Hindawi
Abstract and Applied Analysis
Volume 2021, Article ID 7569209, 12 pages
https://doi.org/10.1155/2021/7569209

Research Article

Uniformly Convergent Nonpolynomial Spline Method for Singularly Perturbed Robin-Type Boundary Value Problems with Discontinuous Source Term

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Received 19 June 2021; Revised 14 September 2021; Accepted 4 October 2021; Published 22 October 2021

Academic Editor: Devendra Kumar

In this paper, a singularly perturbed second-order ordinary differential equation with discontinuous source term subject to mixed-type boundary conditions is considered. A fitted nonpolynomial spline method is suggested. The stability and parameter uniform convergence of the proposed method are proved. To validate the applicability of the scheme, two model problems are considered for numerical experimentation and solved for different values of the perturbation parameter, ε, and mesh size, h. The numerical results are tabulated in terms of maximum absolute errors and rate of convergence, and it is observed that the present method is more accurate and ε-uniformly convergent for $h \geq \varepsilon$ where the classical numerical methods fail to give good result and it also improves the results of the methods existing in the literature.

1. Introduction

Singular perturbation problems (SPPs) are differential equations with a small positive parameter multiplying the highest derivative term. These problems arise in several branches of applied mathematics, including fluid dynamics, quantum mechanics, elasticity, chemical reactor theory, and gas porous electrodes theory. Examples of SPPs include the Navier-Stokes equation of fluid flow at high Reynolds number, the equation governing flow in porous media, the drift-diffusion equation of semiconductor devices, physics and mathematical models of liquid crystal material, and the convection-diffusion and reaction-diffusion equations to mention but a few [1, 2].

Such equations typically exhibit solutions with layers, which cause severe computational difficulties for standard numerical methods. Consequently, a variety of different numerical strategies have been devoted [1–3] and the references are therein to the construction and analysis of accurate numerical methods for SPPs. Recently, authors [4–8] have considered SPPs for second-/third-order ordinary differential equations (ODEs) with discontinuous source term and/or discontinuous convection coefficient. The novel aspect of the problem under consideration is that we take a source term in the differential equation which has a jump discontinuity at one or more points in the interior of the domain. This gives rise to an interior layer in the exact solution of the problem, in addition to the boundary layer at the outflow boundary point.

A single discontinuity is assumed to occur at a point $d \in \Omega = (0, 1)$. The solution of this problem has a boundary layer at $x = 0$ and interior layers with different widths at $x = d$. It is convenient to introduce the notation $\Omega^- = (0, d)$ and $\Omega^+ = (d, 1)$ and to denote the jump at d in any function with $[w](d) = w(d^+) - w(d^-)$.

Recently, Chandru and Shanthi [9] presented a fitted mesh method to solve singularly perturbed robin-type boundary value problems with discontinuous source term. But still, there is a room to increase the accuracy and develop its convergence for the developed scheme because the treatment of singular perturbation problem is not trivial distributions and the solution pended on perturbation parameter ε

and mesh size h [10]. Due to this, numerical treatment of singularly perturbed boundary value problems needs improvement.

Different authors in [11–23] have applied different families of spline method for second- and higher order singular and regular problems. As far as the researchers' knowledge is concerned, numerical solution of fitted nonpolynomial cubic spline method for the problem under consideration is first being considered. Hence, in the present paper, motivated by the works of [9], we developed a fitted nonpolynomial cubic spline method for singularly perturbed robin-type boundary value problems with discontinuous source term. Therefore, it is important to develop more accurate and convergent numerical method for solving singularly perturbed robin boundary value problems. Thus, the purpose of this study is to develop stable, convergent, and more accurate numerical method for solving singularly perturbed robin boundary value problems with discontinuous source term.

In this article, we consider the following class of problems:

$$Ly(x) \equiv \varepsilon y''(x) + a(x)y'(x) - b(x)y(x) = f(x), \quad x \in (\Omega^- \cup \Omega^+), \tag{1}$$

subject to the boundary conditions,

$$\begin{cases} L_1 y(0) = \alpha_1 y(0) - \beta_1 \varepsilon y'(0) = \phi_0, \\ L_2 y(1) = \alpha_2 y(1) + \beta_2 y'(1) = \varphi, \end{cases} \tag{2}$$

where $\alpha_1, \beta_1 \geq 0$, $\alpha_1 + \beta_1 > 0$, $\beta_2 \geq 0$, $\alpha_2 > 0$, and $\varepsilon(0 < \varepsilon \ll 1)$ are a singular perturbation parameter; $a(x)$ and $b(x)$ are sufficiently smooth functions on $\bar{\Omega}^-$ and $\bar{\Omega}^+$, respectively; and $a(x) \geq a > 0$ and $b(x) \geq b \geq 0$. It is assumed that f is a sufficiently smooth function in $\Omega^- \cup \Omega^+ \cup \{0, 1\}$; the left and right limits of f and their derivatives are assumed to exist at $x = d$. The function f is assumed to have simple discontinuity at $x = d$. Hence, the solution y of equations (1) and (2) does not necessarily have a continuous second derivative at the point d; that is, y does not belong to the class of functions $C^2(\Omega)$. Hence, the class of functions, where y belongs to it, is taken as $C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$. Further, the SPPs in equations (1) and (2) have a unique solution $y \in Y \equiv C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ [24].

Boundary value problems of the type in equations (1) and (2) are model confinement of a plasma column by reaction pressure and geophysical fluid dynamics [25].

2. Properties of Continuous Solution

The following lemmas are necessary for the existence and uniqueness of the solution and for the problem to be well-posed.

Lemma 1 (continues minimum principle). *Suppose that the function $y \in C^1(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ satisfies $L_1 y(0) \geq 0$ and $L_2 y(1) \geq 0$ and $Ly(x) \leq 0, \forall x \in \Omega^- \cup \Omega^+$, and $[y'](d) \leq 0$, then $y(x) \geq 0, \forall x \in \bar{\Omega}$..*

Proof. For the proof, refer to [26]. □

Lemma 2 (stability result). *Consider the boundary value problem of equations (1) and (2) subject to the condition $a(x) \geq a > 0, b(x) \geq b \geq 0$. If $y \in C^1(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$, then*

$$\|y\|_{\bar{\Omega}} \leq C \max \left\{ |L_1 y(0)|, |L_2 y(1)|, |Ly|_{\Omega^- \cup \Omega^+} \right\}. \tag{3}$$

Proof. For the proof, refer to [26]. □

Lemma 3. *For each integer k, satisfying $0 \leq k \leq 4$, the solution y of equations (1) and (2) satisfy the bounds*

$$\left\| y^{(k)} \right\|_{\bar{\Omega} \setminus \{d\}} \leq C \varepsilon^{-k}. \tag{4}$$

Proof. To establish the parameter robust properties of the numerical methods involved in this paper, the following decomposition of Y into smooth v and singular w components is required. □

The smooth component v is defined as the solution of

$$\begin{aligned} &a v_0' - b v_0 = f, \quad x \in \Omega^- \cup \Omega^+, L_1 v_0(0) = q, \\ &a v_1' - b v_1 = -v_0'', \quad x \in \Omega^- \cup \Omega^+, L_1 v_1(1) = 0. \end{aligned} \tag{5}$$

