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Research Article

Some Special Ruled Surfaces Generated by a Direction Curve according to the Darboux Frame and their Characterizations

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In this work, we consider the Darboux frame (T, V, U) of a curve lying on an arbitrary regular surface and we construct ruled surfaces having a base curve which is a V-direction curve. Subsequently, a detailed study of these surfaces is made in the case where the directing vector of their generatrices is a vector of the Darboux frame, a Darboux vector field. Finally, we give some examples for special curves such as the asymptotic line, geodesic curve, and principal line, with illustrations of the different cases studied.

1. Introduction

Ruled surfaces are well known as one of the most important surface families in the differential geometry of the surfaces. An important fact about these types of surfaces is that any ruled surface can always result from a continuous movement of a straight line along a curve.

Among the associative curves are the direction curves introduced by Choi and Kim [1] as integral curves of some vector fields generated by Frenet vectors of a given curve. This type of curves is included in several works. In terms of the Frenet frame, the direction curves are studied in [1, 2]; this study has been extended to the alternative frame [3], Darboux frame [4], and Bishop frame [5].

Examen the ruled surfaces constructed by means of direction curves has recently attracted the attention of many differential geometers. In terms of the Frenet frame, Güven [6] defined two ruled surfaces such as normal and binormal surfaces by considering their base curves as the W-direction curve. He get some results about the developability and minimality of these surfaces and the conditions for which their base curve is an asymptotic line, a geodesic curve, or a principal line. In the same way, this two ruled surfaces were also defined with a base curve and adjoint of the base curve in [7]. Moreover, in [8], the authors improve the theory of the ruled surfaces in terms of principal-direction curves of a given curve; they obtained a new representation of these ruled surfaces by slant helices and principal elements of the ruled surface such as the pitch and angle of pitch. After that, in [9] and in terms of the Darboux frame, the authors used the direction curve to define a new ruled surface called the relatively osculating developable surface. Then, they have obtained some results about the existence, uniqueness, and singularity of such surface.

In this work, we consider the Darboux frame (T, V, U) of a curve lying on an arbitrary regular surface and we construct ruled surfaces whose base curve is a V-direction curve. We give some results about the developability, minimality, and condition for which the striction curve is the base curve for the particular cases where the directing vector of the ruled surface is a vector of the Darboux frame, a Darboux vector field. Finally, we give some examples for special curves such as the asymptotic line, geodesic curve, and principal line, with illustrations of the different cases studied.

2. Preliminaries

In this section, we recall some basic concepts and properties on classical differential geometry of curves lying on a regular surface and of ruled surfaces, in the Euclidean 3-space.

(i) We denote by E^3 the Euclidean 3-space, with the usual metric

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3, \tag{1}$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are two vectors of E^3.

Let M be a regular surface and $\alpha = \alpha(s)$: $I \subset \mathbb{R} \longrightarrow M$ be a unit speed curve on the surface M. The Darboux frame along the curve α is an orthonormal frame $(T(s), V(s), U(s))$, where T is the unit tangent, U is the unit normal on the surface M, and $V = U \wedge T$. Then, the Darboux equations are given by the following relations:

$$\frac{d}{ds}\begin{bmatrix} T(s) \\ V(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} 0 & k_g(s) & k_n(s) \\ -k_g(s) & 0 & \tau_g(s) \\ -k_n(s) & -\tau_g(s) & 0 \end{bmatrix}\begin{bmatrix} T(s) \\ V(s) \\ U(s) \end{bmatrix}, \tag{2}$$

or

$$\begin{aligned} \frac{dT}{ds} &= D_n(s) \wedge T(s), \\ \frac{dV}{ds} &= D_r(s) \wedge V(s), \\ \frac{dU}{ds} &= D_o(s) \wedge U(s), \end{aligned} \tag{3}$$

where $D_n(s)$,$D_r(s)$, and $D_o(s)$ are the normal Darboux vectors field, the rectifying Darboux vector field, and the osculator Darboux vectors field, respectively, and are defined by

$$\begin{cases} D_n(s) = -k_n(s)V(s) + k_g(s)U(s), \\ D_r(s) = \tau_g(s)T(s) + k_g(s)U(s), \\ D_o(s) = \tau_g(s)T(s) - k_n(s)V(s), \end{cases} \tag{4}$$

where k_g,k_n, and τ_g are the geodesic curvature, the normal curvature, and the geodesic torsion of the curve α, respectively.

(i) A ruled surface [10] is generated by a one-parameter family of straight lines and has a parametric representation:

$$\Psi(s, v) = \beta(s) + vX(s), v \in \mathbb{R}, \tag{5}$$

where $\beta = \beta(s)$ is called the base curve of the ruled surface and $X(s)$ the unit vectors representing the direction of straight lines. If X is constant, then, the ruled surface is cylindrical; otherwise, the surface is said to be noncylindrical.

Definition 1 [10]. For a curve $\alpha = \alpha(s)$ lying on a regular surface, the following are well known:

(1) $\alpha(s)$ is an asymptotic line if and only if the normal curvature k_n vanishes

(2) $\alpha(s)$ is a geodesic curve if and only if the geodesic curvature k_g vanishes

(3) $\alpha(s)$ is a principal line if and only if the geodesic torsion τ_g vanishes

Definition 2 [4]. Let $\alpha = \alpha(s)$: $I \subset \mathbb{R} \longrightarrow E^3$ be a unit speed curve on the regular surface and $(T(s), V(s), U(s))$ the Darboux frame along the curve α. The integral curve γ of the vector field V is called V -direction curve of α, in other words $\gamma' = V$.

We have

$$\frac{\partial\Psi}{\partial s}(s, v) \wedge \frac{\partial\Psi}{\partial s}(s, v) = \beta'(s) \wedge X(s) + vX'(s) \wedge X(s). \tag{6}$$

Then, the point $\Psi(s_0, v_0)$ is said to be singular if $\beta'(s_0) \wedge X(s_0) + vX'(s_0) \wedge X(s_0) = 0$.

If there exists a common perpendicular to two constructive rulings in the ruled surface, then, the foot of the common perpendicular on the main rulings is called a central point. The locus of the central point is called a striction curve [10].

The parametrization of the striction curve on the ruled surface is given by

$$c(s) = \beta(s) - \frac{\left\langle \beta'(s), X'(s)\right\rangle}{\left\|X'(s)\right\|^2}X(s). \tag{7}$$

Definition 3. A ruled surface Ψ is called developable if det $(\beta', X, X') = 0$.

