Research Article

A Numerical Scheme Based on the Chebyshev Functions to Find Approximate Solutions of the Coupled Nonlinear Sine-Gordon Equations with Fractional Variable Orders

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In this article, a numerical method based on the shifted Chebyshev functions for the numerical approximation of the coupled nonlinear variable-order fractional sine-Gordon equations is shown. The variable-order fractional derivative is considered in the sense of Caputo-Prabhakar. To solve the problem, first, we obtain the operational matrix of the Caputo-Prabhakar fractional derivative of shifted Chebyshev polynomials. Then, this matrix and collocation method are used to reduce the solution of the nonlinear coupled variable-order fractional sine-Gordon equations to a system of algebraic equations which is technically simpler for handling. Convergence and error analysis are examined. Finally, some examples are given to test the proposed numerical method to illustrate the accuracy and efficiency of the proposed method.

1. Introduction

The basic and important applications of differential equations with fractional variable order which are considered as a generalization of ordinary differential equations with integer order in various fields of computational science [1–3], engineering [4–6], physics [7, 8], and chemistry [9, 10] have attracted significant attention in the literature. One of the important discussions in differential equations is the study of nonlinear partial and nonlinear differential equations that are used in physics and applied mathematics such as the sine-Gordon equation which is a significant nonlinear integrable evolution partial differential system in space-time coordinates [11]. The coupled nonlinear partial differential sine-Gordon equations of integer order on the interval $(p, q) \in [a, b] \times [0, T]$ are defined by [12]

$$z_{qq}(p, q) - z_{pp}(p, q) + \theta^2 \sin(z(p, q) - v(p, q)) = h_1(p, q), \quad (1)$$

$$v_{qq}(p, q) - \xi^2 v_{pp}(p, q) - \sin(z(p, q) - v(p, q)) = h_2(p, q), \quad (2)$$

where symbols θ and ξ^2 are the ratios of the acoustic velocities of the functions z, v , and the ratios of masses of particles in the “lower” and “upper” parts of the crystal given in the generalized Frenkel-Kontorova dislocation equation [12–14] are the defined parameters θ and ξ^2 , respectively, which are known, and the functions $z(p, q), v(p, q)$ and $h_1(p, q), h_2(p, q)$ are unknown and known, respectively. If in Equation (2), $\xi = 1$ is considered, then Equation (2) explains the open states in deoxyribonucleic acid (DNA) [15], and if in Equation (2), $\xi^2 \neq 1$ is considered, then Equation (2) shows a gap in the spectrum of velocities of solitary waves [12]. This paper focuses on finding approximate solutions based on the Chebyshev function scheme for fractional differential equations of variable orders which are called coupled nonlinear sine-Gordon equations, and these type of equations of fractional

variable orders $\alpha_1(p, q), \alpha_2(p, q) \in (1, 2]$ are defined as follows:

$${}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} z(p, q) - {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} z(p, q) + \theta^2 \sin(z - v) = h_1(p, q), \quad (3)$$

$${}^{C-P}\mathbb{D}_q^{\alpha_2(p,q)} v(p, q) - \xi^2 {}^{C-P}\mathbb{D}_q^{\alpha_2(p,q)} v(p, q) - \sin(z - v) = h_2(p, q), \quad (4)$$

where ${}^{C-P}\mathbb{D}$ is the Caputo-Prabhakar operator. Also, the initial and boundary conditions for problems (3) are considered as follows:

$$z(p, 0) = l_0(p), z_q(p, 0) = l_1(p), \quad (5)$$

$$v(p, 0) = l_2(p), v_q(p, 0) = l_3(p), \quad (6)$$

$$z(a, q) = f_0(q), z(b, q) = f_1(q), \quad (7)$$

$$v(a, q) = f_2(q), v(b, q) = f_3(q), \quad (8)$$

where functions $l_0(p), l_1(p), l_2(p), l_3(p)$ and $f_0(q), f_1(q), f_2(q), f_3(q)$ are continuous. The Caputo-Prabhakar operator considered in this article is an extension of the Caputo derivative which is introduced in [16], and it is defined by

$$\mathbf{D}_{0^+}^{\mu_1} z(x) = \mathbf{I}_{0^+}^{n-\mu_1} \frac{d^n}{dx^n} z(x), \quad (9)$$

where $n-1 < \mu_1 \leq n, z \in L^1[0, b], b \in \mathbb{R}$ and $\mathbf{I}$ is the Riemann-Liouville integral (RLI). So by putting a function containing the Prabhakar function which is studied in [17], in Equation (9), by developing the RLI kernel with its function, this function is given by [17]

$$e_{q,v}^v(x) = x^{v-1} E_{q,v}^v(\omega x^q), \Re(q) > 0, \Re(v) > 0, \quad x \in \mathbb{R}, q, v, \omega \in \mathbb{C}, \quad (10)$$

where $E_{q,v}^v(\omega x^q)$ is the Prabhakar function and is defined as follows:

$$E_{q,v}^v(\omega x^q) = \sum_{l=0}^{\infty} \frac{\Gamma(v+l) x^{l/l} \Gamma(v)}{\Gamma(ql+v)}. \quad (11)$$

Some applications of the Prabhakar function can be seen in mathematics and physics as a fractional Poisson process [16], Havriliak-Negami relaxation functions [18, 19], irregular case of the dielectric relaxation responses [20], a model of anomalous relaxation in dielectrics of fractional order [21], fractional thermoelasticity [10], telegraph equations [22], thermodynamics [23], and fractal

time random [24]. By placing $\alpha_1(p, q) = 2, \alpha_2(p, q) = 2$ in Equation (3), the coupled nonlinear sine-Gordon equations of fractional variable orders given in (3) change into the classical coupled nonlinear sine-Gordon equations which are defined by Equation (2), and the classical coupled nonlinear sine-Gordon equations have many applications in physics as nonlinear models [25, 26], plasma [27], quantum [28], optics [29], and mathematics [13, 30, 31]. Getting analytic solutions to fractional differential equations in general are not easy; therefore, numerical methods are used to obtain the solutions of this type of equations. For instance, [32] defined a new method based on the linearization formula to find the fuzzy approximate solution of fractional differential equations under uncertainty, a new spectral tau method for solving equations of Kelvin-Voigt equations is proposed in [33], and [34] studied a numerical method based on the Bernstein functions to solve the linear cable equation of variable order. An implicit RBF meshless method [35], spectral method [36], homotopy analysis scheme [37], meshless method [38], collocation and finite difference-collocation methods [39], Ritz-Galerkin method [40], new collocation scheme for solving fractional partial differential equations [41], Sinc-Chebyshev collocation scheme [42], numerical method based on the shifted Chebyshev functions [43], numerical method based on the shifted Legendre functions [44], piecewise integroquadratic spline interpolation and finite difference method [45], wavelet scheme [46], Legendre wavelet method [47], Bernoulli wavelet method [48], and Lagrange multiplier scheme [49] and other methods [50–54]. Several methods are presented for solving variable-order fractional equations. For example, Bhrawy and Zaky [55] used the shifted Jacobi polynomials to obtain solution variable-order fractional Schrödinger equations, Bhrawy and Zaky [56] studied the Jacobi-Gauss-Lobatto collocation method to obtain solution variable-order fractional Schrödinger equations, Mahmoud et al. [57] proposed the Jacobi wavelet collocation method to obtain solution variable-order fractional equations, and Zaky et al. [58] proposed the shifted Chebyshev polynomials to obtain solution variable-order fractional equations; in [59], a proper discrete form of fractional Grönwall-type inequality is introduced and other methods such as shifted Jacobi collocation method [60] and shifted Jacobi method [61]. Since this mathematical system which is given in Equation (3), due to the variable-order fractional operators and nonlinearity, is very complex, we need to reduce it by using a highly accurate and efficient expansion scheme. Thus, for this aim, we first extract an operational matrix of variable-order fractional derivatives for the shifted Chebyshev functions, then we use them for extending the unknown solution. In other words, the problem shown in Equation (3) is converted into an algebraic system of equations by exploiting the operational matrix of variable-order derivative. In this paper, we consider a class of coupled nonlinear variable-order fractional sine-Gordon equations. For solving the given equations, operational matrices based on the shifted Chebyshev functions are applied. First, we approximate the unknown function and its derivatives in terms of the shifted

Chebyshev functions. Then, by substituting these approximations into the equation and applying the properties of the shifted Chebyshev functions together with the collocation points, the main problem is reduced to a set of nonlinear algebraic equations. By solving this system, the approximate solution is calculated. This article is divided into the following sections. In Section 2, we state the essential definitions and lemmas about fractional derivatives and integrals of the variable order, also in this section are introduced the Chebyshev polynomials and the shifted Chebyshev polynomials. The approximate function is obtained in Section 3. In Section 4, an explanation of the suggested method for obtaining approximate solutions of the equations introduced in (3) is provided and four numerical examples are shown in Section 5.

