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Research Article

The Multiple $\mathbb{K}$-Riemann Integral

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The aim of this paper is to extend the notion of $\mathbb{K}$-Riemann integrability of functions defined over $[a, b]$ to functions defined over a rectangular box of $\mathbb{R}^n$. As a generalization of step functions, we introduce a notion of $\mathbb{K}$-step functions which allows us to give an equivalent definition of the $\mathbb{K}$-Riemann integrable functions.

1. Introduction

The motivation of this work results from the notion of the $\mathbb{K}$-Riemann integral of a function defined over an interval I of $\mathbb{R}$ recently introduced by Olbrys' [1]. Our aim of this paper is to give an equivalent definition of the $\mathbb{K}$-Riemann integrability by the use of $\mathbb{K}$-step functions and extend this notion to functions defined over a rectangular box of $\mathbb{R}^n$.After introducing the notion of $\mathbb{K}$-multiple integral, we give its properties and we show a $\mathbb{K}$-version of Fubini's theorem.

Throughout this paper, $\mathbb{K}$ denotes a subfield of the field of real numbers $\mathbb{R}$. We denote the set of positive elements of $\mathbb{K}$ by $\mathbb{K}_+$. For two real numbers $a < b$, we denote

$$\begin{aligned}[a, b]_{\mathbb{K}} &= \{\lambda a + (1-\lambda)b : \lambda \in [0, 1] \cap \mathbb{K}\},\\ ]a, b[_{\mathbb{K}} &= \{\lambda a + (1-\lambda)b : \lambda \in]0, 1[\cap \mathbb{K}\}.\end{aligned} \tag{1}$$

If

$$\mathbf{R} = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \tag{2}$$

is a rectangular box of $\mathbb{R}^n$, we denote

$$\begin{aligned}\mathbf{R}_{\mathbb{K}} &= [a_1, b_1]_{\mathbb{K}} \times [a_2, b_2]_{\mathbb{K}} \times \cdots \times [a_n, b_n]_{\mathbb{K}},\\ \mathring{\mathbf{R}}_{\mathbb{K}} &=]a_1, b_1[_{\mathbb{K}} \times]a_2, b_2[_{\mathbb{K}} \times \cdots \times]a_n, b_n[_{\mathbb{K}}.\end{aligned} \tag{3}$$

Following [1], we define the set of the $\mathbb{K}$-partitions of the interval $[a, b]$ by

$$\mathscr{P}^{\mathbb{K}}_{[a,b]} = \bigcup_{n\geq 1} \left\{\{t_0, t_1, \cdots, t_n\} : \ a = t_0 < t_1 < \cdots < t_n = b, \quad t_i \in [a, b]_{\mathbb{K}}\right\}. \tag{4}$$

As in [1], we give a generalization of the radially $\mathbb{K}$-continuous and uniformly radially $\mathbb{K}$-continuous functions and propositions easy to prove.We say that f is radially $\mathbb{K}$-continuous if it is radially $\mathbb{K}$-continuous at every point.

Definition 1. A function $f : \mathbf{R} \longrightarrow \mathbb{R}$ is called radially $\mathbb{K}$-continuous at a point $(x_1, x_2, \cdots, x_n) \in \mathbf{R}$ if for every $(y_1, y_2, \cdots, y_n) \in \mathbf{R}$

$$f(x_1 + \alpha_1(y_1 - x_1), \cdots, x_n + \alpha_n(y_n - x_n)) \underset{\mathbb{K}^n_+ \ni (\alpha_1, \cdots, \alpha_n) \longrightarrow (0, \cdots, 0)}{\longrightarrow} f(x_1, x_2, \cdots, x_n). \tag{5}$$

Definition 2. We say that a function $f : \mathbf{R} \longrightarrow \mathbb{R}$ is uniformly radially $\mathbb{K}$-continuous if for any $(x_1, x_2, \cdots, x_n) \in \mathbf{R}$ and $(y_1, y_2, \cdots, y_n) \in \mathbf{R}$, the mapping

$$([0,1]\cap\mathbb{K})^n \ni (\alpha_1,\cdots,\alpha_n) \mapsto f(x_1+\alpha_1(y_1-x_1),\cdots,x_n+\alpha_n(y_n-x_n)), \tag{6}$$

is uniformly continuous.

Every continuous and any uniformly continuous function in the usual sense is radially $\mathbb{K}$-continuous and uniformly radially $\mathbb{K}$-continuous, respectively.

Proposition 3.

(i) A function $f:\mathbf{R}\longrightarrow\mathbb{R}$ is radially $\mathbb{K}$-continuous if and only if for every subrectangular box $\mathbf{S}$ of $\mathbf{R}$, the function $f_{|\mathbf{S}_{\mathbb{K}}}$ is continuous

(ii) A function $f:\mathbf{R}\longrightarrow\mathbb{R}$ is uniformly radially $\mathbb{K}$-continuous if and only if for every subrectangular box $\mathbf{S}$ of $\mathbf{R}$, the function $f_{|\mathbf{S}_{\mathbb{K}}}$ is uniformly continuous

2. $\mathbb{K}$-Step Functions and $\mathbb{K}$-Riemann Integral

In the following definition, we introduce the notion of the $\mathbb{K}$-step functions which are a generalization of step functions.

Definition 4. A function $f:[a,b]\longrightarrow\mathbb{R}$ is said to be a $\mathbb{K}$-step function on $[a,b]$, if there exists a $\mathbb{K}$-partition $\pi=\{t_0,t_1,\cdots,t_m\}$ of $[a,b]$ for which f is constant on each set $]t_i,t_{i+1}[_{\mathbb{K}}$. In this case, we say that π is a $\mathbb{K}$-compatible partition of f on $[a,b]$.

We denote the set of the $\mathbb{K}$-step functions on $[a,b]$ by $\mathscr{E}_{\mathbb{K}}([a,b])$.

Remark 5.

(i) Every $\mathbb{K}$-step function on $[a,b]$ is bounded on $[a,b]_{\mathbb{K}}$.

(ii) If $\mathbb{K}=\mathbb{R}$, we get the usual definition of a step function.

Example 1.

(i) Every constant function on $[a,b]$ is a $\mathbb{K}$-step function.

(ii) Consider the following functions:

$$f_1(x)=\begin{cases}1, & x\in[0,1]_{\mathbb{Q}},\\ 0, & x\in[0,1]\setminus[0,1]_{\mathbb{Q}},\end{cases}\quad \text{and } f_2(x)=\begin{cases}1, & x\in\left[0,\sqrt{3}\right[,\\ 0, & x\in\left[\sqrt{3},2\right].\end{cases} \tag{7}$$

It is easy to show that f_1 is a $\mathbb{Q}$-step function but it is not a $\mathbb{R}$-step function on $[0,1]$, and f_2 is a $\mathbb{R}$-step function, but it is not a $\mathbb{Q}$-step function on $[0,1]$.

Now, suppose that $f:[a,b]\longrightarrow\mathbb{R}$ is a $\mathbb{K}$-step function on $[a,b]$ and let $\pi=\{t_0,t_1,\cdots,t_m\}$ be a $\mathbb{K}$-compatible partition of f, with $f(x)=\lambda_i$ whenever $x\in]t_{i-1},t_i[_{\mathbb{K}}$ for all $i=1,2,\cdots,m$. Consider the following number:

$$I_{\mathbb{K}}(f,\pi)=\sum_{i=1}^{m}(t_i-t_{i-1})\lambda_i. \tag{8}$$

We can easily show that the number $I_{\mathbb{K}}(f,\pi)$ does not depend on the choice of the $\mathbb{K}$-compatible partition.

Definition 6. Let $f:[a,b]\longrightarrow\mathbb{R}$ be a $\mathbb{K}$-step function on $[a,b]$, and π a $\mathbb{K}$-compatible partition of f. The real number $I_{\mathbb{K}}(f,\pi)$ is called the $\mathbb{K}$-integral of f on $[a,b]$ and is denoted by

$$\int_a^b \mathrm{f(t)d_{\mathbb{K}}t}. \tag{9}$$

As for the introduction of the usual Riemann integral, we extend the definition of the $\mathbb{K}$-Riemann integral to a broader class of functions than $\mathbb{K}$-step functions.

Definition 7. A function $f:[a,b]\longrightarrow\mathbb{R}$ is said to be $\mathbb{K}$-Riemann integrable on $[a,b]$, if for every $\varepsilon>0$, there exist two $\mathbb{K}$-step functions φ,ψ on $[a,b]$ such that

$$\begin{gathered}\varphi(x)\le f(x)\le\psi(x)\text{ for all } x\in[a,b]_{\mathbb{K}},\\ \int_a^b(\psi(t)-\varphi(t))d_{\mathbb{K}}t<\varepsilon.\end{gathered} \tag{10}$$

We denote the set of the $\mathbb{K}$-Riemann integrable functions by $\mathscr{R}_{\mathbb{K}}([a,b])$.

