y_n} \Gamma(x-y, t) \\ &\quad + 4(1-\delta_{jn}) \partial_{x_j} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(\bar{x}-\bar{z}, x_n) \\ &\quad \cdot \Gamma(\bar{z}-\bar{y}, y_n, t) d\bar{z}.\end{aligned}\quad (\text{A.7})$$

B. Proof of Proposition 3

Define a smooth cut-off function ϕ such that $\phi(r) = 1$ if $0 \leq r \leq 1$ and 0 if $r \geq 2$ with $\int_0^2 \phi(r) dr = 1$. For $x \in \mathbb{R}_+^n$ and $\varepsilon < x_n/2$, also define

$$\phi_{\varepsilon, x}(z) = \phi\left(\frac{|x-z|}{\varepsilon}\right). \quad (\text{B.1})$$

Then, $\phi_{\varepsilon, x}$ is compactly supported in $\mathbb{R}_+^n$. Let us define

$$v_\varepsilon(x) = \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \phi_{\varepsilon, x}(z) \partial_{x_i} N(x-z) f(z) dz. \quad (\text{B.2})$$

Differentiating with respect to x_j for $j \neq n$ yields

$$\partial_{x_j} v_\varepsilon(x) = \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \left[\partial_{x_j} \phi_{\varepsilon, x}(z) \partial_{x_i} N(x-z) + \phi_{\varepsilon, x}(z) \partial_{x_i} \partial_{x_j} N(x-z) \right] f(z) dz. \quad (\text{B.3})$$

If $j = n$, we have

$$\begin{aligned}\partial_{x_n} v_\varepsilon(x) &= \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \left[\partial_{x_n} \phi_{\varepsilon, x}(z) \partial_{x_i} N(x-z) \right. \\ &\quad \left. + \phi_{\varepsilon, x}(z) \partial_{x_i} \partial_{x_n} N(x-z) \right] f(z) \\ &\quad \cdot dz + \int_{\mathbb{R}^{n-1}} \phi_{\varepsilon, x}(\bar{z}, x_n) \partial_{x_i} N(\bar{x}-\bar{z}, 0) f(\bar{z}, x_n) d\bar{z}.\end{aligned}\quad (\text{B.4})$$

Let us set

$$\begin{aligned}I_{1, \varepsilon}(x) &:= \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_j} \phi_{\varepsilon, x}(z) \partial_{x_i} N(x-z) f(z) dz \text{ and } I_{2, \varepsilon} \\ &:= \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \phi_{\varepsilon, x}(z) \partial_{x_i x_j}^2 N(x-z) f(z) dz.\end{aligned}\quad (\text{B.5})$$

It is clear that

$$\lim_{\varepsilon \rightarrow 0} I_{2, \varepsilon} = \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i x_j}^2 N(x-z) f(z) dz. \quad (\text{B.6})$$

Let us denote $I_{1, \varepsilon} = I_{1, \varepsilon}^* + I_{1, \varepsilon}^{**}$ where

$$I_{1, \varepsilon}^*(x) := \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_j} \phi_{\varepsilon, x}(z) \partial_{x_i} N(x-z) [f(z) - f(x)] dz, \quad (\text{B.7})$$

and

$$I_{1, \varepsilon}^{**} := f(x) \int_{\mathbb{R}^{n-1}} \partial_{x_j} \phi_{\varepsilon, x}(z) \partial_{x_i} N(x-z) dz. \quad (\text{B.8})$$

Observe that $I_{1, \varepsilon}^*(x) \rightarrow 0$ as $\varepsilon \rightarrow 0$, since

$$\begin{aligned}|I_{1, \varepsilon}^*(x)| &\leq \int_{|x-z| \leq \varepsilon} \left| \partial_{x_j} \phi_{\varepsilon, x}(z) \right| \left| \partial_{x_i} N(x-z) \right| |f(z) - f(x)| \\ &\quad \cdot dz \leq \int_{|x-z| \leq \varepsilon} \frac{1}{\varepsilon} \frac{1}{|x-z|^{n-1}} |x-z|^\alpha dz [f]_\alpha \leq \varepsilon^\alpha [f]_\alpha,\end{aligned}\quad (\text{B.9})$$

where

$$[f]_\alpha = \sup_{x, z} \frac{|f(x) - f(z)|}{|x - z|^\alpha}. \quad (\text{B.10})$$

Next, we will show that

$$\begin{aligned}\int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_j} \phi_{\varepsilon, x}(z) \partial_{x_i} N(x-z) \\ \cdot dz = \begin{cases} -\frac{\delta_{ij}}{2n} & \text{if } i, j \neq n \text{ or } i = j = n \\ 0 & \text{if } i = n, j \neq n \text{ or } j = n, i \neq n. \end{cases}\end{aligned}\quad (\text{B.11})$$