Note that $v = v_0 = \varepsilon v_1 + \varepsilon^2 v_2$, where $v_2 \in C^1(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ and

$$Lv_2 = -v_2'', L_1 v_2(0) = v_2(d) = L_2 v_2(1) = 0. \tag{6}$$

As in [27], we can obtain the following bounds on the derivatives of v for $k = 1, 2, 3$:

$$\|v\| \leq C \left\| v^{(k)} \right\|_{\Omega^- \cup \Omega^+} \leq C\left(1 + \varepsilon^{(2-k)}\right). \tag{7}$$

Note also that $|[v'](d)| \leq C$, $|[v''](d)| \leq C$. Define the singular components of the decomposition as follows. Find $w \in C^0(\Omega)$ such that $Lw = 0, x \in \Omega^- \cup \Omega^+$,

$$\begin{aligned} &L_1 w(0) = y(0) - v(0), \\ &L_2 w(1) = 0, \\ &\left[w'\right](d) = -\left[v'\right](d), \end{aligned} \tag{8}$$

We can further decompose w as

$$w = w_1 + w_2, \tag{9}$$

where w_1 is the boundary layer function satisfying $w_1 \in C^2(\Omega)$, $Lw_2 = 0, x \in \Omega$,

$$\begin{aligned} &L_1 w(0) = y(0) - v(0), \\ &L_2 w(1) = 0, \end{aligned} \tag{10}$$

and w_2 is the interior layer function satisfying

$$\begin{aligned} &w_2 \in C^0(\Omega),\\ &Lw_2 = 0,\\ &x \in \Omega^- \cup \Omega^+,\\ &L_1 w_2(0) = 0,\\ &L_2 w_2(0) = 0,\\ &\left[w_2'\right](d) = -\left[v'\right](d). \end{aligned} \tag{11}$$

$$\left|y_\varepsilon^{(k)}(x)\right| \le C\left\{1 + \varepsilon^{-k}\left(\exp\left(\frac{-ax}{\varepsilon}\right)\right)\right\}, \quad \text{for all } x \in \Omega^-. \tag{12}$$

Lemma 4. *Let y_ε be the solution of (P_ε). Then, for $k = 0, 1, 2, 3$,*

Proof. For the proof, refer to [26]. □

3. Formulation of the Numerical Method

Consider the following singularly perturbed robin-type boundary value problems with discontinuous source terms of the form

$$\varepsilon y''(x) + a(x)y'(x) - b(x)y(x) = f(x). \tag{13}$$

Rearranging equation (13), we obtain

$$\varepsilon y''(x) + p(x)y'(x) + q(x)y(x) = f(x), \tag{14}$$

where $p(x) = a(x)$ and $q(x) = -b(x)$.

Under boundary conditions,

$$y(0) = \phi_0, \tag{15}$$

$$y(1) = \varphi. \tag{16}$$

Consider a uniform mesh with interval $[0, 1]$ in which $0 = x_0 < x_1 < x_2 < \cdots < x_N = 1$,
where$h = 1/N$and $x_i = ih$, $i = 0, 1, 2, \cdots, N$.

For each segment $[x_i, x_{i+1}]$, $i = 1, 2, \cdots, N-1$, the nonpolynomial cubic spline $S_\Delta(x)$ has the following form:

$$\begin{aligned} S_\Delta(x) &= a_i \sin w(x - x_i) + b_i \cos w(x - x_i)\\ &\quad + c_i\left(e^{w(x-xi)} - e^{-w(x-xi)}\right) + d_i\left(e^{w(x-xi)} + e^{-w(x-xi)}\right), \end{aligned} \tag{17}$$

where $a_i, b_i, c_i,$ and d_i are unknown coefficients and $w \neq 0$ arbitrary parameter which will be used to increase the accuracy of the method of Bawa [28].

To determine the unknown coefficients in equation (17) in terms of $y_i, y_{i+1}, M_i,$ and M_{i+1}, first, we define

$$\begin{cases} S_\Delta(x) = y_i, & S_\Delta(x_{i+1}) = y_{i+1},\\ S_\Delta''(x_i) = M_i, & S_\Delta''(x_{i+1}) = M_{i+1}. \end{cases} \tag{18}$$

By some algebra, the coefficients in equation ((17)) becomes

$$\begin{cases} a_i = \dfrac{w^2 y_{i+1} - M_{i+1} + \left(M_i - y_i w^2\right)\cos\theta}{2w^2 \sin\theta}, & b_i = \dfrac{y_i w^2 - M_i}{2w^2},\\ c_i = \dfrac{2\left(y_{i+1}w^2 + M_{i+1}\right) - \left(y_i w^2 + M_i\right)\left(e^\theta + e^{-\theta}\right)}{4w^2\left(e^\theta - e^{-\theta}\right)}, & d_i = \dfrac{y_i w^2 + M_i}{4w^2}, \end{cases} \tag{19}$$

where $\theta = wh$.

Using the continuity condition of the first derivative at x_i, $S_{\Delta-1}'(x_i) = S_\Delta'(x_i)$, we have

$$a_{i-1}\cos\theta - b_{i-1}\sin\theta + c_{i-1}\left(e^\theta + e^{-\theta}\right) + d_{i-1}\left(e^\theta - e^{-\theta}\right) = a_i + 2c_i. \tag{20}$$

Reducing indices of equation (19) by one and substituting into equation (20), we have

$$\begin{aligned} &\left(\frac{-1}{2\sin\theta} - \frac{1}{e^\theta - e^{-\theta}}\right)y_{i-1} + \left(\frac{\cos\theta}{\sin\theta} + \frac{e^\theta + e^{-\theta}}{e^\theta - e^{-\theta}}\right)y_i\\ &\quad + \left(\frac{-1}{2\sin\theta} - \frac{1}{e^\theta - e^{-\theta}}\right)y_{i+1}\\ &\quad = \left(\frac{-h^2}{2\theta^2\sin\theta} - \frac{h^2}{\theta^2\left(e^\theta - e^{-\theta}\right)}\right)M_{i-1}\\ &\quad + \left(\frac{h^2\cos\theta}{\theta^2\sin\theta} + \frac{h^2\left(e^\theta + e^{-\theta}\right)}{\theta^2\left(e^\theta - e^{-\theta}\right)}\right)M_i\\ &\quad + \left(\frac{-h^2}{2\theta^2\sin\theta} - \frac{h^2}{\theta^2\left(e^\theta - e^{-\theta}\right)}\right)M_{i+1}. \end{aligned} \tag{21}$$

Multiplying both side of equation (21) by $(-2\sin\theta(e^\theta - e^{-\theta}))/(e^\theta - e^{-\theta} + 2\sin\theta)$, we obtain

$$y_{i-1} + ky_i + y_{i+1} = h^2(\alpha M_{i-1} + \beta M_i + \alpha M_{i+1}), \tag{22}$$

where

$$\begin{cases} k = -2\left(\dfrac{e^\theta(\cos\theta + \sin\theta) + e^{-\theta}(\sin\theta - \cos\theta)}{e^\theta - e^{-\theta} + 2\sin\theta}\right),\\ \alpha = \left(\dfrac{e^\theta - e^{-\theta} - 2\sin\theta}{\theta^2\left(e^\theta - e^{-\theta} + 2\sin\theta\right)}\right),\\ \beta = -2\left(\dfrac{e^\theta h^2(\cos\theta - \sin\theta) - e^{-\theta}(\sin\theta + \cos\theta)}{\theta^2\left(e^\theta - e^{-\theta} + 2\sin\theta\right)}\right). \end{cases} \tag{23}$$

As $\theta \longrightarrow 0$, $(\alpha, \beta, k) \longrightarrow (1/6, 4/6, -2)$.

