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Research Article

A Note on Derivative of Sine Series with Square Root

Sergiusz Kęska

Faculty of Exact and Natural Sciences, Siedlce University of Natural Sciences and Humanities, Poland

Correspondence should be addressed to Sergiusz Kęska; keska@se.onet.pl

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Chaundy and Jolliffe proved that if $\{a_n\}$ is a nonnegative, nonincreasing real sequence, then series $\sum a_n \sin(nx)$ converges uniformly if and only if $na_n \longrightarrow 0$. The purpose of this paper is to show that if $\{na_n\}$ is nonincreasing and $na_n \longrightarrow 0$, then the series $f(x)=\sum a_n \sin(\sqrt{n}x)$ can be differentiated term-by-term on $[c, d]$ for $c, d > 0$. However, $f'(0)$ may not exist.

1. Introduction

Chaundy and Jolliffe [1] proved the following.

Theorem 1. *If $\{c_k\}_{k=1}^{\infty} \subset R_+$ is decreasing to zero, then $\sum_{k=1}^{\infty} c_k \sin(kx)$ converges uniformly in x if and only if $kc_k \longrightarrow 0$ as $k \longrightarrow \infty$.*

Theorem 1 has had numerous generalizations.

Leindler [7] verified that in Theorem 1, the monotonicity assumption $c_n \geq c_{n+1}$ can be replaced by $c \in \mathrm{RBVS}$, i.e., if the conditions $c_n \longrightarrow 0$ and $\sum_{k=n}^{\infty}|c_k - c_{k+1}| \leq Kc_n$ hold for all n with constant $K = K(c)$ which depends only upon c.

The next theorem was indicated in [11].

Theorem 2. *If $\{c_k\}$ belongs to the class MVBVS, i.e., if there exist constants C and $\lambda \geq 2$, depending only on the sequence $\{c_k\}$ such that $\sum_{k=n}^{2n}|c_k - c_{k+1}| \leq (C/n)\sum_{k=\lfloor\lambda^{-1}n\rfloor}^{\lfloor\lambda n\rfloor} c_k$ for all $n \geq \lambda$, then series $\sum_{k=1}^{\infty} c_k \sin(kx)$ converges uniformly in x if and only if $\lim_{k\longrightarrow\infty} kc_k = 0$.*

Móricz [8] proves the following theorem.

Theorem 3. *Assume $f : R_+ \longrightarrow [0,\infty)$ with property $xf(x) \in L^1_{loc}(R_+)$. If $f(x)$ is nonincreasing on R_+, then the integral $\int_0^{\infty} f(x) \sin(tx)dx$, $t \in R_+$, converges uniformly in t if and only if $xf(x) \longrightarrow 0$ as $x \longrightarrow \infty$.*

A result due to Žak and Šneider [10] holds for double sine series.

Theorem 4. *If $\{c_{jk}\}_{j,k=1}^{\infty} \subset R_+$ is monotonically decreasing double sequences, i.e., a sequence of real numbers such that for $j, k = 1, 2, \cdots$, $c_{jk} - c_{j+1,k} \geq 0$, $c_{jk} - c_{j,k+1} \geq 0$, and $c_{jk} - c_{j,k+1} - c_{j+1,k} + c_{j+1,k+1} \geq 0$, then $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{jk} \sin(jx)\sin(ky)$ is uniformly regularly convergent in (x, y) if and only if $jk c_{jk} \longrightarrow 0$ as $j + k \longrightarrow \infty$.*

Theorem 4 was generalized by Kórus [6]. He has defined new classes of double sequences (SBVDS_1) to obtain those generalizations.

Duzinkiewicz and Szal [2] introduce a new class of double sequences called $\mathrm{DGM}(\alpha, \beta, \gamma, r)$, which is a generalization of the class considered by Kórus, and they obtain sufficient and necessary conditions for uniform convergence of double sine series.

Dyachenko et al. [3] proved the Chaundy-Jolliffe theorem for $\mathrm{GM}(\beta)$ sequences with majorant β having the following form: $\beta_n = (1/n)F_n(a)$, where F_n is admissible.

In the recent paper [5], it was proved that $\sum a_n \sin(\sqrt{n}x)$ converges uniformly on $[0, \pi]$ if and only if $na_n \longrightarrow 0$.

One of the results of paper [9] is that for any $\alpha \in (0, 2)$, $\sum a_n \sin n^{\alpha}x$ converges uniformly if and only if $na_n \longrightarrow 0$.

2. Main Results

Lemma 5. *Let*

$$I^* = \frac{\cos\left((\pi/x_0)\sqrt{\lfloor (x_0^2(m+1)^2/4)+j+1\rfloor}\right)}{\sqrt{\lfloor (x_0^2(m+1)^2/4)+j+1\rfloor}} - \frac{4x_0}{\pi}\sin\left(\frac{\pi}{2x_0}\left(\sqrt{\left\lfloor \frac{x_0^2(m+1)^2}{4}+j+1\right\rfloor}-\sqrt{\left\lfloor \frac{x_0^2(m+1)^2}{4}+j\right\rfloor}\right)\right) * \cos\left(\frac{\pi}{2x_0}\left(\sqrt{\left\lfloor \frac{x_0^2(m+1)^2}{4}+j+1\right\rfloor}+\sqrt{\left\lfloor \frac{x_0^2(m+1)^2}{4}+j\right\rfloor}\right)\right),$$

$$I^{**} = \frac{\cos\left((\pi/x_0)\sqrt{\lfloor (x_0^2(m+1)^2/4)-j\rfloor}\right)}{\sqrt{\lfloor (x_0^2(m+1)^2/4)-j\rfloor}} - \frac{4x_0}{\pi}\sin\left(\frac{\pi}{2x_0}\left(\sqrt{\left\lfloor \frac{x_0^2(m+1)^2}{4}-j\right\rfloor}-\sqrt{\left\lfloor \frac{x_0^2(m+1)^2}{4}-j-1\right\rfloor}\right)\right) * \cos\left(\frac{\pi}{2x_0}\left(\sqrt{\left\lfloor \frac{x_0^2(m+1)^2}{4}-j\right\rfloor}+\sqrt{\left\lfloor \frac{x_0^2(m+1)^2}{4}-j-1\right\rfloor}\right)\right), \tag{1}$$

where $m \in Odd_+ = \{1, 3, 5, 9, \cdots\}$.

