Hindawi
Abstract and Applied Analysis
Volume 2021, Article ID 6688250, 4 pages
https://doi.org/10.1155/2021/6688250

Research Article

Equivalent Characterization on Besov Space

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Received 8 October 2020; Revised 22 November 2020; Accepted 17 April 2021; Published 26 April 2021

Academic Editor: Micah Osilike

In this paper, we give an equivalent characterization of the Besov space. This reveals the equivalent relation between the mixed derivative norm and single-variable norm. Fourier multiplier, real interpolation, and Littlewood-Paley decomposition are applied.

1. Introduction

In Sobolev spaces, it is known that $\|f\|_{H^2(\mathbb{R}^2)} \sim \|f\|_{L^2(\mathbb{R}^2)} + \sum_{i=1}^{2}\|\partial^2 f/\partial x_i^2\|_{L^2(\mathbb{R}^2)}$, where $\|f\|_{H^2(\mathbb{R}^2)} =: \|f\|_{L^2(\mathbb{R}^2)} + \|\partial_{x_1}\partial_{x_2} f\|_{L^2(\mathbb{R}^2)} + \|\partial_{x_1}^2 f\|_{L^2(\mathbb{R}^2)} + \|\partial_{x_2}^2 f\|_{L^2(\mathbb{R}^2)}$. Note that on the right hand side of the definition $\|f\|_{H^2(\mathbb{R}^2)}$, it contains the mixed derivative norm $\|\partial_{x_1}\partial_{x_2} f\|_{L^2(\mathbb{R}^2)}$. This mixed derivative norm would make the calculation more complicated or even infeasible to estimate partial differential equations with some anisotropy property, like Vlasov-Poisson equation [1, 2], in fractional Sobolev space [3]. So, separating variables becomes necessary and meaningful.

In this paper, we aim to prove $\|f\|_{B^s_{p,r}(\mathbb{R}^n)} \sim \sum_{j=1}^{n} \|f\|_{B^s_{p,r,x_j}(\mathbb{R}^n)}$ which realizes the separation, i.e., the right hand side does not contain the "mixed derivative" term, it only contains fractional derivative with respect to a single variable for each term. Thus, when it comes to estimate $\|f\|_{B^s_{p,r}(\mathbb{R}^n)}$ in solving partial differential equations, it is equivalent to estimate $\|f\|_{B^s_{p,r,x_j}(\mathbb{R}^n)}$ individually. For the other equivalent characterizations for Besov spaces, refer to [4–7] and the references therein.

2. Preliminaries

We first recall definitions on Besov spaces, see [8]. Given $f \in \mathcal{S}$ which is the Schwartz function, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx, \quad (1)$$

and its inverse Fourier transform is defined by $\mathcal{F}^{-1}f(x) = \hat{f}(-x)$.

We consider $\varphi \in \mathcal{S}$ satisfying supp $\varphi \subset \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\}$. Setting $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ with $j = \{1, 2, \cdots,\}$, we can adjust the normalization constant in front of φ and choose $\varphi_0 \in \mathcal{S}$ satisfying supp $\varphi_0 \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2\}$, such that

$$\sum_{j=0}^{\infty} \varphi_j(\xi) = 1, \forall \xi \in \mathbb{R}^n. \quad (2)$$

We observe

$$\text{supp } \varphi_j \cap \text{supp } \varphi_{j'} = \varnothing \quad \text{if } |j - j'| \ge 2. \quad (3)$$

Given $f \in \mathcal{S}'$, we denote $\Delta_j f = \mathcal{F}^{-1}\varphi_j \mathcal{F} f$. For $(s, p, r) \in \mathbb{R} \times [1,\infty] \times [1,\infty]$, then we define the inhomogeneous Besov space by

$$B^s_{p,r} = \left\{ f \in \mathcal{S}' : \|f\|_{B^s_{p,r}} = \left| \sum_{j=0}^{\infty} 2^{js} \left\| \Delta_j f \right\|_p \right|_{l^r_j} < \infty \right\}, \quad (4)$$

with the usual interpretation for $p = \infty$ or $r = \infty$. Throughout this paper, all the function spaces are defined on Euclidean space $\mathbb{R}^n$; we will omit it whenever there is no confusion.

Next, we would like to present some known results which will be used later. The first one is the unit decomposition.

