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Research Article

On the Solvability of a Resonant Third-Order Integral *m*-Point Boundary Value Problem on the Half-Line

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In this work, the existence of at least one solution for the following third-order integral and m-point boundary value problem on the half-line at resonance $(\rho(t)u'(t))'' = w(t, u(t), u'(t), u''(t)), t \in [0, \infty), u(0) = \sum_{j=1}^{m} \alpha_j \int_0^{\eta_j} u(t)dt, u'(0) = 0, \lim_{t \longrightarrow \infty} (\rho(t)u'(t))' = 0$, will be investigated. The Mawhin's coincidence degree theory will be used to obtain existence results while an example will be used to validate the result obatined.

1. Introduction

This work studies the existence of solution for a resonant third-order boundary value problem with integral and m -point boundary conditions on the half-line

$$\left(\rho(t)u'(t)\right)'' = w\left(t, u(t), u'(t), u''(t)\right), \quad t \in [0, \infty), \quad (1)$$

$$u(0) = \sum_{j=1}^{m} \alpha_j \int_0^{\eta_j} u(t)dt, u'(0) = 0, \lim_{t \longrightarrow \infty} \left(\rho(t)u'(t)\right)' = 0, \quad (2)$$

where $w : [0,\infty) \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is an S-Carathéodory function, $\alpha_i \in \mathbb{R}$, $i = 1, 2, \cdots, m$, $0 < \eta_1 < \cdots < \eta_m < 1$, $\rho \in C[0, \infty) \cap C^2(0, \infty)$, $\rho(t) > 0$ for $t \in [0, \infty)$, $1/\rho \in L^1[0, \infty)$ and the resonance condition is $\sum_{j=1}^{m} \alpha_j \eta_j = 1$.

Boundary value problems on the half-line arise in the modeling of various physical processes like the flow of fluid over semi-infinite porous media and wet surfaces [1].

Boundary value problem (1)-(2) is said to be at resonance since the corresponding homogeneous problem

$$\begin{gathered}\left(\rho(t)u'(t)\right)'' = 0, \quad t \in [0, \infty), \\ u(0) = \sum_{j=1}^{m} \alpha_j \int_0^{\eta_j} u(t)dt, u'(0) = 0, \lim_{t \longrightarrow \infty} \left(\rho(t)u'(t)\right)' = 0,\end{gathered} \quad (3)$$

has a nontrivial solution $u(t) = C$, where C is a constant. Boundary value problems at resonance can be expressed in the abstract form as $Lu = Nu$, where L is a linear differential operator that is not invertible. Mawhin's coincidence degree theory [2] is an excellent tool for studying resonant problems of this type.

The problem of existence of solutions for resonant boundary value problems has received the attention of many authors recently, both in the bounded domain and on the half-line. For instance, the authors in [3] studied the third-order problem

$$\begin{gathered}u'''(t) = f\left(t, u(t), u'(t), u''(t)\right), t \in [0, 1], \\ u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), u'(0) = 0, u(1) = \beta u(\eta)\end{gathered} \quad (4)$$

under the resonance conditions: $\beta=1, \sum_{i=1}^{m-2}\alpha_i=1, \sum_{i=1}^{m-2}\alpha_i \xi_i^2=1$ and $\beta=1/\eta^2, \sum_{i=1}^{m-2}\alpha_i=1, \sum_{i=1}^{m-2}\alpha_i\xi_i^2=1$. They applied the coincidence degree arguments to obtain existence results in a bounded domain. For other works on a bounded domain, see [4–7].

In [8], the authors considered the multipoint boundary value problem on the half-line

$$\begin{gathered} u''(t)=f\left(t,u(t),u'(t)\right), t\in[0,\infty), \\ u(0)=0, u'(\infty)=\sum_{i=1}^{m-1}\alpha_i u'(\xi_i), \end{gathered} \tag{5}$$

where $\alpha_i>0$ and $0=\xi_1<\cdots<\xi_{m-1}<\infty$. They applied a perturbation technique in obtaining existence results under the resonant condition $\sum_{i=1}^{m-1}\alpha_i=1$.

Iyase [9] used coincidence degree arguments to study existence of solutions for the multipoint boundary value problem at resonance on the half-line

$$\begin{gathered} \left(q(t)u''(t)\right)'=g\left(t,u(t),u'(t),u''(t)\right), \quad (0,\infty), \\ u'(0)=\sum_{i=1}^{m-1}\alpha_i\int_0^{\xi_i}u(t)dt, \quad u(0)=0, \lim_{t\longrightarrow\infty}q(t)u''(t)=0 \end{gathered} \tag{6}$$

under the resonant condition $\sum_{i=1}^{m-1}\alpha_i=2$ which is different from ours. For other literature on resonant multipoint boundary value problems on the half-line, see [10–14]. Motivated by the results mentioned above, we study the existence of solutions for the resonant third-order boundary value problem with integral and m-point boundary conditions on the half-line. In Section 2 of this work, necessary lemmas, theorems, and definitions will be given; Section 3 will be dedicated to stating and proving the condition for existence of solutions. An example will be given in Section 4 to corroborate the result obtained.

2. Preliminaries

In this section, we will give definitions, theorems, and lemmas that are required for this work.