2. Preliminaries of Fractional Calculus

This section discusses the important topics of fractional derivative and integral of the variable order such as the Prabhakar integral and the Caputo-Prabhakar derivative. The Chebyshev polynomials and the shifted Chebyshev polynomials are introduced in this section.

Definition 1. The Prabhakar integral of variable order $\Re(\mu(p, q)) > 0, q \geq 0$ is defined by [17, 62]

$$\begin{aligned} {}^1\mathcal{E}_{\mathbb{Q}, \mu(p, q), \omega, 0}^{\nu} z(p, q) &= \int_0^q (q - \tau)^{\mu(p, q)-1} E_{\mathbb{Q}, \mu(p, q)}^{\nu}(\omega(q - \tau)^{\omega}) z(p, \tau) d\tau, \\ {}^2\mathcal{E}_{\mathbb{Q}, \mu(p, q), \omega, 0}^{\nu} z(p, q) &= \int_0^p (p - \tau)^{\mu(p, q)-1} E_{\mathbb{Q}, \mu(p, q)}^{\nu}(\omega(p - \tau)^{\omega}) z(\tau, q) d\tau, \end{aligned} \quad (12)$$

where $f \in L^1(0, \infty)$. Also, for $\nu = 0$, the Prabhakar integral coincides with the Riemann-Liouville fractional integral of variable order $\Re(\mu(p, q)) > 0, q \geq 0$.

Definition 2. The Caputo-Prabhakar derivative of variable-order $\alpha_1(p, q), n - 1 < \alpha_1(p, q) \leq n, n \in \mathbb{N}, q \geq 0$ is defined by [17, 62, 63]

$$\begin{aligned} {}^{C-P}\mathbb{D}_q^{\alpha_1(p, q)} z(p, q) &= {}^1\mathcal{E}_{\mathbb{Q}, n - \alpha_1(p, q), \omega, 0}^{-\nu} \frac{d^n}{dq^n} (z(p, q)), \\ {}^{C-P}\mathbb{D}_q^{\alpha_1(p, q)} z(p, q) &= {}^2\mathcal{E}_{\mathbb{Q}, n - \alpha_1(p, q), \omega, 0}^{-\nu} \frac{d^n}{dp^n} (z(p, q)), \end{aligned} \quad (13)$$

where $\Re(\omega) > 0, \Re(\alpha_1(p, q)) > 0, \mathbb{Q}, \alpha_1(p, q), \nu, \omega \in \mathbb{C}$. Here, we consider $n = 2$.

Lemma 3. Let $k > 1$. Then, the following formula based on Definition 1 is established [63]:

$$\begin{aligned} {}^1\mathcal{E}_{\mathbb{Q}, \mu(p, q), \omega, 0}^{\nu} q^{k-1} &= \Gamma(k) q^{k+\mu(p, q)-1} e_{\mathbb{Q}, k+\mu(p, q)+1}^{\nu}(q), \\ {}^2\mathcal{E}_{\mathbb{Q}, \mu(p, q), \omega, 0}^{\nu} p^{k-1} &= \Gamma(k) p^{k+\mu(p, q)-1} e_{\mathbb{Q}, k+\mu(p, q)+1}^{\nu}(p). \end{aligned} \quad (14)$$

Lemma 4. Let $k > 2$. Then the following formula based on the Definition 2 and the Lemma 3 for $n = 2$ are established:

$$\begin{aligned} {}^{C-P}\mathbb{D}_q^{\alpha_1(p, q)} q^{k-1} &= \Gamma(k) q^{k+\mu(p, q)-1} e_{\mathbb{Q}, k+\mu(p, q)+1}^{-\nu}(q), \\ {}^{C-P}\mathbb{D}_p^{\alpha_1(p, q)} p^{k-1} &= \Gamma(k) p^{k+\mu(p, q)-1} e_{\mathbb{Q}, k+\mu(p, q)+1}^{-\nu}(p). \end{aligned} \quad (15)$$

Definition 5. The Chebyshev polynomials on $[-1, 1]$ are defined as follows [64]:

$$\begin{aligned} T_0(p) &= 1, T_1(p) = p, \\ T_{m+1}(p) &= 2pT_m(p) - T_{m-1}(p), \end{aligned} \quad (16)$$

where $T_m(p) = m \sum_{j=0}^{[m/2]} (-1)^j (2^{m-2j-1} (m-j-1)! j! (m-2j)!) p^{m-2j}$. The orthogonality manner for Chebyshev polynomials is given by

$$\int_{-1}^1 \Omega(p) T_i(p) T_j(p) dp = \begin{cases} \pi, & i = j = 0, \\ \frac{\pi}{2}, & i = j \neq 0, \\ 0, & i \neq j, \end{cases} \quad (17)$$

where $\Omega(p) = 1/\sqrt{1-p^2}$ is a weight function for Chebyshev polynomials. Also, the shifted Chebyshev polynomials on $[0, 1]$ are defined as follows:

$$\begin{aligned} \hat{T}_0(q) &= T_0(2p-1) = 1, \hat{T}_1(q) = T_1(2p-1) = q, \\ \underbrace{T_{m+1}(2p-1)}_{\hat{T}_{m+1}(q)} &= \underbrace{2(2p-1)T_m(2p-1) - T_{m-1}(2p-1)}_{2q\hat{T}_m(q) - \hat{T}_{m-1}(q)}, \end{aligned} \quad (18)$$

where $\hat{T}_m(q) = m \sum_{j=0}^m (-1)^{m-j} (2^{2j} (m+j-1)! / 2j! (m-j)!) q^j$. The orthogonality conditions for shifted Chebyshev polynomials are given by

$$\int_0^1 \Omega^*(q) \hat{T}_i(q) \hat{T}_j(q) dq = \begin{cases} \frac{\pi}{2}, & i = j = 0, \\ \frac{\pi}{4}, & i = j \neq 0, \\ 0, & i \neq j, \end{cases} \quad (19)$$

where $\Omega^*(q) = 1/\sqrt{1-q^2}$ is a weight function.

Lemma 6. Let $\Psi(q) = [T_{\wedge_0}(q), T_{\wedge_1}(q), \dots, T_{\wedge_m}(q)]^T$. Then, the function $\Psi(q)$ has a matrix display as follows [64]:

$$\Psi(q) = \mathbb{A} H_m(q), \quad (20)$$

where $\mathbb{A}$ and $H(q)$ are defined by

$$\mathbb{A} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 2 & 0 & \dots & 0 \\ \frac{2(-1)^2 1!}{2!} & \frac{2(-1)^1 2^2 2!}{2!} & \frac{2(-1)^0 2^4 3!}{4!} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{m(-1)^m (m-1)!}{m!} & \frac{m(-1)^{m-1} 2^2 (m)!}{2!(m-1)!} & \frac{m(-1)^{m-2} 2^4 (m+1)!}{4!(m-2)!} & \dots & \frac{m(-1)^0 2^{2m} (2m-1)!}{(2m)!} \end{pmatrix}, \quad (21)$$

$$H_m(q) = \begin{pmatrix} 1 \\ q \\ \vdots \\ q^m \end{pmatrix}.$$

Also, Lemma 6 is established for the function $\Psi(p)$.

