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Research Article

Fixed Point Theorems for $(\alpha, k, \theta, \varphi)$-Contraction Multivalued Mappings in b-Metric Space

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We present the concept of $(\alpha, k, \theta, \varphi)$-contractive multivalued mappings in b-metric spaces and prove some fixed point results for these mappings in this study. Our results expand and refine some of the literature's findings in fixed point theory.

1. Introduction and Preliminaries

Fixed point theory is one of the most active research areas of mathematics. The findings are used to solve problems in a wide variety of disciplines, including transportation theory, economics, and biomathematics. The development of functional analysis methods assisted in the broadening of the well-known Banach contraction theory (see [1–21] and references therein). One of them is the extension of the principle of contraction to cover applications and multivalued applications.

Samet et al. [20] presented the definition of α-admissible mapping in complete metric spaces and partially ordered metric spaces in 2012 and proved several fixed point theorems under the generalized contraction in both. Following that, Hasanzade et al. [22] introduced the idea of α_*-admissible mapping, which is the multivalued edition of α-admissible single-valued mapping established in [20], and they provided a fixed point result for multivalued mappings in complete metric spaces satisfying many generalized contractive conditions.

In 2013, Mohammadi et al. [16] developed the notion of an α_*-admissible mapping to the class of α-admissible mappings.

In 2014, Jleli and Samet [11] expanded the contractive condition by considering the following function Θ:

Definition 1 ([11]). Let Θ denote the set of all functions $\theta : (0, \infty) \longrightarrow (1, \infty)$ fulfilling the following criteria:

(Θ_1): θ is nondecreasing;

(Θ_2): from every sequence $\{z_n\} \subseteq \mathbb{R}^+$, $\lim_{n\longrightarrow\infty}\theta(z_n) = 1$ if and only if $\lim_{n\longrightarrow\infty} z_n = 0$;

(Θ_3): there exists $0 < r < 1$ and $l \in (0, \infty)$ such that $\lim_{z\longrightarrow 0^+}(\theta(z) - 1)/z^r = l$;

(Θ_4): every $\theta \in \Theta$ is continuous.

Definition 2 ([11]). Let (Ω, d) be a metric space. A mapping $F : \Omega \longrightarrow \Omega$ is called a θ-contraction if $\theta : (0, \infty) \longrightarrow (1, \infty)$ fulfills $(\Theta_1) - (\Theta_3)$ and there exists a constant $k \in (0, 1)$ such that for all $x, y \in \Omega$,

$$d(Fx, Fy) \neq 0 \Longrightarrow \theta(d(Fx, Fy)) \leq [\theta(d(x, y))]^k. \qquad (1)$$

Several researchers extended (1) in various ways and proved fixed point theorems for a single and multivalued contractive mappings (see [7, 12, 17, 18, 23]).

Any Banach contraction is a θ-contraction, although this is not so for the inverse ([7]).

Definition 3 ([24, 25]). Let $s \geq 1$ be a given real number and Ω be a nonempty set. A function $d : \Omega \times \Omega \longrightarrow \mathbb{R}^+$ is a b-metric if the following conditions are fulfilled for all $x, y, z \in \Omega$.

(bM_1): $d(x, x) = 0$ if and only if $x = y$;
(bM_2): $d(x, y) = d(y, x)$;
(bM_3): $d(x, z) \leq s[d(x, y) + d(y, z)]$.
The pair (Ω, d) is called a b-metric space.

It is worth noting that b-metric space is a broader category than metric spaces.

The following notations are used in this paper:

$$M(x, y) = \max \left\{ d(x, y), d(x, fx), d(y, fy), \frac{[d(x, fy) + d(y, fx)]}{2s} \right\}. \tag{2}$$

$Q(x, y) = \min \{d(x, fx), d(x, fy), d(y, fx)\}$, and Ψ is the set of all self mapping on $[0, +\infty[$such that ψ is a continuous monotone and nondecreasing, and $\psi(t) = 0 \Longleftrightarrow t = 0$.

Definition 4 ([26]). Let (Ω, d) be a b-metric space. $X \subset \Omega$, and consider a sequence $\{u_n\}$ in Ω. Then,

(i) $\{u_n\}$ is b-convergent if there exists $u \in \Omega$ such that $d(u_n, u) \longrightarrow 0$ as $n \longrightarrow \infty$

In this case, we write $\lim_{n\longrightarrow\infty} u_n = u$.