Definition 1 (see [6]). Let U and Z be Banach spaces. A linear operator $L:\operatorname{dom} L\subset U\longrightarrow Z$ is called a Fredholm mapping of index zero if $\ker L$ and $Z/\operatorname{Im} L$ are finite dimensional.

Take U, Z to be normed spaces, $L:\operatorname{dom} L\subset U\longrightarrow Z$ a Fredholm mapping of zero index and $P:U\longrightarrow U$, $Q:Z\longrightarrow Z$ projectors that are continuous such that $\operatorname{Im} P=\ker L$, $\ker Q=\operatorname{Im} L$, $\operatorname{Im} Q=\ker L$ and $U=\ker L\oplus\ker P$, $Z=\operatorname{Im} L\oplus\operatorname{Im} Q$, then $L|_{\operatorname{dom} L\cap\ker P}:\operatorname{dom} L\cap\ker P\longrightarrow\operatorname{Im} L$ is invertible. The inverse of the mapping L will be denoted by $K_p:\operatorname{Im} L\longrightarrow\operatorname{dom} L\cap\ker P$ while the generalized inverse, $K_{P,Q}:Z\longrightarrow\operatorname{dom} L\cap\ker P$ is defined as $K_{P,Q}=K_p(I-Q)$.

Definition 2 (see [13]). A map $w:[0,+\infty)\times\mathbb{R}^3\longrightarrow\mathbb{R}$ is S-Carathéodory, if the following conditions are satisfied:

(i) For each $(d,e,f)\in\mathbb{R}^3$, the mapping $t\longrightarrow q(t,d,e,f)$ is Lebesgue measurable

(ii) For a.e. $t\in[0,\infty)$, the mapping $(d,e,f)\longrightarrow w(t,d,e,f)$ is continuous on $\mathbb{R}^3$

(iii) For each $k>0$, there exists $\varphi_k(t)$, with $t\varphi_k(t)\in L_1[0,+\infty)$ such that, for a.e. $t\in[0,\infty)$ and every $d,e,f\in[-k,k]$, we have

$$|w(t,d,e,f)|\le\varphi_k(t). \tag{7}$$

Definition 3. Let $L:\operatorname{dom} L\subset U\longrightarrow Z$ be a Fredholm mapping, E a metric space, and $N:E\longrightarrow Z$ a nonlinear mapping. N is said to be L-compact on E if $QN:E\longrightarrow Z$ and $K_{P,Q}N:E\longrightarrow U$ are compact on E. Also, N is L-completely continuous if it is L-compact on every bounded $E\subset U$.

Theorem 4 (see [9]). *Let U be the space of all bounded continuous vector-valued functions on $[0,\infty)$ and $M\subset U$. Then, M is relatively compact on U if the following conditions hold:*

(i) M is a bounded subset of U

(ii) The functions in M are equicontinuous on any compact interval of $[0,\infty)$

(iii) The functions from M are equiconvergent; that is, if given $\varepsilon>0$, there exists $T=T(\varepsilon)>0$ such that $|f(t)-f(\infty)|<\varepsilon$, for all $t>T$ and $f\in M$

Theorem 5 (see [2]). *Let $L:\operatorname{dom} L\subset U\longrightarrow Z$ be a Fredholm map of index zero and let $N:U\longrightarrow Z$ be L-compact on $\bar{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lu\ne\lambda Nu$ for every $(u,\lambda)\in[(\operatorname{dom} L\setminus\ker L)\cap\partial\Omega]\times(0,1)$

(ii) $Nu\notin\operatorname{Im} L$ for every $u\in\ker L\cap\partial\Omega$

(iii) $\deg(QN|_{\ker L},\Omega\cap\ker L,0)\ne 0$, where $Q:Z\longrightarrow Z$ is a projection such that $\operatorname{Im} L=\ker Q$

Then, the abstract equation $Lu=Nu$ has at least one solution in $\operatorname{dom} L\cap\bar{\Omega}$.

Let

$$\begin{aligned} U=\Big\{&u\in C^2[0,+\infty): u,u',\left(\rho u'\right)'\in AC[0,+\infty), \lim_{t\longrightarrow\infty}e^{-t} \\ &\cdot\left|u^{(i)}(t)\right| \text{ exist}, i=0,1,2, \left(\rho u'\right)''\in L^1[0,\infty)\Big\}, \end{aligned} \tag{8}$$

where $AC[0,+\infty)$ is the set of absolutely continuous functions. The norm defined on U is $\|u\|=\max\{\|u\|_\infty,\|u'\|_\infty,\|u''\|_\infty\}$ where $\|u\|_\infty=\sup_{t\in[0,+\infty)}e^{-t}|u(t)|$.