3. Approximate Function

The approximation functions as $z(p, q), v(p, q)$ on $L^2([0, 1] \times [0, 1])$ can be developed in shifted Chebyshev polynomials as follows:

$$z(p, q) \simeq z_m(p, q) = \sum_{i=0}^m \sum_{j=0}^m z_{ij} \hat{T}_i(p) \hat{T}_j(q) = \Psi^T(p) Z \Psi(q), \quad (22)$$

$$v(p, q) \simeq v_m(p, q) = \sum_{i=0}^m \sum_{j=0}^m v_{ij} \hat{T}_i(p) \hat{T}_j(q) = \Psi^T(p) V \Psi(q), \quad (23)$$

where $Z = [z_{ij}]_{(m+1) \times (m+1)}$, $z_{ij} = \langle \hat{T}_i(p), \langle z(p, q), \hat{T}_j(q) \rangle \rangle$ and $V = [v_{ij}]_{(m+1) \times (m+1)}$, $v_{ij} = \langle \hat{T}_i(p), \langle v(p, q), \hat{T}_j(q) \rangle \rangle$. By giving effect Caputo-Prabhakar derivative ${}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)}$ on Equation

(22), we obtain

$${}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} z(p, q) \simeq {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} z_m(p, q) = \Psi^T(p) Z \left[{}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} \Psi(q) \right], \quad (24)$$

using Lemma 6, then from Equation (24), we have

$$\begin{aligned} {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} z_m(p, q) &= \Psi^T(p) Z \left[{}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} \Psi(q) \right] \\ &= \Psi^T(p) Z \mathbb{A} \left[{}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} H_m(p) \right] \\ &= \Psi^T(p) Z \mathbb{A} \begin{pmatrix} {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)_1} \\ {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)_q} \\ \vdots \\ {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)_{q^m}} \end{pmatrix}, \end{aligned} \quad (25)$$

applying Lemma 4 on Equation (25), we have

$$\begin{aligned}
 {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} z_m(p, q) &= \Psi^T(p) Z \mathbb{A} \begin{pmatrix} {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)}_1 \\ {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)}_q \\ \vdots \\ {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)}_{q^m} \end{pmatrix} = \Psi^T(p) Z \mathbb{A} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \Gamma(m+1)q^{m-\alpha_1(p,q)}e^{-v}_{\rho, m-\alpha_1(p,q)+1(q)} \end{pmatrix} \\
 &= \Psi^T(p) Z \mathbb{A} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \Gamma(2)q^{1-\alpha_1(p,q)}e^{-v}_{\rho, 2-\alpha_1(p,q)}(q) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \Gamma(m+1)q^{m-\alpha_1(p,q)}e^{-v}_{\rho, m-\alpha_1(p,q)+1(q)} \end{pmatrix} \times \begin{pmatrix} 1 \\ q \\ \vdots \\ q^m \end{pmatrix} \\
 &= \Psi^T(p) Z \mathbb{A} \mathbb{M}_1 H_m(q) = \Psi^T(p) Z \mathbb{A} \mathbb{M}_1 \mathbb{A}^{-1} \Psi(q), \\
 {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} z(p, q) &\cong {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} z_m(p, q) = \Psi^T(p) Z \mathbb{A} \mathbb{M}_1 \mathbb{A}^{-1} \Psi(q), \tag{26}
 \end{aligned}$$

where

$$\mathbb{M}_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \Gamma(2)q^{1-\alpha_1(p,q)}e^{-v}_{\rho, 2-\alpha_1(p,q)}(q) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \Gamma(m+1)q^{m-\alpha_1(p,q)}e^{-v}_{\rho, m-\alpha_1(p,q)+1(q)} \end{pmatrix}. \tag{27}$$

With a similar process as Equation (26), these results are easily achieved:

$${}^{C-P}\mathbb{D}_q^{\alpha_2(p,q)} v_m(p, q) = \Psi^T(p) V \mathbb{A} \mathbb{M}_2 \mathbb{A}^{-1} \Psi(q), \tag{28}$$

where

$$\begin{aligned}
 \mathbb{M}_2 &= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \Gamma(2)p^{1-\alpha_2(p,q)}e^{-v}_{\rho, 2-\alpha_2(p,q)}(q) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \Gamma(m+1)p^{m-\alpha_2(p,q)}e^{-v}_{\rho, m-\alpha_2(p,q)+1(q)} \end{pmatrix}, \\
 {}^{C-P}\mathbb{D}_p^{\alpha_1(p,q)} z_m(p, q) &= {}^{C-P}\mathbb{D}_p^{\alpha_1(p,q)} [1, p, \dots, p^m] \mathbb{A}^T Z \Psi(q) = [0, 0, \dots, \Gamma(m+1)p^{m-\alpha_1(p,q)}e^{-v}_{\rho, m-\alpha_1(p,q)+1(p)}] \mathbb{A}^T Z \Psi(q) \\
 &= [1, p, \dots, p^m] \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \Gamma(2)p^{1-\alpha_1(p,q)}e^{-v}_{\rho, 2-\alpha_1(p,q)}(p) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \Gamma(m+1)p^{m-\alpha_1(p,q)}e^{-v}_{\rho, m-\alpha_1(p,q)+1(p)} \end{pmatrix} \\
 &= H_m^T(p) \mathbb{M}_3 \mathbb{A}^T V \Psi(q) = \Psi^T(p) (\mathbb{A}^{-1})^T \mathbb{M}_3 \mathbb{A}^T V \Psi(q), \tag{29}
 \end{aligned}$$

where

$$\mathbb{M}_3 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \Gamma(2)p^{1-\alpha_1(p,q)}e_{\rho,2-\alpha_1(p,q)}^{-v}(p) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \Gamma(m+1)p^{m-\alpha_1(p,q)}e_{\rho,m-\alpha_1(p,q)+1}^{-v}(p) \end{pmatrix}, \quad (30)$$

$$\begin{aligned} {}^{C-P}\mathbb{D}_p^{\alpha_2(p,q)}\nu_m(p,q) &= {}^{C-P}\mathbb{D}_p^{\alpha_2(p,q)}[1,p,\dots,p^m]\mathbb{A}^T V\Psi(q) = \begin{bmatrix} 0, 0, \dots, \Gamma(m+1)p^{m-\alpha_1(p,q)}e_{\rho,m-\alpha_1(p,q)+1}^{-v}(p) \end{bmatrix} \mathbb{A}^T V\Psi(q) \\ &= [1,p,\dots,p^m] \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \Gamma(2)p^{1-\alpha_2(p,q)}e_{\rho,2-\alpha_2(p,q)}^{-v}(p) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \Gamma(m+1)p^{m-\alpha_2(p,q)}e_{\rho,m-\alpha_2(p,q)+1}^{-v}(p) \end{pmatrix} \\ &= H_m^T(p)\mathbb{M}_4\mathbb{A}^T V\Psi(q) = \Psi^T(p)(\mathbb{A}^{-1})^T \mathbb{M}_4\mathbb{A}^T V\Psi(q), \end{aligned} \quad (31)$$

where

$$\mathbb{M}_4 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \Gamma(2)p^{1-\alpha_2(p,q)}e_{\rho,2-\alpha_2(p,q)}^{-v}(p) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \Gamma(m+1)p^{m-\alpha_2(p,q)}e_{\rho,m-\alpha_2(p,q)+1}^{-v}(p) \end{pmatrix}. \quad (32)$$

3.1. Description of Convergence. Let $z(p,q), v(p,q)$ be the exact solutions of Equations (3) and $z_m(p,q), \nu_m(p,q)$ be the numerical solutions of Equations (3) which are introduced in (22)–(23), then we show that as $m \rightarrow \infty$, we have $\|z - z_m\|^2 \rightarrow 0$ and $\|v - \nu_m\|^2 \rightarrow 0$. For this purpose, we show that the sequences $z_m(p,q), \nu_m(p,q)$ are a Cauchy sequence in Hilbert space $L^2([a,b] \times [0,T])$. Then, for $\hat{m} > m$, we obtain

$$\begin{aligned} \|z_{\hat{m}} - z_m\|^2 &= \left\| \sum_{i=0}^{\hat{m}} \sum_{j=0}^{\hat{m}} z_{ij} \hat{T}_i(p) \hat{T}_j(q) - \sum_{i=0}^m \sum_{j=0}^m z_{ij} \hat{T}_i(p) \hat{T}_j(q) \right\|^2 \\ &= \left\| \sum_{i=m+1}^{\hat{m}} \sum_{j=m+1}^{\hat{m}} z_{ij} \hat{T}_i(p) \hat{T}_j(q) \right\|^2 \\ &= \left\langle \sum_{i=m+1}^{\hat{m}} \sum_{j=m+1}^{\hat{m}} z_{ij} \hat{T}_i(p) \hat{T}_j(q), \sum_{l=m+1}^{\hat{m}} \sum_{k=m+1}^{\hat{m}} z_{lk} \hat{T}_l(p) \hat{T}_k(q) \right\rangle \\ &= \sum_{i=m+1}^{\hat{m}} \sum_{j=m+1}^{\hat{m}} \sum_{l=m+1}^{\hat{m}} \sum_{k=m+1}^{\hat{m}} z_{ij} \bar{z}_{lk} \langle \hat{T}_i(p) \hat{T}_j(q), \hat{T}_l(p) \hat{T}_k(q) \rangle \\ &= \frac{\pi^2}{16} \sum_{i=m+1}^{\hat{m}} \sum_{j=m+1}^{\hat{m}} |z_{ij}|^2. \end{aligned} \quad (33)$$