Substituting $S''_{\Delta}(x_i) = y''_i = M_i$ in equation (14), we get

$$\begin{cases} \varepsilon M_i = f_i - p_i y'_i - q_i y_i, \\ \varepsilon M_{i-1} = f_{i-1} - p_{i-1} y'_{i-1} - q_{i-1} y_{i-1}, \\ \varepsilon M_{i+1} = f_{i+1} - p_{i+1} y'_{i+1} - q_{i+1} y_{i+1}. \end{cases} \quad (24)$$

Substituting equation (24) into equation (22), we obtain

$$\begin{aligned} \frac{\varepsilon}{h^2}(y_{i-1} + k y_i + y_{i+1}) &= \alpha\left(f_{i-1} - p_{i-1} y'_{i-1} - q_{i-1} y_{i-1}\right) \\ &+ \beta\left(f_i - p_i y'_i - q_i y_i\right) \\ &+ \alpha\left(f_{i+1} - p_{i+1} y'_{i+1} - q_{i+1} y_{i+1}\right) \end{aligned} \quad (25)$$

and the following approximations for the first derivatives of y by Bawa [28]:

$$\begin{cases} y'_{i-1} \approx \dfrac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h}, \\ y'_{i+1} \approx \dfrac{3y_{i+1} - 4y_i + y_{i-1}}{2h}, \\ y'_i \approx \left(\dfrac{1 - 2h^2\psi q_{i+1} - h\psi(3p_{i+1} + p_{i-1})}{2h}\right) y_{i+1} + 2\psi(p_{i+1} + p_{i-1}) y_i, \\ \quad + \left(\dfrac{1 - 2h^2\psi q_{i-1} - h\psi(p_{i+1} + 3p_{i-1})}{2h}\right) y_{i-1} + h\psi(r_{i+1} - r_{i-1}), \end{cases} \quad (26)$$

where ω is the parameter used to raise the accuracy of the method.

Substituting equations (26) into equation (25) and rearranging, we get

$$\begin{aligned} &\frac{\varepsilon}{h^2}[y_{i-1} + k y_i + y_{i+1}] \\ &= \left[\frac{3\alpha p_{i-1}}{2h} - \alpha q_{i-1} - \frac{\beta p_i}{2h}\left(-1 + 2\psi h^2 q_{i-1} - \psi h(p_{i+1} + 3p_{i-1})\right) + \frac{\alpha p_{i+1}}{2h}\right] y_{i-1} \\ &+ \left[\frac{-2\alpha p_{i-1}}{h} - 2\beta p_i \psi(p_{i+1} + p_{i-1}) - \beta p_i q_i \alpha q_{i-1} - \frac{2\alpha p_{i+1}}{h}\right] y_i \\ &+ \left[\frac{3\alpha p_{i+1}}{2h} + \alpha q_{i+1} - \frac{\beta p_i}{2h}\left(1 - 2\psi h^2 q_{i+1} - \psi h(3p_{i+1} + p_{i-1})\right) + \frac{\alpha p_{i-1}}{2h}\right] y_{i+1} \\ &+ (\alpha + \beta p_i \psi h) f_{i-1} + \beta f_i + (\alpha - \beta p_i \psi h) f_{i+1}. \end{aligned} \quad (27)$$

From the theory of singular perturbation, it is known that the solution of equation (14) and equation (15) is of the form (O'Malley [29])

$$y(x) = y_0(x) + \frac{p(0)}{p(x)}(\phi_0 - y_0(0)) \exp\left(-\int_0^x \left(\frac{p(x)}{\varepsilon} - \frac{q(x)}{p(x)}\right) dx\right) + O(\varepsilon), \quad (28)$$

where $y_0(x)$ is the solution of the reduced problem (obtained by setting $\varepsilon = 0$) of equation (14) which is given by

$$p(x) y'_0(x) + q(x) y_0(x) = f(x), \quad \text{with}\ y_0(1) = \varphi. \quad (29)$$

Expanding $p(x)$ and $q(x)$ about point "0" up to the first term using Taylor's series, the discretized form of equation (28) gives

$$y(x_i) = y_0(x_i) + (\phi_0 - y_0(0)) \exp\left(-\left(\frac{p(0)}{\varepsilon} - \frac{q(0)}{p(0)}\right) x_i\right) + O(\varepsilon). \quad (30)$$

Therefore,

$$\lim_{h \longrightarrow \infty} y(ih) = y_0(0) + (\phi_0 - y_0(0)) \exp\left(-\left(\frac{p^2(0) - \varepsilon q(0)}{p(0)}\right) i\rho\right) + O(\varepsilon), \quad (31)$$

where $\rho = h/\varepsilon$.

To handle the effect of the perturbation parameter in equation (27), artificial viscosity ($\sigma(\rho)$) is multiplied on the term containing the perturbation parameter as

$$\begin{aligned} &\frac{\varepsilon\sigma(\rho)}{h^2}[y_{i-1} + k y_i + y_{i+1}] \\ &= \left[\frac{3\alpha p_{i-1}}{2h} - \alpha q_{i-1} - \frac{\beta p_i}{2h}\left(-1 + 2\psi h^2 q_{i-1} - \psi h(p_{i+1} + 3p_{i-1})\right) + \frac{\alpha p_{i+1}}{2h}\right] y_{i-1} \\ &+ \left[\frac{-2\alpha p_{i-1}}{h} - 2\beta p_i \psi(p_{i+1} + p_{i-1}) - \beta p_i q_i \alpha q_{i-1} - \frac{2\alpha p_{i+1}}{h}\right] y_i \\ &+ \left[\frac{3\alpha p_{i+1}}{2h} + \alpha q_{i+1} - \frac{\beta p_i}{2h}\left(1 - 2\psi h^2 q_{i+1} - \psi h(3p_{i+1} + p_{i-1})\right) + \frac{\alpha p_{i-1}}{2h}\right] y_{i+1} \\ &+ (\alpha + \beta p_i \psi h) f_{i-1} + \beta f_i + (\alpha - \beta p_i \psi h) f_{i+1}. \end{aligned} \quad (32)$$