Let $Z = \{z \in L^1[0,+\infty) \cap C^2(0,+\infty): \int_0^{+\infty} s \mid z \mid ds<+\infty\}$ and define the norm $\|z\| = \max\{\|z\|_\infty, \|z\|_1, \|z\|_{L^1}\}$ where $\|z\|_{L^1} = \int_0^\infty \mid z(s) \mid ds$, $\|z\|_1 = \int_0^\infty s \mid z(s) \mid ds$ and $\|z\|_\infty$ is the supremum norm on $[0,\infty)$. The linear operator $L : \text{dom } L \subset U \longrightarrow Z$ will be defined by

$$L : u \mapsto (Lu)(t) = \left(\rho(t)u'(t)\right)'', \tag{9}$$

where

$$\text{dom } L = \left\{u \in U \cap C^3[0,+\infty): u(0) = \sum_{j=1}^m \alpha_j \int_0^{\eta_j} u(t)dt, u'(0) = 0, \lim_{t\longrightarrow\infty} \left(\rho(t)u'(t)\right)' = 0\right\}. \tag{10}$$

Also, the nonlinear operator $N : U \longrightarrow Z$ will be defined by $(Nu)(t) = w(t, u(t), u'(t), u''(t))$, $t \in [0,+\infty)$; thus, problem (1)-(2) may be written in the form $Lu = Nu$.

Lemma 6. *The following conditions hold:*

(i) $\ker L = \{u \in \text{dom } L : u(t) = h, h \in \mathbb{R}, t \in [0,\infty)\}$

$$\text{Im } L = \left\{z \in Z : \sum_{j=1}^m \alpha_j \int_0^{\eta_j} \int_0^t \frac{1}{\rho(s)} \int_0^s \int_\tau^\infty z(v)dvd\tau dsdt = 0\right\} \tag{11}$$

(ii) $L : \text{dom } L \subset U \longrightarrow U$ *is a Fredholm operator of index zero, while the continuous linear projector* $Q : Z \longrightarrow Z$ *may be defined as*

$$Qy = \theta(t) \sum_{j=1}^m \alpha_j \int_0^{\eta_j} \int_0^t \frac{1}{\rho(s)} \int_0^s \int_\tau^\infty z(v)dvd\tau dsdt, \tag{12}$$

where

$$\theta(t) = \frac{e^{-t}}{\sum_{j=1}^m \alpha_j \int_0^{\eta_j} \int_0^t 1 - e^{-s}/\rho(s)dsdt} \tag{13}$$

(iii) The generalized inverse $K_p : \text{Im } L \longrightarrow \text{dom } L \cap \ker P$ *of* L *may be written as*

$$K_P y = -\int_0^t \frac{1}{\rho(s)} \int_0^s \int_\tau^\infty z(v)dvd\tau ds. \tag{14}$$

Furthermore, $\|K_P y\| \le A_1 \|z\|_{L^1}$ *where*

$$A_1 = \max\left\{\left\|\frac{1}{\rho}\right\|_1, \left\|\frac{1}{\rho}\right\|_\infty, 1\right\}. \tag{15}$$

Proof. Let

$$\left(\rho(t)u'(t)\right)'' = 0, \quad t \in [0,\infty), \tag{16}$$

then

$$u(t) = h + \left(\rho(0)u'(0)\right)\int_0^t \frac{1}{\rho(s)}ds + \left(\rho(\infty)u'(\infty)\right)'\int_0^t \frac{s}{\rho(s)}ds. \tag{17}$$

□

From (2), we have $u'(0) = 0$, $\lim_{t\longrightarrow\infty}(\rho(t)u'(t))' = 0$; then, (17) becomes

$$u(t) = h. \tag{18}$$

Therefore, $\ker L = \{u \in \text{dom } L : u(t) = h, h \in \mathbb{R}, t \in [0,\infty)\}$; hence, (i) holds. Next, we show that (ii), (iii), and (iv) also hold.

For $z \in \text{Im } Z$, consider the problem

$$\left(\rho(t)u'(t)\right)'' = z(t), \tag{19}$$

which has a solution $u(t)$ defined as

$$\begin{aligned} u(t) &= u(0) + \rho(0)u'(0)\int_0^t \frac{1}{\rho(s)}ds \\ &\quad + \left(\rho(\infty)u''(\infty)\right)'\int_0^t \frac{s}{\rho(s)}ds \\ &\quad - \int_0^t \frac{1}{\rho(s)}\int_0^s\int_\tau^\infty z(v)dvd\tau ds. \end{aligned} \tag{20}$$

Applying the boundary conditions and using $\sum_{j=1}^m \alpha_j\eta_j = 1$, one obtains

$$\begin{aligned} &\sum_{j=1}^m \alpha_j \int_0^{\eta_j}\int_0^t \frac{1}{\rho(s)}\int_0^s\int_\tau^\infty z(v)dvd\tau dsdt = 0, \\ &u(t) = h - \int_0^t \frac{1}{\rho(s)}\int_0^s\int_\tau^\infty z(v)dvd\tau ds, \end{aligned} \tag{21}$$

where h is an arbitrary constant and $u(t)$ is a solution of (19) satisfying (2). Therefore, condition (ii) holds.

For any $z \in Z$, let the projector

$$Qz = \theta(t)\sum_{j=1}^m \alpha_j \int_0^{\eta_j}\int_0^t \frac{1}{\rho(s)}\int_0^s\int_\tau^\infty z(v)dvd\tau dsdt, \quad t \in (0,\infty). \tag{22}$$

Let $z_1 = z - Qz$, then $z_1 = z(I - Q)$, where I is the identity operator. Since

$$\sum_{j=1}^{m} \alpha_j \int_0^{\eta_j} \int_0^t \frac{1}{\rho(s)} \int_0^s \int_\tau^\infty \theta(v) dv d\tau ds dt = 1, \tag{23}$$

then $z_1 \in \operatorname{Im} L$ and $Z = \operatorname{Im} L + \operatorname{Im} Q$. From $\operatorname{Im} L \cap \operatorname{Im} Q = \{0\}$, we have $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. Therefore, $\dim \ker L = \dim \operatorname{Im} Q = 1$. Thus, L is a Fredholm operator of index zero and (iii) holds.