Letting I and II be the first and the second fundamental forms from the ruled surface Ψ, respectively, we have

$$\begin{aligned} I &= Eds^2 + 2Fdsdv + Gdv^2, \\ II &= Lds^2 + 2Mdsdv + Ndv^2, \end{aligned} \tag{8}$$

where

$$\begin{aligned} E &= \langle \Psi_s, \Psi_s\rangle, \\ F &= \langle \Psi_s, \Psi_v\rangle, \\ G &= \langle \Psi_v, \Psi_v\rangle = 1, \end{aligned} \tag{9}$$

$$L = \frac{\det(\Psi_s, \Psi_v, \Psi_{ss})}{\sqrt{EG - F^2}},$$
$$M = \frac{\det(\Psi_s, \Psi_v, \Psi_{sv})}{\sqrt{EG - F^2}}, \tag{10}$$
$$N = \frac{\det(\Psi_s, \Psi_v, \Psi_{vv})}{\sqrt{EG - F^2}} = 0.$$

The mean curvature H of a ruled surface is given as follows

$$H = \frac{-2FM + L}{2(E - F^2)}. \tag{11}$$

Definition 4. A ruled surface is said to be minimal if its mean curvature vanishes identically.

3. Ruled Surfaces Defined by V-Direction Curves

Letting $\alpha = \alpha(s)$: $I \subset \mathbb{R} \longrightarrow E^3$ be a unit speed curve lying on a regular surface ,$(T(s), V(s), U(s))$, the Darboux frames of α,k_g,k_n,τ_g are the geodesic curvature, the normal curvature, the geodesic torsion of α, and $\beta = \beta(s)$ a V-direction curve, respectively.

We consider the following ruled surface

$$\Psi(s, v) = \beta(s) + vX(s), v \in \mathbb{R}, \tag{12}$$

where $\beta = \beta(s)$ is the base curve and $X(s)$ the unit director vector of the straight line.

3.1. General Study. In this section, we propose to give some properties of the ruled surface Ψ.

Differentiating (12) with respect to s and v, we get

$$\Psi_s = V + vX',$$
$$\Psi_v = X. \tag{13}$$

Then, by using (10), we obtain the components of the first fundamental form E and F as follows:

$$E = 1 + 2v\left\langle V, X'\right\rangle + v^2\left\|X'\right\|^2,$$
$$F = \langle V, X\rangle. \tag{14}$$

Consequently, the condition of regularity of the ruled surface (12) is

$$E - F^2 = 1 - \langle V, X\rangle^2 + 2v\left\langle V, X'\right\rangle + v^2\left\|X'\right\|^2 \neq 0. \tag{15}$$

Differentiating (13) with respect to s and v, we get

$$\Psi_{ss} = V' + vX'',$$
$$\Psi_{sv} = X', \tag{16}$$
$$\Psi_{vv} = 0,$$

which allows, using formula (10), calculating the two components of the second fundamental form M and L, respectively, and we obtain

$$M = \frac{\det\left(V, X, X'\right)}{\sqrt{1 - \langle V, X\rangle^2 + 2v\left\langle V, X'\right\rangle + v^2\left\|X'\right\|^2}},$$

$$L = \frac{\det\left(V, X, V'\right) + v\left[\det\left(V, X, X''\right) + \det\left(X', X, V'\right)\right] + v^2\det\left(X', X, X''\right)}{\sqrt{1 - \langle V, X\rangle^2 + 2v\left\langle V, X'\right\rangle + v^2\left\|X'\right\|^2}}. \tag{17}$$

Therefore, by using (11), the mean curvature of the ruled surface (12) is given as follows

$$H = \frac{\det\left(V, X, V'\right) - 2\det\left(V, X, X'\right)\langle V, X\rangle + v\left[\det\left(V, X, X''\right) + \det\left(X', X, V'\right)\right] + v^2\det\left(X', X, X''\right)}{2\left(1 - \langle V, X\rangle^2 + 2v\left\langle V, X'\right\rangle + v^2\left\|X'\right\|^2\right)^{3/2}}. \tag{18}$$

On the other hand, the striction curve of the surface (12) is

$$c = \beta - \frac{\left\langle V, X'\right\rangle}{\left\|X'\right\|^2}X. \tag{19}$$

3.2. Case Where $X = T$ (Resp. V, U). Let us consider the ruled surfaces defined by

$$\begin{cases} \Psi_1(s, v) = \beta(s) + vT(s), \\ \Psi_2(s, v) = \beta(s) + vV(s), \\ \Psi_3(s, v) = \beta(s) + vU(s). \end{cases} \tag{20}$$

By using (2), we have

$$\begin{aligned} \langle V, T\rangle &= 0, \\ \left\langle V, T'\right\rangle &= k_g, \\ \left\|T'\right\|^2 &= k_g^2 + k_n^2. \end{aligned} \tag{21}$$

If we substitute (21) in (15), we find

$$E_1 - F_1^2 = 1 + 2\nu k_g + \nu^2\left(k_g^2 + k_n^2\right). \tag{22}$$

Likewise, we have

$$\begin{aligned} \langle V, V\rangle &= 1, \\ \left\langle V, V'\right\rangle &= 0, \\ \left\|V'\right\|^2 &= k_g^2 + \tau_g^2. \end{aligned} \tag{23}$$

If we substitute (23) in (15), we find

$$E_2 - F_2^2 = \nu^2\left(k_g^2 + \tau_g^2\right). \tag{24}$$

On the other hand,

$$\begin{aligned} \langle V, U\rangle &= 0, \\ \left\langle V, U'\right\rangle &= -\tau_g, \\ \left\|U'\right\|^2 &= k_n^2 + \tau_g^2. \end{aligned} \tag{25}$$

If we substitute (25) in (15), we find

$$E_3 - F_3^2 = 1 - 2\nu\tau_g + \nu^2\left(k_n^2 + \tau_g^2\right). \tag{26}$$

Corollary 5. *The ruled surfaces Ψ_1 and Ψ_3 are regulars along the curve β, while Ψ_2 is singular along β.*

We denote by c_1 (resp. c_2,c_3) the striction curve of Ψ_1 (resp.Ψ_2, Ψ_3), and from (19), we have

$$\begin{aligned} c_1 &= \beta - \frac{k_g}{k_g^2 + k_n^2} T, \\ c_2 &= \beta, \\ c_3 &= \beta + \frac{\tau_g}{k_n^2 + \tau_g^2} U. \end{aligned} \tag{27}$$

The result follows.

Corollary 6.