Remark 8. One can easily verify that $\mathscr{E}_{\mathbb{K}}([a,b])$ is a part of $\mathscr{R}_{\mathbb{K}}([a,b])$.

Let $f:[a,b]\longrightarrow\mathbb{R}$ be a bounded function on $[a,b]_{\mathbb{K}}$ and set

$$\begin{gathered}\mathbf{E}^+_{\mathbb{K}}(f,[a,b])=\left\{\int_a^b\psi(t)d_{\mathbb{K}}t:\psi\epsilon\mathscr{E}_{\mathbb{K}}([a,b])\text{ and } f(x)\le\psi(x)\text{ for all } x\epsilon[a,b]_{\mathbb{K}}\right\},\\ \mathbf{E}^-_{\mathbb{K}}(f,[a,b])=\left\{\int_a^b\varphi(t)d_{\mathbb{K}}t:\psi\in\mathscr{E}_{\mathbb{K}}([a,b])\text{ and } \varphi(x)\le f(x)\text{ for all } x\in[a,b]_{\mathbb{K}}\right\}.\end{gathered} \tag{11}$$

Note that $\alpha\le\beta$ for all $\alpha\in\mathbf{E}^-_{\mathbb{K}}(f,[a,b])$ and $\beta\in\mathbf{E}^+_{\mathbb{K}}(f,[a,b])$.

Proposition 9. *Let $f:[a,b]\longrightarrow\mathbb{R}$ be a bounded function on $[a,b]_{\mathbb{K}}$. Then, $f\in\mathscr{R}_{\mathbb{K}}([a,b])$ if and only if* $\inf\mathbf{E}^+_{\mathbb{K}}(f,[a,b])=\sup\mathbf{E}^-_{\mathbb{K}}(f,[a,b])$.

Proof. Suppose that $f\in\mathscr{R}_{\mathbb{K}}([a,b])$. Given $\varepsilon>0$, then there exist $\varphi,\psi\in\mathscr{E}_{\mathbb{K}}([a,b])$ such that

$$\varphi(x)\le f(x)\le\psi(x)\text{ for all } x\in[a,b]_{\mathbb{K}}\text{ and }\int_a^b\psi(t)d_{\mathrm{mathbbK}}t-\int_a^b\varphi(t)d_{\mathbb{K}}t<\varepsilon. \tag{12}$$

Since

$$\int_a^b \varphi(t) d_{\mathbb{K}} t \leq \sup \mathbf{E}_{\mathbb{K}}^{-}(f,[a,b]) \leq \inf \mathbf{E}_{\mathbb{K}}^{+}(f,[a,b]) \leq \int_a^b \psi(t) d_{\mathbb{K}} t, \tag{13}$$

then,

$$|\inf \mathbf{E}_{\mathbb{K}}^{+}(f,[a,b]) - \sup \mathbf{E}_{\mathbb{K}}^{-}(f,[a,b])| < \varepsilon, \tag{14}$$

Therefore,

$$\inf \mathbf{E}_{\mathbb{K}}^{+}(f,[a,b]) = \sup \mathbf{E}_{\mathbb{K}}^{-}(f,[a,b]). \tag{15}$$

Conversely, let $\varepsilon > 0$, then there are $\varphi, \psi \in \mathscr{E}_{\mathbb{K}}([a,b])$ such that

$$\varphi(x) \leq f(x) \leq \psi(x) \text{ for all } x \in [a,b]_{\mathbb{K}},$$
$$-\int_a^b \varphi(t) d_{\mathbb{K}} t \leq -\sup \mathbf{E}_{\mathbb{K}}^{-}(f,[a,b]) + \frac{\varepsilon}{2} \text{ and} \int_a^b \psi(t) d_{\mathbb{K}} t \leq \inf \mathbf{E}_{\mathbb{K}}^{+}(f,[a,b]) + \frac{\varepsilon}{2}. \tag{16}$$

Then,

$$\int_a^b \psi(t) d_{\mathbb{K}} t - \int_a^b \varphi(t) d_{\mathbb{K}} t < \varepsilon. \tag{17}$$

Therefore, $f \in \mathscr{R}_{\mathbb{K}}([a,b])$.

Definition 10. Let $f \in \mathscr{R}_{\mathbb{K}}([a,b])$, the common value $\inf \mathbf{E}_{\mathbb{K}}^{+}(f,[a,b]) = \sup \mathbf{E}_{\mathbb{K}}^{-}(f,[a,b])$ is called the $\mathbb{K}$ -integral of f on $[a,b]$; it is denoted by

$$\int_a^b f(t) d_{\mathbb{K}} t. \tag{18}$$

In the case when $\mathbb{K} = \mathbb{R}$, we will use the standard symbol $\int_a^b f(t)dt$ instead of $\int_a^b f(t) d_{\mathbb{R}} t$.

Now, we will show that our definition of the $\mathbb{K}$-integral is equivalent to that recently introduced by Olbrys' ([1], Definition 7)

Proposition 11. *A function $f : [a,b] \longrightarrow \mathbb{R}$ is $\mathbb{K}$ -Riemann integrable on $[a,b]$ in the sense of Definition 7 if and only if it is $\mathbb{K}$ -Riemann integrable on $[a,b]$ in the sense of ([1], Definition 7) (meaning the two definitions are equivalents). Furthermore, when $f \in \mathscr{R}_{\mathbb{K}}([a,b])$, we have*

$$\inf \mathbf{E}_{\mathbb{K}}^{+}(f,[a,b]) = \sup \mathbf{E}_{\mathbb{K}}^{-}(f,[a,b]) = \overline{\int_a^b} f(t) d_{\mathbb{K}} t = \underline{\int_a^b} f(t) d_{\mathbb{K}} t, \tag{19}$$

where

$$\overline{\int_a^b} f(t) d_{\mathbb{K}} t = \inf_{\pi = \{t_0, t_1, \cdots, t_m\} \in \mathscr{P}_{[a,b]}^{\mathbb{K}}} \left\{ \sum_{i=1}^{m} (t_i - t_{i-1}) M_i : M_i = \sup_{x \in [t_i + t_{i+1}]} f(x) \right\},$$
$$\underline{\int_a^b} f(t) d_{\mathbb{K}} t = \sup_{\pi = \{t_0, t_1, \cdots, t_m\} \in \mathscr{P}_{[a,b]}^{\mathbb{K}}} \left\{ \sum_{i=1}^{m} \boxtimes (t_i - t_{i-1}) m_i : m_i = \sup_{x \in [t_i + t_{i+1}]} f(x) \right\}. \tag{20}$$

Proof. Suppose that $f \in \mathscr{R}_{\mathbb{K}}([a,b])$ in the sense of ([1], Definition 7). Given $\varepsilon > 0$, then there exists a partition $\pi = \{t_0, t_1, \cdots, t_m\}$ of $[a,b]$ such that

$$\sum_{i=1}^{m} (t_i - t_{i-1}) M_i - \sum_{i=1}^{m} (t_i - t_{i-1}) m_i < \varepsilon, \tag{21}$$

where $M_i = \sup_{x \in [t_{i-1}, t_i]} f(x)$ and $m_i = \inf_{x \in [t_{i-1}, t_i]} f(x)$, $i = 1, 2, \cdots, m$. We have thus built two functions $\varphi, \psi \in \mathscr{E}_{\mathbb{K}}([a,b])$ such that

$$\varphi(x) \leq f(x) \leq \psi(x) \text{ for all } x \in [a,b]_{\mathbb{K}},$$
$$\int_a^b \psi(t) d_{\mathbb{K}} t - \int_a^b \varphi(t) d_{\mathbb{K}} t < \varepsilon, \tag{22}$$

where φ and ψ are defined on $[a,b]_{\mathbb{K}}$ by $\varphi(b) = \psi(b) = 0$ and

$$\varphi(x) = m_i, \psi(x) = M_i \text{ for all } x \in [t_{i-1}, t_i[_{\mathbb{K}}, i = 1, 2, \cdots, m. \tag{23}$$

Therefore, f is $\mathbb{K}$-Riemann integrable on $[a,b]$ in the sense of Definition 7.