Given $P : U \longrightarrow U$ defined as

$$Pu = u(0), \quad t \in [0,\infty), \tag{24}$$

the generalized inverse $K_P : \operatorname{Im} L \longrightarrow \operatorname{dom} L \cap \ker P$ of L can then be written as

$$-\int_0^t \frac{1}{\rho(s)} \int_0^s \int_\tau^\infty z(v) dv d\tau ds. \tag{25}$$

In fact, for any $z \in \operatorname{Im} L$,

$$(LK_P)z(t) = \left[\rho(t)(K_P z)'(t)\right]'' = z(t), \tag{26}$$

and for $u \in \operatorname{dom} L \cap \ker P$, it follows that

$$\begin{aligned}(K_P L)u(t) &= -\int_0^t \frac{1}{\rho(s)} \int_0^s \int_\tau^\infty \left(\rho(v)u'(v)\right)'' dv d\tau ds \\ &= -\int_0^t \frac{1}{\rho(s)} \int_0^s \left[\left(\rho(\infty)u'(\infty)\right)' - \left(\rho(\tau)u'(\tau)\right)'\right] d\tau ds \\ &= \int_0^t \frac{1}{\rho(s)} \left[\left(\rho(s)u'(s)\right) - \left(\rho(0)u'(0)\right)\right] ds \\ &= u(t) - u(0).\end{aligned} \tag{27}$$

Since $Pu = u(0) = 0$, then

$$(K_P L)u(t) = u(t), \quad t \in [0,\infty) ; \tag{28}$$

thus, $K_P = (L|_{\operatorname{dom} L \cap \ker P})^{-1}$. In addition,

$$\begin{aligned}\|(K_P z)\|_\infty &= \sup_{t\in[0,+\infty)} e^{-t} \left| -\int_0^t \frac{1}{\rho(s)} \int_0^s \int_\tau^\infty z(v) dv d\tau ds \right| \\ &\le \|z\|_{L^1} \int_0^\infty \frac{s}{|\rho(s)|} ds \le \left\|\frac{1}{\rho}\right\|_1 \|z\|_{L^1},\end{aligned} \tag{29}$$

$$\begin{aligned}\|(K_P z)'\|_\infty &= \sup_{t\in[0,\infty)} e^{-t} \left| -\frac{1}{\rho(t)} \int_0^t \int_\tau^\infty z(v) dv d\tau \right| \\ &\le \frac{1}{e} \left\|\frac{1}{\rho}\right\|_\infty \|z\|_{L^1} \le \left\|\frac{1}{\rho}\right\|_\infty \|z\|_{L^1},\end{aligned} \tag{30}$$

$$\begin{aligned}\|(K_P z)''\|_\infty &= \sup_{t\in[0,+\infty)} e^{-t} \left| -\int_\tau^\infty z(v) dv \right| \\ &\le \int_0^\infty |z(v)| dv \le \|z\|_{L^1}.\end{aligned} \tag{31}$$

Hence, (29), (30), and (31) give

$$\|K_P z\| \le \max\left\{\left\|\frac{1}{\rho}\right\|_1, \left\|\frac{1}{\rho}\right\|_\infty, 1\right\} \|z\|_{L^1}. \tag{32}$$

Lemma 7. *The nonlinear operator N is L-compact if w is an S-Carathéodory function.*

Proof. To prove this lemma, we have to show that Theorem 4 satisfied the operator $K_{P,Q}Nu$. This we will do in three steps. Given that $D \subset U$ is bounded and $k = \sup\{\|u\| : u \in D\}$, consider $K_{P,Q}N(D)$. Since $w : [0,\infty) \times \mathbb{R}^3$ satisfies the S-Carathéodory conditions with respect to $L^1[0,\infty)$, there exists a Lebesgue integrable function φ_k, with $t\varphi_k \in L^1[0,\infty)$ such that

$$\begin{aligned}|Nu(t)| &= \left| w\left(t, u(t), u'(t), u''(t)\right) \right| \\ &\le \varphi_k(t), \quad \text{a.e.} t \in (0,\infty).\end{aligned} \tag{33}$$

□

Hence, for all $u \in D$,

$$\begin{aligned}\|QNu\|_\infty &= \sup_{t\in[0,\infty)} e^{-t} |QNu(v)| \\ &= \sup_{t\in[0,\infty)} e^{-t} \Bigg| \theta(t) \sum_{j=1}^m \alpha_j \int_0^{\eta_j} \int_0^t \frac{1}{\rho(s)} \int_0^s \int_\tau^\infty w \\ &\quad \cdot \left(v, ,u(v), u'(v), u''(v)\right) dv d\tau ds dt \Bigg| \\ &\le \|\theta\|_\infty \|\varphi_k\|_{L^1} \sum_{j=1}^m |\alpha_j| \int_0^{\eta_j} \int_0^\infty \frac{s}{|\rho(s)|} ds dt \\ &\le \|\theta\|_\infty \|\varphi_k\|_{L^1} \|\frac{1}{\rho}\|_1 \sum_{j=1}^m |\alpha_j| \eta_j < \infty,\end{aligned}$$