Multiplying equation (32) by h and taking limit as $h \longrightarrow 0$, we get the fitting factor

$$\sigma(\rho) = \rho p(0)\left[2\phi_0 + \frac{\varphi}{2} \coth\left(\left(\frac{p^2(0) - \varepsilon q(0)}{p(0)}\right)\frac{\rho}{2}\right)\right], \quad (33)$$

which is a required fitting factor in the left boundary layer. From equation (32), we get

$$\begin{aligned} &\left(\varepsilon\sigma - \frac{h}{2} u_i + \alpha h^2 q_{i-1}\right) y_{i-1} - \left(-\rho\varepsilon\sigma + \frac{h}{2} v_i - \beta h^2 q_i\right) y_i \\ &+ \left(\varepsilon\sigma - \frac{h}{2} w_i + \alpha h^2 q_{i+1}\right) y_{i+1} \\ &= h^2\left((\alpha + \beta p_i \psi h) r_{i-1} + \beta r_i + (\alpha - \beta p_i \psi h) r_{i+1}\right), \end{aligned} \quad (34)$$

where

$$\begin{cases} u_i = 3\alpha p_{i-1} + \beta p_i \psi h(p_{i+1} + 3p_{i-1}) - \beta p_i \psi h^2 q_{i-1} - \alpha p_{i+1} + \beta p_i, \\ v_i = -4\alpha p_{i-1} - 4\beta h p_i \psi(p_{i+1} + p_{i-1}) + 4\alpha p_{i+1}, \\ wi = \alpha p_{i-1} + \beta p_i \psi h(3p_{i+1} + p_{i-1}) - 3\alpha p_{i+1} - \beta p_i + \beta p_i \psi h^2 q_{i+1}. \end{cases} \quad (35)$$

Then, we get the three term recurrence relation of the form

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, \cdots, N-1, \tag{36}$$

where

$$\begin{cases} E_i = \varepsilon\sigma - \dfrac{h}{2}u_i + \alpha h^2 q_{i-1}, \\ F_i = -\rho\varepsilon\sigma + \dfrac{h}{2}v_i - \beta h^2 q_i, \\ G_i = \varepsilon\sigma - \dfrac{h}{2}w_i + \alpha h^2 q_{i+1}, \\ H_i = h^2((\alpha + \beta p_i \psi h) f_{i-1} + \beta f_i + (\alpha - \beta p_i \psi h) f_{i+1}). \end{cases} \tag{37}$$

To treat the boundary condition, we used forward finite difference formula for $i = 0$ and backward difference formula for $i = N$, respectively, for the first derivative term.

That is, for $i = 0$, from equation (2), we have

$$\begin{aligned} \alpha_1 y(0) - \beta_1 \varepsilon y'(0) = \phi_0 &\Rightarrow \alpha_1 y_0 - \beta_1 \varepsilon y'_0 = \phi_0, \\ &\Rightarrow \left(\alpha_1 + \frac{\beta_1 \varepsilon}{h}\right) y_0 - \frac{\beta_1 \varepsilon}{h} y_1 = \phi_0. \end{aligned} \tag{38}$$

Similarly, for $i = N$, from equation (2), we have

$$\begin{aligned} \alpha_2 y(N) + \beta_2 y'(N) = \varphi &\Rightarrow \alpha_2 y_N + \beta_2 y'_N = \varphi, \\ &\Rightarrow -\frac{\beta_2}{h} y_{N-1} + \left(\alpha_2 + \frac{\beta_2}{h}\right) y_N = \varphi. \end{aligned} \tag{39}$$

Therefore, the problem in equation (1) with given boundary condition in equation (2) can be solved using the scheme equations (36), (38), and (39) which form an $N \times N$ system of algebraic equations.

4. Truncation Error

From equation (26), we have

$$\begin{cases} e'_{i+1} = Y'(x_{i+1}) - y'_{i+1} = \dfrac{h^2 y_i^{(3)}}{3} + \dfrac{h^3 y_i^{(4)}}{12} + \dfrac{h^4 y_i^{(5)}}{30} + O(h^5), \\ e'_{i-1} = Y'(x_{i-1}) - y'_{i-1} = \dfrac{h^2 y_i^{(3)}}{3} - \dfrac{h^3 y_i^{(4)}}{12} + \dfrac{h^4 y_i^{(5)}}{30} + O(h^5), \\ e'_i = Y'(x_i) - y'_i = -h^2\left(\dfrac{1}{6} + 2\psi\varepsilon\sigma\right) y_i^{(3)} - \dfrac{h^4 \psi\varepsilon\sigma y_i^{(5)}}{3} + \dfrac{h^4 y_i^{(5)}}{120} + O(h^5). \end{cases} \tag{40}$$

From equation (24) and equation (22),

$$\begin{aligned} \varepsilon\sigma y_{i-1} + \varepsilon\sigma k y_i + \varepsilon\sigma y_{i+1} = h^2 \Big(&\alpha\left(f_{i-1} - p_{i-1} y'_{i-1} - q_{i-1} y_{i-1}\right) \\ &+ \beta\left(f_i - p_i y'_i - q_i y\right) \\ &+ \alpha\left(f_{i+1} - p_{i+1} y'_{i+1} - q_{i+1} y_{i+1}\right)\Big). \end{aligned} \tag{41}$$

Putting the exact solution in equation (41), we have

$$\begin{aligned} &\varepsilon\sigma Y(x_{i-1}) + \varepsilon\sigma k Y(x_i) + \varepsilon\sigma Y(x_{i+1}) \\ &\quad = h^2 \Big(\alpha\left(f_{i-1} - p_{i-1} Y'(x_{i-1}) - q_{i-1} Y(x_{i-1})\right) \\ &\qquad + \beta\left(f_i - p_i Y'(x_i) - q_i Y(x_i)\right)\Big) \\ &\qquad + h^2 \alpha\left(f_{i+1} - p_{i+1} Y'(x_{i+1}) - q_{i+1} Y(x_{i+1})\right) + T_0(h), \end{aligned} \tag{42}$$

where

$$\begin{aligned} T_0(h) = &-(2+k)\varepsilon\sigma y_i + (2\alpha + \beta - 1)\varepsilon\sigma h^2 y''_i \\ &+ (12\alpha - 1)\frac{\varepsilon\sigma h^4 y_i^{(4)}}{12} + (2\alpha + \beta - 1)\varepsilon\sigma h^6 y_i^{(6)}. \end{aligned} \tag{43}$$

Subtract equation (41) from equation (42) and putting $e_j = Y(x_j) - y_j, j = i \pm 1$, we get

$$\begin{aligned} &\left(\varepsilon\sigma + h^2 \alpha q_{i-1}\right) e_{i-1} + \left(\varepsilon\sigma k + \beta h^2 q_i\right) e_i + \left(\varepsilon\sigma + h^2 \alpha q_{i+1}\right) e_{i+1} \\ &\qquad = -h^2 \left(\alpha p_{i-1} e'_{i-1} + \beta p_i e'_i + \alpha p_{i+1} e'_{i+1}\right) \\ &\qquad + T_0(h). \end{aligned} \tag{44}$$