From Bessel's inequality, we obtain that the series is presented in (33) as $m \rightarrow \infty$ and $\hat{m} \rightarrow \infty$ is convergent, then $\|z_{\hat{m}} - z_m\|^2 \rightarrow 0$. $\|z_{\hat{m}} - z_m\|^2 \rightarrow 0$ show that the sequence z_m is a Cauchy sequence in Hilbert space $L^2([a,b] \times [0,T])$ and it converges to say $z_1(p,q)$. Now, we show $z = z_1(p,q)$, then

$$\begin{aligned} &\langle z - z_1(p,q), \hat{T}_i(p) \hat{T}_j(q) \rangle \\ &= \langle z, \hat{T}_i(p) \hat{T}_j(q) \rangle - \langle z_1(p,q), \hat{T}_i(p) \hat{T}_j(q) \rangle \\ &= \lim_{m \rightarrow \infty} \langle z_{\hat{m}}, \hat{T}_i(p) \hat{T}_j(q) \rangle - z_{ij} = z_{ij} - z_{ij} = 0, \end{aligned} \quad (34)$$

where $z_{ij} = \int_0^T \int_a^b z(p,q) \hat{T}_i(p) \hat{T}_j(q) \omega^*(p) \omega^*(q) dp dq$. Then, $\sum_{i=0}^m \sum_{j=0}^m z_{ij} \hat{T}_i(p) \hat{T}_j(q)$ converges to $z(p,q), v(p,q)$ as $m \rightarrow \infty$. The same process can be considered for the sequence ν_m .

3.2. Error Analysis. Suppose that $z(p,q), v(p,q) \in C(\Omega)$, $\Omega = [0,1] \times [0,1]$ be the exact solutions, respectively. If $z_m(p,q) = \Lambda' \times \Lambda$, $\nu_m(p,q) = \Lambda' \times \Lambda$ be the best numerical solution to $z(p,q)$ that $\Lambda = \text{Span}\{\hat{T}_0(q), \hat{T}_1(q), \dots, \hat{T}_m(q)\}$,

$\Lambda' \in \text{Span}\{\hat{T}_0(p), \hat{T}_1(p), \dots, \hat{T}_m(p)\}$. We define $\mathcal{E}_m$ which is the error as follows:

$$\begin{aligned} & {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} z(p, q) - {}^{C-P}\mathbb{D}_p^{\alpha_1(p,q)} z(p, q) + \theta^2 \sin(z - v) - h_1(p, q) \\ &= {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} z_m(p, q) - {}^{C-P}\mathbb{D}_p^{\alpha_1(p,q)} z_m(p, q) + \theta^2 \sin(z_m - v_m) \\ &\quad - h_1(p, q) + \mathcal{E}_m. \end{aligned} \quad (35)$$

Then, from Eq. (35), we have:

$$\begin{aligned} \|\mathcal{E}_m\|_{L^2(\Omega)} &= \|{}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)}(z_m - z) - {}^{C-P}\mathbb{D}_p^{\alpha_1(p,q)}(z_m - z) \\ &\quad + \theta^2 [\sin(z_m - v_m) - \sin(z - v)]\|_{L^2(\Omega)} \\ &\leq \|{}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)}(z_m - z)\|_{L^2(\Omega)} + \|{}^{C-P}\mathbb{D}_p^{\alpha_1(p,q)}(z_m - z)\|_{L^2(\Omega)} \\ &\quad + \|\theta^2 [\sin(z_m - v_m) - \sin(z - v)]\|_{L^2(\Omega)} \\ &\leq \|{}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)}(z_m - z)\|_{L^2(\Omega)} + \|{}^{C-P}\mathbb{D}_p^{\alpha_1(p,q)}(z_m - z)\|_{L^2(\Omega)} \\ &\quad + \theta^2 \|\sin(z_m - v_m)\|_{L^2(\Omega)} + \theta^2 \|\sin(z - v)\|_{L^2(\Omega)}. \end{aligned} \quad (36)$$

To get Equation (36), we define

$$\begin{aligned} & \Delta_m^{\alpha(p,q), \alpha'(p,q)}(p, q) \\ &= \sum_{i=0}^m \sum_{j=0}^m \frac{p^{i\alpha(p,q)} q^{j\alpha'(p,q)}}{\Gamma(i\alpha(p, q) + 1) \Gamma(j\alpha'(p, q) + 1)} {}^{C-P}\mathbb{D}_p^{i\alpha(p,q)} \\ &\quad \cdot \left({}^{C-P}\mathbb{D}_q^{j\alpha'(p,q)} z(p, q) \right) \Big|_{(0,0)}. \end{aligned} \quad (37)$$

From the multivariable Taylor definition and the Generalized Taylor's formula in [65, 66], we have

$$\begin{aligned} & \left| z(p, q) - \Delta_m^{\alpha(p,q), \alpha'(p,q)}(p, q) \right| \\ & \leq \frac{p^{(m+1)\alpha(p,q)} q^{(m+1)\alpha'(p,q)}}{\Gamma((m+1)\alpha(p, q) + 1) \Gamma((m+1)\alpha'(p, q) + 1)} Y_m^{\alpha(p,q), \alpha'(p,q)}, \end{aligned} \quad (38)$$

where $Y_m^{\alpha(p,q), \alpha'(p,q)} = \sup_{(p,q) \in \Omega} \left| {}^{C-P}\mathbb{D}_p^{i\alpha(p,q)} ({}^{C-P}\mathbb{D}_q^{j\alpha'(p,q)} z(p, q)) \right|$. By using Lemma 4, we have

$$\begin{aligned} & \left| {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} z(p, q) - {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} \Delta_m^{\alpha(p,q), \alpha'(p,q)}(p, q) \right| \\ & \leq \frac{p^{(m+1)\alpha(p,q)} q^{(m+1)\alpha'(p,q) - \alpha_1(p,q)} e^{-v}}{\Gamma((m+1)\alpha(p, q) + 1)} \\ & \quad \times Y_m^{\alpha(p,q), \alpha'(p,q)}, \end{aligned}$$

$$\begin{aligned} & \left| {}^{C-P}\mathbb{D}_p^{\alpha_1(p,q)} z(p, q) - {}^{C-P}\mathbb{D}_p^{\alpha_1(p,q)} \Delta_m^{\alpha(p,q), \alpha'(p,q)}(p, q) \right| \\ & \leq \frac{q^{(m+1)\alpha'(p,q)} p^{(m+1)\alpha(p,q) - \alpha_1(p,q)} e^{-v}}{\Gamma((m+1)\alpha'(p, q) + 1)} \\ & \quad \times Y_m^{\alpha(p,q), \alpha'(p,q)}. \end{aligned} \quad (39)$$

Then,

$$\begin{aligned} \left\| {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)}(z_m - z) \right\|_{L^2(\Omega)}^2 &\leq \left\| {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} z(p, q) - {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} \Delta_m^{\alpha(p,q), \alpha'(p,q)}(p, q) \right\|_{L^2(\Omega)}^2 \\ &\leq \int_0^1 \int_0^1 \left(\frac{x^{(m+1)\alpha(p,q)} t^{(m+1)\alpha'(p,q) - \alpha_1(p,q)} e^{-v}}{\Gamma((m+1)\alpha(p, q) + 1)} Y_m^{\alpha(p,q), \alpha'(p,q)} \right)^2 dt dx. \end{aligned} \quad (40)$$

According to the definition $e_{\rho, (m+1)\alpha'(p,q) - \alpha_1(p,q) + 1}^{-v}(t)$, we have

$$e_{\rho, (m+1)\alpha'(p,q) - \alpha_1(p,q) + 1}^{-v}(t) = t^{(m+1)\alpha'(p,q) - \alpha_1(p,q)} E_{\rho, (m+1)\alpha'(p,q) - \alpha_1(p,q) + 1}^v(\omega t^\rho). \quad (41)$$