Lemma 1 (see [8], page 145). *Assume that $n \geq 2$, and take φ as in the definition of Besov space. Then, there exist functions $\chi_j \in \mathcal{S}(\mathbb{R}^n)(j = 1,\cdots,n)$, such that*

$$\sum_{j=1}^{n} \widehat{\chi}_j = 1 \text{ on } \operatorname{supp} \varphi = \{\xi : 1/2 \leq |\xi| \leq 2\},$$
$$\operatorname{supp} \widehat{\chi}_j \subset \left\{\xi \in \mathbb{R}^n : |\xi_j| \geq \left(3\sqrt{n}\right)^{-1}\right\} (j = 1,\cdots,n). \tag{5}$$

Next, we recall the real interpolation characterization for Besov spaces.

Lemma 2 (see [8], page 142). *Suppose $1 \leq p, q \leq \infty$, $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$, where $s_0 \neq s_1$. We have*

$$\left(H_p^{s_0}, H_p^{s_1}\right)_{\theta,q} = B_{p,q}^s. \tag{6}$$

Remark 3. We also have

$$\left(H_{p,x_j}^{s_0}, H_{p,x_j}^{s_1}\right)_{\theta,q} = B_{p,q,x_j}^s. \tag{7}$$

Its proof can be repeated the process of Lemma 2 completely.

3. Equivalent Characterization

Now, we are in the position to state and prove our theorems. Firstly, we apply the Fourier multiplier [9] to prove that $H_p^s(\mathbb{R}^n) = \bigcap_{j=1}^n H_{p,x_j}^s(\mathbb{R}^n)$ directly; H_p^s space has an advantage that the factor $(1 + |\xi|^2)^{s/2}$ is positive everywhere, which is fundamentally important when applying the Fourier multiplier theorem.

For the sake of brevity, we denote

$$\langle \xi \rangle = \left(1 + |\xi|^2\right)^{1/2}. \tag{8}$$

We have the following equivalent norm theorem in Sobolev spaces.

Theorem 4. *Suppose $1 < p < \infty$, $s > 0$. We have*

$$H_p^s = \bigcap_{j=1}^{n} H_{p,x_j}^s, \tag{9}$$

where

$$\|f\|_{H_p^s} = \|\mathcal{F}^{-1}\langle\xi\rangle^s \widehat{f}\|_{L^p},$$
$$\|f\|_{H_{p,x_j}^s} = \|\mathcal{F}^{-1}\langle\xi_j\rangle^s \widehat{f}\|_{L^p}. \tag{10}$$

Proof. On the one hand, if $f \in H_p^s$, i.e., $\|\mathcal{F}^{-1}\langle\xi\rangle^s \mathcal{F} f\|_p < \infty$ where $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. Note that, for any $j = 1, \cdots, n$, we have

$$\|\mathcal{F}^{-1}\langle\xi_j\rangle^s \mathcal{F} f\|_p = \|\mathcal{F}^{-1}\frac{\langle\xi_j\rangle^s}{\langle\xi\rangle^s}\langle\xi\rangle^s \mathcal{F} f\|_p. \tag{11}$$

Next, we just need to show that $m_1(\xi) = \langle\xi_j\rangle^s / \langle\xi\rangle^s$ is an L^p multiplier. To prove the assertion, we introduce an auxiliary function on $\mathbb{R}^{n+1}$ defined by

$$\widetilde{m}_1(\xi, t) = \left(\frac{t^2 + |\xi_j|^2}{t^2 + |\xi|^2}\right)^{s/2}. \tag{12}$$

It is easy to verify that $\widetilde{m}_1$ is homogeneous of degree 0 and smooth on $\mathbb{R}^{n+1} \setminus \{0\}$. The derivatives $\partial^\beta \widetilde{m}_1$ are homogeneous of degree $-|\beta|$ and satisfy

$$|\partial^\beta \widetilde{m}_1(\xi, t)| \leq C_\beta |(\xi,t)|^{-|\beta|}, \text{ with } C_\beta = \sup_{|\theta|=1}\left|\partial^\beta \widetilde{m}_1(\theta)\right|, \tag{13}$$

whenever $(\xi, t) \neq 0$ and β is a multiindex of $n+1$ variables. In particular, taking $\beta = (\alpha, 0)$, we obtain

$$|\partial^\alpha \widetilde{m}_1(\xi, t)| \leq C_\alpha \left(t^2 + |\xi|^2\right)^{-|\alpha|/2}, \tag{14}$$

and setting $t = 1$, we deduce that $|\partial^\alpha m_1(\xi)| \leq C_\alpha (1+|\xi|^2)^{-|\alpha|/2} \leq C_\alpha |\xi|^{-|\alpha|}$, which implies that $m_1(\xi)$ is an L^p Fourier multiplier by the Mihlin-Hömander theorem [9] (page 446).