Conversely, suppose that f is $\mathbb{K}$-Riemann integrable on $[a,b]$ in the sense of Definition 7. Then, there exist φ and $\psi \in \mathscr{E}_{\mathbb{K}}([a,b])$ such that

$$\varphi(x) \leq f(x) \leq \psi(x) \text{ for all } x \in [a,b]_{\mathbb{K}},$$
$$\int_a^b \psi(t) d_{\mathbb{K}} t - \int_a^b \varphi(t) d_{\mathbb{K}} t < \varepsilon. \tag{24}$$

Let $\pi' = \{a_0, a_1, \cdots, a_n\}$ be a common $\mathbb{K}$-compatible partition of φ and ψ, and let

$$B_i = \sup_{x \in [a_{i-1}, a_i]} f(x) \text{ and } A_i = \inf_{x \in [a_{i-1}, a_i]} f(x), i = 1, 2, \cdots, m. \tag{25}$$

Then,

$$\int_a^b \varphi(t) d_{\mathbb{K}} t \leq \sum_{i=1}^{m} (a_i - a_{i-1}) A_i \leq \sum_{i=1}^{m} (a_i - a_{i-1}) B_i \leq \int_a^b \psi(t) d_{\mathbb{K}} t, \tag{26}$$

which shows

$$\sum_{i=1}^{m}(a_i - a_{i-1})B_i - . \sum_{i=1}^{m}(a_i - a_{i-1})A_i < \int_a^b \psi(t)d_{\mathbb{K}}t - \int_a^b \varphi(t)d_{\mathbb{K}}t < \varepsilon. \tag{27}$$

This means that f is $\mathbb{K}$-Riemann integrable on $[a, b]$ in the sense of ([1], Definition 7).

Finally, if $f \in \mathscr{R}_{\mathbb{K}}([a, b])$, then the equalities (19) are ensured by Proposition 9 and the inequalities (26).

3. $\mathbb{K}$-Multiple Integral

This section is devoted to $\mathbb{K}$-multiple Riemann integral. For the theory of the classical multiple Riemann integral, see for instance [2].

Let $\mathbf{R} = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ be a closed rectangular box in $\mathbb{R}^n$, where a_i, b_i are real numbers such that $a_i < b_i$ for all $i = 1, 2, \cdots, n$.

We denoted by

$$\begin{gathered}\operatorname{vol}(\mathbf{R}) = \prod_{i=1}^{n}(b_i - a_i), \\ \mathbf{R}_{\mathbb{K}} := [a_1, b_1]_{\mathbb{K}} \times [a_2, b_2]_{\mathbb{K}} \times \cdots \times [a_n, b_n]_{\mathbb{K}}.\end{gathered} \tag{28}$$

Definition 12. A rectangular box $\mathbf{S} = [a_1', b_1'] \times [a_2', b_2'] \times \cdots \times [a_n', b_n']$ is said to be a $\mathbb{K}$-subrectangular box of $\mathbf{R}$ if $a_i', b_i' \in [a_i, b_i]_{\mathbb{K}}$ for all $i = 1, 2, \cdots, n$.

Example 2. $[0, 3/4] \times [0, 1/2]$ is a $\mathbb{Q}$-subrectangular box of $[0, 1] \times [0, 1]$.

$[0, 1] \times [0, 2]$ is not a $\mathbb{Q}$-subrectangular box of $[0, \sqrt{3}] \times [0, 2]$.

Definition 13. A $\mathbb{K}$-partition of $\mathbf{R}$ is a finite collection $\pi = \{\mathbf{R}_i\}_{i=1}^{m}$ of $\mathbb{K}$-subrectangular box of $\mathbf{R}$ such that

$$\mathbf{R} = \bigcup_{i=1}^{m} \mathbf{R}_i \text{ and } \overset{\circ}{\mathbf{R}}_i \cap \overset{\circ}{\mathbf{R}}_j = \varnothing, \quad i \neq j, \tag{29}$$

where $\mathbf{R}^{\circ}{}_i$ is the interior of $\mathbf{R}_i$.

Definition 14.

(i) Let $f : \mathbf{R} \longrightarrow \mathbb{R}$ be a bounded function over $\mathbf{R}_{\mathbb{K}}$. f is said to be a $\mathbb{K}$-step function on $\mathbf{R}$ if there exists a $\mathbb{K}$-partition $\pi = \{\mathbf{R}_i\}_{i=1}^{m}$ such that f is constant on $\mathbf{R}^{\circ}{}_{i\mathbb{K}}$ for all $i = 1, 2, \cdots, m$. In this case, π is said to be a $\mathbb{K}$-compatible partition of f on $\mathbf{R}$

We denoted the set of the $\mathbb{K}$-step functions on $\mathbf{R}$ by $\mathscr{E}_{\mathbb{K}}(\mathbf{R})$.

(ii) A $\mathbb{K}$-refinement of a $\mathbb{K}$-partition $\{\mathbf{R}_i\}_{i=1}^{m}$ of $\mathbf{R}$ is another $\mathbb{K}$-partition $\{\mathbf{S}_j\}_{j=1}^{r}$ of $\mathbf{R}$ such that each $\mathbf{S}_j$ is a $\mathbb{K}$-subrectangular box of some $\mathbf{R}_i$

Let $f : \mathbf{R} \longrightarrow \mathbb{R}$ be a $\mathbb{K}$-step function and $\pi = \{\mathbf{R}_i\}_{i=1}^{m}$ be a $\mathbb{K}$-compatible partition of f such that $f(x) = \lambda_i$ for all $x \in \mathbf{R}_i{}^{\circ}{}_{\mathbb{K}}$, $i = 1, 2, \cdots, m$. We define the number

$$I_{\mathbb{K}}(f, \pi) := \sum_{i=1}^{m} \lambda_i \operatorname{vol}(\mathbf{R}_i). \tag{30}$$

We will first show that $I_{\mathbb{K}}(f, \pi)$ does not depend on the choice of the $\mathbb{K}$-compatible partition π.

Lemma 15. *Let $f : \mathbf{R} \longrightarrow \mathbb{R}$ be a $\mathbb{K}$-step function and π be a $\mathbb{K}$-compatible partition of f. Then, for every $\mathbb{K}$-refinement π' of π, we have*

$$I_{\mathbb{K}}(f, \pi) = I_{\mathbb{K}}\left(f, \pi'\right). \tag{31}$$

Proof. Let $\pi = \{\mathbf{R}_i\}_{i=1}^{m}$ be a $\mathbb{K}$-compatible partition of f such that $f(x) = \lambda_i$ for all $x \in \mathbf{R}_i{}^{\circ}{}_{\mathbb{K}}$, $i = 1, 2, \cdots, m$ and $\pi' = \{\mathbf{S}_j\}_{j=1}^{r}$ be a $\mathbb{K}$-refinement of π. For all $i \in \{1, 2, \cdots, m\}$, there is $i_1, i_2, \cdots, i_{k_i} \in \{1, 2, \cdots, r\}$ such that $\mathbf{R}_i = \bigcup_{j=1}^{k_i} \mathbf{S}_{i_j}$. Then,

$$\begin{aligned} I_{\mathbb{K}}(f, \pi) &= \sum_{i=1}^{m} \lambda_i \operatorname{vol}\left(\bigcup_{j=1}^{k_i} \mathbf{S}_{i_j}\right) = \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{k_i} \operatorname{vol}\left(\mathbf{S}_{i_j}\right) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{k_i} \lambda_i \operatorname{vol}\left(\mathbf{S}_{i_j}\right) = I_{\mathbb{K}}\left(f, \pi'\right). \end{aligned} \tag{32}$$

Corollary 16. *Let $f : \mathbf{R} \longrightarrow \mathbb{R}$ be a $\mathbb{K}$-step function and π be a $\mathbb{K}$-compatible partition of f. The number $I_{\mathbb{K}}(f, \pi)$ does not depend on the choice of the $\mathbb{K}$-compatible partition π.*

Proof. Let $\pi = \{\mathbf{R}_i\}_{i=1}^{m}$ and $\pi' = \{\mathbf{S}_j\}_{j=1}^{r}$ be two $\mathbb{K}$-compatible partitions of f, then $\pi'' = \{T_{ij}\}_{1 \le i \le m}^{1 \le j \le r}$, where $T_{ij} = \mathbf{R}_i \cap \mathbf{S}_j$, is a $\mathbb{K}$-refinement of π and π'. Therefore, by using Lemma 15, we get

$$I_{\mathbb{K}}(f, \pi) = I_{\mathbb{K}}\left(f, \pi'\right) = I_{\mathbb{K}}\left(f, \pi''\right). \tag{33}$$

Definition 17. Let $f : \mathbf{R} \longrightarrow \mathbb{R}$ be a $\mathbb{K}$-step function and π be a $\mathbb{K}$-compatible partition of f. We called $I_{\mathbb{K}}(f, \pi)$ the $\mathbb{K}$-Riemann integral of f on $\mathbf{R}$ and we denote it by

$$\int_{\mathbf{R}} f(x) d_{\mathbb{K}}x. \tag{34}$$

In the following proposition, we present some basic properties of $\mathbb{K}$-Riemann multiple integral of $\mathbb{K}$-step functions.