(1) β is the striction curve of the surface Ψ_1 if and only if α is a geodesic curve

(2) β is the striction curve of the surface Ψ_2

(3) β is the striction curve of the surface Ψ_3 if and only if α is a principal line

On the other hand, by using (2), we have

$$\begin{aligned} \det\left(V, T, T'\right) = \det\left(V, U, U'\right) &= -k_n, \\ \det\left(V, V, V'\right) &= 0. \end{aligned} \tag{28}$$

Corollary 7.

(1) The ruled surface Ψ_1 (resp. Ψ_3) is developable if and only if α is an asymptotic line

(2) The ruled surface Ψ_2 is developable

Differentiating (2), we obtain

$$\begin{cases} T'' = -\left(k_g^2 + k_n^2\right)T - \left(k_n\tau_g - k_g'\right)V + \left(k_g\tau_g + k_n'\right)U, \\ V'' = -\left(k_n\tau_g + k_g'\right)T - \left(k_g^2 + \tau_g^2\right)V - \left(k_gk_n - \tau_g'\right)U, \\ U'' = \left(k_g\tau_g - k_n'\right)T - \left(k_gk_n + \tau_g'\right)V - \left(k_n^2 + \tau_g^2\right)U. \end{cases} \tag{29}$$

By using (2) and (29), we have

$$\begin{cases} \det\left(V, T, V'\right) - 2\det\left(V, T, T'\right)\langle V, T\rangle = -\tau_g, \\ \det\left(V, T, T''\right) + \det\left(T', T, V'\right) = -k_n' - 2k_g\tau_g, \\ \det\left(T', T, T''\right) = k_g'k_n - k_gk_n' - \tau_g\left(k_g^2 + k_n^2\right). \end{cases} \tag{30}$$

If we substitute (30) in (18), we find

$$H_1 = \frac{-\tau_g - \left(k_n' + 2k_g\tau_g\right)\nu + \left[k_g'k_n - k_gk_n' - \tau_g\left(k_g^2 + k_n^2\right)\right]\nu^2}{2\left(1 - 2\nu k_g + \nu^2\left(k_g^2 + k_n^2\right)\right)^{3/2}}. \tag{31}$$

Likewise, we have

$$\begin{cases} \det\left(V, U, V'\right) - 2\det\left(V, U, U'\right)\langle V, U\rangle = -k_g, \\ \det\left(V, U, U''\right) + \det\left(U', U, V'\right) = 2k_g\tau_g - k_n', \\ \det\left(U', U, U''\right) = \left(\tau_gk_n' - k_n\tau_g'\right) - k_g\left(k_n^2 + \tau_g^2\right). \end{cases} \tag{32}$$

If we substitute (32) in (18), we find

$$H_3 = \frac{-k_g + \left(2k_g\tau_g - k_n'\right)v + \left[\left(\tau_g k_n' - k_n\tau_g'\right) - k_g\left(k_n^2 + \tau_g^2\right)\right]v^2}{2\left(1 - 2v\tau_g + v^2\left(k_n^2 + \tau_g^2\right)\right)^{3/2}}. \tag{33}$$

Corollary 8. *The ruled surface $\Psi_1(resp.\Psi_3)$ is minimal along its base curve if and only if α is a principal line (resp. a geodesic curve).*

3.3. Case Where X Is a Darboux Vector Field

3.3.1. Case Where $X = X_r$. The ruled surface becomes

$$\Psi_r(s, v) = \beta(s) + vX_r(s), \tag{34}$$

where $X_r = D_r/\|D_r\|$ with $D_r = \tau_g T + k_g U$ and $(\tau_g, k_g) \neq (0, 0)$.

Using (2), we get

$$X_r' = \left(\frac{\tau_g T + k_g U}{\sqrt{\tau_g^2 + k_g^2}}\right)' = \frac{k_g\left[\left(\tau_g' k_g - k_g'\tau_g\right) - k_n\left(\tau_g^2 + k_g^2\right)\right]}{\left(\tau_g^2 + k_g^2\right)^{3/2}} T - \frac{\tau_g\left[\left(\tau_g' k_g - k_g'\tau_g\right) - k_n\left(\tau_g^2 + k_g^2\right)\right]}{\left(\tau_g^2 + k_g^2\right)^{3/2}} U. \tag{35}$$

Hence,

$$X_r' = k_g\sigma_r T - \tau_g\sigma_r U, \tag{36}$$

where

$$\sigma_r = \frac{\left(\tau_g' k_g - k_g'\tau_g\right) - k_n\left(\tau_g^2 + k_g^2\right)}{\left(\tau_g^2 + k_g^2\right)^{3/2}}. \tag{37}$$

We obtain

$$\begin{gathered} \langle V, X_r\rangle = 0, \\ \left\langle V, X_r'\right\rangle = 0, \\ \left\|X_r'\right\|^2 = \sigma_r^2\left(k_g^2 + \tau_g^2\right). \end{gathered} \tag{38}$$

If we substitute (38) in (15), we find

$$E_r - F_r^2 = 1 + v^2\sigma_r^2\left(k_g^2 + \tau_g^2\right). \tag{39}$$

The following corollaries follow:

Corollary 9. *The ruled surface Ψ_r is regular.*

Corollary 10. *In the noncylindrical case (i.e., $\sigma_r \neq 0$), the curve β is the striction curve of the surface Ψ_r.*

On the other hand, we have $\det(\beta', X_r, X_r') = \sigma_r\sqrt{k_g^2 + \tau_g^2}$.

Corollary 11. *The ruled surface Ψ_r is developable if and only if it is cylindrical.*

According to (36), we have

$$X_r' = -\sigma_r V'. \tag{40}$$

Therefore, by using (2) and (29), we obtain

$$\begin{aligned} X_r' = {} & \left[k_g\sigma_r' + \left(k_n\tau_g + k_g'\right)\sigma_r\right]T + \sigma_r\left(k_g^2 + \tau_g^2\right)V \\ & + \left[-\tau_g\sigma_r' + \left(k_g k_n - \tau_g'\right)\sigma_r\right]U. \end{aligned} \tag{41}$$

Then,

$$\begin{cases} \det\left(V, X_r, V'\right) - 2\det\left(V, X_r, X_r'\right) = -\sqrt{k_g^2 + \tau_g^2}, \\ \det\left(V, X_r, X_r'\right) + \det\left(X_r', X_r, V'\right) = -\left[\sigma_r\sqrt{k_g^2 + \tau_g^2}\right]', \\ \det\left(X_r', X_r, X_r'\right) = -\left(k_g^2 + \tau_g^2\right)^{3/2}\sigma_r^2. \end{cases} \tag{42}$$

If we substitute (42) in (18), we find

$$H_r = \frac{-\left[\sqrt{k_g^2 + \tau_g^2} + \left(\sigma_r\sqrt{k_g^2 + \tau_g^2}\right)'v + \left(k_g^2 + \tau_g^2\right)^{3/2}\sigma_r^2 v^2\right]}{2\left(1 + v^2\sigma_r^2\left(k_g^2 + \tau_g^2\right)\right)^{3/2}}. \tag{43}$$

Hence, we have the following corollary:

Corollary 12. *The ruled surface Ψ_r is not minimal.*

3.3.2. Case Where $X = X_o$. The ruled surface becomes

$$\Psi_o(s, v) = \beta(s) + vX_o(s), \tag{44}$$

where $X_o = (D_o/\|D_o\|)$ with $D_o = \tau_g T - k_n V$ and $(\tau_g, k_n) \neq (0, 0)$.

