Research Article

Equivalent Characterization on Besov Space

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Abstract and Applied Analysis

In this paper, we give an equivalent characterization of the Besov space. This reveals the equivalent relation between the mixed derivative norm and single-variable norm. Fourier multiplier, real interpolation, and Littlewood-Paley decomposition are applied.

1. Introduction

In Sobolev spaces, it is known that \(\|f\|_{H^s(\mathbb{R}^2)} \sim \|f\|_{L^2(\mathbb{R}^2)} + \sum_{j=1}^2 \|\partial_x^j \partial_y^j f\|_{L^1(\mathbb{R}^2)}\), where \(\|f\|_{H^s(\mathbb{R}^2)} = \|f\|_{L^2(\mathbb{R}^2)} + \|\partial_x f\|_{L^1(\mathbb{R}^2)} + \|\partial_y f\|_{L^1(\mathbb{R}^2)}\). Note that on the right hand side of the definition \(\|f\|_{H^s(\mathbb{R}^2)}\), it contains the mixed derivative norm \(\|\partial_x \partial_y f\|_{L^2(\mathbb{R}^2)}\). This mixed derivative norm would make the calculation more complicated or even infeasible to estimate partial differential equations with some anisotropy property, like Vlasov-Poisson equation [1, 2], in fractional Sobolev space [3]. So, separating variables becomes necessary and meaningful.

In this paper, we aim to prove \(\|f\|_{B^s_{p,r}(\mathbb{R}^n)} \sim \sum_{j=1}^n \|\partial_x^j f\|_{B^s_{p,r}(\mathbb{R}^n)}\) which realizes the separation, i.e., the right hand side does not contain the “mixed derivative” term, it only contains fractional derivative with respect to a single variable for each term. Thus, when it comes to estimate \(\|f\|_{B^s_{p,r}(\mathbb{R}^n)}\) in solving partial differential equations, it is equivalent to estimate \(\|\partial_x^j f\|_{B^s_{p,r}(\mathbb{R}^n)}\) individually. For the other equivalent characterizations for Besov spaces, refer to [4–7] and the references therein.

2. Preliminaries

We first recall definitions on Besov spaces, see [8]. Given \(f \in \mathcal{S}\) which is the Schwartz function, its Fourier transform \(\mathcal{F}f = \widehat{f}\) is defined by

\[
\widehat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx,
\]

and its inverse Fourier transform is defined by \(\mathcal{F}^{-1}f(x) = \check{f}(-x)\).

We consider \(\varphi \in \mathcal{S}\) satisfying \(\text{supp} \varphi \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}\). Setting \(\varphi_j(x) = \varphi(2^{-j} |x|)\) with \(j = \{1, 2, \cdots\}\), we can adjust the normalization constant in front of \(\varphi\) and choose \(\varphi_0 \in \mathcal{S}\) satisfying \(\text{supp} \varphi_0 \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}\), such that

\[
\sum_{j=0}^{\infty} \varphi_j(\xi) = 1, \forall \xi \in \mathbb{R}^n.
\]

We observe

\[
\text{supp} \varphi_j \cap \text{supp} \varphi_{j'} = \emptyset \quad \text{if} \quad |j - j'| \geq 2.
\]

Given \(f \in \mathcal{S}'\), we denote \(\Delta f = \mathcal{F}^{-1} \varphi \mathcal{F}f\). For \((s, p, r) \in \mathbb{R} \times [1, \infty] \times [1, \infty]\), then we define the inhomogeneous Besov space by

\[
B^s_{p,r} = \left\{ f \in \mathcal{S}' : \|f\|_{B^s_{p,r}} = \sum_{j=0}^{\infty} 2^j \|\Delta f\|_{L^p_{||j||r}} < \infty \right\},
\]

where...
with the usual interpretation for $p = \infty$ or $r = \infty$. Throughout this paper, all the function spaces are defined on Euclidean space $\mathbb{R}^n$; we will omit it whenever there is no confusion.

Next, we would like to present some known results which will be used later. The first one is the unit decomposition.