$$\begin{aligned}\|QNu\|_{L^1} &= \int_0^\infty |QNu(v)| dv \\ &= \int_0^\infty \Bigg| \theta(t) \sum_{j=1}^m \alpha_j \int_0^{\eta_j} \int_0^\zeta \frac{1}{\rho(s)} \int_0^s \int_\tau^\infty w \\ &\quad \cdot \left(v, u(v), u'(v), u''(v)\right) dv d\tau d\zeta ds \Bigg| dt \\ &\le \|\theta\|_{L^1} \|\varphi_k\|_{L^1} \sum_{j=1}^m |\alpha_j| \int_0^{\eta_j} \int_0^\infty \frac{s}{|\rho(s)|} ds d\zeta \\ &\le \|\theta\|_{L^1} \|\varphi_k\|_{L^1} \|\frac{1}{\rho}\|_1 \sum_{j=1}^m |\alpha_j| \eta_j < \infty,\end{aligned}$$

$$\|QNu\|_1 = \int_0^\infty v|QNu(v)|dv = \int_0^\infty t\left|\theta(t)\sum_{j=1}^m \alpha_j \int_0^{\eta_j}\int_0^\zeta \frac{1}{\rho(s)}\int_0^s\int_\tau^\infty w \cdot \left(v, u(v), u'(v), u''(v)\right) dv d\tau ds d\zeta\right| dt \le \|\theta\|_1 \|\varphi_k\|_{L^1} \sum_{j=1}^m |\alpha_j| \int_0^{\eta_j}\int_0^\infty \frac{s}{|\rho(s)|} ds d\zeta \le \|\theta\|_1 \|\varphi_k\|_{L^1} \|\frac{1}{\rho}\|_1 \sum_{j=1}^m |\alpha_j|\eta_j < \infty. \quad (34)$$

Hence, $\|QNu\| = \max\{\|QNu\|_\infty, \|QNu\|_{L^1}, \|QNu\|_1\}$ is bounded. Next, we prove that $K_P(I-Q)N(\bar{D})$ is compact. For $u \in D$, we have

$$\|K_{P,Q}Nu\|_\infty = \sup_{t\in[0,\infty)} e^{-t}|K_{P,Q}Nu| \le \|\varphi_k\|_{L^1}\|\frac{1}{\rho}\|_1 \cdot \left(1 + \|\theta\|_{L^1}\|\frac{1}{\rho}\|_1 \sum_{j=1}^m |\alpha_j|\eta_j\right) < \infty,$$

$$\|(K_{P,Q}Nu)'\|_\infty = \sup_{t\in[0,\infty)} e^{-t}|(K_{P,Q}Nu)'| \le \frac{1}{e}\|\varphi_k\|_{L^1}\|\frac{1}{q}\|_\infty \cdot \left(1 + \|\theta\|_{L^1}\|\frac{1}{\rho}\|_1 \sum_{j=1}^m |\alpha_j|\eta_j\right) < \infty,$$

$$\|(K_{P,Q}Nu)''\|_\infty = \sup_{t\in[0,\infty)} e^{-t}|(K_{P,Q}Nu)''| \le \|\varphi_k\|_{L^1} \cdot \left(1 + \|\theta\|_{L^1}\|\frac{1}{\rho}\|_1 \sum_{j=1}^m |\alpha_j|\eta_j\right) < \infty. \quad (35)$$

Hence, $K_{P,Q}N(D)$ is uniformly bounded in U; thus, (i) of Also, we will prove that (ii) of Theorem 4 holds. Also, we will prove that (ii) of Theorem 4 holds, that is, $K_{P,Q}N(D)$ is equicontinuous on $[0,T]$. Let $t_1, t_2 \in [0,T)$, where $T \in (0,+\infty)$. Then,

$$\left|K_{P,Q}Nu(t_2) - K_{P,Q}Nu(t_1)\right| \le \|\varphi_k\|_{L^1} \cdot \left(1 + \|\theta\|_{L^1}\|\frac{1}{\rho}\|_1 \sum_{j=1}^m |\alpha_j|\eta_j\right)\int_{t_1}^{t_2} \frac{s}{|\rho(s)|} ds \longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2,$$

$$\left|(K_{P,Q}Nu)'(t_2) - (K_{P,Q}Nu)'(t_1)\right| \le \left[\|\varphi_k\|_{L^1}\left(1 + \|\theta\|_{L^1}\|\frac{1}{\rho}\|_1 \sum_{j=1}^m |\alpha_j|\eta_j\right)\left|\frac{1}{\rho(t_1)} - \frac{1}{\rho(t_2)}\right|\int_0^{t_1} d\tau + \|\varphi_k\|_{L^1}\left(1 + \|\theta\|_{L^1}\|\frac{1}{\rho}\|_1 \sum_{j=1}^m |\alpha_j|\eta_j\right)\frac{1}{\rho(t_2)}\int_{t_1}^{t_1} d\tau\right] \longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2,$$

$$\left|(K_{P,Q}Nu)''(t_2) - (K_{P,Q}Nu)''(t_1)\right| \le \int_{t_1}^{t_2}\left(\varphi_k(v) + \|\theta\|_{L^1}\|\frac{1}{\rho}\|_1 \sum_{j=1}^m |\alpha_j|\eta_j\right) \longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2. \quad (36)$$