Let $a, b \in R$ such that $0 < a < b$. Then, for all $x_0 \in [a, b]$ and for all $m > \max\{2/x_0^2, 2/x_0\}$,

$$\sum_{j=0}^{\lfloor (x_0^2/4)(2m+1)-1\rfloor} |I^* + I^{**}| \le \frac{16\pi^2}{x_0^5 m^2}\left(2x_0^4 + x_0^2\right). \tag{2}$$

Proof. Let

$$I_1^* = \frac{\cos\left((\pi/x_0)\sqrt{\lfloor (x_0^2(m+1)^2/4)+j+1\rfloor}\right) - \cos\left((\pi/2x_0)\left(\sqrt{\lfloor (x_0^2(m+1)^2/4)+j+1\rfloor}+\sqrt{\lfloor (x_0^2(m+1)^2/4)+j\rfloor}\right)\right)}{\sqrt{\lfloor (x_0^2(m+1)^2/4)+j+1\rfloor}},$$

$$I_2^* = \cos\left(\frac{\pi}{2x_0}\left(\sqrt{\left\lfloor \frac{x_0^2(m+1)^2}{4}+j+1\right\rfloor}+\sqrt{\left\lfloor \frac{x_0^2(m+1)^2}{4}+j\right\rfloor}\right)\right) * \left\{\frac{1}{\sqrt{\lfloor (x_0^2(m+1)^2/4)+j+1\rfloor}} - \frac{4x_0}{\pi}\sin\left(\frac{\pi}{2x_0}\left(\sqrt{\left\lfloor \frac{x_0^2(m+1)^2}{4}+j+1\right\rfloor}-\sqrt{\left\lfloor \frac{x_0^2(m+1)^2}{4}+j\right\rfloor}\right)\right)\right\}. \tag{3}$$

Note that for all $m > \max\{2/x_0^2, 2/x_0\}$,

$$\frac{\pi}{2x_0}\frac{1}{\sqrt{\lfloor (x_0^2(m+1)^2/4)+j+1\rfloor}+\sqrt{\lfloor (x_0^2(m+1)^2/4)+j\rfloor}} - \frac{1}{3!}\left(\frac{\pi}{2x_0}\frac{1}{\sqrt{\lfloor (x_0^2(m+1)^2/4)+j+1\rfloor}+\sqrt{\lfloor (x_0^2(m+1)^2/4)+j\rfloor}}\right)^3 \le \sin\left(\frac{\pi}{2x_0}\left(\sqrt{\left\lfloor \frac{x_0^2(m+1)^2}{4}+j+1\right\rfloor}-\sqrt{\left\lfloor \frac{x_0^2(m+1)^2}{4}+j\right\rfloor}\right)\right) \le \frac{\pi}{2x_0}\frac{1}{\sqrt{\lfloor (x_0^2(m+1)^2/4)+j+1\rfloor}+\sqrt{\lfloor (x_0^2(m+1)^2/4)+j\rfloor}}. \tag{4}$$

In view of (4), the following inequality is satisfied:

$$|I_2^*| \le \frac{\pi^2}{2x_0^5 3! m^3} + \left|\frac{1}{\sqrt{\lfloor (x_0^2(m+1)^2/4)+j+1\rfloor}} - \frac{2}{\sqrt{\lfloor (x_0^2(m+1)^2/4)+j+1\rfloor}+\sqrt{\lfloor (x_0^2(m+1)^2/4)+j\rfloor}}\right| \le \frac{\pi^2}{2x_0^5 3! m^3} + \frac{2}{m^2 x_0^4}\left(\sqrt{\left\lfloor \frac{x_0^2(m+1)^2}{4}+j\right\rfloor}-\sqrt{\left\lfloor \frac{x_0^2(m+1)^2}{4}+j+1\right\rfloor}\right) \le \frac{8\pi^2}{x_0^5 m^3}. \tag{5}$$

Let

$$I_1^{**} = \frac{\cos\left((\pi/x_0)\sqrt{\lfloor (x_0^2(m+1)^2/4) - j \rfloor}\right) - \cos\left((\pi/2x_0)\left(\sqrt{\lfloor (x_0^2(m+1)^2/4) - j \rfloor} + \sqrt{\lfloor (x_0^2(m+1)^2/4) - j - 1 \rfloor}\right)\right)}{\sqrt{\lfloor (x_0^2(m+1)^2/4) - j \rfloor}}, \quad (6)$$

$$\begin{aligned} I_2^{**} &= \cos\left(\frac{\pi}{2x_0}\left(\sqrt{\left\lfloor \frac{x_0^2(m+1)^2}{4} - j \right\rfloor} + \sqrt{\left\lfloor \frac{x_0^2(m+1)^2}{4} - j - 1 \right\rfloor}\right)\right) \\ &\quad * \left\{ \frac{1}{\sqrt{\lfloor (x_0^2(m+1)^2/4) - j \rfloor}} - \frac{4x_0}{\pi} \sin\left(\frac{\pi}{2x_0}\left(\sqrt{\left\lfloor \frac{x_0^2(m+1)^2}{4} - j \right\rfloor} - \sqrt{\left\lfloor \frac{x_0^2(m+1)^2}{4} - j - 1 \right\rfloor}\right)\right)\right\}, \end{aligned} \quad (7)$$

$$\begin{aligned} I_1^{*} + I_1^{**} &= \frac{-2}{\sqrt{\lfloor (x_0^2/4)(m+1)^2 + j + 1 \rfloor}} \sin\left(\frac{\pi}{4x_0}\left(\sqrt{\left\lfloor \frac{x_0^2}{4}(m+1)^2 + j + 1 \right\rfloor} - \sqrt{\left\lfloor \frac{x_0^2}{4}(m+1)^2 + j \right\rfloor}\right)\right) \\ &\quad * \sin\left(\frac{\pi}{4x_0}\left(3\sqrt{\left\lfloor \frac{x_0^2}{4}(m+1)^2 + j + 1 \right\rfloor} + \sqrt{\left\lfloor \frac{x_0^2}{4}(m+1)^2 + j \right\rfloor}\right)\right) \\ &\quad + \frac{-2}{\sqrt{\lfloor (x_0^2/4)(m+1)^2 - j \rfloor}} \sin\left(\frac{\pi}{4x_0}\left(\sqrt{\left\lfloor \frac{x_0^2}{4}(m+1)^2 - j \right\rfloor} - \sqrt{\left\lfloor \frac{x_0^2}{4}(m+1)^2 - j - 1 \right\rfloor}\right)\right) \\ &\quad * \sin\left(\frac{\pi}{4x_0}\left(3\sqrt{\left\lfloor \frac{x_0^2}{4}(m+1)^2 - j \right\rfloor} + \sqrt{\left\lfloor \frac{x_0^2}{4}(m+1)^2 - j - 1 \right\rfloor}\right)\right) \\ &== \frac{-2}{\sqrt{\lfloor (x_0^2/4)(m+1)^2 + j + 1 \rfloor}} \sin\left(\frac{\pi}{4x_0}\left(\sqrt{\left\lfloor \frac{x_0^2}{4}(m+1)^2 + j + 1 \right\rfloor} - \sqrt{\left\lfloor \frac{x_0^2}{4}(m+1)^2 + j \right\rfloor}\right)\right) \\ &\quad * \left\{ \sin\left(\frac{\pi}{4x_0}\left(3\sqrt{\left\lfloor \frac{x_0^2}{4}(m+1)^2 + j + 1 \right\rfloor} + \sqrt{\left\lfloor \frac{x_0^2}{4}(m+1)^2 + j \right\rfloor}\right)\right) + \sin\left(\frac{\pi}{4x_0}\left(3\sqrt{\left\lfloor \frac{x_0^2}{4}(m+1)^2 - j \right\rfloor} + \sqrt{\left\lfloor \frac{x_0^2}{4}(m+1)^2 - j - 1 \right\rfloor}\right)\right)\right\} \\ &\quad + \sin\left(\frac{\pi}{4x_0}\left(3\sqrt{\left\lfloor \frac{x_0^2}{4}(m+1)^2 - j \right\rfloor} + \sqrt{\left\lfloor \frac{x_0^2}{4}(m+1)^2 - j - 1 \right\rfloor}\right)\right) * \left\{ \frac{-2}{\sqrt{\lfloor (x_0^2/4)(m+1)^2 - j \rfloor}} \sin\left(\frac{\pi}{4x_0}\left(\sqrt{\left\lfloor \frac{x_0^2}{4}(m+1)^2 - j \right\rfloor}\right.\right.\right. \\ &\quad \left.\left.\left. - \sqrt{\left\lfloor \frac{x_0^2}{4}(m+1)^2 - j - 1 \right\rfloor}\right)\right) + \frac{2}{\sqrt{\lfloor (x_0^2/4)(m+1)^2 + j + 1 \rfloor}} \sin\left(\frac{\pi}{4x_0}\left(\sqrt{\left\lfloor \frac{x_0^2}{4}(m+1)^2 + j + 1 \right\rfloor} - \sqrt{\left\lfloor \frac{x_0^2}{4}(m+1)^2 + j \right\rfloor}\right)\right)\right\} = J_1 + J_2. \end{aligned} \quad (8)$$