(ii) $\{u_n\}$ is b-Cauchy sequence if $d(u_n, u_m) \longrightarrow 0$ as n, $m \longrightarrow \infty$

(iii) X is closed if and only if for each sequence $\{u_n\}$ of points in X with $d(u_n, u) \longrightarrow 0$ as $n \longrightarrow \infty$, we have $u \in X$.

Remark 5 ([26]). The following statements hold in a b-metric space (Ω, d).

(1) A b-convergent sequence has a unique limit

(2) Each b-convergent sequence is a b-Cauchy sequence

(3) In general, a b-metric is not continuous

(4) d does not generally induce a topology on Ω.

Consider $(\Omega ; d ; s)$ be a b-metric space and $CL(\Omega)$ the class of nonempty closed subsets of Ω. On $CL(\Omega)$, consider $H(.;.)$ be the generalized Pompeiu-Hausdorff b-metric (see [27]); i.e., for all $G ; R \in CL(\Omega)$,

$$H(G ; R) = \begin{cases} \max \left\{ \sup_{a\in G} d(a, R), \sup_{b\in R} d(G, b) \right\}, & \text{if the maximum exists,} \\ \infty, & \text{otherwise,} \end{cases} \tag{3}$$

where $d(a ; R) = \inf \{d(a ; b): b \in R\}$.

For $G ; R \in CL(\Omega)$, we put $\delta(G ; R) = \sup_{a\in G} d(a ; R)$; we note that $\delta(G ; R) \leq H(G ; R)$.

Definition 6 ([26, 28]). Let (Ω, d) and (Y, δ) be two b-metric spaces.

(D_1): The space (Ω, d) is b-complete if every b-Cauchy sequence in Ωb-converges.

(D_2): A function $f : \Omega \longrightarrow Y$ is b-continuous at a point $u \in \Omega$ if it is b-sequentially continuous at u; that is, whenever $\{u_n\}$ is b-convergent to u, $\{fu_n\}$ is b-convergent to fu.

Definition 7 ([16, 22]). Let Ω be a nonempty set, $\alpha : \Omega \times \Omega \longrightarrow [0, \infty)$ be a given mapping, and $N(\Omega)$ the class of all nonempty subsets of Ω.

(1) If the following condition applies, a mapping $f : \Omega \longrightarrow N(\Omega)$ is called α_*-admissible: for $x, y \in \Omega$, $\alpha(x, y) \geq 1 \Longrightarrow \alpha_*(fx, fy) \geq 1$, where $\alpha_*(fx, fy) := \inf \{\alpha(a, b): a \in fx, b \in fy\}$

(2) For each $x \in \Omega$ and $y \in fx$ with $\alpha(x, y) \geq 1$, we have $\alpha(y, z) \geq 1$; for all $z \in fy$, a mapping $f : \Omega \longrightarrow N(\Omega)$ is said to be α-admissible

Hussain et al. [8] in metric spaces used the concept of α-complete. We expand and apply it in b-metric spaces here.

Definition 8. Let (Ω, d) be a b-metric space and $\alpha : \Omega \times \Omega \longrightarrow [0, \infty)$ be a mapping. The b-metric space Ω is said to be α-complete if and only if every b-Cauchy sequence $\{u_n\}$ in Ω with $\alpha(u_n, u_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, b-converges in Ω.

In 2015, Kutbi and Sintunavarat ([13]) introduced the notion of the α-continuity for multivalued mappings in metric spaces. We expand and apply it in b-metric spaces here.

Definition 9. Let (Ω, d) be a b-metric space and $T : \Omega \longrightarrow CL(\Omega)$ and $\alpha : \Omega \times \Omega \longrightarrow [0, \infty)$ be two given mappings. We say T is an α-continuous multivalued mapping on $(CL(\Omega), H)$ if

(i) for each sequence $\{u_n\}$ with $u_n \longrightarrow u \in \Omega$ as $n \longrightarrow \infty$, and

(ii) $\alpha(u_n, u_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, we have $Tu_n \longrightarrow Tu$ as $n \longrightarrow \infty$

Remark 10.

