Research Article
The Multiple K-Riemann Integral

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Received 27 October 2020; Accepted 10 March 2021; Published 22 April 2021

Academic Editor: Wing-Sum Cheung

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The aim of this paper is to extend the notion of K-Riemann integrability of functions defined over [a, b] to functions defined over a rectangular box of $\mathbb{R}^n$. As a generalization of step functions, we introduce a notion of K-step functions which allows us to give an equivalent definition of the K-Riemann integrable functions.

1. Introduction

The motivation of this work results from the notion of the K-Riemann integral of a function defined over an interval $I$ of $\mathbb{R}$ recently introduced by Olbrys’ [1]. Our aim of this paper is to give an equivalent definition of the K-Riemann integrability by the use of K-step functions and extend this notion to functions defined over a rectangular box of $\mathbb{R}^n$. After introducing the notion of K-multiple integral, we give its properties and we show a K-version of Fubini’s theorem.

Throughout this paper, $K$ denotes a subfield of the field of real numbers $\mathbb{R}$. We denote the set of positive elements of $K$ by $K^+$. For two real numbers $a < b$, we denote

\begin{equation}
[a, b]_K = \{ \lambda a + (1 - \lambda) b : \lambda \in [0, 1] \cap K \},
\end{equation}

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\end{equation}

If

\begin{equation}
\mathbb{R} = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]
\end{equation}

is a rectangular box of $\mathbb{R}^n$, we denote

\begin{equation}
\mathbb{R}_K = [a_1, b_1]_K \times [a_2, b_2]_K \times \cdots \times [a_n, b_n]_K,
\end{equation}

\begin{equation}
\mathbb{R}_K = [a_1, b_1]_K \times [a_2, b_2]_K \times \cdots \times [a_n, b_n]_K.
\end{equation}

Following [1], we define the set of the K-partitions of the interval $[a, b]$ by

\begin{equation}
\mathcal{P}_K^{[a,b]} = \bigcup_{n \geq 1} \{ (t_0, t_1, \ldots, t_n) : a = t_0 < t_1 < \cdots < t_n = b, \ t_i \in [a, b]_K \}.
\end{equation}

As in [1], we give a generalization of the radially K-continuous and uniformly radially K-continuous functions and propositions easy to prove. We say that $f$ is radially K-continuous if it is radially K-continuous at every point.

Definition 1. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called radially K-continuous at a point $(x_1, x_2, \ldots, x_n) \in \mathbb{R}$ if for every $(y_1, y_2, \ldots, y_n) \in \mathbb{R}$

\begin{equation}
f(x_1 + \alpha_1 (y_1 - x_1), \ldots, x_n + \alpha_n (y_n - x_n)) \rightarrow_{K/(\alpha_1, \ldots, \alpha_n) \rightarrow (0, \ldots, 0)} f(x_1, x_2, \ldots, x_n).
\end{equation}

Definition 2. We say that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is uniformly radially K-continuous if for any $(x_1, x_2, \ldots, x_n) \in \mathbb{R}$ and $(y_1, y_2, \ldots, y_n) \in \mathbb{R}$, the mapping
is uniformly continuous.

Every continuous and any uniformly continuous function in the usual sense is radially $|K|$-continuous and uniformly radially $|K|$-continuous, respectively.

**Proposition 3.**

(i) A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is radially $|K|$-continuous if and only if for every subrectangular box $S$ of $\mathbb{R}$, the function $f|_{S_{K}}$ is continuous.

(ii) A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is uniformly radially $|K|$-continuous if and only if for every subrectangular box $S$ of $\mathbb{R}$, the function $f|_{S_{K}}$ is uniformly continuous.

### 2. $|K|$-Step Functions and $|K|$-Riemann Integral

In the following definition, we introduce the notion of the $K$-step functions which are a generalization of step functions.

**Definition 4.** A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be a $K$-step function on $[a, b]$, if there exists a $K$-partition $\pi = \{t_0, t_1, \ldots, t_n\}$ of $[a, b]$ for which $f$ is constant on each set $[t_i, t_{i+1})$. In this case, we say that $\pi$ is a $K$-compatible partition of $f$ on $[a, b]$.

We denote the set of the $K$-step functions on $[a, b]$ by $\mathcal{S}_K([a, b])$.

**Remark 5.**

(i) Every $|K|$-step function on $[a, b]$ is bounded on $[a, b]$, $|K|$.

(ii) If $K = \mathbb{R}$, we get the usual definition of a step function.

**Example 1.**

(i) Every constant function on $[a, b]$ is a $K$-step function.

(ii) Consider the following functions:

$$f_1(x) = \begin{cases} 1, & x \in [0, 1] \setminus \{0, 1\}, \\ 0, & x \in [0, 1] \setminus \{0, 1\}, \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 1, & x \in \left[0, \sqrt{3}\right], \\ 0, & x \in \left[\sqrt{3}, 2\right]. \end{cases}$$

It is easy to show that $f_1$ is a $Q$-step function but it is not a $\mathbb{R}$-step function on $[0, 1]$, and $f_2$ is a $\mathbb{R}$-step function, but it is not a $Q$-step function on $[0, 1]$.

Now, suppose that $f : [a, b] \rightarrow \mathbb{R}$ is a $K$-step function on $[a, b]$ and let $\pi = \{t_0, t_1, \ldots, t_m\}$ be a $K$-compatible partition of $f$, with $f(x) = \lambda_i$ whenever $x \in [t_{i-1}, t_i]$ for all $i = 1, 2, \ldots, m$. Consider the following number:

$$I_{K}(f, \pi) = \sum_{i=1}^{m} (t_i - t_{i-1})\lambda_i.$$

We can easily show that the number $I_{K}(f, \pi)$ does not depend on the choice of the $K$-compatible partition.

**Definition 6.** Let $f : [a, b] \rightarrow \mathbb{R}$ be a $K$-step function on $[a, b]$, and $\pi$ a $K$-compatible partition of $f$. The real number $I_{K}(f, \pi)$ is called the $K$-integral of $f$ on $[a, b]$ and is denoted by

$$\int_{a}^{b} f(t) d_{K}t.$$

As for the introduction of the usual Riemann integral, we extend the definition of the $K$-Riemann integral to a broader class of functions than $|K|$-step functions.