In analogy with (5), we have

$$|I_2^{**}| \le \frac{8\pi^2}{x_0^5 m^3}. \quad (9)$$

If $D \in \{-3,-2,-1,0\}$, then

$$
\begin{aligned}
|J_1| \le{} & \frac{\pi}{x_0^3 m^2}\left|\sin\left(\frac{\pi}{4x_0}\left(2x_0(m+1)+3\sqrt{\left\lfloor\frac{x_0^2}{4}(m+1)^2+j+1\right\rfloor}+\sqrt{\left\lfloor\frac{x_0^2}{4}(m+1)^2+j\right\rfloor}-2x_0(m+1)\right)\right)\right.\\
&\left.+\sin\left(\frac{\pi}{4x_0}\left(2x_0(m+1)+3\sqrt{\left\lfloor\frac{x_0^2}{4}(m+1)^2-j\right\rfloor}+\sqrt{\left\lfloor\frac{x_0^2}{4}(m+1)^2-j-1\right\rfloor}-2x_0(m+1)\right)\right)\right|\\
={} & \frac{\pi}{x_0^3 m^2}\left|(-1)^{(m+1)/2}\sin\left(\frac{\pi}{4x_0}\left(3\sqrt{\left\lfloor\frac{x_0^2}{4}(m+1)^2+j+1\right\rfloor}+\sqrt{\left\lfloor\frac{x_0^2}{4}(m+1)^2+j\right\rfloor}-2x_0(m+1)\right)\right)\right.\\
&\left.+(-1)^{(m+1)/2}\sin\left(\frac{\pi}{4x_0}\left(3\sqrt{\left\lfloor\frac{x_0^2}{4}(m+1)^2-j\right\rfloor}+\sqrt{\left\lfloor\frac{x_0^2}{4}(m+1)^2-j-1\right\rfloor}-2x_0(m+1)\right)\right)\right|\\
={} & \frac{\pi}{x_0^3 m^2}\left|\sin\left(\frac{\pi}{4x_0}\left(\frac{9\lfloor(x_0^2/4)(m+1)^2+j+1\rfloor-(9/4)x_0^2(m+1)^2}{3\sqrt{\lfloor(x_0^2/4)(m+1)^2+j+1\rfloor}+(3/2)x_0(m+1)}+\frac{\lfloor(x_0^2/4)(m+1)^2+j\rfloor-(1/4)x_0^2(m+1)^2}{\sqrt{\lfloor(x_0^2/4)(m+1)^2+j\rfloor}+(1/2)x_0(m+1)}\right)\right)\right.\\
&\left.+\sin\left(\frac{\pi}{4x_0}\left(\frac{9\lfloor(x_0^2/4)(m+1)^2-j\rfloor-(9/4)x_0^2(m+1)^2}{3\sqrt{\lfloor(x_0^2/4)(m+1)^2-j\rfloor}+(3/2)x_0(m+1)}+\frac{\lfloor(x_0^2/4)(m+1)^2-j-1\rfloor-(1/4)x_0^2(m+1)^2}{\sqrt{\lfloor(x_0^2/4)(m+1)^2-j-1\rfloor}+(1/2)x_0(m+1)}\right)\right)\right|\\
\le{} & \frac{\pi}{x_0^3 m^2}\left|2\sin\left(\frac{\pi}{8x_0}\left(\frac{3j+3D}{\sqrt{\lfloor(x_0^2/4)(m+1)^2+j+1\rfloor}+(1/2)x_0(m+1)}+\frac{j+D}{\sqrt{\lfloor(x_0^2/4)(m+1)^2+j\rfloor}+(1/2)x_0(m+1)}\right.\right.\right.\\
&\left.\left.\left.+\frac{-3j+3D}{\sqrt{\lfloor(x_0^2/4)(m+1)^2-j\rfloor}+(1/2)x_0(m+1)}+\frac{-j+D}{\sqrt{\lfloor(x_0^2/4)(m+1)^2-j-1\rfloor}+(1/2)x_0(m+1)}\right)\right)\right|\\
\le{} & \frac{\pi^2}{4x_0^4 m^2}\left\{\left|3j\frac{\sqrt{\lfloor(x_0^2/4)(m+1)^2-j\rfloor}-\sqrt{\lfloor(x_0^2/4)(m+1)^2+j+1\rfloor}}{\left(\sqrt{\lfloor(x_0^2/4)(m+1)^2-j\rfloor}+(1/2)x_0(m+1)\right)\left(\sqrt{\lfloor(x_0^2/4)(m+1)^2+j+1\rfloor}+(1/2)x_0(m+1)\right)}\right|\right.\\
&\left.+\left|j\frac{\sqrt{\lfloor(x_0^2/4)(m+1)^2-j-1\rfloor}-\sqrt{\lfloor(x_0^2/4)(m+1)^2+j\rfloor}}{\left(\sqrt{\lfloor(x_0^2/4)(m+1)^2-j-1\rfloor}+(1/2)x_0(m+1)\right)\left(\sqrt{\lfloor(x_0^2/4)(m+1)^2+j\rfloor}+(1/2)x_0(m+1)\right)}\right|\right\}+\frac{2\pi^2}{x_0^5 m^3}\le\frac{16\pi^2}{x_0^7 m^5}j^2+\frac{2\pi^2}{x_0^5 m^3},
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
|J_2| \le{} & \left|2\frac{\sin\left((\pi/4x_0)\left(\sqrt{\lfloor(x_0^2/4)(m+1)^2+j+1\rfloor}-\sqrt{\lfloor(x_0^2/4)(m+1)^2+j\rfloor}\right)\right)}{\sqrt{\lfloor(x_0^2/4)(m+1)^2-j\rfloor}}\right.\\
&\left.-2\frac{\sin\left((\pi/4x_0)\left(\sqrt{\lfloor(x_0^2/4)(m+1)^2-j\rfloor}-\sqrt{\lfloor(x_0^2/4)(m+1)^2-j-1\rfloor}\right)\right)}{\sqrt{\lfloor(x_0^2/4)(m+1)^2-j\rfloor}}\right|\\
&+\left|2\sin\left(\frac{\pi}{4x_0}\left(\sqrt{\left\lfloor\frac{x_0^2}{4}(m+1)^2+j+1\right\rfloor}-\sqrt{\left\lfloor\frac{x_0^2}{4}(m+1)^2+j\right\rfloor}\right)\right)\left\{\frac{1}{\sqrt{\lfloor(x_0^2/4)(m+1)^2+j+1\rfloor}}-\frac{1}{\sqrt{\lfloor(x_0^2/4)(m+1)^2-j\rfloor}}\right\}\right|\\
\le{} & \frac{4}{x_0 m}\left|2\sin\left(\frac{\pi}{8x_0}\left(\sqrt{\left\lfloor\frac{x_0^2}{4}(m+1)^2+j+1\right\rfloor}-\sqrt{\left\lfloor\frac{x_0^2}{4}(m+1)^2+j\right\rfloor}-\sqrt{\left\lfloor\frac{x_0^2}{4}(m+1)^2-j\right\rfloor}+\sqrt{\left\lfloor\frac{x_0^2}{4}(m+1)^2-j-1\right\rfloor}\right)\right)\right|+\frac{4\pi(j+1)}{x_0^5 m^4}\\
\le{} & \frac{\pi}{x_0^2 m}\left|\frac{1}{\sqrt{\lfloor(x_0^2/4)(m+1)^2+j+1\rfloor}+\sqrt{\lfloor(x_0^2/4)(m+1)^2+j\rfloor}}-\frac{1}{\sqrt{\lfloor(x_0^2/4)(m+1)^2-j\rfloor}+\sqrt{\lfloor(x_0^2/4)(m+1)^2-j-1\rfloor}}\right|+\frac{4\pi(j+1)}{x_0^5 m^4}\\
\le{} & \frac{\pi}{x_0^2 m}\left|\frac{\sqrt{\lfloor(x_0^2/4)(m+1)^2-j\rfloor}+\sqrt{\lfloor(x_0^2/4)(m+1)^2-j-1\rfloor}-\sqrt{\lfloor(x_0^2/4)(m+1)^2+j+1\rfloor}-\sqrt{\lfloor(x_0^2/4)(m+1)^2+j\rfloor}}{x_0^2 m^2}\right|+\frac{4\pi(j+1)}{x_0^5 m^4}\\
\le{} & \frac{\pi}{x_0^4 m^3}\left(\frac{2j}{x_0 m}+\frac{2j+2}{x_0 m}\right)+\frac{4\pi(j+1)}{x_0^5 m^4}\le\frac{6\pi(j+1)}{x_0^5 m^4}.
\end{aligned}
\tag{11}
$$