Proposition 18. *Let $f, g : \mathbf{R} \longrightarrow \mathbb{R}$ be two $\mathbb{K}$-step functions on $\mathbf{R}$ and $\alpha, \beta \in \mathbb{R}$. Then,*

(1) The function $\alpha f+\beta g$ is a $\mathbb{K}$-step function on $\mathbf{R}$; moreover,

$$\int_R [\alpha f(x)+\beta g(x)]d_{\mathbb{K}}x=\alpha\int_R f(x)d_{\mathbb{K}}x+\beta\int_R g(x)d_{\mathbb{K}}x \quad (35)$$

(2) If $f(x)\geq 0, x\in \mathbf{R}_{\mathbb{K}}$, then $\int_{\mathbf{R}} f(x)d_{\mathbb{K}}x\geq 0$

(3) The absolute value $|f|$ is a $\mathbb{K}$-step function on $\mathbf{R}$ and $\int_R f(x)d_{\mathbb{K}}x\leq\int_R |f(x)|d_{\mathbb{K}}x$

(4) If $\pi=\{\mathbf{S_i}\}_{i=1}^{r}$ is a $\mathbb{K}$-partition on $\mathbf{R}$, then $\int_R f(x)d_{\mathbb{K}}x=\sum_{i=1}^{r}\int_{S_i} f(x)d_{\mathbb{K}}x$

Proof. (1) Let $\pi=\{\mathbf{R}_i\}_{i=1}^{m}$(resp $\pi'=\{\mathbf{S}_j\}_{j=1}^{r}$) be a $\mathbb{K}$-compatible partition of f(resp g), then $\pi''=\{T_{ij}\}_{1\leq i\leq m}^{1\leq j\leq r}$, where $T_{ij}=\mathbf{R}_i\cap\mathbf{S}_j$, is a $\mathbb{K}$-compatible partition of $\alpha f+\beta$; moreover, we have

$$\begin{aligned}\int_{\mathrm{R}}[\alpha f(x)+\beta g(x)]d_{\mathbb{K}}x&=I_{\mathbb{K}}\left(\alpha f+\beta g,\pi''\right)\\&=\alpha I_{\mathbb{K}}\left(f,\pi''\right)+\beta I_{\mathbb{K}}\left(g,\pi''\right)\\&=\alpha I_{\mathbb{K}}(f,\pi)+\beta I_{\mathbb{K}}\left(g,\pi'\right)\\&=\alpha\int_{\mathrm{R}} f(x)d_{\mathbb{K}}x+\beta\int_{\mathrm{R}} g(x)d_{\mathbb{K}}x.\end{aligned} \quad (36)$$

(2) and (3) are evident.

(4) Let $\pi=\{\mathbf{R}_i\}_{i=1}^{m}$ be a $\mathbb{K}$-compatible partition of f such that $f(x)=\lambda_i$ for all $x\in\mathbf{R}_i^{\circ}{}_{\mathbb{K}}$, $i=1,2,\cdots,m$. Then,

$$\int_{\mathrm{R}} f(x)d_{\mathbb{K}}x=\sum_{i=1}^{m}\lambda_i\mathrm{vol}(\mathrm{R}_i)=\sum_{i=1}^{m}\lambda_i\sum_{j=1}^{r}\mathrm{vol}(\mathrm{R}_i\cap\mathrm{S}_j)=\sum_{j=1}^{r}\sum_{i=1}^{m}\lambda_i\mathrm{vol}(\mathrm{R}_i\cap\mathrm{S}_j). \quad (37)$$

Since $\{\mathbf{R}_i\cap\mathbf{S}_j\}_{i=1}^{m}$ is a $\mathbb{K}$-compatible partition of f on $\mathbf{S}_j$. Then,

$$\int_{\mathrm{R}} f(x)d_{\mathbb{K}}x=\sum_{j=1}^{r}\int_{\mathrm{S}_j} f(x)d_{\mathbb{K}}x \quad (38)$$

3.1. $\mathbb{K}$-Riemann Integrable Functions Defined over $\mathbf{R}$

Definition 19. Let $f:\mathbf{R}\longrightarrow\mathbb{R}$ be a function, we said that f is a $\mathbb{K}$-Riemann integrable function, if for every $\varepsilon>0$, there exist two $\mathbb{K}$-step functions φ and ψ on $\mathbf{R}$ such that

$$\begin{gathered}\varphi(x)\leq f(x)\leq\psi(x)\text{ for all }x\in R_{\mathbb{K}},\\ \int_R^{\int_{\mathbb{K}}}(\psi(x)-\varphi(x)).\end{gathered} \quad (39)$$

We denote the set of the $\mathbb{K}$-Riemann integrable functions by $\mathscr{R}_{\mathbb{K}}(\mathbf{R})$.

Remark 20.

(i) Every $\mathbb{K}$-Riemann integrable function on $\mathbf{R}$ is bounded on $\mathbf{R}_{\mathbb{K}}$

(ii) $\mathscr{E}_{\mathbb{K}}(\mathbf{R})$ is a part of $\mathscr{R}_{\mathbb{K}}(\mathbf{R})$

Now, suppose that $f:\mathbf{R}\longrightarrow\mathbb{R}$ is a bounded function on $\mathbf{R}_{\mathbb{K}}$. Set

$$\begin{aligned}\mathbf{E}_{\mathbb{K}}^{+}(f,\mathbf{R})&=\left\{\int_{\mathrm{R}}\psi(x)d_{\mathbb{K}}x:\psi\in\varepsilon_{\mathbb{K}}(\mathbf{R})\text{ and }f(x)\leq\psi(x)\text{ for all }x\in\mathbf{R}_{\mathbb{K}}\right\},\\ \mathbf{E}_{\mathbb{K}}^{-}(f,\mathbf{R})&=\left\{\int_{\mathrm{R}}\varphi(x)d_{\mathbb{K}}x:\varphi\in\varepsilon_{\mathbb{K}}(\mathbf{R})\text{ and }\varphi(x)\leq f(x)\text{ for all }x\in\mathbf{R}_{\mathbb{K}}\right\}.\end{aligned} \quad (40)$$

Note that $\alpha\leq\beta$ for all $\alpha\in\mathbf{E}_{\mathbb{K}}^{-}(f,\mathbf{R})$ and $\beta\in\mathbf{E}_{\mathbb{K}}^{+}(f,\mathbf{R})$.

Theorem 21. *Let $f:\mathbf{R}\longrightarrow\mathbb{R}$ be a bounded function on $\mathbf{R}_{\mathbb{K}}$. Then, $f\in\mathscr{R}_{\mathbb{K}}(\mathbf{R})$ if and only if $\inf\mathbf{E}_{\mathbb{K}}^{+}(f,\mathbf{R})=\sup\mathbf{E}_{\mathbb{K}}^{-}(f,\mathbf{R})$.*

Proof. Suppose that $f\in\mathscr{R}_{\mathbb{K}}(\mathbf{R})$. Given $\varepsilon>0$, there exists φ and ψ in $\mathscr{E}_{\mathbb{K}}(\mathbf{R})$ such that

$$\begin{gathered}\varphi(x)\leq f(x)\leq\psi(x)\text{ for all }x\in\mathbf{R}_{\mathbb{K}},\\ \int_{\mathrm{R}}\psi(x)d_{\mathbb{K}}x-\int_{\mathrm{R}}\varphi(x)d_{\mathbb{K}}x<\varepsilon.\end{gathered} \quad (41)$$

Then,

$$\int_{\mathrm{R}}\varphi(x)d_{\mathbb{K}}x\leq\sup\mathbf{E}_{\mathbb{K}}^{-}(f,\mathbf{R})\leq\inf\mathbf{E}_{\mathbb{K}}^{+}(f,\mathbf{R})\leq\int_{\mathrm{R}}\psi(x)d_{\mathbb{K}}x, \quad (42)$$

which shows that

$$\inf\mathbf{E}_{\mathbb{K}}^{+}(f,\mathbf{R})-\sup\mathbf{E}_{\mathbb{K}}^{-}(f,\mathbf{R})<\varepsilon, \quad (43)$$

for all $\varepsilon>0$. Therefore,

$$\inf\mathbf{E}_{\mathbb{K}}^{+}(f,\mathbf{R})=\sup\mathbf{E}_{\mathbb{K}}^{-}(f,\mathbf{R}). \quad (44)$$