Let us apply a change of variables and let $\varepsilon \rightarrow 0$, we get

$$\begin{aligned}\int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_j} \phi_{\varepsilon, x}(z) \partial_{x_i} N(x-z) \\ \cdot dz = \frac{1}{\varepsilon} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{(z_i - x_i)(z_j - x_j)}{n\omega_n |x-z|^{n+1}} \phi' \left(\frac{|x-z|}{\varepsilon} \right) \\ \cdot dz \rightarrow \int_0^\infty \int_{\mathbb{R}^{n-1}} \frac{w_i w_j}{n\omega_n |w|^{n+1}} \phi'(|w|) dw.\end{aligned}\quad (\text{B.12})$$

If $i, j \neq n$ or $i = j = n$, then by the symmetry of $(w_i w_j / |w|^{n+1}) \phi'(|w|)$ in terms of w_n variables, we obtain

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^{n-1}} \frac{w_i w_j}{n \omega_n |w|^{n+1}} \phi'(|w|) dw &= \frac{1}{2} \int_{\mathbb{R}^n} \frac{w_i w_j}{n \omega_n |w|^{n+1}} \phi'(|w|) dw \\ &= \frac{1}{2n \omega_n} \int_0^2 \int_{\mathbb{S}^{n-1}} r w_i w_j \phi'(r) dS_w dr = -\frac{1}{2n} \delta_{ij} \int_0^2 \phi'(r) dr \\ &= -\frac{1}{2n} \delta_{ij}. \end{aligned} \quad (\text{B.13})$$

If $i = n, j \neq n$ or $j = n, i \neq n$, then by the antisymmetry of $(w_i w_j / |w|^{n+1}) \phi'(|w|)$ in terms of w_n variables, we obtain

$$\int_0^\infty \int_{\mathbb{R}^{n-1}} \frac{w_i w_j}{|w|^{n+1}} \phi'(|w|) dw = 0. \quad (\text{B.14})$$

This completes the proof of Proposition 3.

C. Proof of Proposition 5

Denote

$$\begin{aligned} I &= \int_0^{x_n/2} \int_{\mathbb{R}^{n-1}} \partial_{x_i x_j}^2 N(x-z) f(z, y) dz, \\ II &= \int_{x_n/2}^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i x_j}^2 N(x-z) [f(z, y) - f(\bar{x}, z_n, y)] dz, \quad \text{and} \\ III &= \int_{x_n/2}^{x_n} f(\bar{x}, z_n, y) \int_{\mathbb{R}^{n-1}} \partial_{x_i x_j}^2 N(x-z) d\bar{z} dz_n. \end{aligned} \quad (\text{C.1})$$

Then

$$\int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i x_j}^2 N(x-z) f(z, y) dz = I + II + III. \quad (\text{C.2})$$

Notice that

$$|\partial_{x_i x_j}^2 N(x-z)| \leq C|x-z|^{-n}, \quad \int_{\mathbb{R}^{n-1}} |x-z|^{-n} d\bar{z} = C|x_n - z_n|^{-1}. \quad (\text{C.3})$$

Then, we have

$$\begin{aligned} \int_{\mathbb{R}_+^n} I dy &\leq C \left(\int_0^{x_n/2} \frac{1}{x_n - z_n} dz_n \right) \left(\sup_{z \in \mathbb{R}_+^n} |f(z, y)| dy \right) \\ &\leq C \sup_{z \in \mathbb{R}_+^n} \int_{\mathbb{R}_+^n} |f(z, y)| dy. \end{aligned} \quad (\text{C.4})$$

Since, for $0 < \alpha < 1$

$$\begin{aligned} |\partial_{x_i x_j}^2 N(x-z) [f(z, y) - f(\bar{x}, z_n, y)]| \\ \leq C|x-z|^\alpha |x-z|^{-n} \frac{|f(\bar{z}, z_n, y) - f(\bar{x}, z_n, y)|}{|x-z|^\alpha} \\ \leq C|x_n - z_n|^\alpha |x-z|^{-n} \frac{|f(\bar{z}, z_n, y) - f(\bar{x}, z_n, y)|}{|\bar{x} - \bar{z}|^\alpha}, \end{aligned} \quad (\text{C.5})$$