In view of (5) and (8)–(11), the following inequality is satisfied:

$$|I^{*}+I^{**}| \le \frac{16\pi^2}{x_0^5 m^2}\left(\frac{1}{m}+\frac{j}{m^2}+\frac{j^2}{x_0^2 m^3}\right). \tag{12}$$

Then, for all $x_0 \in [a,b]$ and $m > \max\{2/x_0^2, 2/x_0\}$, we have

$$\sum_{j=0}^{\lfloor (x_0^2/4)(2m+1)-1\rfloor} |I^{*}+I^{**}| \le \frac{16\pi^2}{x_0^5 m^2}\left(2x_0^4+x_0^2\right). \tag{13}$$

□

In the recent paper [4], it was proved, among others, that if $\{c_n\}$ is a nonincreasing sequence and $\sum_{n=1} c_n \cos(\sqrt{n}x)$ is convergent at any point $x \neq 0$, then $\lim_{n\longrightarrow\infty}\sqrt{n}c_n = 0$. We show that if $\{\sqrt{n}c_n\}$ is nonincreasing and $\lim_{n\longrightarrow\infty}\sqrt{n}c_n = 0$, then $\sum_{n=1} c_n \cos(x\sqrt{n})$ converges uniformly on $[c,d]$, where $c > 0$.

Theorem 6. *Let $c, d \in R$ such that $0 < c < d$. If $\{na_n\}_{n=1}^{\infty}$ is nonincreasing and $\lim_{n\longrightarrow\infty} na_n = 0$, then the series*

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(x\sqrt{n}), \tag{14}$$

can be differentiated term-by-term on $[c,d]$ (or $[-d,-c]$). The series

$$f'(x) = \sum_{n=1}^{\infty} a_n\sqrt{n}\cos(x\sqrt{n}), \tag{15}$$

converges uniformly on $[c,d]$ (or $[-d,-c]$).

Proof. We can find $a, b \in R$ such that $0 < a < b$, $c = \pi/b$, and $d = \pi/a$. If $\{a_n\}_{n=1}^{\infty}$ is a nonnegative monotone sequence and $\lim_{n\longrightarrow\infty} na_n = 0$, then the series (14) converges uniformly [9]. We show that if, in addition, $\{na_n\}_{n=1}^{\infty}$ is nonincreasing, then (for $x = \pi/x_0$) the series $\sum_{n=1}^{\infty} a_n\sqrt{n}\cos(\pi\sqrt{n}/x_0)$ converges uniformly on $[a,b]$. Let $m \in \mathrm{Odd}+$, $b_n = \cos(\pi\sqrt{n}/x_0)/\sqrt{n}$, $B_k = \sum_{n=1}^{k} b_n$, and $M_m = \{\lfloor (x_0^2/4)m^2\rfloor + 1, \lfloor (x_0^2/4)m^2\rfloor + 2, \cdots, \lfloor (x_0^2/4)(m+2)^2\rfloor\}$. We denote by m_0 the minimal odd number for which there holds the following: $4 \mid m_0 - 1$ and $m_0 \ge \max\{2/x_0^2, 2/x_0\}$. Then,

$$\exists H' \forall x_0 \in [a,b] \sum_{n=1}^{\lfloor x_0^2 m_0^2/4\rfloor} b_n < H'. \tag{16}$$

