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Research Article

Weighted Norm Inequalities for Multilinear Fourier Multipliers with Mixed Norm

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In this paper, weighted norm inequalities for multilinear Fourier multipliers satisfying Sobolev regularity with mixed norm are discussed. Our result can be understood as a generalization of the result by Fujita and Tomita by using the L^r-based Sobolev space, $1 < r \le 2$ with mixed norm.

1. Introduction

For $m \in L^\infty(\mathbb{R}^{Nn})$, the N-linear Fourier multiplier operator T_m is defined by

$$T_m(f_1, \cdots, f_N)(x) = \frac{1}{(2\pi)^{Nn}} \int_{(\mathbb{R}^n)^N} e^{ix\cdot(\xi_1+\cdots+\xi_N)} m(\xi) \widehat{f_1}(\xi_1) \cdots \widehat{f_N}(\xi_N) d\xi, \quad (1)$$

for $f_1, \cdots, f_N \in \mathcal{S}(\mathbb{R}^n)$, where $x \in \mathbb{R}^n$, $\xi = (\xi_1, \cdots, \xi_N) \in (\mathbb{R}^n)^N$ and $d\xi = d\xi_1 \cdots d\xi_N$. Let $\Psi \in \mathcal{S}(\mathbb{R}^d)$ be such that

$$\operatorname{supp} \Psi \subset \left\{ \xi \in \mathbb{R}^d : \frac{1}{2} \le |\xi| \le 2 \right\}, \quad \sum_{k\in\mathbb{Z}} \Psi\left(\xi/2^k\right) = 1, \xi \in \mathbb{R}^d \setminus \{0\}, \quad (2)$$

and set

$$m_j(\xi_1, \cdots, \xi_N) = m\left(2^j\xi_1, \cdots, 2^j\xi_N\right)\Psi(\xi_1, \cdots, \xi_N), \quad j \in \mathbb{Z}, \quad (3)$$

where Ψ is as in (2) with $d = Nn$. We denote by $\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\longrightarrow L^p(w)}$ the smallest constant C satisfying

$$\|T_m(f_1, \cdots, f_N)\|_{L^p(w)} \le C \prod_{i=1}^N \|f_i\|_{L^{p_i}(w_i)}, \quad f_1, \cdots, f_N \in \mathcal{S}(\mathbb{R}^n). \quad (4)$$

See Section 2 for the definition of function spaces.

In the unweighted case, Tomita [1] proved a Hörmander-type multiplier theorem for multilinear operators, namely, if $s > Nn/2$ then

$$\|T_m\|_{L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_N}(\mathbb{R}^n)\longrightarrow L^p(\mathbb{R}^n)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^2_s(\mathbb{R}^{Nn})}, \quad (5)$$

for $1 < p_1, \cdots, p_N, p<\infty$ satisfying $1/p_1+\cdots+1/p_N = 1/p$, where $H^2_s(\mathbb{R}^{Nn})$ is the L^2-based Sobolev space of usual type. Grafakos and Si [2] extended this result to the case $p \le 1$ by using the L^r-based Sobolev space, $1 < r \le 2$. For further results in this direction, see [3–7]. Let $1 < p_1, \cdots, p_N<\infty$ and $1/p_1+\cdots+1/p_N = 1/p$. In the weighted case, Fujita and Tomita [8] proved that if $n/2 < s_i \le n$, $p_i > n/s_i$, and $w_i \in A_{p_i s_i/n}$ for all $i = 1, \cdots, N$, then

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\longrightarrow L^p(w)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^2_{\vec{s}}(\mathbb{R}^{Nn})}, \quad (6)$$

where $w = w_1^{p/p_1} \cdots w_N^{p/p_N}$ and $H^2_{\vec{s}}(\mathbb{R}^{Nn})$ is the L^2-based Sobolev space of product type. This result can also be obtained from another approach of [9]. See [10, 11] for the endpoint cases.

The following is our main result which can be understood as a generalization of the result by Fujita and Tomita [8]. Taking $r_i = 2$ for all $i = 1, \cdots, N$ in (8), we have (6). Si [12] obtained some weighted estimates for multilinear Fourier multipliers with the L^r-based Sobolev regularity, $1 < r \leq 2$.

Theorem 1. *Let $1 < p_1, \cdots, p_N < \infty$, $1/p_1 + \cdots + 1/p_N = 1/p$, $\vec{r} = (r_1, \cdots, r_N) \in (1, 2]^N$, $r_N \leq r_{N-1} \leq \cdots \leq r_2 \leq r_1$, $\vec{s} = (s_1, \cdots, s_N) \in \mathbb{R}^N$, and $n/r_i < s_i \leq n$ for all $i = 1, \cdots, N$. Assume*

$$p_i > \frac{n}{s_i}, \quad w_i \in A_{p_i s_i/n} \quad \text{for all} \quad i = 1, \cdots, N. \quad (7)$$

Then,

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\longrightarrow L^p(w)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^{\vec{r}}_{\vec{s}}((\mathbb{R}^n)^N)}, \quad (8)$$

where $w = w_1^{p/p_1} \cdots w_N^{p/p_N}$ and $H^{\vec{r}}_{\vec{s}}((\mathbb{R}^n)^N)$ is the Sobolev space of product type with mixed norm.

2. Preliminaries

2.1. Notations. Let $n \in \mathbb{N}$ be the fixed dimension of the Euclidean space, and $\mathbb{Z}^n_+$ is defined by $\{0, 1, 2, \cdots\}^n$. The Lebesgue measure on $\mathbb{R}^n$ is denoted by dx (see, for example, Chapters 1 and 2 of [13]). Let N be a natural number, $N \geq 2$. An operator T acting on N-tuples of functions defined on $\mathbb{R}^n$ is called the N-linear operator. For two nonnegative quantities A and B, the notation $A \lesssim B$ means that $A \leq CB$ for some unspecified constant $C > 0$ independent of A and B, and the notation $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. If $x \in \mathbb{R}^d$, we denote $(1 + |x|^2)^{1/2}$ by $\langle x \rangle$. Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz class of all rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx, \quad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)d\xi. \quad (9)$$

See, for example, Chapter 1 of [14]. To distinguish linear and multilinear operators, for $m \in L^\infty(\mathbb{R}^n)$, we denote the linear Fourier multiplier operator by $m(D)$ defined by

$$m(D)f(x) = \mathcal{F}^{-1}\left[m(\xi)\widehat{f}(\xi)\right](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} m(\xi)\widehat{f}(\xi)d\xi, \quad (10)$$

for $f \in \mathcal{S}(\mathbb{R}^n)$, where $x, \xi \in \mathbb{R}^n$. Let $0 < p < \infty$ and $w \geq 0$. The weighted Lebesgue space $L^p(w)$ consists of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(w)} = \|f\|_{L^p(\mathbb{R}^n, w(x)\,dx)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p} < \infty. \quad (11)$$

Let $1 < p < \infty$. We say that a weight w belongs to the Muckenhoupt class A_p if

$$\sup_B \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}dx\right)^{p-1} < \infty, \quad (12)$$

where the supremum is taken over all balls B in $\mathbb{R}^n$, $|B|$ is the Lebesgue measure of B, and p' is the conjugate exponent of p, that is, $1/p + 1/p' = 1$. It is well known that the Hardy-Littlewood maximal operator M is bounded on $L^p(w)$ if and only if $w \in A_p$ (see Theorem 7.3 of [14]).