Using equation (40), we have

$$\begin{aligned} &(\varepsilon\sigma + h^2 \alpha q_{i-1}) e_{i-1} + (k\sigma\varepsilon + \beta h^2 q_i) e_i + (\varepsilon\sigma + h^2 \alpha q_{i+1}) e_{i+1} \\ &\quad = -h^4 \left(\frac{\alpha p_{i-1}}{3} + \beta p_i \left(\frac{1}{6} + 2\psi\varepsilon\sigma\right) + \frac{\alpha p_{i+1}}{3} y_i^{(3)}\right) \\ &\qquad - h^5 \left(\frac{-\alpha p_{i-1}}{12} + \frac{\alpha p_{i+1}}{12}\right) y_i^{(4)} - h^6 \\ &\qquad \cdot \left(\frac{1}{30}(\alpha p_{i-1} + \alpha p_{i+1}) y^{(5)} + \beta p_i \left(-\frac{\psi\varepsilon\sigma}{3} + \frac{1}{120}\right) y_i^{(5)}\right) + T_0(h). \end{aligned} \tag{45}$$

Let

$$\begin{cases} p_{i+1} = p_i + h p'_i + \dfrac{h^2 p''_i}{2} + O(h^3), \\ p_{i-1} = p_i - h p'_i + \dfrac{h^2 p''_i}{2} + O(h^3). \end{cases} \tag{46}$$

Substituting equation (46) into equation (45), we have

$$\left(\varepsilon\sigma + h^2\alpha q_{i-1}\right)e_{i-1} + \left(k\varepsilon\sigma + \beta h^2 q_i\right)e_i + \left(\varepsilon\sigma + h^2\alpha q_{i+1}\right)e_{i+1} = T_i(h), \tag{47}$$

where

$$\begin{aligned} T_i(h) = &-(2+k)\varepsilon\sigma y_i + (2\alpha+\beta-1)\varepsilon\sigma h^2 y_i'' \\ &+ \left(\frac{-2\alpha}{3} + \beta\left(\frac{1}{6} + 2\psi\varepsilon\sigma\right)p_i\right)h^4 y_i^{(3)} \\ &+ (12\alpha-1)\frac{\varepsilon\sigma h^4 y_i^{(4)}}{12} + O\left(h^6\right). \end{aligned} \tag{48}$$

$T_i(h)$ is a local truncation error associated with the scheme developed in equation (36). Thus, for different values of $k, \alpha, \beta,$ and ψ in the scheme of equation (36), the following different orders are indicated:

(1) For $k=-2, 2\alpha+\beta=1$, and for any value of ψ, the scheme of equation (36) gives the second-order method

(2) For $k=-2, \alpha=1/12, \beta=10/12$, and $\psi=-1/20$, from equation (36), the fourth-order method is derived

5. Stability and Convergence Analysis

Writing the tridiagonal system in equation (36) in matrix vector form, we have

$$A\bar{Y} = C, \tag{49}$$

where $A = B_0 + B$ and $B = (B_1 + h^2 B_2 Q)$ is a tridiagonal matrix of order $N-1$. Multiplying both sides of equation (34) by (-1), we get

$$B_0 = \begin{bmatrix} -k\sigma\varepsilon & -\sigma\varepsilon & & & \\ -\sigma\varepsilon & -k\sigma\varepsilon & -\sigma\varepsilon & & \\ & \ddots & \ddots & \ddots & \\ & & -\sigma\varepsilon & -k\sigma\varepsilon & -\sigma\varepsilon \\ & & & -\sigma\varepsilon & -k\sigma\varepsilon \end{bmatrix},$$

$$B_1 = \begin{bmatrix} \frac{hv_1}{2} & \frac{hw_1}{2} & & & \\ \frac{hu_2}{2} & \frac{hv_2}{2} & \frac{hw_2}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{hu_{N-2}}{2} & \frac{hv_{N-2}}{2} & \frac{hw_{N-2}}{2} \\ & & & \frac{hu_{N-1}}{2} & \frac{hv_{N-1}}{2} \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -\beta & -\alpha & & & \\ -\alpha & -\beta & -\alpha & & \\ & \ddots & \ddots & \ddots & \\ & & -\alpha & -\beta & -\alpha \\ & & & -\alpha & -\beta \end{bmatrix},$$

$$Q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{N-2} \\ q_{N-1} \end{bmatrix}. \tag{50}$$

$C = (c_i)$ is the column vector, where

$$\begin{aligned} c_i = &h^2((-\alpha - \beta p_1\psi h)f_0 - \beta f_1 - (\alpha - \beta p_1 \psi h)f_2) \\ &+ \varepsilon\sigma\phi_0 - \frac{h\phi_0}{2}u_1, \quad \text{for } i=1. \end{aligned}$$

$$c_i = h^2((-\alpha - \beta p_i\psi h)f_{i-1} - \beta f_i - (\alpha - \beta p_i\psi h)f_{i+1}), \quad \text{for } 2 \le i \le N-2.$$

$$\begin{aligned} c_i = &h^2\left((-\alpha - \beta p_{n-1}\psi h)f_{N-2} - \beta f_{N-1} - (\alpha - \beta p_{N-1}\psi h)f_N + \alpha h^2 q_N\right) \\ &+ \varepsilon\sigma\varphi - \frac{h\varphi}{2}w_N, \quad \text{for } i=N-1. \end{aligned} \tag{51}$$

$\bar{Y} = [\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_{N-1}]^t$ is an approximation solution; $T(h) = O(h^6)$ for $k=-2$, $\alpha=1/12$, $\beta=10/12$, and $\psi=-1/20\varepsilon$.

Now, considering the above system with exact solution $Y = [y(x_1), y(x_2), \cdots, y(x_{N-1})]^t$, we have

$$AY - T(h) = C, \tag{52}$$

where

$T(h) = [T_1(h), T_2(h), \cdots, T_{N-1}(h)]^t$ is a local truncation error.

From equation (49) and equation (52), we obtain

$$A\left(Y - \bar{Y}\right) = T(h). \tag{53}$$

This gives the error equation

$$AE = T(h), \tag{54}$$

where

$$E = Y - \bar{Y} = (e_1, e_2, \cdots, e_{N-1})^t. \tag{55}$$

Let

$$|p_{i-1}| \le Z_1, |p_i| \le Z_2, |p_{i+1}| \le Z_3, |q_{i-1}| \le K_1, |q_i| \le K_2, |q_{i+1}| \le K_3, \tag{56}$$

where $Z_1, Z_2, Z_3, K_1, K_2,$ and K_3 are positive constants.

Table 1: Maximum absolute errors for Example 1 at different mesh points N.

ε	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
10^{-4}	1.0156e-02	5.0781e-03	2.5391e-03	1.2695e-03	6.2999e-04
10^{-8}	1.0156e-02	5.0781e-03	2.5391e-03	1.2695e-03	6.2999e-04
10^{-12}	1.0156e-02	5.0781e-03	2.5391e-03	1.2695e-03	6.2999e-04
10^{-16}	1.0156e-02	5.0781e-03	2.5391e-03	1.2695e-03	6.2999e-04
10^{-20}	1.0156e-02	5.0781e-03	2.5391e-03	1.2695e-03	6.2999e-04

Table 2: Comparison of maximum absolute errors and order of convergence for Example 1 at different of mesh points N.