Let

$$B''_k = \sum_{n=\lfloor (x_0^2/4)m_0^2+1\rfloor}^{k} b_n. \tag{17}$$

Note that for all m such that $4|m-1$ and for all $k \in M_m$, the following conditions are fulfilled: $b_k \le 0$ and

$$\sum_{n\in\bigcup_{r\in\{m_0,m_0+2,m_0+4,\cdots,m\}} M_r} b_n \le B''_k \le \sum_{n\in\bigcup_{r\in\{m_0,m_0+2,m_0+4,\cdots,m-2\}} M_r} b_n. \tag{18}$$

On the other hand, for all m such that $4 \mid m-3$ and for all $k \in M_m$, we have $b_k \ge 0$ and

$$\sum_{n\in\bigcup_{r\in\{m_0,m_0+2,m_0+4,\cdots,m-2\}} M_r} b_n \le B''_k \le \sum_{n\in\bigcup_{r\in\{m_0,m_0+2,m_0+4,\cdots,m-2,m\}} M_r} b_n. \tag{19}$$

We show that the sequence

$$\sum_{n\in\bigcup_{r\in\{m_0,m_0+2,m_0+4,\cdots,m-2,m\}} M_r} b_n, \tag{20}$$

is bounded for $4|m-1$ and $4|m-3$.

$$\begin{aligned}
\sum_{n\in\bigcup_{r\in\{m_0,m_0+2,m_0+4,\cdots,m-2,m\}} M_r} b_n &= \sum_{r\in\mathrm{Odd}_+, r=m_0,\cdots,m} \sum_{j=1}^{\lfloor (x_0^2/4)(r+2)^2\rfloor - \lfloor (x_0^2/4)r^2\rfloor} \frac{\cos\left((\pi/x_0)\sqrt{\lfloor (x_0^2/4)r^2+j\rfloor}\right)}{\sqrt{\lfloor (x_0^2/4)r^2+j\rfloor}} \\
&= \sum_{r\in\mathrm{Odd}_+, r=m_0,\cdots,m} \left\{ \sum_{j=0}^{\lfloor (x_0^2/4)(2r+1)-1\rfloor} \left(\frac{\cos\left((\pi/x_0)\sqrt{\lfloor (x_0^2/4)(r+1)^2+j+1\rfloor}\right)}{\sqrt{\lfloor (x_0^2/4)(r+1)^2+j+1\rfloor}} + \frac{\cos\left((\pi/x_0)\sqrt{\lfloor (x_0^2/4)(r+1)^2-j\rfloor}\right)}{\sqrt{\lfloor (x_0^2/4)(r+1)^2-j\rfloor}} \right) \right. \\
&\quad \left. + \sum_{j=1}^{\lfloor (x_0^2/4)(r+1)^2\rfloor - \lfloor (x_0^2/4)(2r+1)\rfloor - \lfloor (x_0^2/4)r^2\rfloor} \frac{\cos\left((\pi/x_0)\sqrt{\lfloor (x_0^2/4)r^2+j\rfloor}\right)}{\sqrt{\lfloor (x_0^2/4)r^2+j\rfloor}} + \sum_{j=\lfloor (x_0^2/4)(r+1)^2\rfloor + \lfloor (x_0^2/4)(2r+1)\rfloor - \lfloor (x_0^2/4)r^2\rfloor}^{\lfloor (x_0^2/4)(r+2)^2\rfloor - \lfloor (x_0^2/4)r^2\rfloor} \frac{\cos\left((\pi/x_0)\sqrt{\lfloor (x_0^2/4)r^2+j\rfloor}\right)}{\sqrt{\lfloor (x_0^2/4)r^2+j\rfloor}} \right\} \\
&= \sum_{r\in\mathrm{Odd}_+, r=m_0,\cdots,m} \{S_1+S_2+S_3\}.
\end{aligned} \tag{21}$$

Note that $(x_0^2/2) - 1 \le \lfloor (x_0^2/4)(r+2)^2 \rfloor - \lfloor (x_0^2/4)(r+1)^2 \rfloor - \lfloor (x_0^2/4)(2r+1) \rfloor \le (x_0^2/2) + 2$. Hence,

$$
\begin{aligned}
|S_3| &\le 4\frac{(2+(x_0^2/2))}{x_0 r}\left|\cos\left(\frac{\pi}{x_0}\sqrt{\left\lfloor\frac{x_0^2}{4}(r+1)^2\right\rfloor+\left\lfloor\frac{x_0^2}{4}(2r+1)\right\rfloor}\right)\right| \\
&\le 2\frac{4+x_0}{x_0 r}\left|\cos\left(\frac{\pi}{x_0}\left(\frac{x_0}{2}(r+2)+\sqrt{\left\lfloor\frac{x_0^2}{4}(r+1)^2\right\rfloor+\left\lfloor\frac{x_0^2}{4}(2r+1)\right\rfloor}\right.\right.\right. \\
&\quad \left.\left.\left. -\frac{x_0}{2}(r+2)\right)\right)\right| \le 2\frac{\pi(x_0^2/2+2)(4+x_0)}{x_0^3 r^2}.
\end{aligned} \tag{22}
$$

Moreover,

$$
|S_2| \le \frac{2}{x_0 r/2}\left|\cos\frac{\pi}{x_0}\sqrt{\left\lfloor\frac{x_0^2}{4}r^2+2\right\rfloor}\right| \le \frac{4\pi}{x_0^3 r^2}. \tag{23}
$$