2.2. Function Spaces. To distinguish spaces of usual type and mixed type concerning integrable indices, we use $\mathbb{R}^{Nn}$ and $(\mathbb{R}^n)^N$, respectively.

We recall the definition of L^p-spaces with mixed norm [15]. Let $\vec{p} = (p_1, \cdots, p_N) \in (0, \infty)^N$. The Lebesgue spaces with mixed norm $L^{\vec{p}}((\mathbb{R}^n)^N)$ consists of all measurable functions F on $\mathbb{R}^{Nn}$ such that

$$\|F\|_{L^{\vec{p}}((\mathbb{R}^n)^N)} = \left\|\|F(x_1, \cdots, x_N)\|_{L^{p_1}(\mathbb{R}^n, dx_1)} \cdots \right\|_{L^{p_N}(\mathbb{R}^n, dx_N)} < \infty, \quad (13)$$

where $(x_1, \cdots, x_N) \in (\mathbb{R}^n)^N$ and dx_i is the Lebesgue measure with respect to the variable x_i for all $i = 1, \cdots, N$. In particular, if each p_i is equal to $p \in (0, \infty)$, then we have $\|F\|_{L^{\vec{p}}((\mathbb{R}^n)^N)} = \|F\|_{L^p(\mathbb{R}^{Nn})}$. For $\vec{r} = (r_1, \cdots, r_N) \in (1, \infty)^N$ and $\vec{s} = (s_1, \cdots, s_N) \in \mathbb{R}^N$, the norm of the Sobolev space of product type with mixed norm $H^{\vec{r}}_{\vec{s}}((\mathbb{R}^n)^N)$ for $F \in \mathcal{S}'(\mathbb{R}^{Nn})$ is defined by

$$\|F\|_{H^{\vec{r}}_{\vec{s}}((\mathbb{R}^n)^N)} = \left\|\mathcal{F}^{-1}\left[\langle\xi_1\rangle^{s_1} \cdots \langle\xi_N\rangle^{s_N} \widehat{F}(\xi_1, \cdots, \xi_N)\right]\right\|_{L^{\vec{r}}((\mathbb{R}^n)^N)}, \quad (14)$$

where $\langle\xi_i\rangle = (1 + |\xi_i|^2)^{1/2}$ for $i = 1, \cdots, N$ and $\mathcal{F}^{-1}$ is the inverse Fourier transform of $\mathbb{R}^{Nn}$. Taking $r_i = 2$ for all $i = 1, \cdots, N$, we obtain the L^2-based Sobolev space of product type $H^2_{\vec{s}}(\mathbb{R}^{Nn})$, namely, $\|F\|_{H^2_{\vec{s}}(\mathbb{R}^{Nn})} = \|\langle\xi_1\rangle^{s_1}\cdots\langle\xi_N\rangle^{s_N}\widehat{F}(\xi_1,\cdots,\xi_N)\|_{L^2(\mathbb{R}^{Nn})}$. It should be remarked that if $s_i = s/N$, $s \ge 0$ for all $i = 1,\cdots,N$,

$$H^2_s(\mathbb{R}^{Nn}) \hookrightarrow H^2_{\vec{s}}(\mathbb{R}^{Nn}), \tag{15}$$

where $H^2_s(\mathbb{R}^{Nn})$ is the L^2-based Sobolev space of usual type, that is to say, $\|F\|_{H^2_s(\mathbb{R}^{Nn})} = \|\langle\xi\rangle^s\widehat{F}\|_{L^2(\mathbb{R}^{Nn})}$, where $\xi \in \mathbb{R}^{Nn}$.

For $\vec{p} = (p_1,\cdots,p_N) \in [1,\infty)^N$ and $\vec{s} = (s_1,\cdots,s_N) \in \mathbb{R}^N$, the norm of the weighted Lebesgue space with mixed norm $L^{\vec{p}}_{\vec{s}}((\mathbb{R}^n)^N)$ for $F \in \mathcal{S}'(\mathbb{R}^{Nn})$ is also defined by

$$\|F\|_{L^{\vec{p}}_{\vec{s}}((\mathbb{R}^n)^N)} = \left\|\|F(x_1,\cdots,x_N)\|_{L^{p_1}(\mathbb{R}^n,\langle x_1\rangle^{s_1}dx_1)}\cdots\right\|_{L^{p_N}(\mathbb{R}^n,\langle x_N\rangle^{s_N}dx_N)}, \tag{16}$$

where $(x_1,\cdots,x_N) \in (\mathbb{R}^n)^N$ and $\langle x_i\rangle^{s_i} = (1+|x_i|^2)^{s_i/2}$ for all $i = 1,\cdots,N$. For accuracy, we will frequently write $L^{(p_1,\cdots,p_N)}_{(s_1,\cdots,s_N)}((\mathbb{R}^n)^N)$ instead of $L^{\vec{p}}_{\vec{s}}((\mathbb{R}^n)^N)$ in the proof.

For $\vec{p} = (p_1,\cdots,p_N)$, $\vec{q} = (q_1,\cdots,q_N) \in (0,\infty)^N$, we shall agree that if $a \sim b$ is a relation between numbers a and b, then $\vec{p} \sim \vec{q}$ means that $p_i \sim q_i$ holds for each i.

2.3. Cut-Off Functions. We collect cut-off functions which will be used later on [8]. Let ϕ_0 be a C^∞-function on $[0,\infty)$ satisfying

$$\phi_0(t) = 1 \text{ on } \left[0, \frac{1}{(4N)}\right], \text{supp } \phi_0 \subset \left[0, \frac{1}{(2N)}\right]. \tag{17}$$

We set $\phi_1(t) = 1 - \phi_0(t)$. For $(i_1,\cdots,i_N) \in \{0,1\}^N$, we define the function $\Phi_{(i_1,\cdots,i_N)}$ on $\mathbb{R}^{Nn}\setminus\{0\}$ by

$$\Phi_{(i_1,\cdots,i_N)}(\xi) = \phi_{i_1}\left(\frac{|\xi_1|}{|\xi|}\right)\cdots\phi_{i_N}\left(\frac{|\xi_N|}{|\xi|}\right), \tag{18}$$

where $\xi = (\xi_1,\cdots,\xi_N) \in (\mathbb{R}^n)^N$ and $|\xi| = \sqrt{|\xi_1|^2+\cdots+|\xi_N|^2}$. Note that

$$\Phi_{(0,\cdots,0)}(\xi) = 0. \tag{19}$$

According to the notation of [5] or [6], we also set $\mathcal{A}_0$, $\mathcal{A}_1$: $\mathcal{A}_0$ denotes the set of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ for which supp φ is compact and $\varphi = 1$ on some neighborhood of the origin; $\mathcal{A}_1$ denotes the set of $\tilde{\psi} \in \mathcal{S}(\mathbb{R}^n)$ for which supp $\tilde{\psi}$ is a compact subset of $\mathbb{R}^n\setminus\{0\}$.