From [16, 67, 68], the Prabhakar function is convergent and the convergent boundary for its M is considered, then we obtain

$$\left\| {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)}(z_m - z) \right\|_{L^2(\Omega)}^2 \leq \frac{M^2 \left(Y_m^{\alpha(p,q), \alpha'(p,q)} \right)^2}{(2(m+1)\alpha(p, q) + 1) \left(4 \left((m+1)\alpha'(p, q) - \alpha_1(p, q) \right) + 1 \right) (\Gamma((m+1)\alpha(p, q) + 1))^2}, \quad (42)$$

also

$$\begin{aligned} \|^{C-P}\mathbb{D}_p^{\alpha_1(p,q)}(z_m - z)\|_{L^2(\Omega)}^2 &\leq \|^{C-P}\mathbb{D}_p^{\alpha_1(p,q)}z(p, q) - ^{C-P}\mathbb{D}_p^{\alpha_1(p,q)}\Delta_m^{\alpha(p,q)\kappa(p,q)}(p, q)\|_{L^2(\Omega)}^2 \\ &\leq \int_0^1 \int_0^1 \left(\frac{t^{(m+1)\alpha'(p,q)} x^{(m+1)\alpha'(p,q) - \alpha_1(p,q)} e^{-v}_{\rho, (m+1)\alpha'(p,q) - \alpha_1(p,q) + 1}(x)}{\Gamma((m+1)\alpha'(p, q) + 1)} Y_m^{\alpha(p,q)\kappa(p,q)} \right)^2 dt dx, \end{aligned} \quad (43)$$

from

$$|E_{\rho, (m+1)\alpha'(p,q) - \alpha_1(p,q) + 1}^v(\omega x^p)| \leq M_1, \quad (44)$$

$$e_{\rho, (m+1)\alpha'(p,q) - \alpha_1(p,q) + 1}^{-v}(x) = x^{(m+1)\alpha'(p,q) - \alpha_1(p,q)} E_{\rho, (m+1)\alpha'(p,q) - \alpha_1(p,q) + 1}^v(\omega x^p),$$

we have

$$\|^{C-P}\mathbb{D}_p^{\alpha_1(p,q)}(z_m - z)\|_{L^2(\Omega)}^2 \leq \frac{M_1^2 \left(Y_m^{\alpha(p,q)\kappa(p,q)} \right)^2}{\left(2(m+1)\alpha'(p, q) + 1 \right) \left(4 \left((m+1)\alpha'(p, q) - \alpha_1(p, q) \right) + 1 \right) \left(\Gamma((m+1)\alpha'(p, q) + 1) \right)^2}. \quad (45)$$

So by placing Equations (42) and (45) into (36), we have

$$\begin{aligned} \|\mathcal{E}_m\|_{L^2(\Omega)} &\leq \frac{MY_m^{\alpha(p,q)\kappa(p,q)}}{\sqrt{(2(m+1)\alpha(p, q) + 1) \left(4 \left((m+1)\alpha'(p, q) - \alpha_1(p, q) \right) + 1 \right) (\Gamma((m+1)\alpha(p, q) + 1))}} \\ &\quad + \frac{M_1 Y_m^{\alpha(p,q)\kappa(p,q)}}{\sqrt{(2(m+1)\alpha'(p, q) + 1) \left(4 \left((m+1)\alpha'(p, q) - \alpha_1(p, q) \right) + 1 \right) (\Gamma((m+1)\alpha'(p, q) + 1))}} + \theta^2(R + R_1), \end{aligned} \quad (46)$$

where $R = \sup_{(p,q) \in \Omega} |\sin(z - v)|$ and $R_1 = \sup_{(p,q) \in \Omega} |\sin(z_m - v_m)|$. With a similar process, the following inequality

can be considered for the function $\mathcal{E}'_m$ that is defined as follows:

$$\begin{aligned} &^{C-P}\mathbb{D}_q^{\alpha_2(p,q)}v(p, q) - \xi^2 ^{C-P}\mathbb{D}_p^{\alpha_2(p,q)}v(p, q) - \sin(z - v) - h_1(p, q) \\ &= ^{C-P}\mathbb{D}_q^{\alpha_2(p,q)}v_m(p, q) - \xi^2 ^{C-P}\mathbb{D}_p^{\alpha_2(p,q)}v_m(p, q) - \sin(z_m - v_m) - h_1(p, q) + \mathcal{E}'_m, \end{aligned}$$

$$\begin{aligned} \|\mathcal{E}'_m\|_{L^2(\Omega)} &\leq \frac{MY_m^{\alpha(p,q)\kappa(p,q)}}{\sqrt{(2(m+1)\alpha(p, q) + 1) \left(4 \left((m+1)\alpha'(p, q) - \alpha_2(p, q) \right) + 1 \right) (\Gamma((m+1)\alpha(p, q) + 1))}} \\ &\quad + \xi^2 \frac{M_1 Y_m^{\alpha(p,q)\kappa(p,q)}}{\sqrt{(2(m+1)\alpha'(p, q) + 1) \left(4 \left((m+1)\alpha'(p, q) - \alpha_2(p, q) \right) + 1 \right) (\Gamma((m+1)\alpha'(p, q) + 1))}} + R + R_1. \end{aligned} \quad (47)$$

FIGURE 1: Numerical solution of $z(p, q)$, $v(p, q)$ with $\rho = \omega = v = 1$ for Example 7 when $m = 6$.

4. Description of the Suggested Method

This section describes the suggested method to obtain solutions of the equations considered in (3) with the conditions (6) and (8). Then, we put relations (26)–(30) into Equation (3), and the following results are obtained:

$$\begin{aligned} & \Psi^T(p) Z A M_1 A^{-1} \Psi(q) - \Psi^T(p) (A^{-1})^T M_3 A^T V \Psi(q) \\ & + \theta^2 \Psi^T(p) \sin(Z - V) \Psi(q) - h_1(p, q) \approx 0, \end{aligned} \quad (48)$$

$$\Psi^T(p) \left[Z A M_1 A^{-1} - (A^{-1})^T M_3 A^T V + \theta^2 \sin(Z - V) \right] \Psi(q) - h_1(p, q) \approx 0, \quad (49)$$

$$\begin{aligned} & \Psi^T(p) V A M_2 A^{-1} \Psi(q) - \xi^2 \Psi^T(p) (A^{-1})^T M_4 A^T V \Psi(q) \\ & - \Psi^T(p) \sin(Z - V) \Psi(q) - h_2(p, q) \\ & \approx 0, \Psi^T(p) \left[V A M_2 A^{-1} - \xi^2 (A^{-1})^T M_4 A^T V - \sin(Z - V) \right] \Psi(q) \\ & \times -h_2(p, q) \approx 0, \end{aligned} \quad (50)$$

FIGURE 2: Charts of absolute error z (a) and v (b) with $\rho = \omega = v = 1$ for Example 7 when $m = 6$.

where $\sin(z - v) \approx \sin(z_m - v_m) = \sum_{i=0}^m \sum_{j=0}^m \sin(z_{ij} - v_{ij})$
 $\hat{T}_i(p) \hat{T}_j(q) = \Psi^T(p) \sin(Z - V) \Psi(q)$. Considering the formulas $l_k(p) \approx \Psi^T(p) L_k$, $k = 0, 1, 2, 3$ and $f_k(q) \approx F_k^T \Psi(q)$, $k = 0, 1, 2, 3$, F_k, L_k are known. Then, by substituting functions $l_k(p), f_k(q)$, $k = 0, 1, 2, 3$, and Equations (22) and (23) in the conditions introduced in Equations (6) and (4), we obtain

$$\Psi^T(p) Z \Psi(0) - \Psi^T(p) L_0 \approx 0, \Psi^T(p) Z \mathbb{A} \mathbb{M}_1 \mathbb{A}^{-1} \Psi(0) - \Psi^T(p) L_1 \approx 0, \quad (51)$$

$$\Psi^T(p) V \Psi(0) - \Psi^T(p) L_2 \approx 0, \Psi^T(p) V \mathbb{A} \mathbb{M}_2 \mathbb{A}^{-1} \Psi(0) - \Psi^T(p) L_3 \approx 0, \quad (52)$$

$$\Psi^T(a) Z \Psi(q) - F_0^T \Psi(q) \approx 0, \Psi^T(b) Z \Psi(q) - F_1^T \Psi(q) \approx 0, \quad (53)$$

$$\Psi^T(a) V \Psi(q) - F_2^T \Psi(q) \approx 0, \Psi^T(b) V \Psi(q) - F_3^T \Psi(q) \approx 0. \quad (54)$$

To find solutions of problems (3) which are z, v , we solve the Equations (49)–(50) corresponding to (51)–(54) in collocation points to obtain functions Z, V .