**Definition 7.** A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be $K$-Riemann integrable on $[a, b]$, if for every $\varepsilon > 0$, there exist two $K$-step functions $\varphi, \psi$ on $[a, b]$ such that

$$\varphi(x) \leq f(x) \leq \psi(x) \text{ for all } x \in [a, b],$$

$$\int_{a}^{b} (\psi(t) - \varphi(t)) d_{K}t < \varepsilon.$$

We denote the set of the $K$-Riemann integrable functions by $\mathcal{R}_{K}([a, b])$.

**Remark 8.** One can easily verify that $\mathcal{S}_K([a, b])$ is a part of $\mathcal{R}_{K}([a, b])$.

**Let** $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function on $[a, b]$, $|K|$ and set

$$E_{K}(f, [a, b]) = \left\{ \int_{a}^{b} \varphi(t) d_{K}t : \varphi \in \mathcal{S}_{K}([a, b]) \right\},$$

$$E_{K}(f, [a, b]) = \left\{ \int_{a}^{b} \varphi(t) d_{K}t : \varphi \in \mathcal{S}_{K}([a, b]) \right\}.$$

Note that $\alpha \leq \beta$ for all $a \in E_{K}(f, [a, b])$ and $\beta \in E_{K}(f, [a, b])$.

**Proposition 9.** Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function on $[a, b]$, $|K|$. Then, $f \in \mathcal{R}_{K}([a, b])$ if and only if $\inf E_{K}(f, [a, b]) = \sup E_{K}(f, [a, b])$.

**Proof.** Suppose that $f \in \mathcal{R}_{K}([a, b])$. Given $\varepsilon > 0$, then there exist $\varphi, \psi \in \mathcal{S}_{K}([a, b])$ such that

$$\varphi(x) \leq f(x) \leq \psi(x) \text{ for all } x \in [a, b] \text{ and } \int_{a}^{b} \psi(t) d_{K}t - \int_{a}^{b} \varphi(t) d_{K}t < \varepsilon.$$
Proposition 11. A function \( f : [a, b] \rightarrow \mathbb{R} \) is \( \mathbb{K} \)-Riemann integrable on \([a, b]\) in the sense of Definition 7 if and only if it is \( \mathbb{K} \)-Riemann integrable on \([a, b]\) in the sense of (1), Definition 7 (meaning the two definitions are equivalents). Furthermore, when \( f \in \mathcal{R}_K([a, b]) \), we have

\[
\inf \mathcal{E}_K(f, [a, b]) = \sup \mathcal{E}_K(f, [a, b]) = \int_a^b f(t)\,d\mathcal{K}t.
\]

Proof. Suppose that \( f \in \mathcal{R}_K([a, b]) \) in the sense of (11, Definition 7). Given \( \varepsilon > 0 \), then there exists a partition \( \pi = \{t_0, t_1, \ldots, t_m\} \) of \([a, b]\) such that

\[
\sum_{i=1}^m (t_i - t_{i-1})M_i - \sum_{i=1}^m (t_i - t_{i-1})m_i < \varepsilon,
\]

where \( M_i = \sup f(x) \) and \( m_i = \inf f(x) \), \( i = 1, 2, \ldots, m \). We have thus built two functions \( \varphi, \psi \in \mathcal{E}_K([a, b]) \) such that

\[
\varphi(x) \leq f(x) \leq \psi(x) \quad \text{for all } x \in [a, b],
\]

\[
\int_a^b \varphi(t)\,d\mathcal{K}t - \int_a^b \psi(t)\,d\mathcal{K}t < \varepsilon.
\]

Therefore, \( f \in \mathcal{R}_K([a, b]) \).
which shows
\[
\sum_{i=1}^{m} (a_i - a_{i-1})B_i = \sum_{i=1}^{m} (a_i - a_{i-1}) A_i \int_a^b \psi(t) d\kappa t - \int_a^b \psi(t) d\kappa t < e. 
\] (27)

This means that \( f \) is \( \kappa \)-Riemann integrable on \([a, b]\) in the sense of ([1], Definition 7).

Finally, if \( f \in \mathcal{R}_\kappa([a, b]) \), then the equalities (19) are ensured by Proposition 9 and the inequalities (26).

3. \( \kappa \)-Multiple Integral

This section is devoted to \( \kappa \)-multiple Riemann integral. For the theory of the classical multiple Riemann integral, see for instance [2].

Let \( \mathbf{R} = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \) be a closed rectangular box in \( \mathbb{R}^n \), where \( a_i, b_i \) are real numbers such that \( a_i < b_i \) for all \( i = 1, 2, \ldots, n \).

We denoted by
\[
\text{vol}(\mathbf{R}) = \prod_{i=1}^{n} (b_i - a_i),
\] (28) \( \mathbf{R}_\kappa = [a_1, b_1]_\kappa \times [a_2, b_2]_\kappa \times \cdots \times [a_n, b_n]_\kappa. \)

Definition 12. A rectangular box \( \mathbf{S} = [a'_1, b'_1] \times [a'_2, b'_2] \times \cdots \times [a'_n, b'_n]_\kappa \) is said to be a \( \kappa \)-subrectangular box of \( \mathbf{R} \) if \( a'_i, b'_i \in [a_i, b_i]_\kappa \) for all \( i = 1, 2, \ldots, n \).

Example 2. \([0, 3/4] \times [0, 1/2] \) is a \( \mathbb{Q} \)-subrectangular box of \([0, 1] \times [0, 1] \).

\([0, 1] \times [0, 2] \) is not a \( \mathbb{Q} \)-subrectangular box of \([0, \sqrt{3}] \times [0, 2] \).