2.4. Lemmas. The following lemmas will be used in the proof of Theorem 1.

Lemma 2 (see Lemma 3.1 of [8]). *Let $\Phi_{(i_1,\cdots,i_N)}$ be the same as in (18). Then, the following are true:*

(1) For $(\xi_1,\cdots,\xi_N) \in \mathbb{R}^n\times\cdots\times\mathbb{R}^n\setminus\{(0,\cdots,0)\}$,

$$\sum_{\substack{(i_1,\cdots,i_N)\in\{0,1\}^N,\\(i_1,\cdots,i_N)\neq(0,\cdots,0)}} \Phi_{(i_1,\cdots,i_N)}(\xi_1,\cdots,\xi_N) = 1. \tag{20}$$

(2) For $(i_1,\cdots,i_N) \in \{0,1\}^N$ and $(\alpha_1,\cdots,\alpha_N) \in \mathbb{Z}^n_+\times\cdots\times\mathbb{Z}^n_+$, then there exists a constant $C^{(\alpha_1,\cdots,\alpha_N)}_{(i_1,\cdots,i_N)} > 0$ such that

$$\left|\partial^{\alpha_1}_{\xi_1}\cdots\partial^{\alpha_N}_{\xi_N}\Phi_{(i_1,\cdots,i_N)}(\xi)\right| \le C^{(\alpha_1,\cdots,\alpha_N)}_{(i_1,\cdots,i_N)}(|\xi_1|+\cdots+|\xi_N|)^{-(|\alpha_1|+\cdots+|\alpha_N|)}, \tag{21}$$

for all $\xi = (\xi_1,\cdots,\xi_N) \in \mathbb{R}^n\times\cdots\times\mathbb{R}^n\setminus\{(0,\cdots,0)\}$.

(3) If $i_j = 1$ for some $j = 1,\cdots,N$ and $i_k = 0$ for all $k = 1,\cdots,N$ with $j \ne k$, then $\text{supp}\,\Phi_{(i_1,\cdots,i_N)} \subset \{(\xi_1,\cdots,\xi_N): |\xi_k| \le (|\xi_j|/N) \text{ for } k \ne j\}$. If $i_j = i_{j'} = 1$ for some $j, j' = 1,\cdots,N$ with $j \ne j'$, then $\text{supp}\,\Phi_{(i_1,\cdots,i_N)} \subset \{(\xi_1,\cdots,\xi_N): |\xi_j|/(4N) \le |\xi_{j'}| \le 4N|\xi_j|, |\xi_k| \le 4N|\xi_j| \text{ for } k \ne j, j'\}$.

Lemma 3 (see Chapter 7 of [14]). *Let $1 < p < \infty$ and $w \in A_p$. Then, there exists $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}$.*

Lemma 4 (see [16]). *Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that* supp $\psi \subset$ *$\{\eta \in \mathbb{R}^n : 1/r \le |\eta| \le r\}$ for some $r > 1$. If $1 < p < \infty$ and $w \in A_p$, then*

$$\left\|\left\{\sum_{j\in\mathbb{Z}}\left|\psi\left(\frac{D}{2^j}\right)f\right|^2\right\}^{1/2}\right\|_{L^p(w)} \|f\|_{L^p(w)}, \tag{22}$$

where $\psi(D/2^j)f = \mathcal{F}^{-1}[\psi(\cdot/2^j)\widehat{f}]$.

Lemma 5 (see [17]). *Let $1 < p, q < \infty$ and $w \in A_p$. Then,*

$$\left\|\left\{\sum_{k\in Z}(Mf_k)^q\right\}^{1/q}\right\|_{L^p(w)} \lesssim \left\|\left\{\sum_{k\in\mathbb{Z}}|f_k|^q\right\}^{1/q}\right\|_{L^p(w)}. \tag{23}$$

Lemma 6 (see Proposition 2.7 of [14]). *Let ϕ be a function which is positive, radial, decreasing (as a function on $(0,\infty)$), and integrable. Set $\phi_t(x) = 1/t^n\phi(x/t)$ for $t > 0$. Then,*

$$\sup_{t>0}|\phi_t * f(x)| \lesssim Mf(x), \tag{24}$$

for $x \in \mathbb{R}^n$.

Lemma 7 (see Theorem 1 of Section 10 of [15]). *Let $\overrightarrow{p} = (p_1,\cdots,p_N) \in [1,\infty]^N$, then*

$$\|F * G\|_{L^{\overrightarrow{p}}((\mathbb{R}^n)^N)} \le \|F\|_{L^{\overrightarrow{p}}((\mathbb{R}^n)^N)} \|G\|_{L^1(\mathbb{R}^{Nn})}. \tag{25}$$

Lemma 8 (see Theorem 1 of Section 12 of [15]). *Let $\overrightarrow{p} = (p_1,\cdots,p_N) \in [1,2]^N$ and $p_N \le p_{N-1} \le \cdots \le p_1$, then*

$$\|\widehat{F}\|_{L^{\overrightarrow{(p')}}((\mathbb{R}^n)^N)} \le \|F\|_{L^{\overrightarrow{p}}((\mathbb{R}^n)^N)}, \tag{26}$$

where $\overrightarrow{(p')} = (p_{1'}, \cdots, p_{N'})$ and p'_i is the conjugate exponent of p_i for $i = 1, \cdots, N$.

3. Lemmas

In this section, we prove lemmas which play important roles in the proof of Theorem 1. The proof of the following lemma is based on the argument of Proposition 1.3.2 of [18] or Lemma 3.3 of [1].

Lemma 9. *Let $r > 0$, $\overrightarrow{p} = (p_1,\cdots,p_N)$, $\overrightarrow{q} = (q_1,\cdots,q_N) \in [1,\infty)^N$, $\overrightarrow{s} = (s_1,\cdots,s_N) \in (\mathbb{R}_{\ge 0})^N$, and $\overrightarrow{p} \le \overrightarrow{q}$. Then, the estimate*

$$\begin{aligned} &\left\|\langle\xi_1\rangle^{s_1} \cdots \langle\xi_N\rangle^{s_N} \widehat{F}(\xi_1, \cdots, \xi_N)\right\|_{L^{\overrightarrow{q}}((\mathbb{R}^n)^N)} \\ &\quad \lesssim \left\|\langle\xi_1\rangle^{s_1} \cdots \langle\xi_N\rangle^{s_N} \widehat{F}(\xi_1, \cdots, \xi_N)\right\|_{L^{\overrightarrow{p}}((\mathbb{R}^n)^N)}, \end{aligned} \tag{27}$$

holds, where supp $F \subset \{\xi = (\xi_1, \cdots, \xi_N) \in (\mathbb{R}^n)^N : |\xi| \le r\}$.