5. Numerical Examples

In this section, we show the numerical results of the suggested method on four examples to illustrate the efficiency and accuracy of our scheme. Here, obtained numerical results of some test problems are shown to illustrate the computational efficiency and applicability of the reported method. All numerical examples are done by applying

TABLE 1: The absolute error for different values m for Example 7 at various points (p, q) .

(p, q)	$m = 6$	$m = 7$	$ z_m - z $	$m = 8$	$m = 9$
(0.1,0.1)	$8.9329e-17$	$6.2530e-16$	$7.1463e-16$	$8.0396e-16$	$1.4476e-15$
(0.2,0.2)	$1.6084e-16$	$1.1259e-15$	$1.2867e-15$	$1.4476e-15$	$1.4476e-15$
(0.3,0.3)	$2.1649e-16$	$1.5154e-15$	$1.7319e-15$	$1.9484e-15$	$1.9484e-15$
(0.4,0.4)	$2.5813e-16$	$1.8069e-15$	$2.0650e-15$	$2.3232e-15$	$2.3232e-15$
(0.5,0.5)	$2.8755e-16$	$2.0128e-15$	$2.3004e-15$	$2.5880e-15$	$2.5880e-15$
(0.6,0.6)	$3.0643e-16$	$2.1450e-15$	$2.4515e-15$	$2.7579e-15$	$2.7579e-15$
(0.7,0.7)	$3.1635e-16$	$2.2144e-15$	$2.5308e-15$	$2.8471e-15$	$2.8471e-15$
(0.8,0.8)	$3.1874e-16$	$2.2312e-15$	$2.5499e-15$	$2.8687e-15$	$2.8687e-15$
(0.9,0.9)	$3.1493e-16$	$2.2045e-15$	$2.5195e-15$	$2.8344e-15$	$2.8344e-15$
(1, 1)	$3.0612e-16$	$2.1428e-15$	$2.4489e-15$	$2.7550e-15$	$2.7550e-15$

MATLAB (R2020b) software. Also, the convergence order (C-order) of the depicted method is calculated by the following relation:

$$\text{C-order} = \frac{|\log(E_1/E_2)|}{\log((m_2 + 1)(m_2 + 1)/(m_1 + 1)(m_1 + 1))}. \quad (55)$$

Here, E_1 and E_2 show the first and second values of the maximum absolute error (MAE) provided by the proposed method, respectively. Moreover, $(m_i + 1)(m_i + 1)$ for $i = 1, 2$ are the number of the basis functions used in the first and second implementations, respectively.

TABLE 2: The absolute error for different values m for Example 7 at various points (p, q) .

(p, q)	$m = 6$	$m = 7$	$ v_m - v $	$m = 8$	$m = 9$
(0.1,0.1)	$8.9031e - 16$	$6.2322e - 15$		$7.1225e - 15$	$8.0128e - 15$
(0.2,0.2)	$7.9349e - 16$	$5.5544e - 15$		$6.3479e - 15$	$7.1414e - 15$
(0.3,0.3)	$6.9986e - 16$	$4.8990e - 15$		$5.5989e - 15$	$6.2988e - 15$
(0.4,0.4)	$6.1054e - 16$	$4.2738e - 15$		$4.8843e - 15$	$5.4949e - 15$
(0.5,0.5)	$5.2636e - 16$	$3.6845e - 15$		$4.2109e - 15$	$4.7373e - 15$
(0.6,0.6)	$4.4792e - 16$	$3.1354e - 15$		$3.5833e - 15$	$4.0312e - 15$
(0.7,0.7)	$3.7558e - 16$	$2.6291e - 15$		$3.0047e - 15$	$3.3803e - 15$
(0.8,0.8)	$3.09572e - 16$	$2.1670e - 15$		$2.4765e - 15$	$2.7861e - 15$
(0.9,0.9)	$2.4991e - 16$	$1.7494e - 15$		$1.9993e - 15$	$2.2492e - 15$
(1, 1)	$1.9655e - 16$	$1.3759e - 15$		$1.5724e - 15$	$1.7690e - 15$

Example 7. We consider the following coupled nonlinear variable-order time-fractional sine-Gordon equations:

$$\begin{aligned}
 {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} z(p, q) - {}^{C-P}\mathbb{D}_p^{\alpha_1(p,q)} z(p, q) + \frac{1}{4} \sin(z - v) &= h_1(p, q), \\
 {}^{C-P}\mathbb{D}_q^{\alpha_2(p,q)} v(p, q) - {}^{C-P}\mathbb{D}_p^{\alpha_2(p,q)} v(p, q) - \sin(z - v) &= h_2(p, q),
 \end{aligned} \quad (56)$$

where

$$\begin{aligned}
 h_1(p, q) &= \left(q^{2-\alpha_1(p,q)} \mathbb{E}_{1,3-\alpha_1(p,q)}(-q) + e^{-q} \right) \sin(p) \\
 &\quad + \frac{1}{4} \sin(e^{-q}(\sin(p) - \cos(p))), \\
 h_2(p, q) &= \left(q^{2-\alpha_2(p,q)} \mathbb{E}_{1,3-\alpha_2(p,q)}(-q) + e^{-q} \right) \cos(p) \\
 &\quad - \sin(e^{-q}(\sin(p) - \cos(p))),
 \end{aligned} \quad (57)$$

where $(p, q) \in [0, 1] \times [0, 1]$ and $\mathbb{E}_{\zeta, \eta}(q) = \sum_{k=0}^{\infty} (q^k / \Gamma(k\zeta + \eta))$. For this example, the exact solutions to the case where $v = 0$, $\alpha_1(p, q) = \alpha_2(p, q) = 2$ are $z(p, q) = \sin(p)e^{-q}$, $v(p, q) = \cos(p)e^{-q}$. The numerical results obtained using the proposed method for $\alpha_1(p, q) = 1.5 + 0.25 \sin(pq)$, $\alpha_2(p, q) = 1.75 + 0.25 \sin(pq)$ are shown in Figure 1. An image of the absolute error for value $m = 6$ is shown in Figure 2, and numerical results for absolute error for different values m are shown in Tables 1 and 2. Table 3 shows comparison between the maximum absolute error (MAE) given for different values of m based on the present and the Chebyshev cardinal function (CCF) technique [69]. In other words, the comparison between our numerical method result and the numerical method result in [69] is done using two different operators, i.e., the Caputo-Prabhakar derivative and the Caputo derivative. From the obtained numerical results in Table 3, we see that the proposed method is an effective and good tool for solving this problem.

TABLE 3: Comparison of the numerical results for Example 7.

m	$\alpha_1(p, q) = 1.5 + 0.25 \sin(pq), \alpha_2(p, q) = 1.75 + 0.25 \sin(pq)$	
	MAE(z)	MAE(v)
	Our method	
6	$2.1048e - 16$	$1.328e - 16$
7	$4.2097e - 18$	$2.6561e - 17$
8	$6.3146e - 19$	$3.9842e - 19$
9	$6.3146e - 21$	$3.9842e - 20$
	MA(u)	MA(w)
	Second-kind CCF method [69]	
6	$1.4597e - 07$	$3.4301e - 07$
7	$2.7615e - 09$	$1.2573e - 08$
8	$1.5434e - 10$	$5.3747e - 10$
9	$4.0590e - 12$	$2.0088e - 11$