Lemma 1 (see [8], page 145). Assume that $n \geq 2$, and take $\varphi$ as in the definition of Besov space. Then, there exist functions $\chi_j \in \mathcal{S}(\mathbb{R}^n)$ ($j = 1, \cdots, n$), such that

$$\sum_{j=1}^n \chi_j = 1 \text{ on supp } \varphi = \{ \xi : 1/2 \leq |\xi| \leq 2 \},$$

$$\text{supp } \chi_j \subset \{ \xi \in \mathbb{R}^n : |\xi| \in \left(3^{1/n} - \frac{1}{n} \right) \} (j = 1, \cdots, n).$$

Next, we recall the real interpolation characterization for Besov spaces.

Lemma 2 (see [8], page 142). Suppose $1 \leq p, q \leq \infty$, $0 < \theta < 1$, $s = (1 - \theta)s_0 + \theta s_1$, where $s_0 \neq s_1$. We have

$$\left( H^s_{\theta} \right)_{\theta, p} = B^s_{p, q}.$$

Remark 3. We also have

$$\left( H^s_{\theta, p, x_j} \right)_{\theta, p} = B^s_{p, q, x_j}.$$

Its proof can be repeated the process of Lemma 2 completely.

3. Equivalent Characterization

Now, we are in the position to state and prove our theorems. Firstly, we apply the Fourier multiplier [9] to prove that $H^s_p(\mathbb{R}^n) = \bigcap_{\theta=1}^n H^s_{\theta, p, x_j}(\mathbb{R}^n)$ directly; $H^s_p$ space has an advantage that the factor $(1 + |\xi|^2)^{-\theta/2}$ is positive everywhere, which is fundamentally important when applying the Fourier multiplier theorem.

For the sake of brevity, we denote

$$\langle \xi \rangle = \left( 1 + |\xi|^2 \right)^{1/2}.$$

We have the following equivalent norm theorem in Sobolev spaces.

Theorem 4. Suppose $1 < p < \infty, s > 0$. We have

$$H^s_p = \bigcap_{\theta=1}^n H^s_{\theta, p, x_j},$$

where

$$\|f\|_{H^s_p} = \|\mathcal{F}^{-1} \langle \xi \rangle^s f\|_{L^p},$$

$$\|f\|_{H^s_{\theta, p, x_j}} = \|\mathcal{F}^{-1} \langle \xi \rangle^s f\|_{L^p}.$$  \hspace{1cm} (10)

Proof. On the one hand, if $f \in H^s_p$, i.e., $\|\mathcal{F}^{-1} \langle \xi \rangle^s f\|_p < \infty$ where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. Note that, for any $j = 1, \cdots, n$, we have

$$\|\mathcal{F}^{-1} \langle \xi_j \rangle^s f\|_p = \|\mathcal{F}^{-1} \langle \xi \rangle^s f\|_p.$$  \hspace{1cm} (11)

Next, we just need to show that $m_1(\xi) = \langle \xi \rangle^{s/2} \langle \xi \rangle^s$ is an $L^p$ multiplier. To prove the assertion, we introduce an auxiliary function on $\mathbb{R}^{n+1}$ defined by

$$m_1(\xi, t) = \left( \frac{t^2 + |\xi|^2}{t^2 + |\xi|^2} \right)^{s/2}.$$  \hspace{1cm} (12)

It is easy to verify that $m_1$ is homogeneous of degree 0 and smooth on $\mathbb{R}^{n+1} \setminus \{ 0 \}$. The derivatives $\partial^\beta m_1$ are homogeneous of degree $-|\beta|$ and satisfy

$$|\partial^\beta m_1(\xi, t)| \leq C_{\rho} |\xi|^{|\beta|},$$

where $C_{\rho} = \sup_{|\beta|=1} |\partial^\beta m_1(\theta)|$, and $\beta$ is a multiindex of $n + 1$ variables. In particular, taking $\beta = (\alpha, 0)$, we obtain

$$|\partial^\alpha m_1(\xi, t)| \leq C_{\alpha} \left( t^2 + |\xi|^2 \right)^{-|\alpha|/2},$$

and setting $t = 1$, we deduce that $|\partial^\alpha m_1(\xi)| \leq C_{\alpha} (1 + |\xi|^2)^{-|\alpha|/2} \leq C_{\alpha} |\xi|^{|\alpha|}$, which implies that $m_1(\xi) = \langle \xi \rangle^{s/2} \langle \xi \rangle^s$ is an $L^p$ multiplier by the Mihlin-H"ormander theorem [9] (page 446).