Furthermore,

$$
\begin{aligned}
S_1 = &\sum_{j=0}^{\lfloor (x_0^2/4)(2r+1)-1\rfloor}\left\{\frac{4x_0}{\pi}\sin\left(\frac{\pi}{2x_0}\left(\sqrt{\left\lfloor\frac{x_0}{4}(r+1)^2+j+1\right\rfloor}-\sqrt{\left\lfloor\frac{x_0}{4}(r+1)^2+j\right\rfloor}\right)\right)*\cos\left(\frac{\pi}{2x_0}\left(\sqrt{\left\lfloor\frac{x_0}{4}(r+1)^2+j+1\right\rfloor}+\sqrt{\left\lfloor\frac{x_0}{4}(r+1)^2+j\right\rfloor}\right)\right)\right. \\
&\left.+\frac{4x_0}{\pi}\sin\left(\frac{\pi}{2x_0}\left(\sqrt{\left\lfloor\frac{x_0}{4}(r+1)^2-j\right\rfloor}-\sqrt{\left\lfloor\frac{x_0}{4}(r+1)^2-j-1\right\rfloor}\right)\right)*\cos\left(\frac{\pi}{2x_0}\left(\sqrt{\left\lfloor\frac{x_0}{4}(r+1)^2-j\right\rfloor}+\sqrt{\left\lfloor\frac{x_0}{4}(r+1)^2-j-1\right\rfloor}\right)\right)+I^*+I^{**}\right\} \\
= &\, {S'}_1+\sum_{j=0}^{\lfloor (x_0^2/4)(2r+1)-1\rfloor}(I^*+I^{**}),
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
{S'}_1 = &\sum_{j=0}^{\lfloor (x_0^2/4)(2r+1)-1\rfloor}\left\{\frac{2x_0}{\pi}\left[\sin\left(\frac{\pi}{x_0}\sqrt{\left\lfloor\frac{x_0^2}{4}(r+1)^2+j+1\right\rfloor}\right)-\sin\left(\frac{\pi}{x_0}\sqrt{\left\lfloor\frac{x_0^2}{4}(r+1)^2+j\right\rfloor}\right)\right]+\frac{2x_0}{\pi}\left[\sin\left(\frac{\pi}{x_0}\sqrt{\left\lfloor\frac{x_0^2}{4}(r+1)^2-j\right\rfloor}\right)\right.\right. \\
&\left.\left.-\sin\left(\frac{\pi}{x_0}\sqrt{\left\lfloor\frac{x_0^2}{4}(r+1)^2-j-1\right\rfloor}\right)\right]\right\} == \frac{2x_0}{\pi}\left[\sin\left(\frac{\pi}{x_0}\sqrt{\left\lfloor\frac{x_0^2}{4}(r+1)^2\right\rfloor+\left\lfloor\frac{x_0^2}{4}(2r+1)\right\rfloor}\right)-\sin\left(\frac{\pi}{x_0}\sqrt{\left\lfloor\frac{x_0^2}{4}(r+1)^2\right\rfloor}\right)\right. \\
&\left.-\sin\left(\frac{\pi}{x_0}\sqrt{\left\lfloor\frac{x_0^2}{4}(r+1)^2\right\rfloor-\left\lfloor\frac{x_0^2}{4}(2r+1)\right\rfloor}\right)+\sin\left(\frac{\pi}{x_0}\sqrt{\left\lfloor\frac{x_0^2}{4}(r+1)^2\right\rfloor}\right)\right] \\
== &\frac{2x_0}{\pi}\left[(-1)^{(r+1)/2}\cos\left(\frac{\pi}{x_0}\sqrt{\left\lfloor\frac{x_0^2}{4}(r+1)^2\right\rfloor+\left\lfloor\frac{x_0^2}{4}(2r+1)\right\rfloor}-\frac{(r+2)\pi}{2}\right)+(-1)^{(r+1)/2}\cos\left(\frac{\pi}{x_0}\sqrt{\left\lfloor\frac{x_0^2}{4}(r+1)^2\right\rfloor-\left\lfloor\frac{x_0^2}{4}(2r+1)\right\rfloor}-\frac{r\pi}{2}\right)\right] \\
== &\frac{2x_0}{\pi}(-1)^{(r+1)/2}\left[1-\left(1-\cos\left(\frac{\pi}{x_0}\sqrt{\left\lfloor\frac{x_0^2}{4}(r+1)^2\right\rfloor+\left\lfloor\frac{x_0^2}{4}(2r+1)\right\rfloor}-\frac{(r+2)\pi}{2}\right)\right)+1-\left(1-\cos\left(\frac{\pi}{x_0}\sqrt{\left\lfloor\frac{x_0^2}{4}(r+1)^2\right\rfloor-\left\lfloor\frac{x_0^2}{4}(2r+1)\right\rfloor}-\frac{r\pi}{2}\right)\right)\right] \\
= &\frac{4x_0}{\pi}(-1)^{(r+1)/2}-\frac{2x_0}{\pi}(-1)^{(r+1)/2}\left[\frac{1-\cos^2\left((\pi/x_0)\sqrt{\lfloor (x_0^2/4)(r+1)^2\rfloor+\lfloor (x_0^2/4)(2r+1)\rfloor}-((r+2)\pi/2)\right)}{1+\cos\left((\pi/x_0)\sqrt{\lfloor (x_0^2/4)(r+1)^2\rfloor+\lfloor (x_0^2/4)(2r+1)\rfloor}-((r+2)\pi/2)\right)}++\frac{1-\cos^2\left((\pi/x_0)\sqrt{\lfloor (x_0^2/4)(r+1)^2\rfloor-\lfloor (x_0^2/4)(2r+1)\rfloor}-(r\pi/2)\right)}{1+\cos\left((\pi/x_0)\sqrt{\lfloor (x_0^2/4)(r+1)^2\rfloor-\lfloor (x_0^2/4)(2r+1)\rfloor}-(r\pi/2)\right)}\right].
\end{aligned} \tag{25}
$$

Note that

$$
\sum_{r\in\mathrm{Odd}_+\ r=m_0,\cdots,m}(-1)^{(r+1)/2}=\begin{cases}-1 & \text{if } 4|m-1,\\ 0 & \text{if } 4|m-3.\end{cases} \tag{26}
$$

In view of Lemma 5 and (22)–(26), the following inequality is satisfied:

$$\begin{aligned}\left|\sum_{r\in \mathrm{Odd}_+, r=m_0,\cdots,m}(S_1+S_2+S_3)\right| &\le \left|\sum_{r\in \mathrm{Odd}_+, r=m_0,\cdots,m}\frac{4x_0}{\pi}(-1)^{(r+1)/2}\right| + \sum_{r\in \mathrm{Odd}_+, r=m_0,\cdots,m}\left\{\frac{2x_0}{\pi}\left[\sin^2\left(\frac{\pi}{x_0}\sqrt{\left\lfloor\frac{x_0^2}{4}(r+1)^2\right\rfloor+\left\lfloor\frac{x_0^2}{4}(2r+1)\right\rfloor}-\frac{(r+2)\pi}{2}\right)\right.\right.\\ &\quad\left.\left.+\sin^2\left(\frac{\pi}{x_0}\sqrt{\left\lfloor\frac{x_0^2}{4}(r+1)^2\right\rfloor-\left\lfloor\frac{x_0^2}{4}(2r+1)\right\rfloor}-\frac{r\pi}{2}\right)\right]+\left|\sum_{j=0}^{\lfloor (x_0^2/4)(2r+1)-1\rfloor}(I^*+I^{**})\right|+|S_2|+|S_3|\right\}\\ &\le \frac{4x_0}{\pi}+\sum_{r\in \mathrm{Odd}_+, r=m_0,\cdots,m}\left\{\frac{16\pi^2}{r^2x_0^5}\left(2x_0^4+x_0^2\right)+\frac{16x_0}{r^2\pi}\left(\frac{\pi^2}{4}+\frac{\pi^2}{x_0}\right)^2+\frac{4\pi}{r^2x_0^3}+\frac{2\pi\left((x_0^2/2)+2\right)(4+x_0)}{r^2x_0^3}\right\}<H,\end{aligned} \tag{27}$$

for all $x_0 \in [a, b]$. Using (17)–(19), (21), and (27), we obtain

$$\exists H''\forall x_0 \in [a,b] B''_k < H''. \tag{28}$$

In view of (16) and (28), we get

$$\exists G\forall x_0 \in [a,b] B_k < G. \tag{29}$$

We know that $\{na_n\}$ is nonincreasing and $\lim_{n\longrightarrow\infty} na_n = 0$ and (29) holds. Hence, the series $\sum na_n(\cos(\pi\sqrt{n}/x_0)/\sqrt{n})$ converges uniformly. This follows from Dirichlet's test. □

Remark 7. The conditions ($\{na_n\}$ is nonincreasing, and $\lim_{n\longrightarrow\infty} na_n = 0$) are not sufficient for the differentiability of (14) at the point $x = 0$.