Example 8. We consider the following coupled nonlinear variable-order time-fractional sine-Gordon equations:

$$\begin{aligned}
 {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} z(p, q) - {}^{C-P}\mathbb{D}_p^{\alpha_1(p,q)} z(p, q) + \sin(z - v) &= h_1(p, q), \\
 {}^{C-P}\mathbb{D}_q^{\alpha_2(p,q)} v(p, q) - {}^{C-P}\mathbb{D}_p^{\alpha_2(p,q)} v(p, q) - \sin(z - v) &= h_2(p, q),
 \end{aligned} \quad (58)$$

where

$$\begin{aligned}
 h_1(p, q) &= - \left(q^{3-\alpha_1(p,q)} \mathbb{E}_{2,4-\alpha_1(p,q)}(-q^2) + \sin(q) \right) \cosh(p) \\
 &\quad + \sin(\sin(q)(\cosh(p) - \sinh(p))), \\
 h_2(p, q) &= - \left(q^{3-\alpha_2(p,q)} \mathbb{E}_{2,4-\alpha_2(p,q)}(-q^2) + \sin(q) \right) \sinh(p) \\
 &\quad - \sin(\sin(q)(\cosh(p) - \sinh(p))),
 \end{aligned} \quad (59)$$

FIGURE 3: Numerical solution of $z(p, q)$, $v(p, q)$ with $\rho = \omega = v = 1$ for Example 8 when $m = 6$.

where $(p, q) \in [0, 1] \times [0, 2]$. For this example, the exact solutions to the case where $v = 0$, $\alpha_1(p, q) = \alpha_2(p, q) = 2$ are $z(p, q) = \sin(q) \cosh(p)$, $v(p, q) = \sin(q) \sinh(p)$. The numerical results obtained using the proposed method for $\alpha_1(p, q) = 1.5 + 0.25p^2q$, $\alpha_2(p, q) = 1.75 + 0.25 \sin(pq)$ are shown in Figure 3. An image of the absolute error for value $m = 6$ is shown in Figure 4, and numerical results for the absolute error for different values m are shown in Tables 4 and 5. The values of the MAE, C-order, and CPU time of the presented method are listed in Table 6.

Example 9. We consider the following coupled nonlinear variable-order time-fractional sine-Gordon equations:

$$\begin{aligned}
 {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} z(p, q) - {}^{C-P}\mathbb{D}_p^{\alpha_1(p,q)} z(p, q) + \sin(z - v) &= h_1(p, q), \\
 {}^{C-P}\mathbb{D}_q^{\alpha_2(p,q)} v(p, q) - \frac{1}{4} {}^{C-P}\mathbb{D}_p^{\alpha_2(p,q)} v(p, q) - \sin(z - v) &= h_2(p, q),
 \end{aligned} \tag{60}$$

FIGURE 4: Charts of absolute error z (a) and v (b) with $\rho = \omega = v = 1$ for Example 8 when $m = 6$.

TABLE 4: The absolute error for different values m for Example 8 at various points (p, q) .

(p, q)	$m = 6$	$m = 7$	$ z_m - z $	$m = 8$	$m = 9$
(0.2,0.2)	$1.9744e-16$	$1.3821e-15$	$1.5795e-15$	$1.7770e-15$	$1.7770e-15$
(0.4,0.4)	$3.9281e-16$	$2.7497e-15$	$3.1425e-15$	$3.5353e-15$	$3.5353e-15$
(0.6,0.6)	$5.8368e-16$	$4.0857e-15$	$4.6694e-15$	$5.2531e-15$	$5.2531e-15$
(0.8,0.8)	$7.6689e-16$	$5.3682e-15$	$6.1351e-15$	$6.9020e-15$	$6.9020e-15$
(1, 1)	$9.3832e-16$	$6.5682e-15$	$7.5065e-15$	$8.4449e-15$	$8.4449e-15$
(0.2,1.2)	$1.0926e-15$	$7.6483e-15$	$8.7409e-15$	$9.8336e-15$	$9.8336e-15$
(0.4,1.4)	$1.2231e-15$	$8.5621e-15$	$9.7853e-15$	$1.1008e-14$	$1.1008e-14$
(0.6,1.6)	$1.3220e-15$	$9.2540e-15$	$1.0576e-14$	$1.1898e-14$	$1.1898e-14$
(0.8,1.8)	$1.3801e-15$	$9.6607e-15$	$1.1040e-14$	$1.2420e-14$	$1.2420e-14$
(1, 2)	$1.3875e-15$	$9.7127e-15$	$1.1100e-14$	$1.2487e-14$	$1.2487e-14$

TABLE 5: The absolute error for different values m for Example 8 at various points (p, q) .

(p, q)	$m = 6$	$m = 7$	$ v_m - v $	$m = 8$	$m = 9$
(0.2,0.2)	$1.9678e-17$	$1.3775e-16$	$1.5743e-16$	$1.7711e-16$	$1.7711e-16$
(0.4,0.4)	$7.7532e-17$	$5.4272e-16$	$6.2026e-16$	$6.9779e-16$	$6.9779e-16$
(0.6,0.6)	$1.7003e-16$	$1.1902e-15$	$1.3602e-15$	$1.5303e-15$	$1.5303e-15$
(0.8,0.8)	$2.9138e-16$	$2.0396e-15$	$2.3310e-15$	$2.6224e-15$	$2.6224e-15$
(1, 1)	$4.3361e-16$	$3.0353e-15$	$3.4689e-15$	$3.9025e-15$	$3.9025e-15$
(0.2,1.2)	$5.8679e-16$	$4.1075e-15$	$4.6943e-15$	$5.2811e-15$	$5.2811e-15$
(0.4,1.4)	$7.3924e-16$	$5.1746e-15$	$5.9139e-15$	$6.6531e-15$	$6.6531e-15$
(0.6,1.6)	$8.7786e-16$	$6.1450e-15$	$7.0229e-15$	$7.9007e-15$	$7.9007e-15$
(0.8,1.8)	$9.8856e-16$	$6.9199e-15$	$7.9085e-15$	$8.8970e-15$	$8.8970e-15$
(1, 2)	$1.0567e-15$	$7.3971e-15$	$8.4538e-15$	$9.5106e-15$	$9.5106e-15$

where

$$\begin{aligned}
 h_1(p, q) &= p^3(p-2)q^{2-\alpha_1(p,q)}E_{2,3-\alpha_1(p,q)}(-q^2) + 6p(2p-2)\cos(q) \\
 &\quad + \sin(p^3(2-p)(\cos(q) - \sin(q))), \\
 h_2(p, q) &= p^3(p-2)q^{3-\alpha_2(p,q)}E_{2,4-\alpha_2(p,q)}(-q^2) \\
 &\quad + \frac{3}{2}p(2p-2)\sin(q) \\
 &\quad - \sin(p^3(2-p)(\cos(q) - \sin(q))),
 \end{aligned} \tag{61}$$

TABLE 6: The MA-error, C-order, and CPU time of the presented method in Example 8.

m	$z(p, q)$		$v(p, q)$		CPU-time (s)
	MAE	C-order	MAE	C-order	
6	$1.1801e-15$	—	$1.0547e-15$	—	6.15
7	$7.5127e-16$	5.7033	$7.3971e-16$	07.6726	18.14
8	$1.0100e-16$	15.0260	$6.4538e-16$	11.3146	44.13
9	$1.0487e-17$	16.9570	$9.5106e-17$	11.4025	87.14

FIGURE 5: Numerical solution of $z(p, q)$, $v(p, q)$ with $\rho = \omega = v = 1$ for Example 9 when $m = 6$.

FIGURE 6: Charts of absolute error z (a) and v (b) with $\rho = \omega = \nu = 1$ for Example 9 when $m = 6$.TABLE 7: The absolute error for different values m for Example 9 at various points (p, q) .

(p, q)	$ z_m - z $			
	$m = 6$	$m = 7$	$m = 8$	$m = 9$
(0.2,0.2)	$1.4168e-17$	$9.9182e-17$	$1.1335e-16$	$1.2751e-16$
(0.4,0.4)	$9.9243e-17$	$6.9470e-16$	$7.9395e-16$	$8.9319e-16$
(0.6,0.6)	$2.8568e-16$	$1.9997e-15$	$2.2854e-15$	$2.5711e-15$
(0.8,0.8)	$5.5961e-16$	$3.9172e-15$	$4.4769e-15$	$5.0365e-15$
(1, 1)	$8.6783e-16$	$6.0748e-15$	$6.9426e-15$	$7.8104e-15$
(1.2,0.2)	$1.1282e-15$	$7.8978e-15$	$9.0261e-15$	$1.0154e-14$
(1.4,0.4)	$1.2452e-15$	$8.7167e-15$	$9.9619e-15$	$1.1207e-14$
(1.6,0.6)	$1.1288e-15$	$7.9016e-15$	$9.0304e-15$	$1.0159e-14$
(1.8,0.8)	$7.1699e-16$	$5.0189e-15$	$5.7359e-15$	$6.4529e-15$

TABLE 8: The absolute error for different values m for Example 9 at various points (p, q) .