Definition 13. A \( \kappa \)-partition of \( \mathbf{R} \) is a finite collection \( \pi = \{\mathbf{R}_i\}_{i=1}^{m} \) of \( \kappa \)-subrectangular box of \( \mathbf{R} \) such that
\[
\mathbf{R} = \bigcup_{i=1}^{m} \mathbf{R}_i \text{ and } \mathbf{R}_i \cap \mathbf{R}_j = \emptyset, \quad i \neq j,
\] (29)\( \mathbf{R}^*_i \) is the interior of \( \mathbf{R}_i \).

Definition 14.

(i) Let \( f : \mathbf{R} \longrightarrow \mathbb{R} \) be a bounded function over \( \mathbf{R}_\kappa. \) \( f \) is said to be a \( \kappa \)-step function on \( \mathbf{R} \) if there exists a \( \kappa \)-partition \( \pi = \{\mathbf{R}_i\}_{i=1}^{m} \) such that \( f \) is constant on \( \mathbf{R}_i \) for all \( i = 1, 2, \ldots, m. \) In this case, \( \pi \) is said to be a \( \kappa \)-compatible partition of \( f \) on \( \mathbf{R} \)

We denoted the set of the \( \kappa \)-step functions on \( \mathbf{R} \) by \( \mathfrak{B}_\kappa(\mathbf{R}). \)

(ii) A \( \kappa \)-refinement of a \( \kappa \)-partition \( \{\mathbf{R}_i\}_{i=1}^{m} \) of \( \mathbf{R} \) is another \( \kappa \)-partition \( \{\mathbf{S}_j\}_{j=1}^{n} \) of \( \mathbf{R} \) such that each \( \mathbf{S}_j \) is a \( \kappa \)-subrectangular box of some \( \mathbf{R}_i \).

Let \( f : \mathbf{R} \longrightarrow \mathbb{R} \) be a \( \kappa \)-step function and \( \pi = \{\mathbf{R}_i\}_{i=1}^{m} \) be a \( \kappa \)-compatible partition of \( f \) such that \( f(x) = \lambda_i \) for all \( x \in \mathbf{R}^*_i \), \( i = 1, 2, \ldots, m. \) We define the number
\[
I_\kappa(f, \pi) = \sum_{i=1}^{m} \lambda_i \text{vol}(\mathbf{R}_i). \] (30)

We will first show that \( I_\kappa(f, \pi) \) does not depend on the choice of the \( \kappa \)-compatible partition \( \pi \).

Lemma 15. Let \( f : \mathbf{R} \longrightarrow \mathbb{R} \) be a \( \kappa \)-step function and \( \pi \) be a \( \kappa \)-compatible partition of \( f \). Then, for every \( \kappa \)-refinement \( \pi' \) of \( \pi \), we have
\[
I_\kappa(f, \pi) = I_\kappa(f, \pi'). \] (31)

Proof. Let \( \pi = \{\mathbf{R}_i\}_{i=1}^{m} \) be a \( \kappa \)-compatible partition of \( f \) such that \( f(x) = \lambda_i \) for all \( x \in \mathbf{R}^*_i \), \( i = 1, 2, \ldots, m \). Let \( \pi' = \{\mathbf{S}_j\}_{j=1}^{n} \) be a \( \kappa \)-refinement of \( \pi \). For all \( i \in \{1, 2, \ldots, m\} \), there is \( i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, r\} \) such that \( \mathbf{R}_i = \bigcup_{j=1}^{k} \mathbf{S}_j \). Then,
\[
I_\kappa(f, \pi) = \sum_{i=1}^{m} \lambda_i \text{vol}(\bigcup_{j=1}^{k} \mathbf{S}_j) = \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{k} \text{vol}(\mathbf{S}_j) = \sum_{j=1}^{k} \sum_{i=1}^{m} \lambda_i \text{vol}(\mathbf{S}_j) = I_\kappa(f, \pi'). \] (32)

Corollary 16. Let \( f : \mathbf{R} \longrightarrow \mathbb{R} \) be a \( \kappa \)-step function and \( \pi \) be a \( \kappa \)-compatible partition of \( f \). The number \( I_\kappa(f, \pi) \) does not depend on the choice of the \( \kappa \)-compatible partition \( \pi \).

Proof. Let \( \pi = \{\mathbf{R}_i\}_{i=1}^{m} \) and \( \pi' = \{\mathbf{S}_j\}_{j=1}^{n} \) be two \( \kappa \)-compatible partitions of \( f \), then \( \pi'' = \{T_{ij}\}_{1 \leq i \leq m, j} \) where \( T_{ij} = \mathbf{R}_i \cap \mathbf{S}_j \), is a \( \kappa \)-refinement of \( \pi \) and \( \pi''. \) Therefore, by using Lemma 15, we get
\[
I_\kappa(f, \pi) = I_\kappa(f, \pi') = I_\kappa(f, \pi''). \] (33)

Definition 17. Let \( f : \mathbf{R} \longrightarrow \mathbb{R} \) be a \( \kappa \)-step function and \( \pi \) be a \( \kappa \)-compatible partition of \( f \). We called \( I_\kappa(f, \pi) \) the \( \kappa \)-Riemann integral of \( f \) on \( \mathbf{R} \) and we denote it by
\[
\int_{\mathbf{R}} f(x) d\kappa x. \] (34)

In the following proposition, we present some basic properties of \( \kappa \)-Riemann multiple integral of \( \kappa \)-step functions.