On the other hand, assume $f \in \bigcap_{\theta=1}^n H^s_{\theta, p, x_j}$, that is, $\|\mathcal{F}^{-1} \langle \xi \rangle^s f\|_p < \infty$, for any $j = 1, \cdots, n$. Note that

$$\|\mathcal{F}^{-1} \langle \xi \rangle^s f\|_p = \|\mathcal{F}^{-1} \sum_{j=1}^n \langle \xi_j \rangle^s f\|_p.$$  \hspace{1cm} (13)

Similarly, we can verify that $m_2(\xi) = \langle \xi \rangle^{s/2} \langle \xi \rangle^s$ is an $L^p$ multiplier which finishes the proof of Theorem 4.

We return to prove the equivalent characterization on Besov spaces. However, we cannot do the same trick as in $H^s_p$ space since $\varphi_j(\xi)$ is not positive everywhere as $\langle \xi \rangle$. Fortunately, we have $(H^s_{p, x_j})_{\theta, p} = B^s_{p,q}$, see Lemma 2. This observation is favourable to prove the equivalent relation in one direction; however, for the other direction, we need a more delicate technique, in fact, we establish an identity by
applying the Littlewood-Paley decomposition [10], which is very important in our proof. In what follows, \( A \leq B \) means there exists a constant \( c \) independent of the main parameters such that \( A \leq cB \). \( A \sim B \) means \( A \leq B \) and \( B \leq A \).

**Theorem 5.** Suppose \( 1 < p < \infty, 1 \leq q \leq \infty, s > 0 \). We have

\[
B^p_{p,q} = \bigcap_{j=1}^n B^p_{p,q,\xi_j},
\]

(16)

where \( \|f\|_{B^p_{p,q}} = (\sum_{k=0}^\infty 2^{ksq} \|\mathcal{F}^{-1} \phi_k^x \mathcal{F}f\|_{L^p}^q)^{1/q} \) and \( \phi_k^x \) is the dyadic block of the unit decomposition for the \( j \)th variable as in the definition of Besov spaces.

**Proof.** We split the proof into the following two steps:

Step I. To prove

\[
B^p_{p,q} \subset \bigcap_{j=1}^n B^p_{p,q,\xi_j}
\]

(17)

Assume \( f \in B^p_{p,q} \), by the real interpolation Lemma 2, we have

\[
\|f\|_{B^p_{p,q}} \sim \|f\|_{\left( \begin{array}{c} H^0_p, \infty \end{array} \right)_{\theta,s}}
\]

\[
\sim \left( \int_0^\infty \left( \inf_{f=\sum f_j \xi_j} \left\{ \left\| f_0 \|_{L^p} + \| f_1 \|_{L^p} \right\} \right\}^{\theta} dt \right)^{1/q},
\]

(18)

where \( 0 < \theta < 1, s = (1 - \theta)\delta_0 + \theta \delta_s \), and we applied the equivalent norm for the interpolation space \( (H^0_p, H^s_p)_{\theta,q} \) see [8] ((3) page 39 and (5) page 40).

By Remark 3, we obtain, for any \( j = 1, \ldots, n \),

\[
\|f\|_{B^p_{p,q,\xi_j}} \sim \|f\|_{\left( \begin{array}{c} H^0_p, H^s_p \end{array} \right)_{\theta,s}}
\]

\[
\sim \left( \int_0^\infty \left( \inf_{f=\sum f_j \xi_j} \left\{ \left\| f_0 \|_{H^0_p} + \| f_1 \|_{H^s_p} \right\} \right\}^{\theta} dt \right)^{1/q},
\]

(19)

combining (18) and (19), it follows that

\[
\|f\|_{B^p_{p,q,\xi_j}} \leq \|f\|_{B^p_{p,q}},
\]

(20)

the arbitrariness of \( j \) implies that (17) holds.