Conversely, suppose that $\inf\mathbf{E}_{\mathbb{K}}^{+}(f,\mathbf{R})=\sup\mathbf{E}_{\mathbb{K}}^{-}(f,\mathbf{R})$. Let $\varepsilon>0$, by the definition of the greatest lower bound and

the least upper bound, there are $\varphi, \psi \in \mathscr{E}_{\mathbb{K}}(\mathbf{R})$ such that

$$\begin{aligned} &\varphi(x) \le f(x) \le \psi(x) \text{ for all } x \in \mathbf{R}_{\mathbb{K}}, \\ &-\int_{\mathrm{R}} \varphi(x) d_{\mathbb{K}} x \le -\sup \mathbf{E}_{\mathbb{K}}^{-}(f, \mathbf{R}) + \frac{\varepsilon}{2}, \\ &\int_{\mathrm{R}} \psi(x) d_{\mathbb{K}} x \le \inf \mathbf{E}_{\mathbb{K}}^{+}(f, \mathbf{R}) + \frac{\varepsilon}{2}. \end{aligned} \tag{45}$$

Then,

$$\int_{\mathrm{R}} \psi(x) d_{\mathbb{K}} x - \int_{\mathrm{R}} \varphi(x) d_{\mathbb{K}} x < \varepsilon. \tag{46}$$

Therefore, $f \in \mathscr{R}_{\mathbb{K}}(\mathrm{R})$.

Definition 22. Let $f \in \mathscr{R}_{\mathbb{K}}(\mathbf{R})$, the common value $\inf \mathbf{E}_{\mathbb{K}}^{+}(f, \boldsymbol{R}) = \sup \mathbf{E}_{\mathbb{K}}^{-}(f, \mathbf{R})$ is called the $\mathbb{K}$ -Riemann integral of f on $\mathbf{R}$, and we denote it by

$$\int_{\mathrm{R}} f(x) d_{\mathbb{K}} x. \tag{47}$$

In the case when $\mathbb{K} = \mathbb{R}$, we will use the standard symbol $\int_{\mathrm{R}} f(x) dx$ instead of $\int_{\mathrm{R}} f(x) d_{\mathbb{R}} x$.

Proposition 23. *Let $f : \mathbf{R} \longrightarrow \mathbb{R}$ be a function. Then, f is $\mathbb{K}$ -Riemann integrable on $\mathbf{R}$ if and only if there exist two sequences of $\mathbb{K}$-step functions $(\varphi_m)_m$ and $(\psi_m)_m$ such that*

$$\begin{aligned} &\varphi_m(x) \le f(x) \le \psi_m(x) \; m \in \mathbb{N}, \quad x \in \mathbf{R}_{\mathbb{K}}, \\ &\lim_{m \mapsto \infty} \int_{R} (\psi_m(x) - \varphi_m(x)) d_{\mathbb{K}} t = 0. \end{aligned} \tag{48}$$

Moreover, we have

$$\int_{\mathrm{R}} f(x) d_{\mathbb{K}} x = \lim_{m \to \infty} \int_{\mathrm{R}} \varphi_m(x) d_{\mathbb{K}} x = \lim_{m \to \infty} \int_{\mathrm{R}} \psi_m(x) d_{\mathbb{K}} x. \tag{49}$$

Before presenting a version of the Fubini theorem for $\mathbb{K}$ -Riemann integrable functions. We start by giving a lemma that is essential for the proof of this result.

Proof. It runs in a similar way as for the case of one variable. Suppose that f is $\mathbb{K}$-integrable. Then, $\inf \mathbf{E}_{\mathbb{K}}^{+}(f, \mathbf{R}) = \sup \mathbf{E}_{\mathbb{K}}^{-}(f, \mathbf{R})$. For all $m \in \mathbb{N}^*$, there exist two $\mathbb{K}$-step functions φ_m and ψ_m on $\mathbf{R}$ such that

$$\varphi_m \le f \le \psi_m \text{ on } \mathbf{R}_{\mathbb{K}} \text{ and } \int_{\mathbf{R}} (\psi_m - \varphi_m)(x) d_{\mathbb{K}} x < \frac{1}{m}. \tag{50}$$

Hence, $\lim_{m \mapsto +\infty} \int_{\mathbf{R}} (\psi_m - \varphi_m)(x) d_{\mathbb{K}} x = 0$. Now, let us show that

$$\int_{\mathbf{R}} f(x) d_{\mathbb{K}} x = \lim_{m \to \infty} \int_{\mathbf{R}} \varphi_m(x) d_{\mathbb{K}} x. \tag{51}$$

Let $m \in \mathbb{N}$. We have

$$\int_{\mathbf{R}} \varphi_m(x) d_{\mathbb{K}} x \le \int_{\mathbf{R}} f(x) d_{\mathbb{K}} x \le \int_{\mathbf{R}} \psi_m(x) d_{\mathbb{K}} x. \tag{52}$$

Then,

$$0 \le \int_{\mathbf{R}} f(x) d_{\mathbb{K}} x - \int_{\mathbf{R}} \varphi_m(x) d_{\mathbb{K}} x \le \int_{\mathbf{R}} (\psi_m(x) - \varphi_m(x)) d_{\mathbb{K}} x. \tag{53}$$

Hence,

$$\begin{aligned} 0 &\le \lim_{m \to \infty} \left(\int_{\mathbf{R}} f(x) d_{\mathbb{K}} x - \int_{\mathbf{R}} \varphi_m(x) d_{\mathbb{K}} x \right) \\ &\le \lim_{m \to \infty} \left(\int_{\mathbf{R}} (\psi_m(m) - \varphi_m(x)) d_{\mathbb{K}} x \right) = 0. \end{aligned} \tag{54}$$

The converse is trivial.

Lemma 24. *Let $\mathbb{K}_1 \subseteq \mathbb{K}_2$ be two subfields of $\mathbb{R}$ and let $f : \mathbf{R} \longrightarrow \mathbb{R}$ be a $\mathbb{K}_2$-step function. Then, f is a $\mathbb{K}_1$-Riemann integrable function. Moreover, we have*

$$\int_{\mathbf{R}} f(x) d_{\mathbb{K}_1} x = \int_{\mathbf{R}} f(x) d_{\mathbb{K}_2} x. \tag{55}$$

Proof. Let $\{\mathrm{R}_i\}_{i=1}^{m}$ be a $\mathbb{K}_2$-compatible partition of f such that $f(x) = \lambda_i$ for all $x \in \mathbf{R}_{i_{\mathbb{K}}}{}^{\circ}$. For all $i \in \{1, 2, \cdots, m\}$, there is a sequence of $\mathbb{K}_1$-subrectangular $\{\mathbf{S}_i^p\}_p$ of $\mathbf{R}$ such that

$$\begin{aligned} &S_i^p \subset S_i^{(p+1)}, p \in \mathbb{N}, \\ &\lim_{p \longrightarrow \infty} \mathbf{S}_i^p = \bigcup_{p=0}^{\infty} \mathbf{S}_i^p = \mathbf{R}_i, \quad i \in \{1, 2, \cdots, m\}. \end{aligned} \tag{56}$$

Set

$$T_p = cl\left(\mathbf{R} \setminus \bigcup_{i=1}^{m} \mathbf{S}_i^p\right), \tag{57}$$

where $cl(\mathbf{R} \setminus \bigcup_{i=1}^{m} \mathbf{S}_i^p)$ is the closure of $\mathbf{R} \setminus \bigcup_{i=1}^{m} \mathbf{S}_i^p$. Then, there is a sequence $\{T_j^p\}_{j=1}^{k_p}$ of $\mathbb{K}_1$-subrectangular of $\mathbf{R}$ such that

$$T_p = \bigcup_{j=1}^{k_p} \mathbf{T}_j^p \text{ and } \overset{\circ}{T}{}_j^p \cap \overset{\circ}{T}{}_{j'}^p = \varnothing, \quad j \ne j'. \tag{58}$$

Now, consider the functions

$$\psi_p(x) = \begin{cases} f(x), & x \in \bigcup_{j=1}^{m} \left(\mathbf{S}_j^p\right)_{\mathbb{K}_1}, \\ M = \sup_{x \in \mathbf{R}_{\mathbb{K}_2}} f(x), & \text{else}, \end{cases}$$
$$\varphi_p(x) = \begin{cases} f(x), & x \in \bigcup_{j=1}^{m} \left(\mathbf{S}_j^p\right)_{\mathbb{K}_1}, \\ m = \inf_{x \in \mathbf{R}_{\mathbb{K}_2}} f(x), & \text{else}. \end{cases} \tag{59}$$