(1) It is easy to see that α_*-admissibility implies α-admissibility. But the converse may not be true as shown in example 15 of [15]

(2) If Ω is b-complete b-metric space, then Ω is also α-complete b-metric space. But the converse is not true (see Example 1)

(3) Note that the b-continuity implies the α-continuity. In general, the converse is not true (see in Example 2)

Example 1. Let $\Omega=(0,\infty), d(x,y)=|x-y|^2$ for all $x, y\in\Omega$, and Λ be a closed subset of Ω. Define $\alpha:\Omega\times\Omega\longrightarrow[0,\infty)$ by the following:

$$\alpha(x,y)=\begin{cases}(x-y)^2+1, & \text{if } x, y\in\Lambda,\\ 0, & \text{otherwise.}\end{cases} \tag{4}$$

Clearly, (Ω, d) is not a b-complete b-metric space, but it is an α-complete b-metric space.

Example 2. Let $\Omega=[0,\infty)$ and $d(x,y)=|x-y|^2$ for all $x, y\in\Omega$; let $T:\Omega\longrightarrow CL(\Omega)$ and $\alpha:\Omega\times\Omega\longrightarrow[0,\infty)$ defined by the following:

$$\begin{aligned}Tx&=\begin{cases}\{x^3\}, & \text{if } x\in[0,1],\\ \{\cos(\pi x)+3\}, & \text{if } x\in(1,\infty),\end{cases}\\ \alpha(x,y)&=\begin{cases}e^{xy}, & \text{if } x, y\in[0,1],\\ 0, & \text{otherwise.}\end{cases}\end{aligned} \tag{5}$$

Clearly, T is not a b-continuous multivalued mapping on $(CL(\Omega), H)$. Indeed, if $\{z_n\}$ in Ω defined by $z_n=1+1/n$ for all $n\geq 1$, we see that $z_n=1+1/n\longrightarrow^d 1$ but $Tz_n=\{\cos(\pi+\pi/n)+3\}\longrightarrow^H\{2\}\neq\{1\}=T1$; however, T is a α-continuous multivalued mapping on $(CL(\Omega), H)$. Indeed, if $z_n\longrightarrow^d z$ as $n\longrightarrow\infty$ with $\alpha(z_n, z_{n+1})\geq 1$ for all $n\in\mathbb{N}$, then $z, z_n\in[0,1]$ for all $n\in\mathbb{N}$ and so $Tz_n=\{z_n^3\}\longrightarrow^H\{z^3\}=Tz$ as $n\longrightarrow\infty$.

Lemma 11 [29]. *Let $(\Omega; d; s)$ be a b-metric space, and let $R\in CL(\Omega)$. If $\varsigma\in\Omega$ and $d(\varsigma; R)<\eta$, then there exists $\nu\in R$ such that $d(\varsigma;\nu)<\eta$.*

Lemma 12 [1]. *Let $(\Omega; d; s)$ be a b-metric space, and let $R\in CL(\Omega)$ and $\varsigma\in\Omega$. Then, we have*

$$d(\varsigma; R)=0 \text{ if and only if } \varsigma\in\bar{R}=R, \tag{6}$$

where $\bar{R}$ denotes the closure of R.

Lemma 13 ([30]). *Let $\{u_n\}$ be a sequence in a b-metric space (Ω, d, s) with $s\geq 1$ such that*

$$d(u_n, u_{n+1})\leq\lambda d(u_{n-1}, u_n), \tag{7}$$

for some $\lambda\in[0,1)$, and each $n\in\mathbb{N}$. Then, $\{u_n\}$ is a b-Cauchy sequence in (Ω, d, s).

Lemma 14 [31]. *If (Ω, d) is a b-metric space with $s\geq 1$ and $\{u_n\}$ and $\{y_n\}$ are b-convergent to u, y, respectively, we have*

$$\begin{aligned}\frac{1}{s^2}d(u,y)&\leq\liminf_{n\longrightarrow\infty}d(u_n,y_n)\leq\limsup_{n\longrightarrow\infty}d(u_n,y_n)\\&\leq s^2 d(u,y).\end{aligned} \tag{8}$$

In particular, if $u=y$, then we have $\lim_{n\longrightarrow\infty}d(u_n,y_n)=0$. Moreover, for each $z\in\Omega$, we have

$$\frac{1}{s}d(u,z)\leq\liminf_{n\longrightarrow\infty}d(u_n,z)\leq\limsup_{n\longrightarrow\infty}d(u_n,z)\leq sd(u,z). \tag{9}$$

We will prove some new results of the fixed point in α-complete b-metric spaces, we will add a new kind of contraction for multivalued mappings called $(\alpha,k,\theta,\varphi)$-contraction multivalued mappings, and we give some examples to illustrate the main results of this paper.