Differentiating X_o, we obtain

$$X_o' = \left(\frac{\tau_g T - k_n V}{\sqrt{\tau_g^2 + k_n^2}}\right)' = \frac{k_n\left[\left(\tau_g' k_n - k_n' \tau_g\right) + k_g\left(\tau_g^2 + k_n^2\right)\right]}{\left(\tau_g^2 + k_n^2\right)^{3/2}} T + \frac{\tau_g\left[\left(\tau_g' k_n - k_n' \tau_g\right) + k_g\left(\tau_g^2 + k_n^2\right)\right]}{\left(\tau_g^2 + k_n^2\right)^{3/2}} V. \tag{45}$$

Thus,

$$X_o' = k_n \sigma_o T + \tau_g \sigma_o V, \tag{46}$$

where

$$\sigma_o = \frac{\left(\tau_g' k_n - k_n' \tau_g\right) + k_g\left(\tau_g^2 + k_n^2\right)}{\left(\tau_g^2 + k_n^2\right)^{3/2}}. \tag{47}$$

We obtain

$$\begin{gathered} \langle V, X_o\rangle = \frac{-k_n}{\sqrt{k_n^2 + \tau_g^2}}, \\ \left\langle V, X_o'\right\rangle = \tau_g \sigma_o, \\ \left\|X_o'\right\|^2 = \sigma_o^2\left(k_n^2 + \tau_g^2\right). \end{gathered} \tag{48}$$

If we substitute (48) in (15), we get

$$E_o - F_o^2 = \frac{\left[\tau_g + \nu\left(\tau_g^2 + k_n^2\right)\sigma_o\right]^2}{\tau_g^2 + k_n^2}. \tag{49}$$

On the other hand, the striction curve of Ψ_o is

$$c_o = \beta - \frac{\tau_g}{\left(\tau_g^2 + k_n^2\right)\sigma_o} X_o. \tag{50}$$

Hence, we have the following corollary:

Corollary 13. *In the noncylindrical case (i.e., $\sigma_o \neq 0$), we have the following:*

(1) Ψ_o is singular along its striction curve

(2) β is the striction curve of the surface Ψ_o if and only if α is a principal line

We have $\det(\beta', X_o, X_o') = 0$. Hence, the result is as follows.

Corollary 14. *The ruled surface Ψ_o is developable.*

From (46), we have

$$X_o' = -\sigma_o U'. \tag{51}$$

Then, by using (2) and (29), we obtain

$$X_o' = \left[k_n \sigma_o' - \left(k_g \tau_g - k_n'\right)\sigma_o\right] T + \left[\tau_g \sigma_o' + \left(k_g k_n + \tau_g'\right)\sigma_o\right] V + \sigma_o\left(k_n^2 + \tau_g^2\right) U. \tag{52}$$

Hence,

$$\begin{cases} \det\left(V, X_o, V'\right) - 2\det\left(V, X_o, X_o'\right)\langle V, X_o\rangle = -\dfrac{\tau_g^2}{\sqrt{k_n^2 + \tau_g^2}} \\ \det\left(V, X_o, X_o'\right) + \det\left(X_o', X_o, V'\right) = -2\tau_g \sigma_o \sqrt{k_n^2 + \tau_g^2}, \\ \det\left(X_o', X_o, X_o'\right) = -\left(k_n^2 + \tau_g^2\right)^{3/2} \sigma_o^2. \end{cases} \tag{53}$$

If we substitute (53) in (18), we obtain

$$H_o = -\frac{\tau_g^2 + k_n^2}{2\left(\tau_g + \nu\sigma_o\left(k_n^2 + \tau_g^2\right)\right)}. \tag{54}$$

Corollary 15. *The surface Ψ_o is not minimal.*

3.3.3. Case Where $X = X_n$. The ruled surface becomes

$$\Psi_n(s, v) = \beta(s) + v X_n(s), \tag{55}$$

where $X_n = D_n / \|D_n\|$ with $D_n = -k_n V + k_g U$ and $(k_g, k_n) \neq (0, 0)$.

By using (2), we get

$$X_n' = \left(\frac{-k_n V + k_g U}{\sqrt{k_n^2 + k_g^2}}\right)' = \frac{k_g\left[\left(k_g' k_n - k_g k_n'\right) - \tau_g\left(k_n^2 + k_g^2\right)\right]}{\left(k_n^2 + k_g^2\right)^{3/2}} V + \frac{k_n\left[\left(k_g' k_n - k_g k_n'\right) - \tau_g\left(k_n^2 + k_g^2\right)\right]}{\left(k_n^2 + k_g^2\right)^{3/2}} U. \tag{56}$$

Thus,

$$X_n' = k_g \sigma_n V + k_n \sigma_n U, \tag{57}$$

where

$$\sigma_n = \frac{\left(k_g' k_n - k_g k_n'\right) - \tau_g\left(k_n^2 + k_g^2\right)}{\left(k_n^2 + k_g^2\right)^{3/2}}. \tag{58}$$

It follows that

$$\begin{aligned} \langle V, X_n \rangle &= \frac{-k_n}{\sqrt{k_n^2 + k_g^2}}, \\ \left\langle V, X_n' \right\rangle &= k_g \sigma_n, \\ \left\| X_n' \right\|^2 &= \sigma_n^2 \left(k_n^2 + k_g^2 \right). \end{aligned} \tag{59}$$

If we substitute (59) in (15), we obtain

$$E_n - F_n^2 = \frac{\left[k_g + \nu \left(k_n^2 + k_g^2 \right) \sigma_n \right]^2}{k_n^2 + k_g^2}. \tag{60}$$

On the other hand, the striction curve of Ψ_n is

$$c_n = \beta - \frac{k_g}{\left(k_n^2 + k_g^2 \right) \sigma_n} X_n. \tag{61}$$

Hence, we have the following corollary:

Corollary 16. *In the noncylindrical case (i.e., $\sigma_n \neq 0$), we have the following:*

(1) Ψ_n is singular along its striction curve

(2) β is the striction curve of the surface Ψ_n if and only if α is a geodesic curve

We have $\det(\beta', X_n, X_n') = 0$. Hence, we have the following result.