We have $\pi = \{T_j^p\}_{j=1}^{k_p} \cup \{\mathbf{S}_j^p\}_{j=1}^{m}$ is a $\mathbb{K}_1$-compatible partition of φ_p and ψ_p on $\mathbf{R}$; moreover,

$$\varphi_p(x) \le f(x) \le \psi_p(x) \text{ for all } x \in \mathbf{R}_{\mathbb{K}_1},$$
$$\int_{\mathbf{R}} \left(\psi_P(x) - \varphi_p(x)\right) d_{\mathbb{K}_1} x = \sum_{j=1}^{k_p} (M-m)\operatorname{vol}\left(T_j^p\right) = (M-m)\operatorname{vol}\left(T_p\right). \tag{60}$$

Since $(T_p)_p$ is a decreasing sequence, then $\lim_{p \longrightarrow \infty} \mathbf{T}_p = \bigcap_{p=0}^{\infty} \mathbf{T}_p = \varnothing$ then $\lim_{p \longrightarrow \infty} \operatorname{vol}(T_p) = 0$. Hence, f is $\mathbb{K}_1$-Riemann integrable. Moreover,

$$\lim_{p \to \infty} \int_{\mathbf{R}} \psi_p(x) d_{\mathbb{K}_1} x = \lim_{p \to \infty} \left(\sum_{j=1}^{k_p} M \operatorname{vol}\left(T_j^p\right) + \sum_{j=1}^{m} \lambda_j \operatorname{vol}\left(S_j^p\right) \right)$$
$$= \sum_{j=1}^{m} \lambda_j \lim_{p \to \infty} \operatorname{vol}\left(S_j^p\right) = \sum_{j=1}^{m} \lambda_j \operatorname{vol}(R_j) = \int_{\mathbf{R}} f(x) d_{\mathbb{K}_2} x. \tag{61}$$

Theorem 25. *Let $\mathbb{K}_1 \subseteq \mathbb{K}_2$ be two subfields of $\mathbb{R}$. Then, $\mathscr{R}_{\mathbb{K}_2}(\mathbf{R}) \subset \mathscr{R}_{\mathbb{K}_1}(\mathbf{R})$ and*

$$\int_{\mathbf{R}} f(x) d_{\mathbb{K}_1} x = \int_{\mathbf{R}} f(x) d_{\mathbb{K}_2} x \text{ for all } f \in \mathscr{R}_{\mathbb{K}_2}(\mathbf{R}). \tag{62}$$

Proof. Let $f \in \mathscr{R}_{\mathbb{K}_2}(\mathbf{R})$, given $\varepsilon > 0$, there exist $\varphi, \psi \in \mathscr{E}_{\mathbb{K}_2}(\mathbf{R})$ such that

$$\varphi(x) \le f(x) \le \psi(x) \text{ for all } x \in \mathbb{R}_{\mathbb{K}_2},$$
$$\int_{\mathbb{R}} (\psi(x) - \varphi(x)) d_{\mathbb{K}_2} x < \varepsilon. \tag{63}$$

By Lemma 24, there exist two sequences $(\tilde{\varphi}_n)_n$ and $(\tilde{\psi}_n)_n$ of $\mathbb{K}_1$-step functions such that

$$\tilde{\varphi}_n(x) \le \varphi(x) \le \psi(x) \le \tilde{\psi}_n(x), \quad x \in \mathbf{R}_{\mathbb{K}_1}, \tag{64}$$

with

$$\lim_{n \longrightarrow \infty} \int_{\mathbb{R}} \tilde{\varphi}_n(x) d_{\mathbb{K}_1} x = \int_{\mathbb{R}} \varphi(x) d_{\mathbb{K}_2} x \text{ and}$$
$$\lim_{n \longrightarrow \infty} \int_{\mathbb{R}} \tilde{\psi}_n(x) d_{\mathbb{K}_1} x = \int_a^b \varphi(x) d_{\mathbb{K}_2} x. \tag{65}$$

Then, there is $n_0 \in \mathbb{N}$ such that

$$-\int_{\mathbb{R}} \tilde{\varphi}_{n_0}(x) d_{\mathbb{K}_1} x + \varepsilon \le -\int_{\mathbb{R}} \varphi(x) d_{\mathbb{K}_2} x,$$
$$\int_{\mathbf{R}} \tilde{\psi}_{n_0}(x) d_{\mathbf{K}_1} x - \varepsilon \le \int_{\mathbf{R}} \psi(x) d_{\mathbf{K}_2} x. \tag{66}$$

Hence,

$$\int_{\mathbf{R}} \tilde{\psi}_{n_0}(x) d_{\mathbb{K}_1} x - \int_{\mathbf{R}} \tilde{\varphi}_{n_0}(x) d_{\mathbb{K}_1} x < \varepsilon. \tag{67}$$

Therefore, $f \in \mathscr{R}_{\mathbb{K}_1}|(\mathbf{R})$ and

$$\int_{\mathbf{R}} f(x) d_{\mathbb{K}_1} x = \int_{\mathbf{R}} f(x) d_{\mathbb{K}_2} x. \tag{68}$$

Corollary 26. *Let $f : \mathbf{R} \longrightarrow \mathbb{R}$ be a uniformly radially $\mathbb{K}$-continuous function, then it is $\mathbb{K}$-Riemann integrable on any subrectangular $\mathbf{S}$ of $\mathbf{R}$.*

Proof. Let $\mathbf{S}$ be a subrectangular of $\mathbf{R}$. By Proposition 3, f is uniformly continuous on $\mathbf{S}_{\mathbb{K}}$. Since $cl(\mathbf{S}_{\mathbb{K}}) = \mathbf{S}$, where $cl(\mathbf{S}_{\mathbb{K}})$ is the closure of $\mathbf{S}_{\mathbb{K}}$, then there exists a unique continuous function $g : \mathbf{S} \longrightarrow \mathbb{R}$ such that $g(x) = f(x), x \in \mathbf{S}_{\mathbb{K}}$. Since g is Riemann integrable on $\mathbf{S}$ then, using Theorem 25, g is $\mathbb{K}$-Riemann integrable on $\mathbf{S}$; moreover,

$$\int_{\mathbf{S}} f(x) d_{\mathbb{K}} x = \int_{\mathbf{S}} g(x) dx. \tag{69}$$

Lemma 27. *Let $f : \mathbf{R} = \mathbf{R}_1 \times \mathbf{R}_2 \longrightarrow \mathbb{R}$ be a $\mathbb{K}$-step function on $\mathbf{R}$. Then, there is a $\mathbb{K}$-partition $\{\mathbf{R}_1^i\}_{i=1}^{r}$ of $\mathbf{R}_1$ and a $\mathbb{K}$-partition $\{\mathbf{R}_2^j\}_{j=1}^{s}$ of $\mathbf{R}_2$ such that the function*

$$f_x : \quad \mathbf{R}_2 \longrightarrow \mathbb{R}, \quad y \mapsto f(x, y) \tag{70}$$

is a $\mathbb{K}$-step function and $\{\mathbf{R}_2^j\}_{j=1}^{s}$ is a $\mathbb{K}$-compatible partition of f_x whenever $x \in \bigcup_{i=1}^{r} \mathbf{R}^{i\circ}{}_{1\mathbb{K}}$. Moreover, the function

$$F(x) = \begin{cases} \int_{\mathbf{R}_2} f(x, y) d_{\mathbb{K}} y, & x \in \bigcup_{i=1}^{r} \overset{\circ}{\mathbf{R}}{}^i{}_{1\mathbb{K}}, \\ 0, & \text{else} \end{cases} \tag{71}$$

is a $\mathbb{K}$-step function on $\mathbf{R}_1$ and

$$\int_{\mathbf{R}_1} F(x)d_{\mathbb{K}}x = \int_{\mathbf{R}} f(x,y)d_{\mathbb{K}}xd_{\mathbb{K}}y. \tag{72}$$