Proof. We consider only the case $N = 2$. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\phi(y) = 1 \text{ on } \{y \in \mathbb{R}^n : |y| \le r\}, \text{supp } \phi \subset \{y \in \mathbb{R}^n : |y| \le 2r\}. \tag{28}$$

Since supp $F \subset \{x = (x_1, x_2) \in (\mathbb{R}^n)^2 : |x_i| \le r, i = 1, 2\}$, we see that $F(x_1, x_2) = \phi(x_1)\phi(x_2)F(x_1, x_2)$. Then, it is easy to see that

$$\widehat{F}(\xi_1, \xi_2) = \int_{(\mathbb{R}^n)^2} \widehat{\phi}(\xi_1 - \eta_1)\widehat{\phi}(\xi_2 - \eta_2)\widehat{F}(\eta_1, \eta_2) d\eta_1 d\eta_2. \tag{29}$$

For fixed $\xi_2 \in \mathbb{R}^n$, by Minkowski's inequality for integrals, it follows that

$$\begin{aligned} &\left\|\langle\xi_1\rangle^{s_1} \widehat{F}(\xi_1, \xi_2)\right\|_{L^{q_1}(\mathbb{R}^n_{\xi_1})} \\ &\quad \le \int_{\mathbb{R}^n_{\eta_2}} \left|\widehat{\phi}(\xi_2 - \eta_2)\right| \left\| \langle\xi_1\rangle^{s_1} \int_{\mathbb{R}^n_{\eta_1}} \left|\widehat{\phi}(\xi_1 - \eta_1)\right| \cdot \left|\widehat{F}(\eta_1, \eta_2)\right| d\eta_1 \right\|_{L^{q_1}(\mathbb{R}^n_{\xi_1})} d\eta_2. \end{aligned} \tag{30}$$

Since $\langle\xi_1\rangle^{s_1} \lesssim \langle\xi_1 - \eta_1\rangle^{s_1}\langle\eta_1\rangle^{s_1}$, for fixed $\eta_2 \in \mathbb{R}$, we obtain

$$\begin{aligned} &\left\|\langle\xi_1\rangle^{s_1} \int_{\mathbb{R}^n_{\eta_1}} \left|\widehat{\phi}(\xi_1 - \eta_1)\right| \left|\widehat{F}(\eta_1, \eta_2)\right| d\eta_1\right\|^{q_1}_{L^{q_1}(\mathbb{R}^n_{\xi_1})} \\ &= \int_{\mathbb{R}^n_{\xi_1}} \left(\int_{\mathbb{R}^n_{\eta_1}} \langle\xi_1\rangle^{s_1} \left|\widehat{\phi}(\xi_1 - \eta_1)\right| \left|\widehat{F}(\eta_1, \eta_2)\right| d\eta_1\right)^{q_1} d\xi_1 \\ &\lesssim \int_{\mathbb{R}^n_{\xi_1}} \left(\int_{\mathbb{R}^n_{\eta_1}} \langle\xi_1 - \eta_1\rangle^{s_1}\langle\eta_1\rangle^{s_1} \left|\widehat{\phi}(\xi_1 - \eta_1)\right| \left|\widehat{F}(\eta_1, \eta_2)\right| d\eta_1\right)^{q_1} d\xi_1 \\ &\le \sup_{\xi_1 \in \mathbb{R}^n} \left(\int_{\mathbb{R}^n_{\eta_1}} \langle\xi_1 - \eta_1\rangle^{s_1}\langle\eta_1\rangle^{s_1} \left|\widehat{\phi}(\xi_1 - \eta_1)\right| \left|\widehat{F}(\eta_1, \eta_2)\right| d\eta_1\right)^{q_1 - p_1} \\ &\quad \times \int_{\mathbb{R}^n_{\xi_1}} \left(\int_{\mathbb{R}^n_{\eta_1}} \langle\xi_1 - \eta_1\rangle^{s_1}\langle\eta_1\rangle^{s_1} \left|\widehat{\phi}(\xi_1 - \eta_1)\right| \left|\widehat{F}(\eta_1, \eta_2)\right| d\eta_1\right)^{p_1} d\xi_1, \end{aligned} \tag{31}$$

where we have used the fact $p_1 \le q_1$.

For the first term on the right-hand side of (31), by Hölder's inequality and a change of variables, we have

$$\begin{aligned} &\int_{\mathbb{R}^n_{\eta_1}} \langle\xi_1 - \eta_1\rangle^{s_1}\langle\eta_1\rangle^{s_1} \left|\widehat{\phi}(\xi_1 - \eta_1)\right| \left|\widehat{F}(\eta_1, \eta_2)\right| d\eta_1 \\ &\quad \le \left\|\langle\cdot\rangle^{s_1}\widehat{\phi}\right\|_{L^{p_1'}(\mathbb{R}^n_{\eta_1})} \left\|\langle\cdot\rangle^{s_1}\widehat{F}(\cdot, \eta_2)\right\|_{L^{p_1}(\mathbb{R}^n_{\eta_1})}, \end{aligned} \tag{32}$$

where $\xi_1 \in \mathbb{R}^n$. Thus, we see that

$$\begin{aligned} &\sup_{\xi_1 \in \mathbb{R}^n} \left(\int_{\mathbb{R}^n_{\eta_1}} \langle\xi_1 - \eta_1\rangle^{s_1}\langle\eta_1\rangle^{s_1} \left|\widehat{\phi}(\xi_1 - \eta_1)\right| \left|\widehat{F}(\eta_1, \eta_2)\right| d\eta_1\right)^{q_1 - p_1} \\ &\quad \le \left\|\langle\cdot\rangle^{s_1}\widehat{\phi}\right\|^{q_1 - p_1}_{L^{p_1'}(\mathbb{R}^n_{\eta_1})} \left\|\langle\cdot\rangle^{s_1}\widehat{F}(\cdot, \eta_2)\right\|^{q_1 - p_1}_{L^{p_1}(\mathbb{R}^n_{\eta_1})}. \end{aligned} \tag{33}$$