Step II. To prove

\[
\bigcap_{j=1}^n B^p_{p,q,\xi_j} \subset B^p_{p,q}
\]

(21)

For \( n = 1 \), it is trivial.

For \( n \geq 2 \), we need the following key claim.

**Claim.** There exists a positive integer \( m \) depending on \( n \) only such that

\[
\sum_{|l-k| \leq m} \| \varphi_k \mathcal{F} \mathcal{F}^{-1} f \|_{L^p}^q = \varphi_k \mathcal{F} \mathcal{F}^{-1} f
\]

where

\[
\varphi_k(\xi) = \varphi(2^{-k} \xi), \quad \varphi_k(\xi) = \varphi(2^{-k} \xi),
\]

(22)

which is the dyadic block for \( j \)th variable, \( \varphi_k \) is the usual dyadic block as in the definition of Besov spaces, and \( \chi_j \) is the same as in Lemma 1.

**Proof of Claim.** By Lemma 1, we have \( \varphi_k = \sum_{j=1}^n \varphi_k \mathcal{F} \mathcal{F}^{-1} f \).

Note

\[
\sum_{|l-k| \leq m} \| \varphi_k \mathcal{F} \mathcal{F}^{-1} f \|_{L^p}^q = \varphi_k \mathcal{F} \mathcal{F}^{-1} f
\]

(24)

In order to get \( \varphi_k \mathcal{F} \mathcal{F}^{-1} f \neq 0 \), for any chosen \( j \) and \( k \), we must have

\[
\begin{cases}
2^{k-1} \leq |\xi| \leq 2^{k+1}, \\
2^{l-1} \leq |\xi| \leq 2^{l+1}, \\
|\xi| \leq 2 \left( \frac{3}{\sqrt{n}} \right)^{-1},
\end{cases}
\]

(25)

which implies that \( |l-k| \leq m \) with \( m = \lfloor \log_2 (3/\sqrt{n}) \rfloor + 1 \), ending the proof of the claim. With this claim in mind, we get

\[
2^{k_2} \| \mathcal{F}^{-1} \varphi_k \mathcal{F} \mathcal{F}^{-1} f \|_{L^p}^q = \sum_{j=1}^n \| \mathcal{F}^{-1} \varphi_k \mathcal{F} \mathcal{F}^{-1} f \|_{L^p}^q
\]

\[
\leq 2^{k_2} \sum_{j=1}^n \| \mathcal{F} \mathcal{F}^{-1} \varphi_k \mathcal{F} \mathcal{F}^{-1} f \|_{L^p}^q
\]

\[
\leq 2^{k_2} \sum_{j=1}^n \sum_{|l-k| \leq m} \| \mathcal{F} \mathcal{F}^{-1} \varphi_k \mathcal{F} \mathcal{F}^{-1} f \|_{L^p}^q
\]

\[
(26)
\]

\[
\leq 2^{k_2} \sum_{j=1}^n \sum_{|l-k| \leq m} \| \mathcal{F} \mathcal{F}^{-1} \varphi_k \mathcal{F} \mathcal{F}^{-1} f \|_{L^p}^q
\]

\[
\leq 2^{k_2} \sum_{j=1}^n \sum_{|l-k| \leq m} \| \mathcal{F} \mathcal{F}^{-1} \varphi_k \mathcal{F} \mathcal{F}^{-1} f \|_{L^p}^q,
\]

where we used the fact that \( \varphi_k \mathcal{F} \mathcal{F}^{-1} f \) is the Fourier multiplier.
With Young’s inequality [11], taking the $l^q$ norm on both sides of (26) yields that

$$\|f\|_{B_{p,q}^s} \leq C(n, s, q) \sum_{j=1}^{n} \|f\|_{B_{p,q}^{s,x_j}},$$

which implies (21) holds; thus, we complete the proof of our main theorem.

**Remark 6.** The methods could be adapted to the weighted Sobolev spaces and weighted Besov space, or even in the anisotropic function space.

**Data Availability**

The data in this paper is available on request. Please contact Jingchun Chen at jingchun.chen@utoledo.edu.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**References**