Corollary 17. *The ruled surface Ψ_n is developable.*

From (57), we have

$$X_n' = \sigma_n T'. \tag{62}$$

Then, by using (2) and (29), we obtain

$$\begin{aligned} X_n' = {} & -\sigma_n \left(k_g^2 + k_n^2 \right) T + \left[k_g \sigma_n' - \left(k_n \tau_g - k_g' \right) \sigma_n \right] V \\ & + \left[k_n \sigma_n' + \left(k_g \tau_g + k_n' \right) \sigma_n \right] U. \end{aligned} \tag{63}$$

Then,

$$\begin{cases} \det\left(V, X_n, V' \right) - 2 \det\left(V, X_n, X_n' \right) \langle V, X_n \rangle = -\dfrac{k_g^2}{\sqrt{k_g^2 + k_n^2}} \\ \det\left(V, X_n, X_n' \right) + \det\left(X_n', X_n, V' \right) = -2 k_g \sigma_n \sqrt{k_g^2 + k_n^2}, \\ \det\left(X_n', X_n, X_n' \right) = -\left(k_g^2 + k_n^2 \right)^{3/2} \sigma_n^2. \end{cases} \tag{64}$$

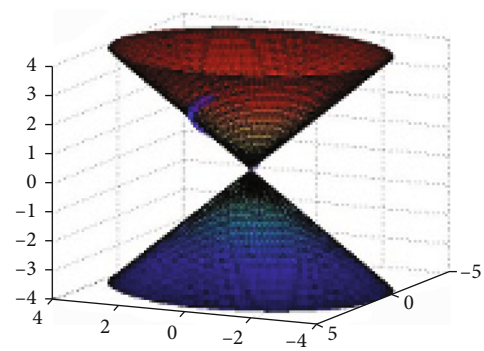

Figure 1: ${}^1M$.

If we substitute (64) in (18), we obtain

$$H_n = -\frac{k_g^2 + k_n^2}{2\left(k_g + \nu \sigma_n \left(k_n^2 + k_g^2 \right) \right)}. \tag{65}$$

Corollary 18. *The ruled surface Ψ_n is not minimal.*

4. Examples

In this section, we reinforce the previous study by the given four examples. The first one corresponds to the general case, and the three others represent the particular cases where the initial curve is an asymptotic line, a geodesic curve, and a line of curvature.

In the examples which follow, the same notations as in the preceding paragraphs are retained. We denote by α the initial curve lying on a surface M defined by φ,(T, V, U) the Darboux frame of α,k_n,k_g,τ_g, the normal curvature, the geodesic curvature, and the geodesic torsion of the curve α, respectively; by β the V-direction curve; and by X_r (resp., X_o,X_n) the unit rectifying (resp., osculator, normal) vector field. We give for each example the illustrations of the ruled surfaces denoted by Ψ_1,Ψ_2, Ψ_3,Ψ_r,Ψ_o,Ψ_n.

Example 1. Let ${}^1\alpha(s) = ((s/2)\cos(\sqrt{2}\ln(s/2)), (s/2)\sin(\sqrt{2}\ln(s/2)), (s/2))$ be a curve lying on the surface ${}^1M$ given by the following parametrization:

$${}^1\varphi(u, v) = (u \cos v, u \sin v, u), \tag{66}$$

which can be seen in Figure 1. The Darboux frame of ${}^1\alpha$ is

$${}^1T = \begin{pmatrix} \frac{1}{2} \cos\left(\sqrt{2} \ln \frac{s}{2} \right) - \frac{1}{\sqrt{2}} \sin\left(\sqrt{2} \ln \frac{s}{2} \right) \\ \frac{1}{2} \sin\left(\sqrt{2} \ln \frac{s}{2} \right) + \frac{1}{\sqrt{2}} \cos\left(\sqrt{2} \ln \frac{s}{2} \right) \\ \frac{1}{2} \end{pmatrix},$$

$$
{}^1V = \begin{pmatrix} -\frac{1}{\sqrt{2}} \sin\left(\sqrt{2}\ln\frac{s}{2}\right) - \frac{1}{\sqrt{2}} \cos\left(\sqrt{2}\ln\frac{s}{2}\right) \\ \frac{1}{\sqrt{2}} \cos\left(\sqrt{2}\ln\frac{s}{2}\right) - \frac{1}{2} \sin\left(\sqrt{2}\ln\frac{s}{2}\right) \\ -\frac{1}{2} \end{pmatrix},
$$

$$
{}^1U = \begin{pmatrix} -\frac{1}{\sqrt{2}} \cos\left(\sqrt{2}\ln\frac{s}{2}\right) \\ -\frac{1}{\sqrt{2}} \sin\left(\sqrt{2}\ln\frac{s}{2}\right) \\ \frac{1}{2} \end{pmatrix}. \tag{67}
$$

Then,

$$
\begin{aligned}
{}^1k_n &= \frac{1}{\sqrt{2}s}, \\
{}^1k_g &= \frac{1}{s}, \\
{}^1\tau_g &= \frac{1}{\sqrt{2}s}.
\end{aligned} \tag{68}
$$

It follows that

$$
{}^1X_r = \begin{pmatrix} -\frac{1}{2\sqrt{3}} \cos\left(\sqrt{2}\ln\frac{s}{2}\right) - \frac{1}{\sqrt{6}} \sin\left(\sqrt{2}\ln\frac{s}{2}\right) \\ -\frac{1}{2\sqrt{3}} \sin\left(\sqrt{2}\ln\frac{s}{2}\right) + \frac{1}{\sqrt{6}} \cos\left(\sqrt{2}\ln\frac{s}{2}\right) \\ \frac{\sqrt{3}}{2} \end{pmatrix},
$$

$$
{}^1X_o = \begin{pmatrix} \frac{1}{\sqrt{2}} \cos\left(\sqrt{2}\ln\frac{s}{2}\right) \\ \frac{1}{\sqrt{2}} \sin\left(\sqrt{2}\ln\frac{s}{2}\right) \\ \frac{1}{\sqrt{2}} \end{pmatrix},
$$

$$
{}^1X_n = \begin{pmatrix} -\frac{1}{2\sqrt{3}} \cos\left(\sqrt{2}\ln\frac{s}{2}\right) + \frac{1}{\sqrt{6}} \sin\left(\sqrt{2}\ln\frac{s}{2}\right) \\ -\frac{1}{2\sqrt{3}} \sin\left(\sqrt{2}\ln\frac{s}{2}\right) - \frac{1}{\sqrt{6}} \cos\left(\sqrt{2}\ln\frac{s}{2}\right) \\ \frac{\sqrt{3}}{2} \end{pmatrix}. \tag{69}
$$

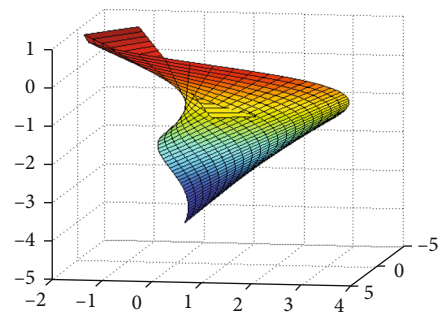

Figure 2: ${}^1\Psi_1$.