For the second term on the right-hand side of (31), by Young's inequality, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n_{\xi_1}} \left(\int_{\mathbb{R}^n_{\eta_1}} \langle\xi_1 - \eta_1\rangle^{s_1}\langle\eta_1\rangle^{s_1} \left|\widehat{\phi}(\xi_1 - \eta_1)\right| \left|\widehat{F}(\eta_1, \eta_2)\right| d\eta_1\right)^{p_1} d\xi_1 \\ &\quad \le \left\|\langle\cdot\rangle^{s_1}\widehat{\phi}\right\|^{p_1}_{L^1(\mathbb{R}^n_{\xi_1})} \left\|\langle\cdot\rangle^{s_1}\widehat{F}(\cdot, \eta_2)\right\|^{p_1}_{L^{p_1}(\mathbb{R}^n_{\xi_1})}. \end{aligned} \tag{34}$$

By (33) and (34), it follows that

$$\left\| \langle \xi_1 \rangle^{s_1} \int_{\mathbb{R}^n_{\eta_1}} \left| \widehat{\phi}(\xi_1 - \eta_1) \right| \left| \widehat{F}(\eta_1, \eta_2) \right| d\eta_1 \right\|_{L^{q_1}\left(\mathbb{R}^n_{\xi_1}\right)} \lesssim \left\| \langle \cdot \rangle^{s_1} \widehat{F}(\cdot, \eta_2) \right\|_{L^{p_1}\left(\mathbb{R}^n_{\eta_1}\right)}, \tag{35}$$

where we have used the fact that $\| \langle \cdot \rangle^{s_1} \widehat{\phi} \|_{L^{p_{1'}}}, \| \langle \cdot \rangle^{s_1} \widehat{\phi} \|_{L^1} < \infty$. By the same way for $\xi_2 \in \mathbb{R}^n$, we have the desired estimate with $N = 2$.□ □

The following is a key lemma in the proof of Theorem 1. Fujita and Tomita (Proposition A.2 of [6]) proved (6) using the fact that $H_2^{\vec{s}}(\mathbb{R}^{Nn})$ is a multiplication algebra when $s_i > n/2$ for all $i = 1, \cdots, N$. Instead of this, we shall use the following lemma.

Lemma 10. *Let $N_0 \in \mathbb{N}$, $\vec{r} = (r_1, \cdots, r_N) \in (1, 2]^N$, $r_N \le r_{N-1} \le \cdots \le r_2 \le r_1$, $\vec{s} = (s_1, \cdots, s_N) \in \mathbb{R}^N$, $n/r_i < s_i \le n$, and $n/s_i < q_i < r_i$ for all $i = 1, \cdots, N$. Then, the estimate*

$$\begin{aligned} &\left\| \mathscr{F}\left[m\left(2^j \cdot\right) \Psi\left(\cdot/2^k\right) \right] \right\|_{L^{\left(q_1', \cdots, q_N'\right)}_{\left(s_1 q_1', \cdots, s_N q_N'\right)}\left((\mathbb{R}^n)^N\right)} \\ &\quad \lesssim \sup_{j \in \mathbb{Z}} \left\| m_j \right\|_{H^{\vec{r}}_{\vec{s}}\left((\mathbb{R}^n)^N\right)}, \end{aligned} \tag{36}$$

holds for all $j \in \mathbb{Z}$, $-N_0 \le k \le N_0$, and $m \in H^{\vec{r}}_{\vec{s}}((\mathbb{R}^n)^N)$.

Proof. We consider only the case $N = 2$. By the change of variables and k satisfies $-N_0 \le k \le N_0$, we see that

$$\begin{aligned} &\left\| \mathscr{F}\left[m\left(2^j \cdot\right) \Psi\left(\frac{\cdot}{2^k}\right) \right] \right\|_{L^{\left(q_1', q_2'\right)}_{\left(s_1 q_1{}', s_2 q_2{}'\right)}\left((\mathbb{R}^n)^2\right)} \\ &\quad \lesssim \left\| \widehat{m_{j+k}} \right\|_{L^{\left(q_1', q_2'\right)}_{\left(s_1 q_1{}', s_2 q_2{}'\right)}\left((\mathbb{R}^n)^2\right)}, \end{aligned} \tag{37}$$

where m_j is defined by (3). By Lemma 9 and Hausdorff-Young's inequality with mixed type (Lemma 8), it follows that

$$\begin{aligned} &\left\| \langle \xi_1 \rangle^{s_1} \langle \xi_2 \rangle^{s_2} \widehat{m_{j+k}}(\xi_1, \xi_2) \right\|_{L^{\overrightarrow{(q')}}\left((\mathbb{R}^n)^2\right)} \\ &\quad \lesssim \left\| \langle \xi_1 \rangle^{s_1} \langle \xi_2 \rangle^{s_2} \widehat{m_{j+k}}(\xi_1, \xi_2) \right\|_{L^{\overrightarrow{(r')}}\left((\mathbb{R}^n)^2\right)} \\ &\quad \le \left\| \mathscr{F}^{-1}\left[\langle \xi_1 \rangle^{s_1} \langle \xi_2 \rangle^{s_2} \widehat{m_{j+k}}(\xi_1, \xi_2) \right] \right\|_{L^{\vec{r}}\left((\mathbb{R}^n)^2\right)}, \end{aligned} \tag{38}$$

where we have used the fact $\vec{q} \le \vec{r}$. This completes the proof with $N = 2$.□ □

The following lemma is known, but we shall give a proof for the reader's convenience.

Lemma 11. *Let $r > 0$, $1 < q_i < \infty$, and $n/q_i < s_i < \infty$ for all $i = 1, \cdots, N$. Then, the estimate*

$$\begin{aligned} &\left| T_{m(\cdot/2^j)}(f_1, \cdots, f_N)(x) \right| \\ &\quad \le \| \widehat{m} \|_{L^{\left(q_1', \cdots, q_N'\right)}_{\left(s_1 q_1', \cdots, s_N q_N'\right)}\left((\mathbb{R}^n)^N\right)} \prod_{i=1}^{N} M\left(|f_i|^{q_i}\right)(x)^{1/q_i}, \end{aligned} \tag{39}$$

holds for all $x \in \mathbb{R}^n$, $j \in \mathbb{Z}$, $m \in H^{\vec{r}}_{\vec{s}}((\mathbb{R}^n)^N)$ with supp $m \subset \{\xi = (\xi_1, \cdots, \xi_N) \in (\mathbb{R}^n)^N : |\xi| \le r\}$ *and $f_1, \cdots, f_N \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. We consider only the case $N = 2$. For all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, by Fubini's theorem and the change of variables, we see that

$$\begin{aligned} T_{m(\cdot/2^j)}(f_1, f_2)(x) = &\left(2^{jn}\right)^2 \int_{(\mathbb{R}^n)^2} \mathscr{F}^{-1}[m]\left(2^j(x - y_1), 2^j(x - y_2)\right) \\ &\cdot f_1(y_1) f_2(y_2)\, dy_1 dy_2, \end{aligned} \tag{40}$$