Example 1. Let $f(x) = \sum_{n=4}(\sin(x\sqrt{n})/n\ln n)$. It converges uniformly on $[0, \pi]$ [5]. We show that f is not differentiable at $x = 0$. Let $f(\Delta x) = \sum_{n=4}(1/n\ln n)\sin(\pi\sqrt{n}/2\alpha)$ for $\Delta x = \pi/2\alpha$ and $\alpha \in \mathrm{Odd}_+ : 4|\alpha - 1, \alpha \ge 5$. Then, by Lagrange's theorem,

$$\begin{aligned}\sum_{n=4}^{\alpha^2}\frac{1}{n\ln n}\sin\frac{\pi\sqrt{n}}{2\alpha} &\ge \frac{1}{\alpha\ln\alpha^2}\sum_{n=4}^{\alpha^2}\frac{1}{\sqrt{n}}\sin\frac{\pi\sqrt{n}}{2\alpha}\\ &\ge \frac{1}{\alpha\ln\alpha^2}\int_3^{\alpha^2}\frac{1}{\sqrt{n}}\sin\frac{\pi\sqrt{n}}{2\alpha}dn\\ &= \frac{4}{\pi\ln\alpha^2}\cos\frac{\pi\sqrt{3}}{2\alpha},\end{aligned} \tag{30}$$

$$\begin{aligned}&\sum_{n=\alpha^2+1}^{(2\alpha)^2}\frac{1}{n\ln n}\sin\frac{\pi\sqrt{n}}{2\alpha} \ge \frac{1}{2\alpha\ln(2\alpha)^2}\sum_{n=\alpha^2+1}^{(2\alpha)^2}\frac{1}{\sqrt{n}}\sin\frac{\pi\sqrt{n}}{2\alpha}\\ &\ge \frac{1}{2\alpha\ln(2\alpha)^2}\int_{\alpha^2+1}^{(2\alpha)^2+1}\frac{1}{\sqrt{n}}\sin\frac{\pi\sqrt{n}}{2\alpha}dn\\ &= \frac{-4}{2\pi\ln(2\alpha)^2}\left[\cos\frac{\pi\sqrt{(2\alpha)^2+1}}{2\alpha}-\cos\frac{\pi\sqrt{(\alpha)^2+1}}{2\alpha}\right].\end{aligned} \tag{31}$$

The function $\sin x/x$ has a relative minimum at the point $x_1 \in (\pi, 3\pi/2)$. Then,

$$\begin{aligned}&\sum_{n=(2\alpha)^2+1}^{\lfloor (x_1 2\alpha/\pi)^2\rfloor}\frac{1}{n\ln n}\sin\frac{\pi\sqrt{n}}{2\alpha}\\ &\ge \frac{-4}{2\pi\ln(2\alpha)^2}\left[\cos\frac{\pi\sqrt{\lfloor (x_1 2\alpha/\pi)^2+1\rfloor}}{2\alpha}-\cos\frac{\pi\sqrt{(2\alpha)^2+1}}{2\alpha}\right],\end{aligned} \tag{32}$$

$$\begin{aligned}&\sum_{n=\lfloor (x_1 2\alpha/\pi)^2\rfloor+1}^{(3\alpha)^2}\frac{1}{n\ln n}\sin\frac{\pi\sqrt{n}}{2\alpha}\\ &= \sum_{n=\lfloor (x_1 2\alpha/\pi)^2\rfloor+2}^{(3\alpha)^2}\frac{1}{n\ln n}\sin\frac{\pi\sqrt{n}}{2\alpha}\\ &\quad+\frac{\sin\left(\pi\sqrt{\lfloor (x_1 2\alpha/\pi)^2+1\rfloor}/2\alpha\right)}{\lfloor (x_1 2\alpha/\pi)^2+1\rfloor\ln\lfloor (x_1 2\alpha/\pi)^2+1\rfloor}\\ &\ge \frac{4}{2\pi\ln(2\alpha)^2}\cos\frac{\pi\sqrt{\lfloor (x_1 2\alpha/\pi)^2+1\rfloor}}{2\alpha}-\frac{1}{\alpha^2\ln\alpha^2},\end{aligned} \tag{33}$$

$$\sum_{n=(3\alpha)^2+1}^{(4\alpha)^2}\frac{1}{n\ln n}\sin\frac{\pi\sqrt{n}}{2\alpha} \ge \frac{-4}{3\pi\ln(3\alpha)^2}. \tag{34}$$

In view of (30)–(34), we obtain

$$\sum_{n=4}^{(4\alpha)^2}\frac{1}{n\ln n}\sin\frac{\pi\sqrt{n}}{2\alpha} \ge \frac{1}{3\pi\ln\alpha}-\frac{1}{\alpha^2\ln\alpha^2}. \tag{35}$$

If x'_r and x''_r are the relative minimum and maximum (respectively) points of $\sin x/x$, such that

$$x_r'' \in \left(2\pi + 2r\pi, \frac{5\pi}{2} + 2r\pi\right),$$
$$x'_r \in \left(3\pi + 2r\pi, \frac{7\pi}{2} + 2r\pi\right), \tag{36}$$