Thus, $K_{P,Q}N(D)$ is equicontinuous every compact subset of $[0,\infty)$. Finally, we show that (iii) of Theorem 4 holds; that is, $K_{P,Q}N(D)$ is equiconvergent at ∞:

$$\left|K_{P,Q}Nu(t) - K_{P,Q}Nu(\infty)\right| \le \|\varphi_k\|_{L^1}\left(1 + \|\theta\|_{L^1}\|\frac{1}{\rho}\|_1 \sum_{j=1}^m |\alpha_j|\eta_j\right)\int_t^\infty \frac{s}{\rho(s)} ds \longrightarrow 0, \quad \text{uniformly as } t \longrightarrow \infty,$$

$$|(K_{P,Q}Nu)'(t) - (K_{P,Q}Nu)'(\infty)| + \|\varphi_k\|_{L^1}\left(1 + \|\theta\|_{L^1}\|\frac{1}{\rho}\|_1 \sum_{j=1}^m |\alpha_j|\eta_j\right)\frac{1}{\rho(\infty)}\int_t^\infty d\tau] \longrightarrow 0, \quad \text{uniformly as } t \longrightarrow \infty,$$

$$\left|(K_{P,Q}Nu)''(t) - (K_{P,Q}Nu)''(\infty)\right| \le \int_t^\infty\left(\varphi_k(v) + |\theta(v)|\|\varphi_k\|_{L^1}\|\frac{1}{\rho}\|_1 \sum_{j=1}^m |\alpha_j|\eta_j\right) \longrightarrow 0, \quad \text{uniformly as } t \longrightarrow \infty. \quad (37)$$

Hence, $K_{P,Q}N(D)$ is equiconvergent at ∞. Thus, from Definition 3 and Theorem 4, we see that the nonlinear operator N is L-compact.

3. Existence Result

Theorem 8. *Let $w : [0,+\infty) \times R^3 \longrightarrow R$ be an S-Carathéodory function. If the following hold:*

(E_1) There exists constant $\sigma \in [0,1)$ and functions $b, c, d, e, g \in L^1[0,\infty)$, satisfying $\int_0^{+\infty} s|b(s)|ds < +\infty$, $\int_0^{+\infty} s|c(s)|ds < +\infty$, $\int_0^{+\infty} s|d(s)|ds < +\infty$, $\int_0^{+\infty} s|e(s)|ds < +\infty$, $\int_0^{+\infty} s|g(s)|ds < +\infty$, such that for all $(x,y,z) \in \mathbb{R}^3$ and $t \in [0,\infty)$,

$$|w(t,x,y,z)| \le e^{-t}[b(t)|x| + c(t)|y| + d(t)|z| + e(t)|z|^\sigma] + g(t). \quad (38)$$

(E_2) There exists a constant $B > 0$, such that for $u \in \operatorname{dom} L$, if $|u(t)| > B$, for all $t \in [0,\infty)$, then

$$\theta(t)\sum_{j=1}^m \alpha_j \int_0^{\eta_j}\int_0^t \frac{1}{\rho(s)}\int_0^s\int_\tau^\infty w\left(v, u(v), u'(v), u''(v)\right) dv d\tau ds dt \ne 0. \quad (39)$$

(E_3) There exists a constant $B^* > 0$, such that for any $u(t) = h \in \mathbb{R}$, if $|h| > B^*$, then

$$h \cdot \sum_{j=1}^{m} \alpha_j \int_0^{\eta_j} \int_0^t \frac{1}{\rho(s)} \int_0^s \int_\tau^\infty w(v, h, 0, 0)\,dv\,ds\,d\tau\,dt < 0 \quad (40)$$

$$\text{or } h \cdot \sum_{j=1}^{m} \alpha_j \int_0^{\eta_j} \int_0^t \frac{1}{\rho(s)} \int_0^s \int_\tau^\infty w(v, h, 0, 0)\,dv\,ds\,d\tau\,dt > 0. \quad (41)$$

Then, problem (1) subject to (2) has at least one solution if

$$\|b\|_{L^1} + \|c\|_{L^1} + \|d\|_{L^1} < \frac{1}{A}, \quad (42)$$

where $A = \max\{\|1/\rho\|_1 + \|1/\rho\|_\infty, 2\|1/\rho\|_\infty, \|1/\rho\|_\infty + 1\}$.

In order to prove Theorem 8, the following lemmas are required. These lemmas will help us prove that condition (iii) of Theorem 5 holds.