where $\mathscr{F}^{-1}$ is the inverse Fourier transform of $(\mathbb{R}^n)^2$. For fixed $y_2 \in \mathbb{R}^n$, by Hölder's inequality and Lemma 6 with $\phi(x) = (1+|x|)^{-s_1 q_1}$, it follows that

$$\begin{aligned} &\left| \int_{\mathbb{R}^n_{y_1}} \left(1 + 2^j |x - y_1|\right)^{s_1} \mathscr{F}^{-1}[m]\left(2^j(x - y_1), 2^j(x - y_2)\right) \right. \\ &\quad \left. \cdot \left(1 + 2^j |x - y_1|\right)^{-s_1} f_1(y_1)\, 2^{jn}\, dy_1 \right| \\ &\le \left\{ \int_{\mathbb{R}^n_{y_1}} \left(1 + 2^j |x - y_1|\right)^{s_1 q_{1'}} \right. \\ &\quad \left. \cdot \left| \mathscr{F}^{-1}[m]\left(2^j(x - y_1), 2^j(x - y_2)\right) \right|^{q_{1'}} 2^{jn}\, dy_1 \right\}^{1/q_{1'}} \\ &\quad \times \left\{ \int_{\mathbb{R}^n_{y_1}} \frac{|f_1(y_1)|^{q_1}}{\left(1 + 2^j |x - y_1|\right)^{s_1 q_1}}\, 2^{jn}\, dy_1 \right\}^{1/q_1} \\ &\lesssim \left\{ \int_{\mathbb{R}^n_{y_1}} \left(1 + 2^j |x - y_1|\right)^{s_1 q_{1'}} \right. \\ &\quad \left. \cdot \left| \mathscr{F}^{-1}[m]\left(2^j(x - y_1), 2^j(x - y_2)\right) \right|^{q_{1'}} 2^{jn}\, dy_1 \right\}^{1/q_{1'}} \\ &\quad \cdot M\left(|f_1|^{q_1}\right)(x)^{1/q_1}, \end{aligned} \tag{41}$$

where we have used the fact that $s_1 q_1 > n$. By the same way for y_2 and the change of variables, we have the desired estimate with $N = 2$.□ □

4. Proof of Theorem 1

In this section, we prove Theorem 1. Let $1 < p_1, \cdots, p_N < \infty$, $1/p_1 + \cdots + 1/p_N = 1/p$, $\vec{r} = (r_1, \cdots, r_N) \in (1,2]^N$, $r_N \le r_{N-1} \le \cdots \le r_2 \le r_1$, $\vec{s} = (s_1, \cdots, s_N) \in \mathbb{R}^N$, and $n/r_i < s_i \le n$ for all $i = 1, \cdots, N$. Assume $p_i > n/s_i$ and $w_i \in A_{p_i s_i/n}$ for all $i = 1, \cdots, N$ and set $w = w_1^{p/p_1} \cdots w_N^{p/p_N}$. We also assume that $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies

$$\sup_{j\in\mathbb{Z}} \|m_j\|_{H^{\vec{r}}_{\vec{s}}((\mathbb{R}^n)^N)} < \infty. \tag{42}$$

Since $n/s_i < \min\{r_i, p_i\}$ and $w_i \in A_{p_i/(n/s_i)}$ for all $i = 1, \cdots, N$, by Lemma 3, we can take $n/s_i < q_i < \min\{r_i, p_i\}$ satisfying $w_i \in A_{p_i/q_i}$ for all $i = 1, \cdots, N$. By Lemma 2 (1), we decompose m as follows:

$$\begin{aligned} m(\xi) &= \sum_{\substack{(i_1,\cdots,i_N)\in\{0,1\}^N,\\ (i_1,\cdots,i_N)\neq(0,\cdots,0)}} \Phi_{(i_1,\cdots,i_N)}(\xi) m(\xi) \\ &= \sum_{\substack{(i_1,\cdots,i_N)\in\{0,1\}^N,\\ (i_1,\cdots,i_N)\neq(0,\cdots,0)}} m_{(i_1,\cdots,i_N)}(\xi). \end{aligned} \tag{43}$$

4.1. Estimate for $m_{(1,0,\cdots,0)}$ Type. We first consider the case where $(i_1, \cdots, i_N)$ satisfies $\#\{j : i_j = 1\} = 1$. Without loss of generality, we may assume that $i_1 = 1$. We simply write m instead of $m_{(1,0,\cdots,0)}$. Note that by Lemma 2 (3),

$$\operatorname{supp}\ m \subset \left\{\xi = (\xi_1, \cdots, \xi_N) \in (\mathbb{R}^n)^N : |\xi_i| \le |\xi_1|/N, i = 2, \cdots, N\right\}. \tag{44}$$

It is easy to see that if $\xi = (\xi_1, \cdots, \xi_N) \in \operatorname{supp}\ m$, then $|\xi_1 + \cdots + \xi_N| \approx |\xi_1|$.

Let ψ be as in (2) with $d = n$. Since $w \in A_{Np} \subset A_\infty$ (see p. 1232 of [19]), we can use the results of Grafakos and Si (see Lemma 2.4 of [9]) and Fujita and Tomita (see Remark 2.6 of [6]); hence,

$$\|T_m(f_1, \cdots, f_N)\|_{L^p(w)} \lesssim \left\| \left\{ \sum_{j\in\mathbb{Z}} \left|\Delta_j T_m(f_1, \cdots, f_N)\right|^2 \right\}^{1/2} \right\|_{L^p(w)}, \tag{45}$$

where $\Delta_j g = \psi(D/2^j) g$.

By Fubini's theorem and the Fourier inversion formula, it is easy to see that

$$\begin{aligned} \Delta_j T_m(f_1, \cdots, f_N)(x) &= \frac{1}{(2\pi)^{Nn}} \int_{(\mathbb{R}^n)^N} e^{ix\cdot(\xi_1+\cdots+\xi_N)} m(\xi) \psi \\ &\quad \cdot \left((\xi_1+\cdots+\xi_N)/2^j\right) \widehat{f_1}(\xi_1) \cdots \widehat{f_N}(\xi_N)\, d\xi. \end{aligned} \tag{46}$$

We shall prove that we can find functions $\varphi \in \mathscr{A}_0$ and $\tilde{\psi} \in \mathscr{A}_1$ independent of j such that

$$\begin{aligned} m(\xi)\psi\left(\frac{\xi_1+\cdots+\xi_N}{2^j}\right) &= m(\xi)\psi\left(\frac{\xi_1+\cdots+\xi_N}{2^j}\right) \tilde{\psi}\left(\xi_1/2^j\right)^2 \varphi\left(\xi_2/2^j\right) \cdots \varphi\left(\xi_N/2^j\right). \end{aligned} \tag{47}$$