Proposition 18. Let \( f, g : \mathbf{R} \longrightarrow \mathbf{R} \) be two \( \kappa \)-step functions on \( \mathbf{R} \) and \( \alpha, \beta \in \mathbb{R}. \) Then,
The function $af + \beta g$ is a $\mathbb{K}$-step function on $\mathbb{R}$; moreover,

\[
\int_{\mathbb{R}} [af(x) + \beta g(x)]d\mathbb{K}x = a\int_{\mathbb{R}} f(x)d\mathbb{K}x + \beta \int_{\mathbb{R}} g(x)d\mathbb{K}x \quad (35)
\]

(2) If $f(x) \geq 0$, $x \in \mathbb{R}_{\mathbb{K}}$, then $\int_{\mathbb{R}} f(x)d\mathbb{K}x \geq 0$

(3) The absolute value $|f|$ is a $\mathbb{K}$-step function on $\mathbb{R}$ and

$$\int_{\mathbb{R}} |f(x)|d\mathbb{K}x \leq \int_{\mathbb{R}} |f(x)|d\mathbb{K}x$$

(4) If $\pi = \{S_j\}_{j=1}^m$ is a $\mathbb{K}$-partition on $\mathbb{R}$, then

$$\int_{\mathbb{R}} f(x)d\mathbb{K}x = \sum_{j=1}^m \int_{S_j} f(x)d\mathbb{K}x$$

Proof. (1) Let $\pi = \{R_j\}_{j=1}^m$ (resp $\pi' = \{S_j\}_{j=1}^r$) be a $\mathbb{K}$-compatible partition of $f$ (resp $g$), then $\pi'' = \{T_{ij}\}_{j,i=1}^{r,s}$, where $T_{ij} = R_j \cap S_i$, is a $\mathbb{K}$-compatible partition of $af + \beta g$; moreover, we have

\[
\int_{\mathbb{R}} [af(x) + \beta g(x)]d\mathbb{K}x = I_{\mathbb{K}}(af + \beta g, \pi'')
\]

\[
= aI_{\mathbb{K}}(f, \pi') + \beta I_{\mathbb{K}}(g, \pi')
\]

\[
= a\int_{\mathbb{R}} f(x)d\mathbb{K}x + \beta \int_{\mathbb{R}} g(x)d\mathbb{K}x \quad (36)
\]

(2) and (3) are evident.

(4) Let $\pi = \{R_j\}_{j=1}^m$ be a $\mathbb{K}$-compatible partition of $f$ such that $f(x) = \lambda_i$ for all $x \in R_i$, $i = 1, 2, \ldots, m$. Then,

\[
\int_{\mathbb{R}} f(x)d\mathbb{K}x = \sum_{i=1}^m \lambda_i \text{vol}(R_i) = \sum_{j=1}^r \lambda_j \text{vol}(R_i \cap S_j).
\]

\[
\int_{\mathbb{R}} f(x)d\mathbb{K}x = \sum_{i=1}^m \lambda_i \text{vol}(R_i \cap S_j) \quad (37)
\]

Since $\{R_i \cap S_j\}_{i=1}^m$ is a $\mathbb{K}$-compatible partition of $f$ on $S_j$, Then,

\[
\int_{\mathbb{R}} f(x)d\mathbb{K}x = \sum_{j=1}^r \int_{S_j} f(x)d\mathbb{K}x \quad (38)
\]

3.1. $\mathbb{K}$-Riemann Integrable Functions Defined over $\mathbb{R}$

Definition 19. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function, we said that $f$ is a $\mathbb{K}$-Riemann integrable function, if for every $\varepsilon > 0$, there exist two $\mathbb{K}$-step functions $\varphi$ and $\psi$ on $\mathbb{R}$ such that

$$\varphi(x) \leq f(x) \leq \psi(x) \quad \text{for all} \ x \in \mathbb{R}_{\mathbb{K}},$$

$$\int_{\mathbb{R}} (\psi(x) - \varphi(x)). \quad (39)$$

We denote the set of the $\mathbb{K}$-Riemann integrable functions by $\mathcal{R}_{\mathbb{K}}(\mathbb{R})$.

Remark 20.

(i) Every $\mathbb{K}$-Riemann integrable function on $\mathbb{R}$ is bounded on $\mathbb{R}_{\mathbb{K}}$

(ii) $\mathcal{B}_{\mathbb{K}}(\mathbb{R})$ is a part of $\mathcal{R}_{\mathbb{K}}(\mathbb{R})$

Now, suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded function on $\mathbb{R}_{\mathbb{K}}$. Set

\[
E_{\mathbb{K}}^{\text{r}}(f, \mathbb{R}) = \left\{ \int_{\mathbb{R}} \psi(x)d\mathbb{K}x : \psi \in \mathcal{E}_{\mathbb{K}}(\mathbb{R}) \text{ and } f(x) \leq \psi(x) \quad \text{for all} \ x \in \mathbb{R}_{\mathbb{K}} \right\},
\]

\[
E_{\mathbb{K}}^*(f, \mathbb{R}) = \left\{ \int_{\mathbb{R}} \psi(x)d\mathbb{K}x : \varphi \in \mathcal{E}_{\mathbb{K}}(\mathbb{R}) \text{ and } f(x) \leq \varphi(x) \quad \text{for all} \ x \in \mathbb{R}_{\mathbb{K}} \right\}.
\]

Note that $\alpha \leq \beta$ for all $\alpha \in E_{\mathbb{K}}^{\text{r}}(f, \mathbb{R})$ and $\beta \in E_{\mathbb{K}}^*(f, \mathbb{R})$.

Theorem 21. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded function on $\mathbb{R}_{\mathbb{K}}$. Then, $f \in \mathcal{R}_{\mathbb{K}}(\mathbb{R})$ if and only if $\inf E_{\mathbb{K}}^{\text{r}}(f, \mathbb{R}) = \sup E_{\mathbb{K}}^*(f, \mathbb{R})$.