2. Results

Definition 15. Let (Ω, d, s) be a b-metric space. A mapping $f:\Omega\longrightarrow CL(\Omega)$ is called weak $(\alpha,k,\theta,\varphi)$-contractive multivalued mapping if there exist $\alpha:\Omega\times\Omega\longrightarrow[0,\infty), \theta\in\Theta$, $k\in(0,1), L\geq 0, \varphi\in\Psi$ and constant $\lambda\in(0,1]$ such that

$$\begin{aligned}\alpha(x,y)\geq 1\Longrightarrow&\,\theta(sH(fx,fy))\\&\leq\lambda\theta(M(x,y))^k-\varphi(M(x,y))+LQ(x,y),\end{aligned} \tag{10}$$

for all $x, y\in\Omega$.

Example 3. Let $\Omega=[12/10,\infty)$, and take the b-metric $d(x,y)=|x-y|^2$ for all $x, y\in\Omega$. Define $f:\Omega\longrightarrow CL(\Omega)$, $\varphi:[0,\infty)\longrightarrow[0,\infty)$, $\alpha:\Omega\times\Omega\longrightarrow[0,\infty)$, and $\theta:[0,\infty)\longrightarrow[1,\infty)$ by the following:

$$\begin{aligned}fx&=\left[\frac{12}{10},\frac{12}{10}+\frac{x}{4}\right],\\ \varphi(x)&=\frac{1}{2}x,\\ \alpha(x,y)&=\begin{cases}e^{x-y}, & \text{if } x\in\left[\frac{12}{10}+y,\infty\right),\\ 0, & \text{otherwise,}\end{cases}\end{aligned} \tag{11}$$

and $\theta(t)=e^{te^t}$. Note that for all $t\geq 12/10$, one has

$$e^{t/8e^{t/8}}\leq\frac{3}{10}e^{t/2e^t}-\frac{7}{10}t. \tag{12}$$

Now, for all $x, y\in\Omega$ such that $x\in[12/10+y,\infty)$, we have $H(fx,fy)=|x-y|^2/16$.

Then,

$$\begin{aligned}\theta(sH(fx,fy))&=e^{|x-y|^2/8e^{|x-y|^2/8}}=e^{d(x,y)/8e^{d(x,y)/8}}\leq e^{M(x,y)/8e^{M(x,y)/8}}\\&\leq\frac{3}{10}e^{M(x,y)/2e^{M(x,y)}}-\frac{7}{10}M(x,y)\\&=\frac{3}{10}(\theta(M(x,y)))^{1/2}-\varphi(M(x,y))\\&=\lambda(\theta(M(x,y)))^k-\varphi(M(x,y)).\end{aligned} \tag{13}$$

And thus,

$$\theta(sH(fx,fy)) \le \lambda(\theta(M(x,y)))^k - \varphi(M(x,y)) + LQ(x,y). \tag{14}$$

Thus, f is weak $(\alpha, k, \theta, \varphi)$-contractive multivalued mapping, with $\lambda = 3/10$ and $k = 1/2$.

The following is our primary result:

Theorem 16. *Let (Ω, d, s) be a α-complete b-metric space and $f : \Omega \longrightarrow CL(\Omega)$ is a weak $(\alpha, k, \theta, \varphi)$-contractive multivalued mapping. Assume that the following conditions hold:*

(i) f is a α-admissible

(ii) There exists $x_0, x_1 \in \Omega$ such that $x_1 \in fx_0$ and $\alpha(x_0, x_1) \ge 1$

(iii) f is α-continuous

Then, f has a fixed point. Moreover, if u and v are fixed points of f such that $\alpha(u,v) \ge 1$, then $u = v$.