we have

$$\begin{aligned} \int_{\mathbb{R}_+^n} (II) dy &\leq C \int_{x_n/2}^{x_n} (x_n - z_n)^\alpha \int_{\mathbb{R}^{n-1}} |x-z|^{-n} \\ &\quad \times \left[\sup_{\bar{z} \in \mathbb{R}^{n-1}} \int_{\mathbb{R}_+^n} \frac{|f(\bar{z}, z_n, y) - f(\bar{x}, z_n, y)|}{|\bar{x} - \bar{z}|^\alpha} dy \right] d\bar{z} dz_n \leq C x_n^{\alpha+1} \\ &\quad \times \left(\int_{\mathbb{R}^{n-1}} |x-z|^{-n} d\bar{z} \right) \left(\sup_{x_n/2 < z_n < x_n} \sup_{\bar{z} \in \mathbb{R}^{n-1}} \int_{\mathbb{R}_+^n} \frac{|f(\bar{z}, z_n, y) - f(\bar{x}, z_n, y)|}{|\bar{x} - \bar{z}|^\alpha} dy \right) \\ &\leq C x_n^\alpha \left(\sup_{x_n/2 < z_n < x_n} \sup_{\bar{z} \in \mathbb{R}^{n-1}} \int_{\mathbb{R}_+^n} \frac{|f(\bar{z}, z_n, y) - f(\bar{x}, z_n, y)|}{|\bar{x} - \bar{z}|^\alpha} dy \right). \end{aligned} \quad (\text{C.6})$$

Next, we want to show $(III) = 0$. For that, notice

$$PV \int_{\mathbb{R}^{n-1}} \partial_{y_i y_j}^2 N(x-y) d\bar{y} = \lim_{\varepsilon \rightarrow 0, R \rightarrow \infty} \int_{\varepsilon \leq |\bar{x} - \bar{y}| \leq R} \partial_{y_i y_j}^2 N(x-y) dS_{\bar{y}}, \quad (\text{C.7})$$

and

$$\begin{aligned} \int_{\varepsilon \leq |\bar{x} - \bar{y}| \leq R} \partial_{y_i y_j}^2 N(x-y) d\bar{y} &= \int_{S_R(\bar{x})} \partial_{y_i} N(x-y) n_j dS_{\bar{y}} \\ &\quad - \int_{S_\varepsilon(\bar{x})} \partial_{y_i} N(x-y) n_j dS_{\bar{y}}. \end{aligned} \quad (\text{C.8})$$

Here, $S_R(\bar{x}) = \{\bar{y} \in \mathbb{R}^{n-1} : |\bar{x} - \bar{y}| = R\}$, and $n_j = y_j - x_j / |\bar{y} - \bar{x}|$ is the j th component of the unit outer normal vector. If $i = 1, \dots, n-1$, then

$$\begin{aligned} \left| \int_{S_R(\bar{x})} \partial_{y_i} N(x-y) n_j dS_{\bar{y}} \right| &= \left| \int_{S_R(\bar{x})} \frac{y_i - x_i}{|x-y|^n} \frac{y_j - x_j}{|\bar{y} - \bar{x}|} dS_{\bar{y}} \right| \\ &= \left| \int_{S_{n-2}} \frac{w_i w_j}{n \omega_n (R^2 + (x_n - y_n)^2)^{n/2}} R^{n-1} dS_w \right| \\ &\leq \frac{C}{R} \rightarrow 0 \quad \text{as } R \rightarrow \infty, \end{aligned} \quad (\text{C.9})$$

where S_{n-2} is the unit sphere in $\mathbb{R}^{n-1}$, $w_i = y_i/|y|$ is the outward unit normal vector to S_{n-2} , and

$$\begin{aligned} \left| \int_{S_\varepsilon(\bar{x})} \partial_{y_i} N(x-y) n_j dS_{\bar{y}} \right| &= \left| \int_{S_{n-2}} \frac{w_i w_j}{n \omega_n (\varepsilon^2 + (x_n - y_n)^2)^{n/2}} \varepsilon^{n-1} dS_w \right| \\ &\leq C \frac{\varepsilon^{n-1}}{|x_n - y_n|^n} \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0. \end{aligned} \quad (\text{C.10})$$

If $i = n$, then

$$\int_{S_R(\bar{x})} \partial_{y_i} N(x-y) n_j dS_{\bar{y}} = 0, \quad (\text{C.11})$$

since

$$\begin{aligned} \int_{S_R(\bar{x})} \partial_{y_i} N(x-y) n_j dS_{\bar{y}} &= \int_{S_{n-2}} \frac{(x_n - y_n) w_j}{n \omega_n (R^2 + (x_n - y_n)^2)^{n/2}} R^{n-2} dS_w \\ &= \frac{(x_n - y_n) R^{n-2}}{n \omega_n (R^2 + (x_n - y_n)^2)^{n/2}} \int_{S_{n-2}} w_j dS_w = 0. \end{aligned} \quad (\text{C.12})$$

Similarly

$$\int_{S_\varepsilon(\bar{x})} \partial_{y_i} N(x-y) d\bar{y} = 0. \quad (\text{C.13})$$

This implies that

$$\int_{\mathbb{R}^{n-1}} \partial_{y_i y_j}^2 N(x-y) d\bar{y} = 0, \quad (\text{C.14})$$

Hence, we finally show that (III) = 0.

Data Availability

I have provided all the essential references that I have used in this research article in the reference section.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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