Proof. Suppose that $f \in \mathcal{R}_{\mathbb{K}}(\mathbb{R})$. Given $\varepsilon > 0$, there exists $\varphi$ and $\psi$ in $\mathcal{E}_{\mathbb{K}}(\mathbb{R})$ such that

$$\varphi(x) \leq f(x) \leq \psi(x) \quad \text{for all} \ x \in \mathbb{R}_{\mathbb{K}},$$

$$\int_{\mathbb{R}} \psi(x)d\mathbb{K}x - \int_{\mathbb{R}} \varphi(x)d\mathbb{K}x < \varepsilon. \quad (41)$$

Then,

$$\int_{\mathbb{R}} \varphi(x)d\mathbb{K}x \leq \sup E_{\mathbb{K}}(f, \mathbb{R}) \leq \inf E_{\mathbb{K}}^{\text{r}}(f, \mathbb{R}) \leq \int_{\mathbb{R}} \psi(x)d\mathbb{K}x,$$

which shows that

$$\inf E_{\mathbb{K}}^{\text{r}}(f, \mathbb{R}) - \sup E_{\mathbb{K}}^*(f, \mathbb{R}) < \varepsilon,$$

for all $\varepsilon > 0$. Therefore,

$$\inf E_{\mathbb{K}}^{\text{r}}(f, \mathbb{R}) = \sup E_{\mathbb{K}}^*(f, \mathbb{R}). \quad (44)$$

Conversely, suppose that $\inf E_{\mathbb{K}}^{\text{r}}(f, \mathbb{R}) = \sup E_{\mathbb{K}}^*(f, \mathbb{R})$. Let $\varepsilon > 0$, by the definition of the greatest lower bound and
the least upper bound, there are $\varphi, \psi \in \mathcal{F}_K(R)$ such that
\[
\varphi(x) \leq f(x) \leq \psi(x) \quad \text{for all } x \in R_K,
\]
and
\[
\int_R \varphi(x) \, dx \leq -\sup E^+_K(f, R) + \frac{\varepsilon}{2},
\]
\[
\int_R \psi(x) \, dx \leq \inf E^+_K(f, R) + \frac{\varepsilon}{2}.
\]
Then,
\[
\int_R \varphi(x) \, dx - \int_R \psi(x) \, dx < \varepsilon.
\]
Therefore, $f \in \mathcal{R}_K(R)$.

**Definition 22.** Let $f \in \mathcal{R}_K(R)$, the common value $\inf E^+_K(f, R) = \sup E^-_K(f, R)$ is called the $K$-Riemann integral of $f$ on $R$, and we denote it by
\[
\int_R f(x) \, dx.
\]

In the case when $K = R$, we will use the standard symbol $\int_R f(x) \, dx$ instead of $\int_R f(x) \, dx$.

**Proposition 23.** Let $f : R \to \mathbb{R}$ be a function. Then, $f$ is $K$-Riemann integrable on $R$ if and only if there exist two $K$-step functions $(\varphi_m)_m$ and $(\psi_m)_m$ such that
\[
\varphi_m(x) \leq f(x) \leq \psi_m(x) \quad \text{for all } x \in R_K,
\]
\[
\lim_{m \to \infty} \int_R (\psi_m(x) - \varphi_m(x)) \, dx = 0.
\]
Moreover, we have
\[
\int_R f(x) \, dx = \lim_{m \to \infty} \int_R \varphi_m(x) \, dx = \lim_{m \to \infty} \int_R \psi_m(x) \, dx.
\]

Before presenting a version of the Fubini theorem for $K$-Riemann integrable functions. We start by giving a lemma that is essential for the proof of this result.

**Proof.** It runs in a similar way as for the case of one variable. Suppose that $f$ is $K$-integrable. Then, $\inf E^+_K(f, R) = \sup E^-_K(f, R)$. For all $m \in \mathbb{N}^*$, there exist two $K$-step functions $\varphi_m$ and $\psi_m$ on $R$ such that
\[
\varphi_m \leq f \leq \psi_m \quad \text{on } R_K
\]
and
\[
\int_R (\psi_m - \varphi_m)(x) \, dx < \frac{1}{m}.
\]
Hence,
\[
\lim_{m \to \infty} \int_R (\psi_m - \varphi_m)(x) \, dx = 0.
\]
Now, let us show that
\[
\int_R f(x) \, dx = \lim_{m \to \infty} \int_R \varphi_m(x) \, dx = \lim_{m \to \infty} \int_R \psi_m(x) \, dx.
\]

Let $m \in \mathbb{N}$. We have
\[
\int_R \varphi_m(x) \, dx \leq \int_R f(x) \, dx \leq \int_R \psi_m(x) \, dx.
\]
Then,
\[
0 \leq \int_R f(x) \, dx - \int_R \varphi_m(x) \, dx \leq \int_R (\psi_m(x) - \varphi_m(x)) \, dx.
\]
Hence,
\[
0 \leq \lim_{m \to \infty} \left( \int_R f(x) \, dx - \int_R \varphi_m(x) \, dx \right)
\]
\[
\leq \lim_{m \to \infty} \left( \int_R (\psi_m(x) - \varphi_m(x)) \, dx \right) = 0.
\]

The converse is trivial.

**Lemma 24.** Let $\mathcal{K}_1 \subseteq \mathcal{K}_2$ be two subfields of $\mathbb{R}$ and let $f : R \to \mathbb{R}$ be a $\mathcal{K}_2$-step function. Then, $f$ is a $\mathcal{K}_1$-Riemann integrable function. Moreover, we have
\[
\int_R f(x) \, dx = \int_R f(x) \, dx.
\]

**Proof.** Let $\{R_i\}_{i=1}^m$ be a $\mathcal{K}_2$-compatible partition of $f$ such that $f(x) = \lambda_i$ for all $x \in R_{K_i}$. For all $i \in \{1, 2, \ldots, m\}$, there is a sequence of $\mathcal{K}_1$-subrectangular $\{S_i^p\}_p$ of $R$ such that
\[
S_i^p \subseteq S_{i+1}^{(p+1)}, p \in \mathbb{N},
\]
\[
\lim_{p \to \infty} S_i^p = \bigcup_{p=0}^\infty S_i^p = R, \quad i \in \{1, 2, \ldots, m\}.
\]
Set
\[
T_p = \overline{\left( R \setminus \bigcup_{i=1}^m S_i^p \right)},
\]
where $\overline{\left( R \setminus \bigcup_{i=1}^m S_i^p \right)}$ is the closure of $R \setminus \bigcup_{i=1}^m S_i^p$. Then, there is a sequence $\{T_j^p\}_{j=1}^{k_j}$ of $\mathcal{K}_1$-subrectangular of $R$ such that
\[
T_j^p = \bigcup_{j=1}^{k_j} T_j^p \quad \text{and} \quad T_j^p \cap T_j^q = \emptyset, \quad j \neq j'.
\]
Now, consider the functions

\[
\psi_p(x) = \begin{cases} 
  f(x), & x \in \bigcup_{j=1}^m \left( S_p^j \right)_{K_1}^c, \\
  M = \sup_{x \in K_1} f(x), & \text{else},
\end{cases}
\]

\[
\phi_p(x) = \begin{cases} 
  f(x), & x \in \bigcup_{j=1}^m \left( S_p^j \right)_{K_1}, \\
  m = \inf_{x \in K_1} f(x), & \text{else}.
\end{cases}
\]