Once this is proven, setting

$$m^{(j)}(\xi) = m(2^j\xi)\psi(\xi_1+\cdots+\xi_N)\tilde{\psi}(\xi_1)\varphi(\xi_2)\cdots\varphi(\xi_N), \tag{48}$$

we have

$$\Delta_j T_m(f_1, \cdots, f_N)(x) = T_{m^{(j)}(\cdot/2^j)}\left(\tilde{\Delta}_j f_1, f_2, \cdots, f_N\right)(x), \tag{49}$$

where $\tilde{\Delta}_j f_1 = \tilde{\psi}(D/2^j) f_1$. Let $\xi = (\xi_1, \cdots, \xi_N) \in \operatorname{supp}\ m$ satisfying $2^{j-1} \le |\xi_1 + \cdots + \xi_N| \le 2^{j+1}$. We take functions $\tilde{\psi} \in \mathscr{A}_1$ such that $\tilde{\psi}(\eta) = 1$ on $\{\eta \in \mathbb{R}^n : N/2(2N-1) \le |\eta| \le 2N\}$ and $\varphi \in \mathscr{A}_0$ such that $\varphi(\eta) = 1$ on $\{\eta \in \mathbb{R}^n : |\eta| \le 2\}$. Hence, we obtain (47).

Since $\operatorname{supp}\ \Psi(\cdot/2^\ell) \subset \{\xi \in (\mathbb{R}^n)^N : 2^{\ell-1} \le |\xi| \le 2^{\ell+1}\}$ and

$$\begin{aligned} &\operatorname{supp}\ \tilde{\psi}(\xi_1)\varphi(\xi_2)\cdots\varphi(\xi_N) \\ &\quad \subset \left\{\xi = (\xi_1, \cdots, \xi_N) \in (\mathbb{R}^n)^N : 2^{-j_0} \le |\xi| \le 2^{j_0}\right\}, \end{aligned} \tag{50}$$

for some $j_0 \in \mathbb{N}$, we have

$$\begin{aligned} &T_{m^{(j)}(\cdot/2^j)}\left(\tilde{\Delta}_j f_1, f_2, \cdots, f_N\right)(x) \\ &\quad = \sum_{k=-N_0}^{N_0} T_{m^{(j,k)}(\cdot/2^j)}\left(\tilde{\Delta}_j f_1, f_2, \cdots, f_N\right)(x), \end{aligned} \tag{51}$$

for some $N_0 \in \mathbb{N}$, where $m^{(j,k)}(\xi) = m^{(j)}(\xi)\Psi(\xi/2^k)$. By Lemma 11, we see that

$$\begin{aligned} &\left|T_{m^{(j,k)}(\cdot/2^j)}\left(\tilde{\Delta}_j f_1, f_2, \cdots, f_N\right)(x)\right| \\ &\quad \lesssim \|\widehat{m^{(j,k)}}\|_{L^{(q_1',\cdots,q_N')}_{(s_1q_1',\cdots,s_Nq_N')}((\mathbb{R}^n)^N)} M\left(\left|\tilde{\Delta}_j f_1\right|^{q_1}\right) \\ &\quad\quad \cdot (x)^{1/q_1} \prod_{i=2}^N M(|f_i|^{q_i})(x)^{1/q_i}. \end{aligned} \tag{52}$$

We shall prove that the estimate

$$\|\widehat{m^{(j,k)}}\|_{L^{(q_1',\cdots,q_N')}_{(s_1q_1',\cdots,s_Nq_N')}((\mathbb{R}^n)^N)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^{\vec{r}}_{\vec{s}}((\mathbb{R}^n)^N)}, \tag{53}$$

holds. Once this is proved, combining (51), (52), and (53), we have

$$\begin{aligned}&\left|T_{m^{(j)}(\cdot/2^j)}\left(\tilde{\Delta}_j f_1, f_2, \cdots, f_N\right)(x)\right|\\&\quad\lesssim\left(\sup_{j\in\mathbb{Z}}\|m_j\|_{H^{\overrightarrow{r}}_{\overrightarrow{s}}((\mathbb{R}^n)^N)}\right)M\left(\left|\tilde{\Delta}_j f_1\right|^{q_1}\right)\\&\qquad\cdot(x)^{1/q_1}\prod_{i=2}^{N}M(|f_i|^{q_i})(x)^{1/q_i}.\end{aligned}\tag{54}$$

Since $m^{(j)}(\xi)$ is defined by (48) and we have used m instead of $m_{(1,0,\cdots,0)}$, by Young's inequality with mixed type (Lemma 7), we see that

$$\begin{aligned}&\left\|\mathscr{F}\left[m^{(j)}(\cdot)\Psi\left(\cdot/2^k\right)\right]\right\|_{L^{(q_1',\cdots,q_N')}_{(s_1q_1',\cdots,s_Nq_N')}((\mathbb{R}^n)^N)}\\&\quad\leq\left\|\mathscr{F}\left[m(2^j\cdot)\Psi\left(\cdot/2^k\right)\right]\right\|_{L^{(q_1',\cdots,q_N')}_{(s_1q_1{}',\cdots,s_Nq_N{}')}((\mathbb{R}^n)^N)}\\&\qquad\times\left\|\langle\xi_1\rangle^{s_1}\cdots\langle\xi_N\rangle^{s_N}\widehat{B_j}\right\|_{L^1(\mathbb{R}^{Nn})},\end{aligned}\tag{55}$$

where $B_j(\xi_1,\cdots,\xi_N)=\Phi_{(1,0,\cdots,0)}(2^j\xi)\psi(\xi_1+\cdots+\xi_N)\tilde{\psi}(\xi_1)\varphi(\xi_2)\cdots\varphi(\xi_N)$. By Lemma 10, we obtain (53), where we have used the fact that

$$\left\|\langle\xi_1\rangle^{s_1}\cdots\langle\xi_N\rangle^{s_N}\widehat{B_j}\right\|_{L^1(\mathbb{R}^{Nn})}<\infty.\tag{56}$$

By Hölder's inequality, we have

$$\begin{aligned}&\left\|\left\{\sum_{j\in\mathbb{Z}}\left|T_{m^{(j)}(\cdot/2^j)}\left(\tilde{\Delta}_j f_1, f_2, \cdots, f_N\right)\right|^2\right\}^{1/2}\right\|_{L^p(w)}\\&\quad\lesssim\left(\sup_{j\in\mathbb{Z}}\|m_j\|_{H^{\overrightarrow{r}}_{\overrightarrow{s}}((\mathbb{R}^n)^N)}\right)\\&\qquad\times\left\|\left\{\sum_{j\in\mathbb{Z}}M\left(\left|\tilde{\Delta}_j f_1\right|^{q_1}\right)^{2/q_1}\right\}^{1/2}\right\|_{L^{p_1}(w_1)}\\&\qquad\cdot\prod_{i=2}^{N}\left\|M(|f_i|^{q_i})^{1/q_i}\right\|_{L^{p_i}(w_i)}.\end{aligned}\tag{57}$$