Lemma 9. *The set* $\Omega_1 = \{u \in \operatorname{dom} L \setminus \ker L : Lu = \lambda N u \text{ for some } \lambda \in [0, 1]\}$ *is bounded.*

Proof. Assume that $u \in \Omega_1$ and $Lu = \lambda Nu$, then $\lambda \neq 0$, $QNu = 0$ and by condition (E_2) of Theorem 8, there exists $t_0 \in [0, \infty)$, such that $|u(t_0)| \leq B$. By the absolute continuity of u, it follows from $u(0) = u(t_0) - \int_0^{t_0} u'(t)dt$ that

$$|u(0)| \leq |u(t_0)| + \int_0^{t_0} |u'(v)|dv \leq B + \|u'\|_\infty. \quad (43)$$

From $u'(t) = 1/\rho(t)\int_0^t u''(v)dv$, one gets

$$\begin{aligned}\|u'\|_\infty &\leq \sup_{t\in[0,\infty)} e^{-t} \frac{1}{|\rho(t)|} \int_0^t |u''|(v)\,dv\,ds \\ &\leq \left\|\frac{1}{\rho}\right\|_\infty \|u''\|_\infty,\end{aligned} \quad (44)$$

and from $u'' = \int_t^\infty u'''(v)dv$, we obtain

$$\begin{aligned}\|u''\|_\infty &\leq \sup_{t\in[0,\infty)} e^{-t} \int_t^\infty |u'''(v)|dv \leq \int_0^\infty |Lu(v)|dv \\ &\leq \int_0^\infty |Nu(v)|dv \leq \|Nu\|_{L^1}.\end{aligned} \quad (45)$$

ence, from (43) and (45)

$$\|Pu\| = |u(0)| \leq B + \|u'\|_\infty \leq B + \|u'\|_{L^1} \leq B + \left\|\frac{1}{\rho}\right\|_\infty \|Nu\|_{L^1}. \quad (46)$$

Also, for $u \in \Omega_1$, $u \in \operatorname{dom} L \setminus \ker L$, then $(I - P)u \in \operatorname{dom} L \cap \ker P$ and $LPu = 0$; then, from Lemma 6, one gets

$$\begin{aligned}\|(I-P)u\| &= \|K_P L(I-P)u\| \\ &\leq \max\left\{\left\|\frac{1}{\rho}\right\|_1, \left\|\frac{1}{\rho}\right\|_\infty, 1\right\}\|L(I-P)u\|_{L^1} \\ &\leq \max\left\{\left\|\frac{1}{\rho}\right\|_1, \left\|\frac{1}{\rho}\right\|_\infty, 1\right\}\|Nu\|_{L^1}.\end{aligned} \quad (47)$$

Hence, from (46) and (47),

$$\begin{aligned}\|u\| &\leq \|Pu\| + \|(I-P)u\| \leq B + \max \\ &\cdot \left\{\left\|\frac{1}{\rho}\right\|_1 + \left\|\frac{1}{\rho}\right\|_\infty, 2\left\|\frac{1}{\rho}\right\|_\infty, \left\|\frac{1}{\rho}\right\|_\infty + 1\right\}\|Nu\|_{L^1}.\end{aligned} \quad (48)$$

Setting $\max\{\|1/\rho\|_1 + \|1/\rho\|_\infty, 2\|1/\rho\|_\infty, \|1/\rho\|_\infty + 1\} = A$ gives

$$\|u\| \leq B + A\|Nu\|_{L^1}. \quad (49)$$

From (38), we see that

$$\begin{aligned}\|Nu\|_{L^1} &\leq \int_0^\infty \left|w\left(t, u(v), u'(v), u''(z)\right)\right|dv \\ &\leq \|u\|[\|b\|_{L^1} + \|c\|_{L^1} + \|d\|_{L^1}] + \|e\|_{L^1}\|u''\|_\infty^\sigma.\end{aligned} \quad (50)$$

From (49), we obtain

$$\|u\| \leq \frac{B + A\|e\|_{L^1}\|u''\|_\infty^\sigma}{1 - A[\|b\|_{L^1} + \|c\|_{L^1} + \|d\|_{L^1}]}. \quad (51)$$

Hence, Ω_1 is bounded. □

Lemma 10. *Given that* $\Omega_2 = \{u \in \ker L : J^{-1}u - [(1-\lambda)/\lambda] QNu = 0, \lambda \in [0, 1]\}$. *Suppose that (40) holds, for all* $|h| > B^*$ *where* $J : \operatorname{Im} Q \longrightarrow \ker L$ *is the linear isomorphism defined by* $J(h) = h$, *then* Ω_2 *is bounded.*

Proof. For all $u(t) = h \in \Omega_2$, one obtains

$$h = \frac{1-\lambda}{\lambda} \sum_{j=1}^{m} \alpha_j \int_0^{\eta_j} \int_0^t \frac{1}{\rho(s)} \int_0^s \int_\tau^\infty w(v, h, 0, 0)\,dv\,ds\,d\tau\,dt. \quad (52)$$

When $\lambda = 1$, then $h = 0$. However, when $|h| > B^*$, then in view of (40), one gets

$$\lambda h^2 = h(1-\lambda) \sum_{j=1}^{m} \alpha_j \int_0^{\eta_j} \int_0^t \frac{1}{\rho(s)} \int_0^s \int_\tau^\infty w(v, h, 0, 0)\,dv\,ds\,d\tau\,dt < 0, \quad (53)$$

contradicting $\lambda h^2 > 0$. Then, $\|u\| = |x| \leq B^*$, implying that $\|u\| \leq B^*$. Hence, Ω_2 is bounded. □