With

\[
\lim_{n \to \infty} \int_{\mathbb{R}} \tilde{\psi}_n(x) d\mathcal{E}_{K_2} x = \int_{\mathbb{R}} \psi(x) d\mathcal{E}_{K_2} x
\]

and

\[
\lim_{n \to \infty} \int_{\mathbb{R}} \tilde{\phi}_n(x) d\mathcal{E}_{K_2} x = \int_{\mathbb{R}} \phi(x) d\mathcal{E}_{K_2} x.
\]

Then, there is \( n_0 \in \mathbb{N} \) such that

\[
\int_{\mathbb{R}} \tilde{\psi}_n(x) d\mathcal{E}_{K_2} x - \varepsilon \leq \int_{\mathbb{R}} \psi(x) d\mathcal{E}_{K_2} x.
\]

Hence,

\[
\int_{\mathbb{R}} \tilde{\phi}_n(x) d\mathcal{E}_{K_2} x < \varepsilon.
\]

Therefore, \( f \in \mathcal{K}_{\mathcal{E}}(\mathbb{R}) \) and

\[
\int_{\mathbb{R}} f(x) d\mathcal{E}_{K_2} x = \int_{\mathbb{R}} f(x) d\mathcal{E}_{K_2} x.
\]

**Corollary 26.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a uniformly radially \( \mathcal{K} \)-continuous function, then it is \( \mathcal{K} \)-Riemann integrable on any subrectangular \( S \) of \( \mathbb{R} \).

**Proof.** Let \( S \) be a subrectangular of \( \mathbb{R} \). By Proposition 3, \( f \) is uniformly continuous on \( S_{K_1} \). Since \( \text{cl}(S_{K_1}) = S \), where \( \text{cl}(S_{K_1}) \) is the closure of \( S_{K_1} \), then there exists a unique continuous function \( g : S \to \mathbb{R} \) such that \( g(x) = f(x), x \in S_{K_1} \). Since \( g \) is Riemann integrable on \( S \) then, using Theorem 25, \( g \) is \( \mathcal{K} \)-Riemann integrable on \( S \); moreover,

\[
\int_{S} f(x) d\mathcal{E}_{K_2} x = \int_{S} g(x)dx.
\]

**Lemma 27.** Let \( f : \mathbb{R}_1 \times \mathbb{R}_2 \to \mathbb{R} \) be a \( \mathcal{K} \)-step function on \( \mathbb{R} \). Then, there is a \( \mathcal{K} \)-partition \( \{R_i^1\}_{i=1}^r \) of \( \mathbb{R}_1 \) and a \( \mathcal{K} \)-partition \( \{R_j^2\}_{j=1}^s \) of \( \mathbb{R}_2 \) such that the function

\[
f_x : \mathbb{R}_1 \to \mathbb{R},
\]

\[
y \mapsto f(x, y)
\]

is a \( \mathcal{K} \)-step function and \( \{R_j^2\}_{j=1}^s \) is a \( \mathcal{K} \)-compatible partition of \( f_x \) whenever \( x \in \bigcup_{i=1}^r R_i^1 \). Moreover, the function

\[
F(x) = \begin{cases} 
  \int_{R_i^1} f(x, y) d\mathcal{E}_{K_2} y, & x \in \bigcup_{i=1}^r R_i^1, \\
  0, & \text{else}
\end{cases}
\]

is a \( \mathcal{K} \)-step function on \( \mathbb{R}_1 \) and
\[
\int_{R_i} F(x)d\lambda x = \int_{R} f(x, y)d\lambda x d\lambda y. \quad (72)
\]

Proof. Let \( \pi = \{ T_i \}_{i=1}^{m} \) be a \( \mathbb{K} \)-partition of \( \mathbb{R} \), then there exists a \( \mathbb{K} \)-refinement \( \{ T^1_i \times T^2_j \} \subseteq \mathbb{K}^{1+2} \) of \( \pi \) such that \( \{ T^1_i \}_{i=1}^{m} \) and \( \{ T^2_j \}_{j=1}^{n} \) are two \( \mathbb{K} \)-partitions of \( \mathbb{R}_1 \) and \( \mathbb{R}_2 \), respectively. Since \( f \) is a \( \mathbb{K} \)-step function on \( \mathbb{R} \), then there are two \( \mathbb{K} \)-partitions \( \{ R^1_i \}_{i=1}^{s} \) and \( \{ R^2_j \}_{j=1}^{t} \) of \( \mathbb{R}_1 \) and \( \mathbb{R}_2 \), respectively, such that \( \{ R^1_i \times R^2_j \} \subseteq \mathbb{K}^{1+2} \) is a \( \mathbb{K} \)-partition of \( \mathbb{R} \) and \( f \) is constant on each \( R^1_i \times R^2_j \); hence, \( f \) is a \( \mathbb{K} \)-step function on \( \mathbb{R}_2 \). Now, we can say that \( F \) is a \( \mathbb{K} \)-step function on \( \mathbb{R}_1 \). Let us show the last equality in the theorem. Consider the two \( \mathbb{K} \)-partitions \( \{ R^1_i \}_{i=1}^{s} \) and \( \{ R^2_j \}_{j=1}^{t} \) given in the first step, then for all \( i = 1, 2, \ldots, r \) and \( j = 1, 2, \ldots, s \), there exist \( \lambda_{ij} \) such that \( f(x, y) = \lambda_{ij} \) whenever \( (x, y) \in R^1_i \times R^2_j \); therefore, we have

\[
F(x) = \left\{ \begin{array}{ll}
\sum_{i=1}^{s} \lambda_{ij} \text{vol}(R^1_i) & , x \in R^1_i, \\
0, & \text{else}
\end{array} \right. \quad (73)
\]

\[
\int_{R} F(x)d\lambda x = \int_{R} f(x, y)d\lambda x d\lambda y. \quad (74)
\]

\[
\Phi(x)d\lambda x = \int_{R} \varphi(x, y)d\lambda x d\lambda y. \quad (75)
\]

\[
\Psi(x)d\lambda x = \int_{R} \psi(x, y)d\lambda x d\lambda y. \quad (76)
\]

\[
\Phi(x) \leq F(x) \leq \Psi(x), \quad x \in R_{1\mathbb{K}}. \quad (77)
\]