For the second term on the right-hand side of (57), since $1<2/q_1, p_1/q_1$, and $w_1\in A_{p_1/q_1}$, it follows from Lemmas 5 and 4 that

$$\begin{aligned}&\left\|\left\{\sum_{j\in\mathbb{Z}}M\left(\left|\tilde{\Delta}_j f_1\right|^{q_1}\right)^{2/q_1}\right\}^{1/2}\right\|_{L^{p_1}(w_1)}\\&\quad=\left\|\left\{\sum_{j\in\mathbb{Z}}M\left(\left|\tilde{\Delta}_j f_1\right|^{q_1}\right)^{2/q_1}\right\}^{q_1/2}\right\|_{L^{p_1/q_1}(w_1)}^{1/q_1}\\&\quad\lesssim\left\|\left(\sum_{j\in\mathbb{Z}}\left|\tilde{\Delta}_j f_1\right|^2\right)^{q_1/2}\right\|_{L^{p_1/q_1}(w_1)}^{1/q_1}\\&\quad=\left\|\left(\sum_{j\in\mathbb{Z}}\left|\tilde{\Delta}_j f_1\right|^2\right)^{1/2}\right\|_{L^{p_1}(w_1)}\lesssim\|f\|_{L^{p_1}(w_1)}.\end{aligned}\tag{58}$$

For the third term on the right-hand side of (57), since $p_i>q_i$ and $w_i\in A_{p_i/q_i}$ for all $i=2,\cdots,N$, we see that

$$\begin{aligned}\prod_{i=2}^{N}\left\|M(|f_i|^{q_i})^{1/q_i}\right\|_{L^{p_i}(w_i)}&=\prod_{i=2}^{N}\|M(|f_i|^{q_i})\|_{L^{p_i/q_i}(w_i)}^{1/q_i}\\&\lesssim\prod_{i=2}^{N}\||f_i|^{q_i}\|_{L^{p_i/q_i}(w_i)}^{1/q_i}\\&=\prod_{i=2}^{N}\|f_i\|_{L^{p_i}(w_i)},\end{aligned}\tag{59}$$

where we have used the boundedness of M on $L^{p_i/q_i}(w_i)$ for all $i=2,\cdots,N$. By (45), (49), (54), (57), (58), and (59), we obtain the desired estimate.

4.2. Estimate for $m_{(1,1,i_3,\cdots,i_N)}$ Type. Next, we consider the case where $(i_1,\cdots,i_N)$ satisfies $\#\{j: i_j=1\}\geq 2$. Without loss of generality, we may assume that $i_1=i_2=1$. We simply write m instead of $m_{(1,1,i_3,\cdots,i_N)}$, where $i_3,\cdots,i_N\in\{0,1\}$. Note that by Lemma 2 (3),

$$\begin{aligned}\operatorname{supp} m\subset\Big\{&\xi\in(\mathbb{R}^n)^N:|\xi_1|/(4N)\leq|\xi_2|\\&\leq 4N|\xi_1|,|\xi_i|\leq 4N|\xi_1|, i=3,\cdots,N\Big\},\end{aligned}\tag{60}$$

where $\xi=(\xi_1,\cdots,\xi_N)\in(\mathbb{R}^n)^N$.

Since ψ is in (2) with $d=n$, we see that

$$\begin{aligned}T_m(f_1,\cdots,f_N)(x)=\sum_{j\in\mathbb{Z}}\frac{1}{(2\pi)^{Nn}}\int_{(\mathbb{R}^n)^N}&e^{ix\cdot(\xi_1+\cdots+\xi_N)}m(\xi)\\&\cdot\psi\left(\frac{\xi_1}{2^j}\right)\widehat{f_1}(\xi_1)\cdots\widehat{f_N}(\xi_N)\,d\xi.\end{aligned}\tag{61}$$

We shall prove that we can find functions $\varphi\in\mathscr{A}_0$ and $\tilde{\psi}\in\mathscr{A}_1$ independent of j such that

$$m(\xi)\psi\left(\frac{\xi_1}{2^j}\right) = m(\xi)\psi\left(\frac{\xi_1}{2^j}\right)\tilde{\psi}\left(\frac{\xi_1}{2^j}\right)\tilde{\psi}\left(\frac{\xi_2}{2^j}\right)^2 \cdot \varphi\left(\frac{\xi_3}{2^j}\right)\cdots\varphi\left(\frac{\xi_N}{2^j}\right). \tag{62}$$

Once this is proved, setting

$$m^{(j)}(\xi) = m\left(2^j\xi\right)\psi(\xi_1)\tilde{\psi}(\xi_2)\varphi(\xi_3)\cdots\varphi(\xi_N), \tag{63}$$

we have

$$T_m(f_1,\cdots,f_N)(x) = \sum_{j\in\mathbb{Z}} T_{m^{(j)}(\cdot/2^j)}\left(\tilde{\Delta}_j f_1, \tilde{\Delta}_j f_2, f_3, \cdots, f_N\right)(x), \tag{64}$$

where $\tilde{\Delta}_j f_i = \tilde{\psi}(D/2^j) f_i$ for $i=1,2$. Let $\xi=(\xi_1,\cdots,\xi_N)\in$ supp m and $\xi_1\in$ supp $\psi(\cdot/2^j)$. We take functions $\tilde{\psi}\in\mathscr{A}_1$ such that $\tilde{\psi}(\eta)=1$ on $\{\eta\in\mathbb{R}^n : 1/(8N)\le|\eta|\le 8N\}$ and $\varphi\in\mathscr{A}_0$ such that $\varphi(\eta)=1$ on $\{\eta\in\mathbb{R}^n : |\eta|\le 8N\}$. Hence, we obtain (62).

As in Section 4.1, by Lemmas 11 and 10, and Schwarz's inequality, we have

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\left|T_{m^{(j)}(\cdot/2^j)}\left(\tilde{\Delta}_j f_1, \tilde{\Delta}_j f_2, f_3,\cdots,f_N\right)(x)\right| \\ &\lesssim \left(\sup_{j\in\mathbb{Z}}\|m_j\|_{H^{\overrightarrow{r}}_{\overrightarrow{s}}((\mathbb{R}^n)^N)}\right)\left\{\sum_{j\in\mathbb{Z}} M\left(\left|\tilde{\Delta}_j f_1\right|^{q_1}\right)(x)^{2/q_1}\right\}^{1/2} \\ &\quad\cdot\left\{\sum_{j\in\mathbb{Z}} M\left(\left|\tilde{\Delta}_j f_2\right|^{q_2}\right)(x)^{2/q_2}\right\}^{1/2}\times\prod_{i=3}^{N} M(|f_i|^{q_i})(x)^{1/q_i}.\end{aligned} \tag{65}$$

The rest of the proof is similar to that of $m_{(1,0,\cdots,0)}$, and we omit it.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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