Proof of Theorem 8. We will now show that all the conditions of Theorem 5 hold. Take $\cup_{k=1}^{2} \bar{\Omega}_k$ as an open bounded subset of U. Lemma 6 shows that L is a Fredholm operator of index zero, and from Lemma 7, N is L-compact on $\bar{\Omega}$. Thus, conditions (i) and (ii) of Theorem 5 are satisfied based on the definition of Ω. Finally, we show that condition (iii) of Theorem 5 holds. Set $E(u,\lambda) = -\lambda Ju + (1-\lambda)QNu$. Since $\bar{\Omega}_2 \subset \Omega$, then for every $u \in \ker L \cap \partial\Omega$, $E(u,\lambda) \neq 0$. Therefore, the homotopy property of the Brouwer degree gives

$$\begin{aligned} &\deg\left(QN|_{\ker L}, \Omega \cap \ker L, 0\right) \\ &\quad= \deg\left(E(\cdot,0), \Omega \cap \ker L, 0\right) \\ &\quad= \deg\left(E(\cdot,1), \Omega \cap \ker L, 0\right) \\ &\quad= \deg\left(\pm J, \Omega \cap \ker L, 0\right) \neq 0. \end{aligned} \tag{54}$$

□

Hence, condition (iii) of Theorem 5 holds, and problem (1)-(2) has at least one solution in U.

4. Example

Consider the following problem

$$\left(\rho(t)u'(t)\right)'' = e^{-t}\left(\frac{u}{20} + \frac{1}{10}\sin\left(u''\right)^{1/16} + \frac{1}{20}u''\right), \tag{55}$$

$$\begin{aligned} u(0) &= 18\int_0^{1/6} u(t)dt - 2\int_0^{1/2} u(t)dt, u'(0) \\ &= 0, \lim_{t\longrightarrow\infty}\left(\rho(t)u'(t)\right)' = 0, \end{aligned} \tag{56}$$

where

$$\rho(t) = e^t, t \in [0,\infty), \alpha_1 = 18, \alpha_2 = -2, \eta_1 = \frac{1}{6}, \eta_2 = \frac{1}{2},$$

$$\sum_{j=1}^{2}\alpha_j\eta_j = 18\left(\frac{1}{6}\right) - 2\left(\frac{1}{2}\right) = 3 \neq 1,$$

$$\begin{aligned} \sum_{j=1}^{2}\alpha_j\int_0^{\eta_j}\int_0^t \frac{1-e^{-s}}{e^s}dsdt &= \sum_{j=1}^{2}\alpha_j\left(\frac{\eta_j}{2} + e^{-\eta_j} - \frac{e^{-2\eta_j}}{4} - \frac{3}{4}\right) \\ &= -0.01684 \neq 0. \end{aligned} \tag{57}$$

Given that $(t,x,y,z) \in [0,\infty) \times \mathbb{R}^3$, then $w(t,x,y,z) > 0$ and

$$\begin{aligned} &\theta(t)\sum_{j=1}^{m}\alpha_j\int_0^{\eta_j}\int_0^t \frac{1}{\rho(s)}\int_0^s\int_\tau^\infty w\left(v,u(v),u'(v),u''(v)\right) \\ &\quad\cdot dvd\tau dsdt \neq 0; \end{aligned} \tag{58}$$

hence, condition (E_2) is satisfied. For any $h \in \mathbb{R}$ such that $|h| > B^*$, where $B^* > 0$,

$$\begin{aligned} &h\cdot\sum_{j=1}^{m}\alpha_j\int_0^{\eta_j}\int_0^t \frac{1}{\rho(s)}\int_0^s\int_\tau^\infty w(v,h,0,0)dvd\tau dsdt \\ &\quad= h\sum_{j=1}^{m}\alpha_j\int_0^{\eta_j}\int_0^t \frac{1}{e^s}\int_0^s\int_\tau^\infty \frac{h}{20}e^{-v}dvd\tau dsdt \\ &\quad= -\frac{0.01684h^2}{20} \neq 0 \text{ since } |h| > 0. \end{aligned} \tag{59}$$

Now, $|w(t,x,y,z)| \le e^{-t}(|x|/20 + 1/10\cdot(1)^{1/16} + |z|/20)$. Taking $b(t) = 1/20e^{-t}$, $c(t) = 0$, $d(t) = 1/20e^{-t}$, $e(t) = 0$, $t \in [0,\infty)$, then $b,c,d,e,g \in L^1[0,\infty)$ satisfy condition (E_1). Finally,

$$\|b\|_{L^1} = \int_0^\infty \frac{1}{20}|e^{-t}|dt = \frac{1}{20}, \|d\|_{L^1} = \int_0^\infty \frac{1}{20}e^{-t}dt = \frac{1}{20},$$

$$\begin{aligned} A &= \max\left\{\left\|\frac{1}{\rho}\right\|_1 + \left\|\frac{1}{\rho}\right\|_\infty, 2\left\|\frac{1}{\rho}\right\|_\infty, \left\|\frac{1}{\rho}\right\|_\infty + 1\right\} \\ &= \max\{2,2,2\} = 2, \end{aligned}$$

$$\|b\|_{L^1} + \|c\|_{L^1} + \|d\|_{L^1} = \frac{1}{20} + \frac{1}{20} = \frac{1}{10} < \frac{1}{2}. \tag{60}$$

All the conditions are satisfied, and we conclude from Theorem 8 that there exists at least one solution $u(t)$ for problem (55) subject to (56).

Data Availability

N/A.

Conflicts of Interest

The authors declare they have no competing interest.

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