$N \longrightarrow$	64	128	256	512	1024
Present method					
E^N	5.0781e-03	2.5382e-03	1.2467e-03	5.5910e-04	2.3203e-04
R^N	1.0005	1.2667	1.1569	1.2688	
Method in [9]					
E^N	2.5658e−02	1.4128e−02	7.5359e−03	3.8225e−03	1.7536e−03
R^N	0.86085	0.90671	0.97927	1.1242	

Let r_{ij} be the $(i,j)^{\text{th}}$ element of the matrix B; then, for $i=1,2,\cdots,N-1$,

$$\begin{aligned}
\left|r_{i,i+1}\right| &= \left|\frac{h}{2}w_i - \alpha h^2 q_{i+1}\right| \\
&\le \frac{h}{2}\left(\alpha Z_1 + \beta Z_2\left(1+2\psi h^2 K_3 + \psi h(3Z_3+Z_1)\right) + 3\alpha Z_3 + 2\alpha h K_3\right), \\
\left|r_{i,i-1}\right| &= \left|\frac{h}{2}u_i - \alpha h^2 q_{i-1}\right| \\
&\le \frac{h}{2}\left(3\alpha Z_1 + \beta Z_2\left(1+2\omega h^2 K_1 + \omega h(Z_3+3Z_1)\right) + \alpha Z_3 + 2\alpha h K_1\right)..
\end{aligned} \tag{57}$$

Thus, for sufficiently small h, we have

$$\begin{aligned}
-\varepsilon\sigma + \left|r_{i,i+1}\right| \ne 0 \quad \text{for } i=1,2,\cdots,N-2, \\
-\varepsilon\sigma + \left|r_{i,i-1}\right| \ne 0 \quad \text{for } i=2,\cdots,N-1.
\end{aligned} \tag{58}$$

Hence, the matrix A is irreducible (Varga [30]).

From equation (36), for sufficiently small h and $k=-2$ and $\varepsilon, 0<\varepsilon\ll 1$, and we get $|E_i+G_i|<|F_i|$.

Hence, A is diagonally dominant. Under this condition, Thomas algorithm method is stable (Kadalbajoo and Reddy [31]).

Let S_i be the sum of the elements of the i^{th} row of the matrix A; then, we have

$$\begin{aligned}
S_i &= -k\sigma\varepsilon - \varepsilon\sigma + \frac{h}{2}(-3\alpha p_{i-1} - \beta p_i + \alpha p_{i+1}) \\
&\quad + \frac{h^2}{2}(\beta\psi p_i(p_{i+1}+3p_{i-1}) - 2(\beta q_i + \alpha q_{i+1})) - h^3\beta\psi p_i q_{i+1}, \text{for } i=1, \\
S_i &= -k\sigma\varepsilon - 2\sigma\varepsilon + h^2(-\alpha q_{i-1} - \beta q_i - \alpha q_{i+1})) + O(h^3), \text{for } 2\le i\le N-2, \\
S_i &= -k\sigma\varepsilon - \varepsilon\sigma + \frac{h}{2}(-\alpha p_{i-1} + \beta p_i + 3\alpha p_{i+1}) \\
&\quad + \frac{h^2}{2}(\beta\psi p_i(p_{i+1}+p_{i-1}) - 2(\beta q_i + \alpha q_{i-1})) \\
&\quad + h^3\beta\psi p_i q_{i-1}, \text{for } i=N-1.
\end{aligned} \tag{59}$$

Let $Z_{2*} = \min_{1\le i\le N-1}|p_i|$, $Z_2^* = \max_{1\le i\le N-1}|p_i|$, $K_{2*} = \min_{1\le i\le N-1}|q_i|$, and $K_2^* = \max_{1\le i\le N-1}|q_i|$, then $0<Z_{2*}\le Z_2\le Z_2^*$ and $0<K_{2*}\le K_2\le K_2^*$.

For sufficiently small h, A is monotone (Young [32]).

Hence, A^{-1} exist and $A^{-1}\ge 0$. From equation (63), we have

$$\begin{aligned}
E &= A^{-1}T(h), \\
\|E\| &\le \left\|A^{-1}\right\|\|T(h)\|.
\end{aligned} \tag{60}$$

For sufficiently small h, we have

$$\begin{cases}
S_1 > h^2 K_{2*} D_1, \text{for } i=1, \\
S_i > h^2 K_{2*} D_2, \text{for } 2\le i\le N-2, \\
S_{n-1} > h^2 K_{2*} D_1, \text{for } i=N-1,
\end{cases} \tag{61}$$

where $D_1=\alpha+\beta$ and $D_2 = 2\alpha+\beta$.

Table 3: Maximum absolute errors for Example 2 at different mesh points.

$\varepsilon \downarrow N \longrightarrow$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
10^{-4}	4.1730e-02	2.0931e-02	1.0482e-02	5.2450e-03	2.4856e-03
10^{-8}	4.1730e-02	2.0931e-02	1.0482e-02	5.2450e-03	2.4856e-03
10^{-12}	4.1730e-02	2.0931e-02	1.0482e-02	5.2450e-03	2.4856e-03
10^{-16}	4.1730e-02	2.0931e-02	1.0482e-02	5.2450e-03	2.4856e-03
10^{-20}	4.1730e-02	2.0931e-02	1.0482e-02	5.2450e-03	2.4856e-03

Table 4: Comparison of maximum absolute errors and order of convergence for Example 2 at number of mesh points N.

$N \longrightarrow$	64	128	256	512	1024
Present method					
E^N	2.0929e-02	1.0400e-02	4.7763e-03	1.9081e-03	7.6337e-04
R^N	1.0089	1.1226	1.3238	1.3217	
Method in [9]					
E^N	9.6698e−01	5.8056e−01	3.2795e−01	1.7313e−01	8.1501e−02
R^N	0.73604	0.82399	0.92161	1.08700	

Let $A_{i,k}^{-1}$ be the $(i,k)^{\text{th}}$ element of A^{-1}, and we define

$$\|A_{i,k}^{-1}\| = \max_{1\le i\le N-1} \sum_{k=1}^{n-1} |A_{i,k}^{-1}|, \tag{62}$$

$$\|T(h)\| = \max_{1\le i\le N-1} |T_i(h)|. \tag{63}$$

Since $A_{i,k}^{-1} \ge 0$, then from the theory of matrices, we have

$$\sum_{k=1}^{N-1} A_{i,k}^{-1} \cdot S_k = 1 \quad \text{for } i = 1, 2, \cdots, N-1. \tag{64}$$

Hence,

$$A_{i,1}^{-1} \le \frac{1}{S_1} < \frac{1}{h^2 K_{2*} D_1} \quad \text{for } k = 1, \tag{65}$$

$$A_{i,N-1}^{-1} \le \frac{1}{S_{N-1}} < \frac{1}{h^2 K_{2*} D_1} \quad \text{for } k = N-1, \tag{66}$$

$$\sum_{k=2}^{N-2} A_{i,k}^{-1} \le \frac{1}{\min_{2\le k\le N-2} S_k} < \frac{1}{h^2 K_{2*} D_2} \quad \text{for } k = 2, 3, \cdots, N-2. \tag{67}$$

From equations (54)–(68) and (48), we get

$$\|E\| \le \left(\frac{1}{h^2 K_{2*} D_1} + \frac{1}{h^2 K_{2*} D_1} + \frac{1}{h^2 K_{2*} D_2}\right) O\left(h^6\right) = D^* h^4, \tag{68}$$

where

$$D^* = \left(\frac{1}{K_{2*} D_1} + \frac{1}{K_{2*} D_1} + \frac{1}{K_{2*} D_2}\right), \tag{69}$$

which is independent of mesh size h. This establishes that the present method is fourth-order convergent for $k = -2$, $\alpha = 1/12$, $\beta = 10/12$, and $\psi = -1/20\varepsilon$.