for $r = 0, 1, 2, 3, \cdots$, then

$$\begin{aligned}
\sum_{n=\alpha^2+1}^{\infty} \frac{\sin(\pi\sqrt{n}/2\alpha)}{n \ln n} &= \sum_{r=0} \left\{ \sum_{n=(4(r+1)\alpha)^2+1}^{\lfloor (x_r'' 2\alpha/\pi)^2 \rfloor} \frac{\sin(\pi\sqrt{n}/2\alpha)}{n \ln n} + \sum_{n=\lfloor (x_r'' 2\alpha/\pi)^2+1 \rfloor}^{((4r+5)\alpha)^2} \frac{\sin(\pi\sqrt{n}/2\alpha)}{n \ln n} + \sum_{n=((4r+5)\alpha)^2+1}^{((4r+6)\alpha)^2} \frac{\sin(\pi\sqrt{n}/2\alpha)}{n \ln n} \right. \\
&\quad \left. + \sum_{n=(4r+6)\alpha)^2+1}^{\lfloor (x_r' 2\alpha/\pi)^2 \rfloor} \frac{\sin(\pi\sqrt{n}/2\alpha)}{n \ln n} + \sum_{n=\lfloor (x_r' 2\alpha/\pi)^2+1 \rfloor}^{((4r+7)\alpha)^2} \frac{\sin(\pi\sqrt{n}/2\alpha)}{n \ln n} + \sum_{n=((4r+7)\alpha)^2+1}^{((4r+8)\alpha)^2} \frac{\sin(\pi\sqrt{n}/2\alpha)}{n \ln n} \right\} \\
&\geq \sum_{r=0} \left\{ \frac{1}{(4r+5)\alpha \ln((4r+5)\alpha)^2} \left(\sum_{n=(4(r+1)\alpha)^2+1}^{\lfloor (x_r'' 2\alpha/\pi)^2 \rfloor} \frac{\sin(\pi\sqrt{n}/2\alpha)}{\sqrt{n}} ++ \sum_{n=\lfloor (x_r'' 2\alpha/\pi)^2+1 \rfloor}^{((4r+5)\alpha)^2} \frac{\sin(\pi\sqrt{n}/2\alpha)}{\sqrt{n}} \right) \right. \\
&\quad + \frac{1}{(4r+6)\alpha \ln((4r+6)\alpha)^2} \left(\sum_{n=((4r+5)\alpha)^2+1}^{((4r+6)\alpha)^2} \frac{\sin(\pi\sqrt{n}/2\alpha)}{\sqrt{n}} + \sum_{n=(4r+6)\alpha)^2+1}^{\lfloor (x_r' 2\alpha/\pi)^2 \rfloor} \frac{\sin(\pi\sqrt{n}/2\alpha)}{\sqrt{n}} + \sum_{n=\lfloor (x_r' 2\alpha/\pi)^2+1 \rfloor}^{((4r+7)\alpha)^2} \frac{\sin(\pi\sqrt{n}/2\alpha)}{\sqrt{n}} \right) \\
&\quad \left. + \frac{1}{(4r+7)\alpha \ln((4r+7)\alpha)^2} \sum_{n=((4r+7)\alpha)^2+1}^{((4r+8)\alpha)^2} \frac{\sin(\pi\sqrt{n}/2\alpha)}{\sqrt{n}} \right\}.
\end{aligned} \tag{37}$$

In view of (36), we obtain

$$\begin{aligned}
&\sum_{n=(4(r+1)\alpha)^2+1}^{\lfloor (x_r'' 2\alpha/\pi)^2 \rfloor} \frac{\sin(\pi\sqrt{n}/2\alpha)}{\sqrt{n}} + \sum_{n=\lfloor (x_r'' 2\alpha/\pi)^2+1 \rfloor}^{((4r+5)\alpha)^2} \frac{\sin(\pi\sqrt{n}/2\alpha)}{\sqrt{n}} \\
&\geq \int_{(4(r+1)\alpha)^2}^{\lfloor (x_r'' 2\alpha/\pi)^2 \rfloor} \frac{\sin(\pi\sqrt{n}/2\alpha)}{\sqrt{n}} dn + \int_{\lfloor (x_r'' 2\alpha/\pi)^2+1 \rfloor}^{((4r+5)\alpha)^2-1} \frac{\sin(\pi\sqrt{n}/2\alpha)}{\sqrt{n}} dn \\
&= \int_{(4(r+1)\alpha)^2}^{((4r+5)\alpha)^2} \frac{\sin(\pi\sqrt{n}/2\alpha)}{\sqrt{n}} dn - \int_{\lfloor (x_r'' 2\alpha/\pi)^2 \rfloor}^{\lfloor (x_r'' 2\alpha/\pi)^2 \rfloor+1} \frac{\sin(\pi\sqrt{n}/2\alpha)}{\sqrt{n}} dn \\
&\quad - \int_{((4r+5)\alpha)^2-1}^{((4r+5)\alpha)^2} \frac{\sin(\pi\sqrt{n}/2\alpha)}{\sqrt{n}} dn \geq \frac{4\alpha}{\pi} - \frac{2}{(r+1)\alpha},
\end{aligned} \tag{38}$$

$$\begin{aligned}
&\sum_{n=((4r+5)\alpha)^2+1}^{((4r+6)\alpha)^2} \frac{\sin(\pi\sqrt{n}/2\alpha)}{\sqrt{n}} + \sum_{n=(4r+6)\alpha)^2+1}^{\lfloor (x_r' 2\alpha/\pi)^2 \rfloor} \frac{\sin(\pi\sqrt{n}/2\alpha)}{\sqrt{n}} \\
&\quad + \sum_{n=\lfloor (x_r' 2\alpha/\pi)^2+1 \rfloor}^{((4r+7)\alpha)^2} \frac{\sin(\pi\sqrt{n}/2\alpha)}{\sqrt{n}} \\
&\geq \int_{((4r+5)\alpha)^2}^{((4r+6)\alpha)^2} \frac{\sin(\pi\sqrt{n}/2\alpha)}{\sqrt{n}} dn - \frac{4}{(r+1)\alpha} = \frac{-4}{(r+1)\alpha},
\end{aligned} \tag{39}$$

$$\sum_{n=((4r+7)\alpha)^2+1}^{((4r+8)\alpha)^2} \frac{\sin(\pi\sqrt{n}/2\alpha)}{\sqrt{n}} \geq \int_{((4r+7)\alpha)^2}^{((4r+8)\alpha)^2} \frac{\sin(\pi\sqrt{n}/2\alpha)}{\sqrt{n}} dn = -\frac{4\alpha}{\pi}. \tag{40}$$

In view of (37)–(40), we get

$$\begin{aligned}
&\sum_{n=\alpha^2+1}^{\infty} \frac{\sin(\pi\sqrt{n}/2\alpha)}{n \ln n} \\
&\geq \sum_{r=0} \left\{ \frac{4}{\pi(4r+5)\ln((4r+5)\alpha)^2} - \frac{4}{\pi(4r+7)\ln((4r+7)\alpha)^2} - \frac{6}{(r+1)^2\alpha^2} \right\} \\
&\geq -\frac{\pi^2}{\alpha^2}.
\end{aligned} \tag{41}$$

Hence,

$$\begin{aligned}
&\sum_{n=4} \frac{1}{n \ln n} \sin \frac{\pi\sqrt{n}}{2\alpha} \geq \frac{1}{3\pi \ln \alpha} - \frac{\pi^2+1}{\alpha^2}, \\
&\lim_{\alpha \longrightarrow \infty} \frac{f(\Delta x)}{\Delta x} = +\infty.
\end{aligned} \tag{42}$$

This follows from (35) and (41).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they do not have any conflicts of interest.

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