On the other hand, assume $f \in \bigcap_{j=1}^n H_{p,x_j}^s$, that is, $\|\mathcal{F}^{-1}\langle\xi_j\rangle^s \mathcal{F} f\|_p < \infty$, for any $j = 1, \cdots, n$. Note that,

$$\|\mathcal{F}^{-1}\langle\xi\rangle^s \mathcal{F} f\|_p = \|\mathcal{F}^{-1}\frac{\langle\xi\rangle^s}{\sum_{j=1}^n \langle\xi_j\rangle^s}\sum_{j=1}^{n}\langle\xi_j\rangle^s \mathcal{F} f\|_p. \tag{15}$$

Similarly, we can verify that $m_2(\xi) = \langle\xi\rangle^s / \sum_{j=1}^n \langle\xi_j\rangle^s$ is an L^p Fourier multiplier which finishes the proof of Theorem 4.

We return to prove the equivalent characterization on Besov spaces. However, we cannot do the same trick as in H_p^s space since $\varphi_j(\xi)$ is not positive everywhere as $\langle\xi\rangle$. Fortunately, we have $(H_p^{s_0}, H_p^{s_1})_{\theta,q} = B_{p,q}^s$, see Lemma 2. This observation is favourable to prove the equivalent relation in one direction; however, for the other direction, we need a more delicate technique, in fact, we establish an identity by applying the Littlewood-Paley decomposition [10], which is very important in our proof. In what follows, $A \lesssim B$ means there exists a constant c independent of the main parameters such that $A \leq cB. A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

Theorem 5. *Suppose $1 < p < \infty$, $1 \leq q \leq \infty$, $s > 0$. We have*

$$B^s_{p,q} = \bigcap_{j=1}^{n} B^s_{p,q,x_j}, \tag{16}$$

where $\|f\|_{B^s_{p,q,x_j}} = \left(\sum_{k=0}^{\infty} 2^{ksq} \|\mathcal{F}^{-1}\varphi^j_k \mathcal{F} f\|^q_{L^p}\right)^{1/q}$ and φ^j_k is the dyadic block of the unit decomposition for the jth variable as in the definition of Besov spaces.

Proof. We split the proof into the following two steps:

Step I. To prove

$$B^s_{p,q} \subset \bigcap_{j=1}^{n} B^s_{p,q,x_j} \tag{17}$$

Assume $f \in B^s_{p,q}$, by the real interpolation Lemma 2, we have

$$\begin{aligned} \|f\|_{B^s_{p,q}} &\sim \|f\|_{(H^{s_0}_p, H^{s_1}_p)_{\theta,q}} \\ &\sim \left(\int_0^{\infty} \left(t^{-\theta} \left(\inf_{f=f_0+f_1, f_0 \in H^{s_0}_p, f_1 \in H^{s_1}_p} \|f_0\|_{H^{s_0}_p} + t\|f_1\|_{H^{s_1}_p}\right)\right)^q \frac{dt}{t}\right)^{1/q}, \end{aligned} \tag{18}$$

where $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$, and we applied the equivalent norm for the interpolation space $(H^{s_0}_p, H^{s_1}_p)_{\theta,q}$, see [8] ((3) page 39 and (5) page 40).

By Remark 3, we obtain, for any $j = 1, \cdots, n$,

$$\begin{aligned} \|f\|_{B^s_{p,q,x_j}} &\sim \|f\|_{\left(H^{s_0}_{p,x_j}, H^{s_1}_{p,x_j}\right)_{\theta,q}} \\ &\sim \left(\int_0^{\infty} \left(t^{-\theta} \left(\inf_{f=f_0+f_1, f_0 \in H^{s_0}_{p,x_j}, f_1 \in H^{s_1}_{p,x_j}} \|f_0\|_{H^{s_0}_{p,x_j}} + t\|f_1\|_{H^{s_1}_{p,x_j}}\right)\right)^q \frac{dt}{t}\right)^{1/q} \\ &\cdot \left(\int_0^{\infty} \left(t^{-\theta} \left(\inf_{f=f_0+f_1, f_0 \in H^{s_0}_p, f_1 \in H^{s_1}_p} \|f_0\|_{H^{s_0}_p} + t\|f_1\|_{H^{s_1}_p}\right)\right)^q \frac{dt}{t}\right)^{1/q}, \end{aligned} \tag{19}$$

combining (18) and (19), it follows that

$$\|f\|_{B^s_{p,q,x_j}} \lesssim \|f\|_{B^s_{p,q}}, \tag{20}$$

the arbitrariness of j implies that (17) holds.