3.2. Biconvex Function. Now, we give the notion of \( \mathbb{K} \)-biconvex function and we will prove that any \( \mathbb{K} \)-biconvex function is \( \mathbb{K} \)-Riemann integrable. Since a function is Jensen-convex if and only if it is \( \mathbb{Q} \)-convex, for more details, see [3, 4]. Then, we can easily show that a function is Jensen-biconvex if and only if it is \( \mathbb{Q} \)-biconvex.

Definition 29. A mapping \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is said to be \( \mathbb{K} \)-bilinear, if it satisfies

(i) \( f(\lambda x + y, z) = \lambda f(x, z) + f(y, z) \)

(ii) \( f(x, \lambda y + z) = \lambda f(x, y) + f(x, z) \)

for every \( \lambda \in \mathbb{K} \) and for every \( x, y, z \in \mathbb{R} \).

Definition 30. A function \( f : [a, b] \times [c, d] \rightarrow \mathbb{R} \) is said to be

(1) Jensen-biconvex if

(i) \( f((x + y)/2, z) \leq (f(x, z) + f(y, z))/2 \) for every \( x, y \in [a, b] \) and \( z \in [c, d] \).

(ii) \( f(x, (y + z)/2) \leq (f(x, y) + f(x, z))/2 \) for every \( x \in [a, b] \) and \( y, z \in [c, d] \).

(2) \( \mathbb{K} \)-biconvex if

(i) \( f(\lambda x + (1 - \lambda)y, z) \leq \lambda f(x, z) + (1 - \lambda)f(y, z), \) for every \( x, y \in [a, b] \) and \( z \in [c, d] \) and \( \lambda \in [0, 1]_{\mathbb{K}} \).

(ii) \( f(x, \lambda y + (1 - \lambda)z) \leq \lambda f(x, y) + (1 - \lambda)f(x, z), \) for every \( x \in [a, b] \) and \( y, z \in [c, d] \) and \( \lambda \in [0, 1]_{\mathbb{K}} \).

Now, we claim that every \( \mathbb{K} \)-biconvex function is \( \mathbb{K} \)-Riemann integrable. For the proof, we need the following lemma (see [5] p.143).
Lemma 31. Let $I \subset \mathbb{R}$ be an open interval and $g : I \rightarrow \mathbb{R}$ be a convex function. Then,
\[
g(x) - g(x - nd) \leq \frac{g(x) - g(x - md)}{m} \leq \frac{g(x + md) - g(x)}{m} \leq \frac{g(x + nd) - g(x)}{n},
\]
for every $x \in I$, $d \in \mathbb{R}$ and $m, n \in \mathbb{N}$ such that $0 < m < n$ and $x \pm nd \in I$.

Theorem 32. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded and $\mathbb{K}$-biconvex function, and let $S$ be a closed rectangular box such that $S \subset \mathbb{R}^r$. Then, $f$ is uniformly continuous on $S_{\mathbb{K}}$.