The V-direction curve is given by

$$
{}^1\beta = \begin{pmatrix} -\frac{\sqrt{2}s}{3} \sin\left(\sqrt{2}\ln\frac{s}{2}\right) + \frac{s}{6} \cos\left(\sqrt{2}\ln\frac{s}{2}\right) + a_1 \\ \frac{\sqrt{2}s}{3} \cos\left(\sqrt{2}\ln\frac{s}{2}\right) + \frac{s}{6} \sin\left(\sqrt{2}\ln\frac{s}{2}\right) + a_2 \\ -\frac{s}{2} + a_3 \end{pmatrix}, \tag{70}
$$

where a_1, a_2, a_3 are integration constants. Consequently, the surfaces ${}^1\Psi_1$, ${}^1\Psi_2$, ${}^1\Psi_3$, ${}^1\Psi_r$, ${}^1\Psi_o$, ${}^1\Psi_n$ are given as illustrate in Figures 2–7.

Example 2. Let ${}^2\alpha(s) = (\cos(s/\sqrt{2}), \sin(s/\sqrt{2}), (s/\sqrt{2}))$ be the curve lying on the surface ${}^2M$ given by the following parametrization:

$$
{}^2\varphi(s,v) = \left(\cos\frac{s}{\sqrt{2}} - \frac{v}{\sqrt{2}} \sin\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}} + \frac{v}{\sqrt{2}} \cos\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} + \frac{v}{\sqrt{2}}\right), \tag{71}
$$

which can be seen in Figure 8. The Darboux frame of ${}^2\alpha$ is

$$
\begin{cases}
{}^2T(s) = \left(-\frac{1}{\sqrt{2}} \sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \\
{}^2V(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, 0\right), \\
{}^2U(s) = \left(-\frac{1}{\sqrt{2}} \sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \cos\frac{s}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).
\end{cases} \tag{72}
$$

Then, ${}^2k_n = 0, {}^2k_g = -(1/2)$, and ${}^2\tau_g = 1/2$ and

$$
\begin{cases}
{}^2X_r = (0, 0, 1), \\
{}^2X_o = \left(-\frac{1}{\sqrt{2}} \sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \cos\frac{s}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \\
{}^2X_n = \left(\frac{1}{\sqrt{2}} \sin\frac{s}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).
\end{cases} \tag{73}
$$

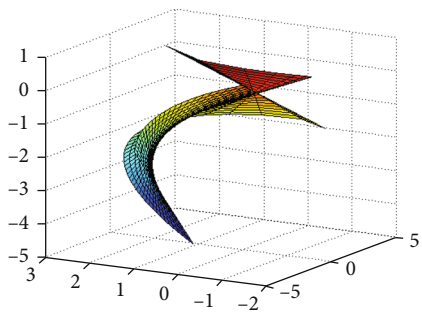

Figure 3: ${}^{1}\Psi_{2}$.

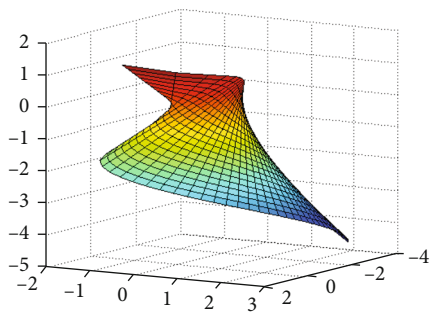

Figure 4: ${}^{1}\Psi_{3}$.

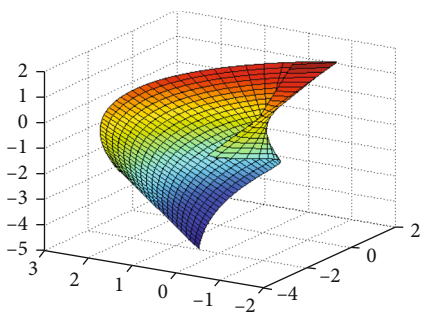

Figure 5: ${}^{1}\Psi_{r}$.

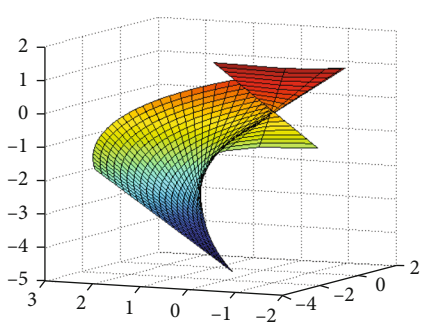

Figure 6: ${}^{1}\Psi_{o}$.

Let ${}^{2}\beta(s) = (\sqrt{2}\sin(s/\sqrt{2}) + b_1, -\sqrt{2}\cos(s/\sqrt{2}) + b_2, b_3)$, where b_1, b_2, b_3 are integration costants. Consequently, the surfaces ${}^{2}\Psi_{1}$, ${}^{2}\Psi_{2}$, ${}^{2}\Psi_{3}$, ${}^{2}\Psi_{r}$, ${}^{2}\Psi_{o}$, ${}^{2}\Psi_{n}$ are given as illustrated in Figures 9–13.

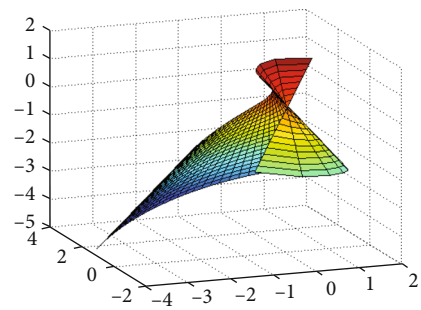

Figure 7: ${}^{1}\Psi_{n}$.