Step II. To prove

$$\bigcap_{j=1}^{n} B^s_{p,q,x_j} \subset B^s_{p,q} \tag{21}$$

For $n = 1$, it is trival.

For $n \geq 2$, we need the following key claim.

Claim. There exists a positive integer m depending on n only such that

$$\sum_{|l-k| \leq m} \varphi_k \widehat{\chi}_{j,k} \varphi^j_l = \varphi_k \widehat{\chi}_{j,k}, \tag{22}$$

where

$$\begin{aligned} &\varphi_k(\xi) = \varphi\left(2^{-k}\xi\right), \widehat{\chi}_{j,k}(\xi) = \widehat{\chi}_j\left(2^{-k}\xi\right), \\ &\varphi^j_l(\xi) = \varphi\left(2^{-l}\xi_j\right), \end{aligned} \tag{23}$$

which is the dyadic block for jth variable, φ_k is the usual dyadic block as in the definition of Besov spaces, and χ_j is the same as in Lemma 1.

Proof of Claim. By Lemma 1, we have $\varphi_k = \sum_{j=1}^{n} \varphi_k \widehat{\chi}_{j,k}$.

Note

$$\sum_{l=0}^{\infty} \varphi_k \widehat{\chi}_{j,k} \varphi^j_l = \varphi_k \widehat{\chi}_{j,k}. \tag{24}$$

In order to get $\varphi_k \widehat{\chi}_{j,k} \varphi^j_l \neq 0$, for any chosen j and k, we must have

$$\begin{cases} 2^{k-1} \leq |\xi| \leq 2^{k+1}, \\ 2^{l-1} \leq |\xi_j| \leq 2^{l+1}, \\ |\xi_j| \geq 2^k \left(3\sqrt{n}\right)^{-1}, \end{cases} \tag{25}$$

which implies that $|l - k| \leq m$ with $m = [\log_2 3\sqrt{n}] + 1$, ending the proof of the claim. With this claim in mind, we get

$$\begin{aligned} 2^{ks} \|\mathcal{F}^{-1}\varphi_k \mathcal{F}^{-1} f\|_{L^p} &= 2^{ks} \|\sum_{j=1}^{n} \mathcal{F}^{-1}\varphi_k \widehat{\chi}_{j,k} \mathcal{F} f\|_{L^p} \\ &\leq 2^{ks} \sum_{j=1}^{n} \|\mathcal{F}^{-1}\varphi_k \widehat{\chi}_{j,k} \mathcal{F} f\|_{L^p} \\ &\leq 2^{ks} \sum_{j=1}^{n} \|\sum_{|l-k| \leq m} \mathcal{F}^{-1}\varphi_k \widehat{\chi}_{j,k} \varphi^j_l \mathcal{F} f\|_{L^p} \\ &\leq 2^{ks} \sum_{j=1}^{n} \sum_{|l-k| \leq m} \|\mathcal{F}^{-1}\varphi_k \widehat{\chi}_{j,k} \varphi^j_l \mathcal{F} f\|_{L^p} \\ &\leq 2^{ms} \sum_{j=1}^{n} 2^{ls} \sum_{|l-k| \leq m} \|\mathcal{F}^{-1}\varphi^j_l \mathcal{F} f\|_{L^p}, \end{aligned} \tag{26}$$

where we used the fact that $\varphi_k \widehat{\chi}_{j,k}$ is the Fourier multiplier.

With Young's inequality [11], taking the l^q norm on both sides of (26) yields that

$$\|f\|_{B^s_{p,q}} \le C(n,s,q) \sum_{j=1}^{n} \|f\|_{B^s_{p,q,x_j}}, \quad (27)$$

which implies (21) holds; thus, we complete the proof our main theorem.

Remark 6. The methods could be adapted to the weighted Sobolev spaces and weighted Besov space, or even in the anisotropic function space.

Data Availability

The data in this paper is available on request. Please contact Jingchun Chen at jingchun.chen@utoledo.edu.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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