Proof. Put $S = [a, b] \times [c, d] \subset \mathbb{R}^r$, there exists a positive number $\rho \in \mathbb{K}$, such that
\[
[a_1, b_1] \times [c_1, d_1] \subset \mathbb{R},
\]
with $a_1 = a - \rho(b - a)$, $b_1 = b + \rho(b - a)$, $c_1 = c - \rho(d - c)$ and $d_1 = d + \rho(d - c)$. We show easily that
\[
[a, b]_{\mathbb{K}} \times [c, d]_{\mathbb{K}} \subset [a_1, b_1]_{\mathbb{K}} \times [c_1, d_1]_{\mathbb{K}}.
\]
It is sufficient to show that $[a, b]_{\mathbb{K}} \subset [a_1, b_1]_{\mathbb{K}}$. If $u \in [a, b]_{\mathbb{K}}$, then there exists $\lambda \in [0, 1] \cap \mathbb{K}$ such that $u = \lambda a + (1 - \lambda)b$. Put $v = (\rho + \lambda)/(1 + 2\rho)$, then $v \in [0, 1] \cap \mathbb{K}$. Moreover,
\[
va_1 + (1 - v)b_1 = \frac{\rho + \lambda}{1 + 2\rho} [a - \rho(b - a)] + \frac{1 + \rho - \lambda}{1 + 2\rho} [b + \rho(b - a)] = u.
\]
Then,
\[
[a, b]_{\mathbb{K}} \times [c, d]_{\mathbb{K}} \subset [a_1, b_1]_{\mathbb{K}} \times [c_1, d_1]_{\mathbb{K}}.
\]
On the other hand, since $f$ is bounded, there exist two real numbers $m \leq M$ such that
\[
m \leq f(x, y) \leq M \quad \text{for all} \quad (x, y) \in \mathbb{R}.
\]
Now, given an arbitrary $\varepsilon > 0$, put $n = [2((M - m)/\varepsilon) + 1]$ (the integral part of $2((M - m)/\varepsilon) + 1$). We have $n \in \mathbb{N}$ and
\[
2 \frac{M - m}{n} < \varepsilon.
\]
Further, put $\eta = \rho/n \min \{ |b - a|, |d - c| \}$. Take arbitrary $(x, y), (x', y') \in [a, b]_{\mathbb{K}} \times [c, d]_{\mathbb{K}}$ such that
\[
\|(x, y) - (x', y')\|_{\infty} = \max \{ |x - x'|, |y - y'| \} < \eta.
\]
Then, there exist $\alpha_i, \beta_i \in [0, 1] \cap \mathbb{K}$, $i = 1, 2$ such that
\[
x = \alpha_1 a + (1 - \alpha_1) b, \quad x' = \alpha_2 a + (1 - \alpha_2) b,
\]
\[
y = \beta_1 c + (1 - \beta_1) d, \quad y' = \beta_2 c + (1 - \beta_2) d.
\]
Whence
\[
x - x' = (\alpha_2 - \alpha_1)(b - a), \quad y - y' = (\beta_2 - \beta_1)(d - c),
\]
and by (88), we get
\[
n|\alpha_1 - \alpha_2| < \rho, \quad n|\beta_1 - \beta_2| < \rho.
\]
Consider the numbers
\[
\mu_{12} = \frac{\rho + \alpha_1 \pm n(\alpha_2 - \alpha_1)}{1 + 2\rho}, \quad \lambda_{12} = \frac{\rho + \beta_1 \pm n(\beta_2 - \beta_1)}{1 + 2\rho}.
\]
By (92), we have
\[
0 \leq \frac{\alpha_1}{1 + 2\rho} < \mu_{12} < \frac{\alpha_1 + 2\rho}{1 + 2\rho} \leq 1, \quad 0 \leq \frac{\beta_1}{1 + 2\rho} < \lambda_{12} < \frac{\beta_1 + 2\rho}{1 + 2\rho} \leq 1.
\]
Then, $\mu_{12}, \lambda_{12} \in [0, 1] \cap \mathbb{K}$. Now,
\[
\mu_{12} a_1 + (1 - \mu_{12}) b_1 = x \pm n (x - x') \in [a_1, b_1]_{\mathbb{K}}, \quad \lambda_{12} c_1 + (1 - \mu_{12}) d_1 = y \pm n (y - y') \in [c_1, d_1]_{\mathbb{K}}.
\]
Since the functions
\[
f_y : I \rightarrow \mathbb{R}, \quad f_{x'} : I \rightarrow \mathbb{R}, \quad x \mapsto f(x, y), \quad y \mapsto f(x', y)
\]
are $\mathbb{K}$-convex, taking in Lemma 31 $x^- = x, d = x - x'$, and $m = 1$, we get
\[
\frac{f(x, y) - f(x - nd, y)}{n} \leq f(x, y) - f(x', y) \leq \frac{f(x' + nd, y) - f(x', y)}{n}.
\]
and taking \( y' = y, d' = y - y' \), and \( m = 1 \), we find that
\[
\frac{f\left(x', y\right) - f\left(x', y - nd\right)}{n} \leq f\left(x', y\right) - f\left(x', y'\right) \leq \frac{f\left(x', y + nd\right) - f\left(x', y\right)}{n}
\]
\[
(98)
\]
Then
\[
-2 \frac{M - m}{n} \leq f\left(x, y\right) - f\left(x', y'\right) \leq 2 \frac{M - m}{n},
\]
\[
(100)
\]
which mean, by using \( (87) \), that
\[
-\varepsilon < f\left(x, y\right) - f\left(x', y'\right) < \varepsilon,
\]
\[
(101)
\]
Hence, \( f \) is uniformly continuous on \( S_k \).

**Corollary 33.** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a bounded and \( k \)-biconvex function. Then, \( f \) is \( k \)-Riemann integrable on any subrectangular \( S \) such that \( S \subset \mathbb{R}^k \).

**Proposition 34.** Let \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) be a \( k \)-bilinear function. Then, it is \( k \)-Riemann integrable on every rectangular box \( \mathbb{R} = [a, b] \times [c, d] \). Moreover,
\[
\int_{\mathbb{R}} f\left(x, y\right) d_k x d_k y = \int_a^b \left( \int_c^d f\left(x, y\right) d_k y \right) d_k x
\]
\[
= \int_a^b \left( d - c \right) f\left(x, \frac{c + d}{2}\right) d_k x
\]
\[
= f\left(\frac{a + b}{2}, \frac{c + d}{2}\right)(b - a)(d - c).
\]
\[
(105)
\]
**Proof.** \( f \) is a \( k \)-bilinear function; then, it is a \( k \)-convex function; hence, \( f \) is a \( k \)-Riemann integrable function. Since
\[
\begin{align*}
\quad f_y : [a, b] & \rightarrow \mathbb{R}, \quad y \in [c, d], \\
\quad y & \mapsto f\left(x, y\right),
\end{align*}
\]
\[
(103)
\]
and
\[
\begin{align*}
\quad f_x : [c, d] & \rightarrow \mathbb{R}, \quad x \in [a, b], \\
\quad x & \mapsto f\left(x, y\right),
\end{align*}
\]
\[
(103)
\]
are \( k \)-convex function, then by \([11, \text{Theorem } 7]\), they are \( k \)-Riemann integrable functions; moreover,
\[
\begin{align*}
\int_c^d f\left(x, y\right) d_k y & = (d - c) f\left(x, \frac{c + d}{2}\right), \quad x \in [a, b], \\
\int_a^b f\left(x, y\right) d_k x & = (b - a) f\left(\frac{a + b}{2}, y\right), \quad y \in [c, d].
\end{align*}
\]
\[
(104)
\]
By Fubini Theorem 3.18, we get

Remark 35. In this last part, we only took the case of two variables. By the same way, we can extend these results to the case of \( n \) variables.

\section{Conclusion}

We plan, thereafter, to extend the definition of \( k \)-integrability to functions defined on bounded subsets and to generalize Fubini’s theorem.

\section{Data Availability}

Data are available in https://link.springer.com/article/10.1007/s00010-017-0472-0.

\section{Conflicts of Interest}

The authors declare that they have no conflicts of interest.

\section{References}