6. Numerical Examples and Results

To validate the applicability of the method, two model problems are considered for numerical experimentation for different values of the perturbation parameter and mesh points. The numerical results are tabulated in terms of maximum absolute errors, numerical rate of convergence, and uniform errors (see Tables 1–4) and compared with the results of the previously developed numerical methods existing in the literature (Tables 2 and 4). Further, behavior of the numerical solution without and with fitted operator (Figures 1 and 2), point-wise absolute errors (Figure 3), and the ε-uniform convergence of the method are shown by the log-log plot of the ε-uniform error (Figure 4).

Having $y_j \equiv y_j^N$ (the approx. solution obtained via the present method) for different values of N and ε since the exact solution is not available, the maximum errors (denoted by E_ε^N) are evaluated using the formula given by the double-mesh principle:

$$E_\varepsilon^N \coloneqq \max_{0\le j\le N} \left| y_j^N - y_{2j}^{2N} \right|. \tag{70}$$

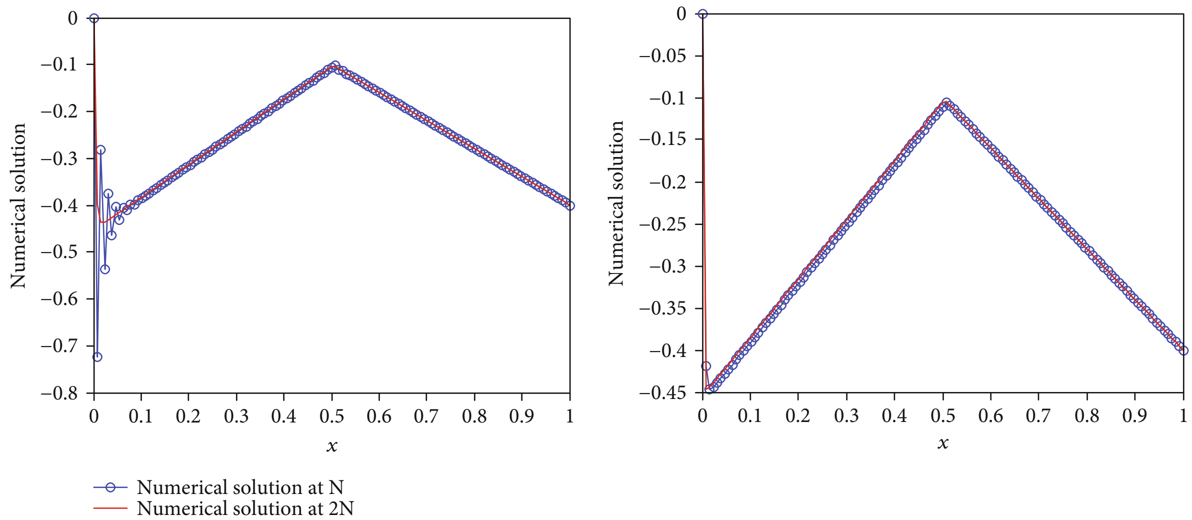


FIGURE 1: Numerical behavior of Example 1 without and with fitted operator using nonpolynomial cubic spline method at $N = 128$ and $\varepsilon = 2^{-10}$.

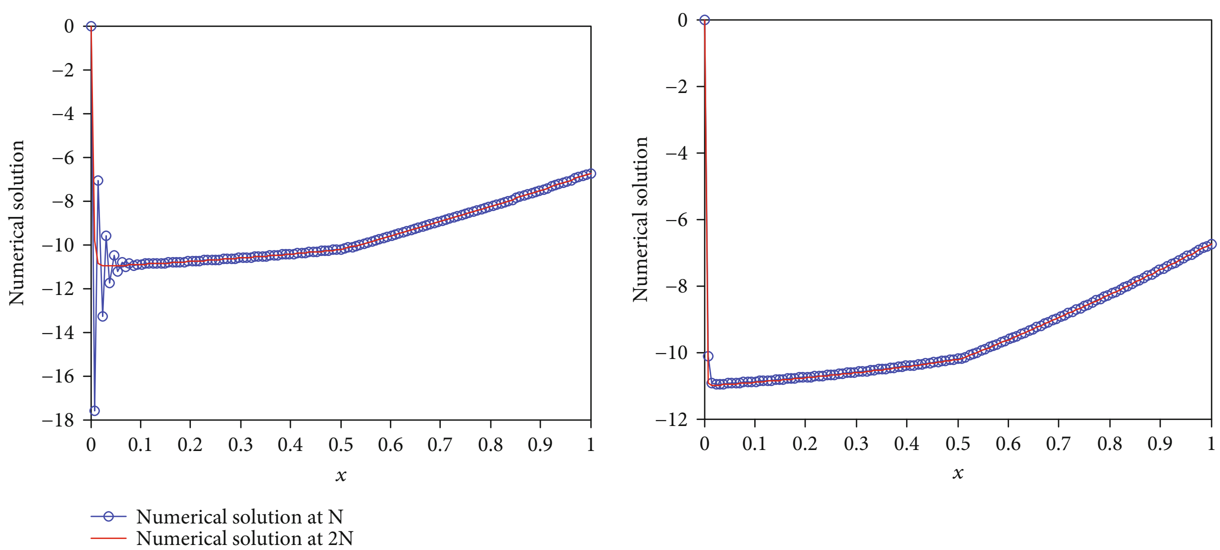


FIGURE 2: Numerical behavior of Example 2 without and with fitted operator using nonpolynomial cubic spline method at $N = 128$ and $\varepsilon = 2^{-10}$.

Further, we will tabulate the errors

$$E^N = \max_{0<\varepsilon\leq 1} E_\varepsilon^N. \tag{71}$$

The numerical rates of convergence are computed using the formula

$$r_\varepsilon^N := \log\ \log_2^{\left(E_\varepsilon^N/E_\varepsilon^{2N}\right)}, \tag{72}$$

and the numerical rate of "ε-uniform convergence" is computed using

$$R^N = \log_2^{\left(E^N/E^{2N}\right)}. \tag{73}$$

Example 1.

$$\begin{cases} \varepsilon y''(x) + y'(x) = f(x), & x \in \Omega^- \cup \Omega^+, \\ y(0) - \varepsilon y'(0) = 1, y(1) + y'(1) = -1, \end{cases} \tag{74}$$

where

$$f(x) = \begin{cases} 0.7, & \text{for } 0 \leq x \leq 0.5, \\ -0.6, & \text{for } 0.5 < x \leq 1. \end{cases} \tag{75}$$

Example 2.