Proof. Let $\pi = \{\mathbf{T}_i\}_{i=1}^m$ be a $\mathbb{K}$-partition of $\mathbf{R}$, then there exists a $\mathbb{K}$-refinement $\{\mathbf{T}_1^i \times \mathbf{T}_2^j\}_{1\le i\le r}^{1\le j\le s}$ of π such that $\{\mathbf{T}_1^i\}_{i=1}^r$ and $\{\mathbf{T}_2^j\}_{j=1}^s$ are two $\mathbb{K}$-partitions of $\mathbf{R}_1$ and $\mathbf{R}_2$, respectively. Since f is a $\mathbb{K}$-step function on $\mathbf{R}$, then there are two $\mathbb{K}$-partitions $\{\mathbf{R}_1^i\}_{i=1}^r$ and $\{\mathbf{R}_2^j\}_{j=1}^s$ of $\mathbf{R}_1$ and $\mathbf{R}_2$, respectively, such that $\{\mathbf{R}_1^i \times \mathbf{R}_2^j\}_{1\le i\le r}^{1\le j\le s}$ is a $\mathbb{K}$-partition of $\mathbf{R}$ and f is constant on each $\mathbf{R}_1^{i\,\circ}{}_{\mathbb{K}} \times \mathbf{R}_2^{j\,\circ}{}_{\mathbb{K}}$; hence, f_x is a $\mathbb{K}$-step function on $\mathbf{R}_2$.Now, we can say that F is a $\mathbb{K}$-step function on $\mathbf{R}_1$.Let us show the last equality in the theorem. Consider the two $\mathbb{K}$-partitions $\{\mathbf{R}_1^i\}_{i=1}^r$ and $\{\mathbf{R}_2^j\}_{j=1}^s$ given in the first step, then for all $i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, s$, there exist λ_{ij} such that $f(x,y) = \lambda_{ij}$ whenever $(x,y) \in \mathbf{R}_1^{i\,\circ}{}_{\mathbb{K}} \times \mathbf{R}_2^{j\,\circ}{}_{\mathbb{K}}$; therefore, we have

$$F(x) = \begin{cases} \sum_{j=1}^{s} \lambda_{ij}\mathrm{vol}\left(\mathbf{R}_2^j\right), & x \in \overset{\circ}{\mathbf{R}}{}_{1\,\mathbb{K}}^{\,i}, \quad i = 1, 2, \cdots, r, \\ 0, & \text{else}, \end{cases}$$

$$\int_{\mathbf{R}_1} F(x)d_{\mathbb{K}}x = \sum_{i=1}^{r}\left(\sum_{j=1}^{s}\lambda_{ij}\mathrm{vol}\left(\mathbf{R}_2^j\right)\right)\mathrm{vol}\left(\mathbf{R}_1^i\right) = \sum_{i=1}^{r}\sum_{j=1}^{s}\lambda_{ij}\mathrm{vol}\left(\mathbf{R}_2^j \times \mathbf{R}_1^i\right) = \int_{\mathbf{R}} f(x,y)d_{\mathbb{K}}xd_{\mathbb{K}}y. \tag{73}$$

Theorem 28 (Fubini theorem). *Let $f : \mathbf{R} = \mathbf{R}_1 \times \mathbf{R}_2 \longrightarrow \mathbb{R}$ be a $\mathbb{K}$-Riemann integrable function on $\mathbf{R}$. If the function*

$$\begin{aligned} f_x : \quad & \mathbf{R}_2 \longrightarrow \mathbb{R}, \\ & y \mapsto f(x,y) \end{aligned} \tag{74}$$

is $\mathbb{K}$-Riemann integrable on $\mathbf{R}_2$ for all $x \in \mathbf{R}_{1\,\mathbb{K}}$, then the function

$$F(x) = \begin{cases} \int_{\mathbf{R}_2} f(x,y)d_{\mathbb{K}}y, & x \in \mathbf{R}_{1\mathbb{K}}, \\ 0, & \text{else} \end{cases} \tag{75}$$

is a $\mathbb{K}$-Riemann integrable function on $\mathbf{R}_1$ and

$$\int_{R} f(x,y)d_{\mathbb{K}}xd_{\mathbb{K}}y = \int_{\mathbf{R}_1} F(x)d_{\mathbb{K}}x. \tag{76}$$

Proof. Let $\varepsilon > 0$, there exist two $\mathbb{K}$-step functions φ and ψ on $\mathbf{R}$ such that

$$\begin{gathered} \varphi(x,y) \le f(x,y) \le \psi(x,y) \text{ for all } (x,y) \in \mathbf{R}_{1\mathbb{K}} \times \mathbf{R}_{2\mathbb{K}}, \\ \int_{\mathrm{R}} (\psi(x,y) - \varphi(x,y))d_{\mathbb{K}}xd_{\mathbb{K}}y < \varepsilon. \end{gathered} \tag{77}$$

Now, we can find two $\mathbb{K}$-step functions Φ and Ψ such that

$$\begin{gathered} \int_{\mathbf{R}_1} \Phi(x)d_{\mathbb{K}}x = \int_{\mathbf{R}} \varphi(x,y)d_{\mathbb{K}}xd_{\mathbb{K}}y, \\ \int_{\mathbf{R}_1} \Psi(x)d_{\mathbb{K}}x = \int_{\mathbf{R}} \psi(x,y)d_{\mathbb{K}}xd_{\mathbb{K}}y, \\ \Phi(x) \le F(x) \le \Psi(x), \quad x \in \mathbf{R}_{1\mathbb{K}}. \end{gathered} \tag{78}$$

Since

$$\int_{\mathbf{R}_1} (\Psi - \Phi)(x)d_{\mathbb{K}}x = \int_{\mathbf{R}} (\psi - \varphi)(x,y)d_{\mathbb{K}}xd_{\mathbb{K}}y < \varepsilon, \tag{79}$$

then F is $\mathbb{K}$-Riemann integrable on $\mathbf{R}$; moreover,

$$\int_{\mathbf{R}_1} F(x)d_{\mathbb{K}}x = \int_{\mathbf{R}} f(x,y)d_{\mathbb{K}}xd_{\mathbb{K}}y. \tag{80}$$

3.2. Biconvex Function. Now, we give the notion of $\mathbb{K}$-biconvex function and we will prove that any $\mathbb{K}$-biconvex function is $\mathbb{K}$-Riemann integrable.Since a function is Jensen-convex if and only if it is $\mathbb{Q}$-convex, for more details, see [3, 4]. Then, we can easily show that a function is Jensen-biconvex if and only if it is $\mathbb{Q}$-biconvex.

Definition 29. A mapping $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is said to be $\mathbb{K}$-bilinear, if it satisfies

(i) $f(\lambda x + y, z) = \lambda f(x,z) + f(y,z)$

(ii) $f(x, \lambda y + z) = \lambda f(x,y) + f(x,z)$

for every $\lambda \in \mathbb{K}$ and for every $x, y, z \in \mathbb{R}$.

Definition 30. A function $f : [a,b] \times [c,d] \longrightarrow \mathbb{R}$ is said to be

(1) Jensen-biconvex if

(i) $f((x+y)/2, z) \le (f(x,z) + f(y,z))/2$ for every $x, y \in [a,b]$ and $z \in [c,d]$

(ii) $f(x, (y+z)/2) \le (f(x,y) + f(x,z))/2$ for every $x \in [a,b]$ and $y, z \in [c,d]$

(2) $\mathbb{K}$-biconvex if

(i) $f(\lambda x + (1-\lambda)y, z) \le \lambda f(x,z) + (1-\lambda)f(y,z)$, for every $x, y \in [a,b]$ and $z \in [c,d]$ and $\lambda \in [0,1]_{\mathbb{K}}$

(ii) $f(x, \lambda y + (1-\lambda)z) \le \lambda f(x,y) + (1-\lambda)f(x,z)$ for every $x \in [a,b]$ and $y, z \in [c,d]$ and $\lambda \in [0,1]_{\mathbb{K}}$

Now, we claim that every $\mathbb{K}$-biconvex function is $\mathbb{K}$-Riemann integrable. For the proof, we need the following lemma (see [5] p.143)

Lemma 31. *Let $I \subset \mathbb{R}$ be an open interval and $g : I \longrightarrow \mathbb{R}$ be a convex function. Then,*

$$\begin{aligned}\frac{g(x)-g(x-nd)}{n} &\leq \frac{g(x)-g(x-md)}{m}\\ &\leq \frac{g(x+md)-g(x)}{m}\\ &\leq \frac{g(x+nd)-g(x)}{n},\end{aligned} \tag{81}$$

for every $x \in I, d \in \mathbb{R}$ and $m, n \in \mathbb{N}$ such that $0 < m < n$ and $x \pm nd \in I$.