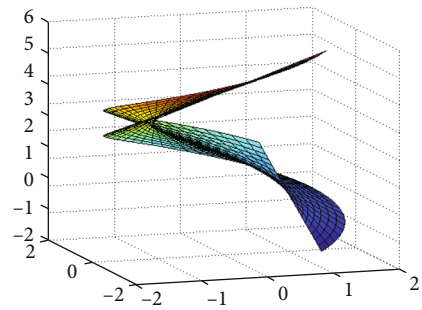

Figure 8: ${}^{2}M$.

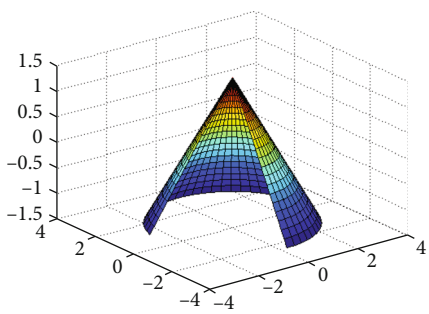

Figure 9: ${}^{2}\Psi_{1}$ and ${}^{2}\Psi_{o}$.

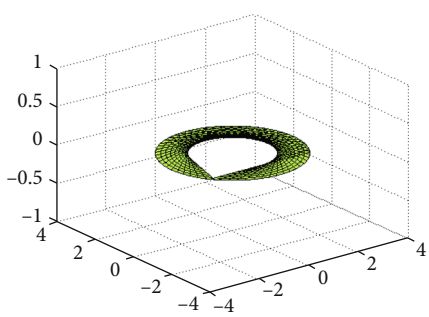

Figure 10: ${}^{2}\Psi_{2}$.

Example 3. We consider the curve ${}^{3}\alpha(s) = ((1/\sqrt{2})\cos s, (1/\sqrt{2})\sin s, (s/\sqrt{2}))$, lying on the surface ${}^{3}M$ defined by

$$ {}^{3}\varphi(s, v) = \left(\frac{1}{\sqrt{2}}\cos s, \frac{1}{\sqrt{2}}\sin s, \frac{s}{\sqrt{2}} - 2v\right), \tag{74} $$

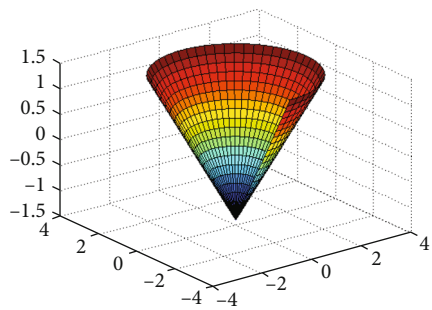

Figure 11: ${}^{2}\Psi_{3}$.

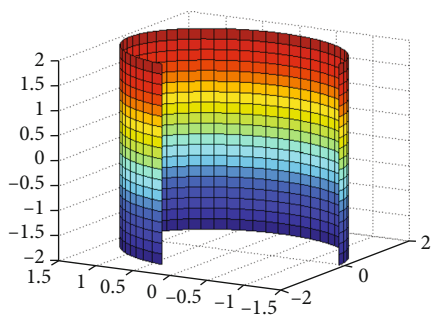

Figure 12: ${}^{2}\Psi_{r}$.

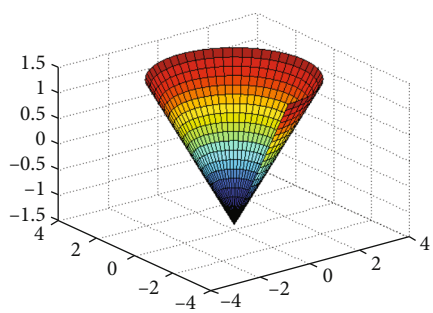

Figure 13: ${}^{2}\Psi_{n}$.

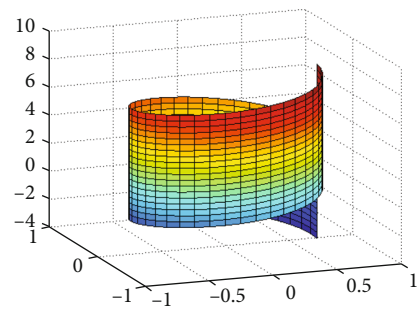

Figure 14: ${}^{3}M$.

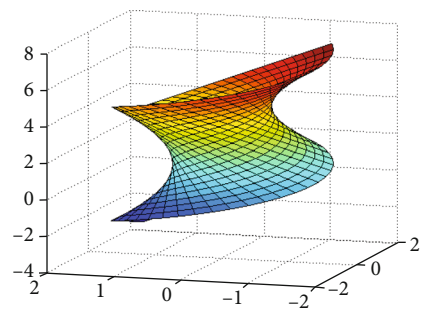

Figure 15: ${}^{3}\Psi_{1}$ and ${}^{3}\Psi_{r}$.

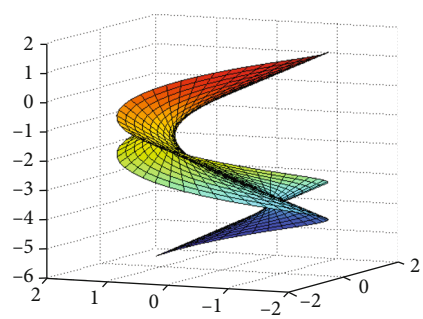

Figure 16: ${}^{3}\Psi_{2}$.

which can be seen in Figure 14. The Darboux frame of ${}^{3}\alpha$ is

$$
\begin{aligned}
{}^{3}T(s) &= \left(-\frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}\cos s, \frac{1}{\sqrt{2}}\right),\\
{}^{3}V(s) &= \left(-\frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}\cos s, -\frac{1}{\sqrt{2}}\right),\\
{}^{3}U(s) &= (-\cos s, -\sin s, 0).
\end{aligned}
\tag{75}
$$

Then, ${}^{3}k_{n} = 1/\sqrt{2}$, ${}^{3}k_{g} = 0$,, and ${}^{3}\tau_{g} = 1/\sqrt{2}$ and

$$
\begin{cases}
{}^{3}X_{r} = \left(-\frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}\cos s, \frac{1}{\sqrt{2}}\right),\\
{}^{3}X_{o} = (0, 0, 1),\\
{}^{3}X_{n} = \left(\frac{1}{\sqrt{2}}\sin s, -\frac{1}{\sqrt{2}}\cos s, \frac{1}{\sqrt{2}}\right).
\end{cases}
\tag{76}
$$

Let ${}^{3}\beta(s) = ((1/\sqrt{2})\cos s + c_{1}, (1/\sqrt{2})\sin s + c_{2}, -(s/\sqrt{2}) + c_{3})$, where c_{1}, c_{2}, c_{3} are integration costants. Consequently, the surfaces ${}^{3}\Psi_{1}$, ${}^{3}\Psi_{2}$, ${}^{3}\Psi_{3}$, ${}^{3}\Psi_{r}$, , ${}^{3}\Psi_{n}$ are given as illustrated in Figures 15–19.