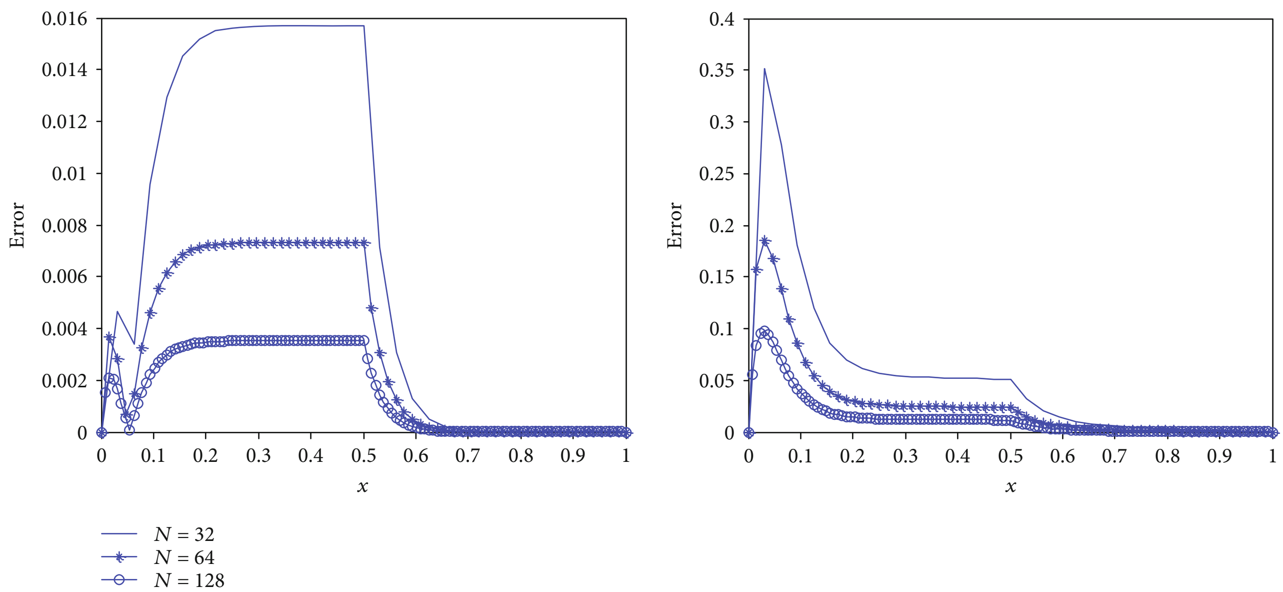
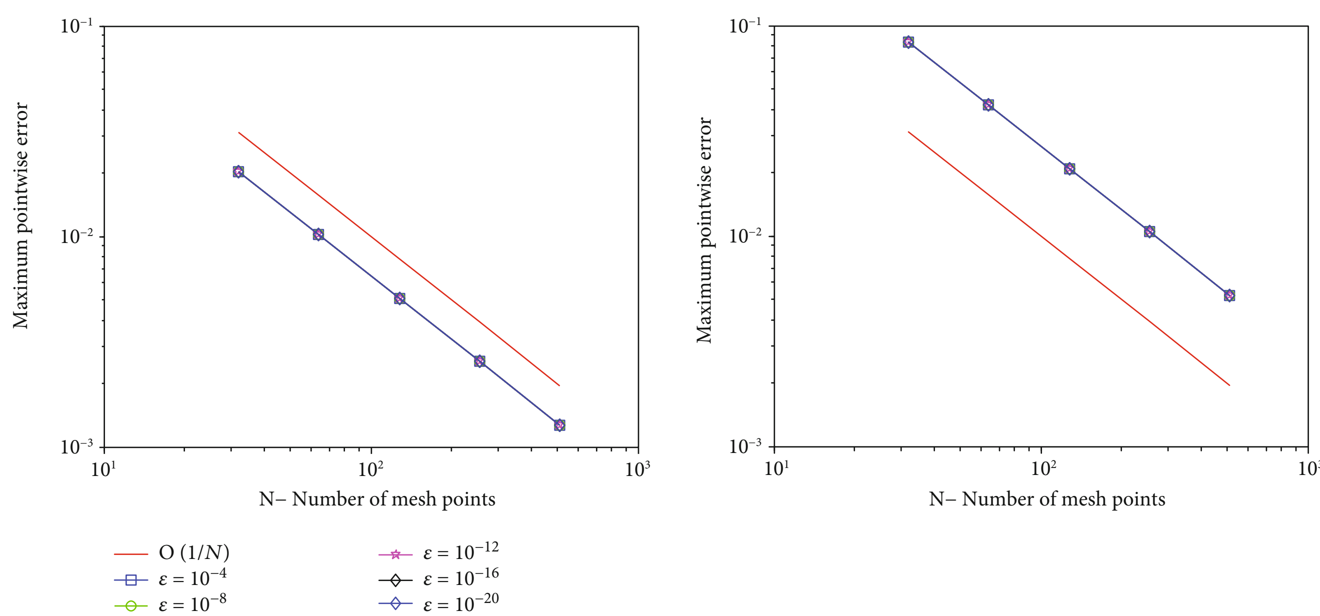


FIGURE 3: Point-wise absolute error of Example 1 and Example 2 with fitted operator using nonpolynomial cubic spline method, respectively, for $N = 128$ and $\varepsilon = 2^{-10}$.



(a) (b)

FIGURE 4: ε-uniform convergence with log-log scale for Example 1 and Example 2, respectively.

$$\begin{cases} \varepsilon y''(x) + \dfrac{1}{1+x} y'(x) = f(x), x \in \Omega^{-} \cup \Omega^{+}, \\ y(0) - \varepsilon y'(0) = 1, y(1) + y'(1) = 1, \end{cases} \tag{76}$$

where

$$f(x) = \begin{cases} 1 + x, & \text{for } 0 \leq x \leq 0.5, \\ 4.0, & \text{for } 0.5 < x \leq 1. \end{cases} \tag{77}$$

7. Discussion and Conclusion

This study introduces uniformly convergent nonpolynomial cubic spline method based on exponential fitted operator for solving singularly perturbed second-order ODEs of Robin-type BVPs with discontinuous source term. Due to discontinuity in the source term, there is an interior layer occurring. To fit the interior and boundary layer, a suitable fitted operator finite difference method on uniform mesh is constructed. The behavior of the continuous solution of the problem is studied and shown that it satisfies the continuous stability estimate and the derivatives of the solution are also bounded. The numerical scheme is developed on uniform mesh. The Robin-type BVPs are treated using numerical finite difference techniques, and the results are compared accordingly. The stability of the developed scheme is established, and its uniform convergence is proved. Unlike other fitted operator finite difference methods constructed in standard ways, the method that we presented in this paper is fairly simple to construct.

Data Availability

All data generated or analyzed during this study are included.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

HGD proposed the main idea of this paper. HGD and GFD prepared the manuscript and performed all the steps of the proofs in this research. Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

Acknowledgments

The authors wish to express their thanks to Jimma University, College of Natural Sciences, for the technical support and the authors of literatures for the provided scientific aspects and idea for this work.

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