(p, q)	$ v_m - v $			
	$m = 6$	$m = 7$	$m = 8$	$m = 9$
(0.2,0.2)	$1.4216e-18$	$9.9514e-18$	$1.1373e-17$	$1.2794e-17$
(0.4,0.4)	$2.0117e-17$	$1.4082e-16$	$1.6094e-16$	$1.8105e-16$
(0.6,0.6)	$8.8372e-17$	$6.1860e-16$	$7.0697e-16$	$7.9535e-16$
(0.8,0.8)	$2.3660e-16$	$1.6562e-15$	$1.8928e-15$	$2.1294e-15$
(1, 1)	$4.7409e-16$	$3.3186e-15$	$3.7927e-15$	$4.2668e-15$
(1.2,0.2)	$7.7188e-16$	$5.4032e-15$	$6.1751e-15$	$6.9470e-15$
(1.4,0.4)	$1.0488e-15$	$7.3419e-15$	$8.3908e-15$	$9.4397e-15$
(1.6,0.6)	$1.1622e-15$	$8.1358e-15$	$9.2980e-15$	$1.0460e-14$
(1.8,0.8)	$9.0352e-16$	$6.3246e-15$	$7.2281e-15$	$8.1316e-15$

TABLE 9: The MAE, C-order, and CPU time of the presented method in Example 9.

m	$z(p, q)$		$v(p, q)$		CPU-time (s)
	MAE	C-order	MAE	C-order	
6	$7.2352e-15$	—	$1.1522e-15$	—	8.17
7	$4.7067e-15$	2.7023	$8.1258e-16$	6.6706	20.04
8	$1.9519e-16$	12.0250	$7.2880e-16$	10.3156	46.10
9	$6.8004e-17$	13.9560	$9.4297e-17$	10.4015	89.04

where $(p, q) \in [0, 2] \times [0, 1]$. For this example, the exact solutions to the case where $v = 0$, $\alpha_1(p, q) = \alpha_2(p, q) = 2$ are $z(p, q) = p^3(2 - p) \cos(q)$, $v(p, q) = p^3(2 - p) \sin(q)$. The numerical results obtained using the proposed method for $\alpha_1(p, q) = 1.5 + 0.25 \sin(pq)$, $\alpha_2(p, q) = 1.75 + 0.25 \sin(pq)$ are shown in Figure 5. An image of the absolute error for value $m = 6$ is shown in Figure 6, and numerical results for absolute error for different values m are shown in Tables 7 and 8. The values of the MAE, C-order, and CPU time of the presented method are listed in Table 9.

Example 10. We consider the following coupled nonlinear variable-order time-fractional sine-Gordon equations:

$$\begin{aligned}
 {}^{C-P}\mathbb{D}_q^{\alpha_1(p,q)} z(p, q) - {}^{C-P}\mathbb{D}_p^{\alpha_1(p,q)} z(p, q) + 4 \sin(z - v) &= h_1(p, q), \\
 {}^{C-P}\mathbb{D}_q^{\alpha_2(p,q)} v(p, q) - {}^{C-P}\mathbb{D}_p^{\alpha_2(p,q)} v(p, q) - \sin(z - v) &= h_2(p, q),
 \end{aligned} \tag{62}$$

FIGURE 7: Numerical solution of $z(p, q)$, $v(p, q)$ with $\rho = \omega = v = 1$ for Example 10 when $m = 6$.

where

$$\begin{aligned}
 h_1(p, q) &= \frac{\Gamma(7/2)}{\Gamma(7/2) - \alpha_1(p, q)} p^2 q^{5/2 - \alpha_1(p, q)} - 2q^{5/2} \\
 &\quad + 4 \sin((p^2 - p^3)q^{5/2}), \\
 h_2(p, q) &= \frac{\Gamma(7/2)}{\Gamma(7/2) - \alpha_2(p, q)} p^3 q^{5/2 - \alpha_2(p, q)} - 54pq^{5/2} \\
 &\quad - \sin((p^2 - p^3)q^{5/2}),
 \end{aligned} \tag{63}$$

where $(p, q) \in [0, 1] \times [0, 1]$. For this example, the exact solutions to the case where $v = 0$, $\alpha_1(p, q) = \alpha_2(p, q) = 2$ are $z(p, q) = p^2 q^{5/2}$, $v(p, q) = p^3 q^{5/2}$. The numerical results obtained using the proposed method for $\alpha_1(p, q) = 1.65 + 0.25 \sin(pq)$, $\alpha_2(p, q) = 1.55 + 0.35 \sin(pq)$ are shown in Figure 7. An image of the absolute error for value $m = 6$ is shown in Figure 8, and numerical results for absolute error for different values m are shown in Tables 10–12. Table 12 shows the comparison between the maximum absolute error (MAE) given for different values of m based on the present and the Chebyshev cardinal function (CCF)

FIGURE 8: Charts of absolute error z (a) and v (b) with $\rho = \omega = \nu = 1$ for Example 10 when $m = 6$.

TABLE 10: The absolute error for different values m for Example 10 at various points (p, q) .

(p, q)	$m = 6$	$m = 7$	$ z_m - z $	$m = 8$	$m = 9$
(0.1,0.1)	$3.1271e-20$	$2.1890e-19$		$2.50179e-19$	$2.8144e-19$
(0.3,0.3)	$4.3872e-18$	$3.0710e-17$		$3.5098e-17$	$3.9485e-17$
(0.5,0.5)	$4.3703e-17$	$3.0592e-16$		$3.4962e-16$	$3.9332e-16$
(0.7,0.7)	$1.9865e-16$	$1.3905e-15$		$1.5892e-15$	$1.7878e-15$
(0.9,0.9)	$6.1551e-16$	$4.3086e-15$		$4.9241e-15$	$5.5396e-15$

TABLE 11: The absolute error for different values m for Example 10 at various points (p, q) .

(p, q)	$m = 6$	$ v_m - v $	$m = 7$	$m = 8$	$m = 9$
(0.1,0.1)	$3.1271e-21$	$2.1890e-20$	$2.5017e-20$	$2.8144e-20$	
(0.3,0.3)	$1.3161e-18$	$9.2132e-18$	$1.0529e-17$	$1.1845e-17$	
(0.5,0.5)	$2.1851e-17$	$1.5296e-16$	$1.7481e-16$	$1.9666e-16$	
(0.7,0.7)	$1.3905e-16$	$9.7338e-16$	$1.1124e-15$	$1.2514e-15$	
(0.9,0.9)	$5.5396e-16$	$3.8777e-15$	$4.4317e-15$	$4.9856e-15$	

technique [69]. In other words, the comparison between our numerical method result and the numerical method result in [69] is done with using two different operators, i.e., the Caputo-Prabhakar derivative and the Caputo derivative.

TABLE 12: Comparison of the numerical results for Example 10.

m	$\alpha_1(p, q) = 1.65 + 0.25 \sin(pq), \alpha_2(p, q) = 1.55 + 0.35 \sin(pq)$	
	MAE(z)	MAE(v)
	Our method	
6	$3.1131e-18$	$2.1440e-19$
7	$1.3161e-18$	$7.2122e-20$
8	$2.1941e-19$	$1.4596e-20$
9	$1.2605e-19$	$8.5238e-21$
	MAE(u)	MAE(w)
	Second-kind CCF method [69]	
6	$6.5157e-04$	$1.7826e-05$
7	$4.1287e-04$	$9.8284e-06$
8	$2.7169e-04$	$5.8208e-06$
9	$2.0393e-04$	$2.9727e-06$

6. Conclusion

This paper is focused on a numerical method based on the orthogonal polynomials known as the shifted Chebyshev polynomials of degree m . The fractional sine-Gordon equations of variable order which includes a fractional derivative in terms of the Caputo-Prabhakar sense in this paper are studied. Using this numerical method, we obtain the numerical solutions of this type of equations. In this article, the convergence and error analysis for numerical solutions are studied. Some numerical examples are shown for the application of numerical methods.

Data Availability

No data were used to support the study.

Conflicts of Interest

The author declares that he has no conflict of interest.

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