Theorem 32. *Let $f : \mathbf{R} \longrightarrow \mathbb{R}$ be a bounded and $\mathbb{K}$-biconvex function, and let $\mathbf{S}$ be a closed rectangular box such that $\mathbf{S} \subset \mathbf{R}^{\circ}$. Then, f is uniformly continuous on $\mathbf{S}_{\mathbb{K}}$.*

Proof. Put $\mathbf{S} = [a, b] \times [c, d] \subset \mathbf{R}^{\circ}$, there exists a positive number $\rho \in \mathbb{K}_+$ such that

$$[a_1, b_1] \times [c_1, d_1] \subset \overset{\circ}{\mathbf{R}}, \tag{82}$$

with $a_1 = a - \rho(b-a), b_1 = b + \rho(b-a), c_1 = c - \rho(d-c)$ and $d_1 = d + \rho(d-c)$. We show easily that

$$[a, b]_{\mathbb{K}} \times [c, d]_{\mathbb{K}} \subset [a_1, b_1]_{\mathbb{K}} \times [c_1, d_1]_{\mathbb{K}}. \tag{83}$$

It is sufficient to show that $[a, b]_{\mathbb{K}} \subset [a_1, b_1]_{\mathbb{K}}$. If $u \in [a, b]_{\mathbb{K}}$, then there exists $\lambda \in [0, 1] \cap \mathbb{K}$ such that $u = \lambda a + (1-\lambda)b$. Put $\nu = (\rho + \lambda)/(1 + 2\rho)$, then $\nu \in [0, 1] \cap \mathbb{K}$. Moreover,

$$\nu a_1 + (1-\nu)b_1 = \frac{\rho+\lambda}{1+2\rho}[a - \rho(b-a)] + \frac{1+\rho-\lambda}{1+2\rho}[b + \rho(b-a)] = u. \tag{84}$$

Then,

$$[a, b]_{\mathbb{K}} \times [c, d]_{\mathbb{K}} \subset [a_1, b_1]_{\mathbb{K}} \times [c_1, d_1]_{\mathbb{K}}. \tag{85}$$

On the other hand, since f is bounded, there exist two real numbers $m \leq M$ such that

$$m \leq f(x, y) \leq M \text{ for all } (x, y) \in \mathbf{R}. \tag{86}$$

Now, given an arbitrary $\varepsilon > 0$, put $n = [2((M-m)/\varepsilon) + 1]$ (the integral part of $2((M-m)/\varepsilon) + 1$). We have $n \in \mathbb{N}$ and

$$2\frac{M-m}{n} < \varepsilon. \tag{87}$$

Further, put $\eta = \rho/n \min\{|b-a|, |d-c|\}$. Take arbitrary $(x, y), (x', y') \in [a, b]_{\mathbb{K}} \times [c, d]_{\mathbb{K}}$ such that

$$\left\|(x, y) - \left(x', y'\right)\right\|_{\infty} = \max\left\{|x - x'|, |y - y'|\right\} < \eta. \tag{88}$$

Then, there exist $\alpha_i, \beta_i \in [0, 1] \cap \mathbb{K}$, $i = 1, 2$ such that

$$\begin{aligned} x &= \alpha_1 a + (1-\alpha_1)b,\\ x' &= \alpha_2 a + (1-\alpha_2)b,\\ y &= \beta_1 c + (1-\beta_1)d,\\ y' &= \beta_2 c + (1-\beta_2)d.\end{aligned} \tag{89}$$

Whence

$$\begin{aligned} x - x' &= (\alpha_2 - \alpha_1)(b-a),\\ y - y' &= (\beta_2 - \beta_1)(d-c),\end{aligned} \tag{90}$$

and by (88), we get

$$n|\alpha_1 - \alpha_2| < \rho, \tag{91}$$

$$n|\beta_1 - \beta_2| < \rho. \tag{92}$$

Consider the numbers

$$\begin{aligned} \mu_{12} &= \frac{\rho + \alpha_1 \pm n(\alpha_2 - \alpha_1)}{1+2\rho},\\ \lambda_{12} &= \frac{\rho + \beta_1 \pm n(\beta_2 - \beta_1)}{1+2\rho}.\end{aligned} \tag{93}$$

By (92), we have

$$\begin{aligned} 0 \leq \frac{\alpha_1}{1+2\rho} < \mu_{12} < \frac{\alpha_1 + 2\rho}{1+2\rho} \leq 1,\\ 0 \leq \frac{\beta_1}{1+2\rho} < \lambda_{12} < \frac{\beta_1 + 2\rho}{1+2\rho} \leq 1.\end{aligned} \tag{94}$$

Then, $\mu_{12}, \lambda_{12} \in [0, 1] \cap \mathbb{K}$. Now,

$$\begin{aligned} \mu_{12}a_1 + (1-\mu_{12})b_1 &= x \pm n\left(x - x'\right) \in [a_1, b_1]_{\mathbb{K}},\\ \lambda_{12}c_1 + (1-\mu_{12})d_1 &= y \pm n\left(y - y'\right) \in [c_1, d_1]_{\mathbb{K}}.\end{aligned} \tag{95}$$

Since the functions

$$\begin{aligned} f_y : \quad & I \longrightarrow \mathbb{R},\\ & x \mapsto f(x, y),\end{aligned} \quad \text{and} \quad \begin{aligned} f_{x'} : \quad & J \longrightarrow \mathbb{R},\\ & y \mapsto f\left(x', y\right)\end{aligned} \tag{96}$$

are $\mathbb{K}$-convex, taking in Lemma 31 $x'' = x, d = x - x'$, and $m = 1$, we get

$$\frac{f(x, y) - f(x - nd, y)}{n} \leq f(x, y) - f\left(x', y\right) \leq \frac{f\left(x' + nd, y\right) - f\left(x', y\right)}{n}, \tag{97}$$

and taking $y'' = y, d' = y - y'$, and $m = 1$, we find that

$$\frac{f\left(x', y\right) - f\left(x', y - nd'\right)}{n} \leq f\left(x', y\right) - f\left(x', y'\right) \leq \frac{f\left(x', y + nd'\right) - f\left(x', y\right)}{n}. \tag{98}$$

Since

$$m \leq f(x, y) \leq M \text{ for all } (x, y) \in \mathbf{R}, \tag{99}$$

then

$$-2\frac{M - m}{n} \leq f(x, y) - f\left(x', y'\right) \leq 2\frac{M - m}{n}, \tag{100}$$

which mean, by using (87), that

$$-\varepsilon < f(x, y) - f\left(x', y'\right) < \varepsilon, \tag{101}$$

Hence, f is uniformly continuous on $\mathbf{S}_{\mathbb{K}}$.

Corollary 33. *Let $f : \mathbf{R} \longrightarrow \mathbb{R}$ be a bounded and $\mathbb{K}$-biconvex function. Then, f is $\mathbb{K}$-Riemann integrable on any subrectangular $\mathbf{S}$ such that $\mathbf{S} \subset \mathbf{R}^{\circ}$.*

Proposition 34. *Let $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a $\mathbb{K}$-bilinear function. Then, it is $\mathbb{K}$-Riemann integrable on every rectangular box $\mathbf{R} = [a, b] \times [c, d]$. Moreover,*

$$\int_{\mathbf{R}} f(x, y) d_{\mathbb{K}} x d_{\mathbb{K}} y = f\left(\frac{a + b}{2}, \frac{c + d}{2}\right)(b - a)(d - c). \tag{102}$$

Proof. f is a $\mathbb{K}$-bilinear function; then, it is a $\mathbb{K}$-biconvex function; hence, f is a $\mathbb{K}$-Riemann integrable function. Since

$$\begin{aligned} f_y : \quad & [a, b] \longrightarrow \mathbb{R}, \quad y \in [c, d], \\ & y \mapsto f(x, y), \\ f_x : \quad & [c, d] \longrightarrow \mathbb{R}, \quad x \in [a, b], \\ & y \mapsto f(x, y) \end{aligned} \tag{103}$$

are $\mathbb{K}$-convex function, then by ([1], Theorem 7), they are $\mathbb{K}$-Riemann integrable functions; moreover,

$$\begin{aligned} \int_c^d f(x, y) d_{\mathbb{K}} y &= (d - c) f\left(x, \frac{c + d}{2}\right), \quad x \in [a, b], \\ \int_a^b f(x, y) d_{\mathbb{K}} x &= (b - a) f\left(\frac{a + b}{2}, y\right), \quad y \in [c, d]. \end{aligned} \tag{104}$$

By Fubini Theorem 3.18, we get

$$\begin{aligned} \int_{\mathbf{R}} f(x, y) d_{\mathbb{K}} x d_{\mathbb{K}} y &= \int_a^b \left(\int_c^d f(x, y) d_{\mathbb{K}} y\right) d_{\mathbb{K}} x \\ &= \int_a^b (d - c) f\left(x, \frac{c + d}{2}\right) d_{\mathbb{K}} x \\ &= f\left(\frac{a + b}{2}, \frac{c + d}{2}\right)(b - a)(d - c). \end{aligned} \tag{105}$$

Remark 35. In this last part, we only took the case of two variables. By the same way, we can extend these results to the case of n variables.

4. Conclusion

We plan, thereafter, to extend the definition of k-integrability to functions defined on bounded subsets and to generalize Fubini's theorem.

Data Availability

Data are available in https://link.springer.com/article/10.1007/s00010-017-0472-0.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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