Example 4. We consider the curve ${}^{4}\alpha(s) = (\cos s, \sin s, 0)$ lying on the surface ${}^{4}M$ given by the following parametrization:

$$
{}^{4}\varphi(s, v) = \left(\cos s - \frac{v}{\sqrt{2}}\cos s, \sin s - \frac{v}{\sqrt{2}}\sin s, \frac{v}{\sqrt{2}}\right),
\tag{77}
$$

which can be seen in Figure 20. The Darboux frame of ${}^{4}\alpha$ is

$$
\begin{cases}
{}^{4}T(s) = (-\sin s, \cos s, 0),\\
{}^{4}V(s) = \left(-\frac{1}{\sqrt{2}}\cos s, -\frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}\right),\\
{}^{4}U(s) = \left(\frac{1}{\sqrt{2}}\cos s, \frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}\right).
\end{cases}
\tag{78}
$$

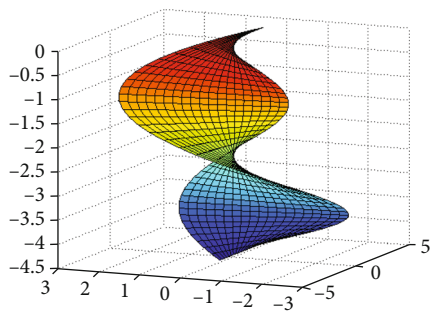

Figure 17: ${}^{3}\Psi_{3}$.

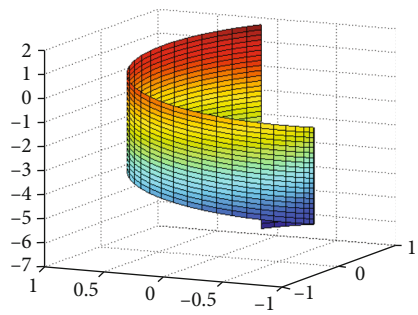

Figure 18: ${}^{3}\Psi_{o}$.

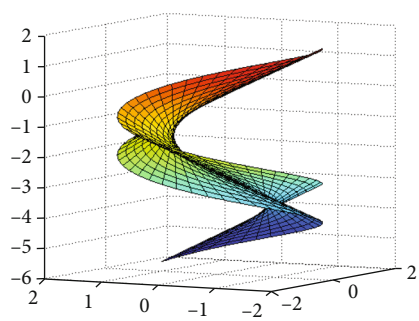

Figure 19: ${}^{3}\Psi_{n}$.

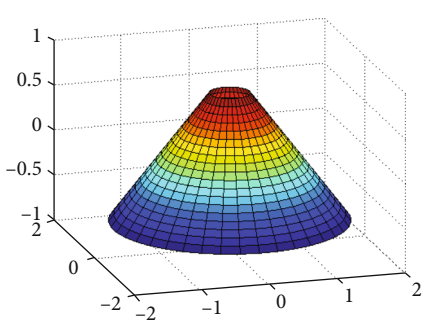

Figure 20: ${}^{4}M$.

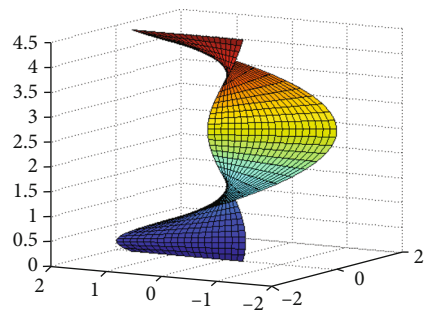

Figure 21: ${}^{4}\Psi_{1}$.

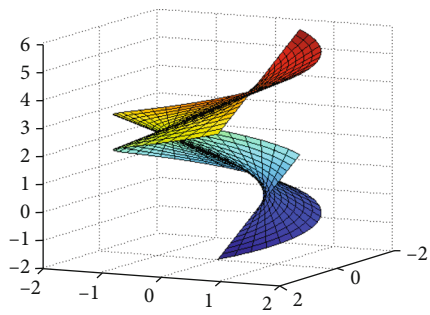

Figure 22: ${}^{4}\Psi_{2}$ and ${}^{4}\Psi_{o}$.

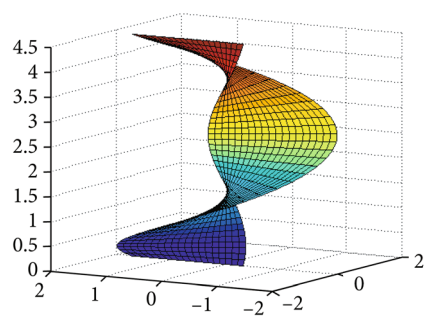

Figure 23: ${}^{4}\Psi_{3}$ and ${}^{4}\Psi_{r}$.

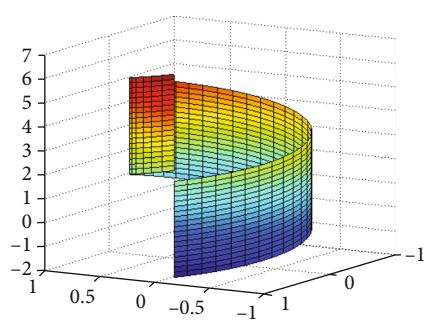

Figure 24: ${}^{4}\Psi_{n}$.

Then, ${}^4k_n = -1/\sqrt{2}$, ${}^4k_g = 1/\sqrt{2}$, and ${}^4\tau_g = 0$ and

$$\begin{cases} {}^4X_r = \left(\frac{1}{\sqrt{2}}\cos s, \frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}\right), \\ {}^4X_o = \left(\frac{1}{-\sqrt{2}}\cos s, -\frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}\right) \\ {}^4X_n = (0, 0, 1). \end{cases} \tag{79}$$

Let ${}^4\beta(s) = (-(1/\sqrt{2})\sin s + d_1, (1/\sqrt{2})\cos s + d_2, (s/\sqrt{2}) + d_3)$, where d_1, d_2, d_3 are integration costants. Consequently, the surfaces ${}^4\Psi_1$, ${}^4\Psi_2$, ${}^4\Psi_3$, ${}^4\Psi_r$, ${}^4\Psi_o$, ${}^4\Psi_n$ are given as illustrated in Figures 21–24.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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