Proof. If $x_0 = x_1$ or $x_1 \in fx_1$, then x_1 is a fixed point of f. Assume that $x_0 \ne x_1$ and $x_1 \in fx_1$.

Since f is weak $(\alpha, k, \theta, \varphi)$-contractive multivalued mapping, we obtain

$$\begin{aligned} 1 < \theta(d(x_1, fx_1)) &\le \theta(sH(fx_0, fx_1)) \\ &\le \lambda\theta(M(x_0,x_1))^k - \varphi(M(x_0,x_1)) + LQ(x_0,x_1) \\ &\le \theta(M(x_0,x_1))^k + LQ(x_0,x_1), \end{aligned} \tag{15}$$

with

$$\begin{aligned} M(x_0,x_1) &= \max\left\{d(x_0,x_1), d(x_0,fx_0), d(x_1,fx_1), \frac{d(x_0,fx_1)+d(x_1,fx_0)}{2s}\right\} \\ &\le \max\{d(x_0,x_1), d(x_1,fx_1)\}, \\ Q(x_0,x_1) &= \min\{d(x_1,fx_1), d(x_0,fx_1), d(x_1,fx_0)\} = 0. \end{aligned} \tag{16}$$

Assume that $M(x_0, x_1) = d(x_1, fx_1)$, and from (15), we get

$$1 < \theta(d(x_1,fx_1)) \le \theta(sH(fx_0,fx_1)) \le \theta(d(x_1,fx_1))^k, \tag{17}$$

which is a contradiction; it follows that $M(x_0,x_1) = d(x_0,x_1)$, and (15) becomes

$$1 < \theta(d(x_1,fx_1)) \le \theta(sH(fx_0,fx_1)) \le \theta(d(x_0,x_1))^k. \tag{18}$$

By Lemma 11, there exists $x_2 \in fx_1$ such that $d(x_1,x_2) \le d(x_0,x_1)$, and by the monotonicity of θ, we obtain

$$1 < \theta(d(x_1,x_2)) \le \theta(d(x_0,x_1))^k. \tag{19}$$

Since f is a α-admissible and $\alpha(x_0,x_1) \ge 1$, we have $\alpha(x_1,x_2) \ge 1$.

If $x_2 \in fx_2$, this ends the proof; if $x_2 \notin fx_2$, we have $d(x_2, fx_2) > 0$.

From (10), we have

$$\begin{aligned} 1 < \theta(d(x_2,fx_2)) &\le \theta(sH(fx_1,fx_2)) \\ &\le \lambda\theta(M(x_1,x_2))^k - \varphi(M(x_1,x_2)) + LQ(x_1,x_2) \\ &\le \lambda\theta(M(x_1,x_2))^k + LQ(x_1,x_2) \\ &\le \theta(M(x_1,x_2))^k + LQ(x_1,x_2). \end{aligned} \tag{20}$$

As above, we obtain

$$1 < \theta(d(x_2,fx_2)) \le \theta(sH(fx_1,fx_2)) \le \theta(d(x_1,x_2))^k, \tag{21}$$

and then, there exists $x_3 \in fx_2$ such that

$$1 < \theta(d(x_2,x_3)) \le \theta(d(x_1,x_2))^k \le \theta(d(x_0,x_1))^{k^2}. \tag{22}$$

We will assume that there exists the sequence $\{x_n\}$ in X such that $x_n \ne x_{n+1} \in fx_n, \alpha(x_n,x_{n+1}) \ge 1$, using an inductive method.

And

$$1 < \theta(d(x_{n+1},x_{n+2})) \le \theta(d(x_0,x_1))^{k^n}, \tag{23}$$

for all $n \in \mathbb{N}$. This shows that $\lim_{n\longrightarrow\infty} \theta(d(x_n,x_{n+1})) = 1$ and so $\lim_{n\longrightarrow\infty} d(x_n,x_{n+1}) = 0$.

Prove that $\{x_n\}$ is a b-Cauchy sequence.

For $s > 1$, we have

$$\begin{aligned} 1 < \theta(d(x_{n+1},x_{n+2})) &\le \theta(sd(x_{n+1},x_{n+2})) \\ &\le \theta(sH(fx_n,fx_{n+1})) \le \theta(d(x_n,x_{n+1}))^k \\ &\le \theta(d(x_n,x_{n+1})), \end{aligned} \tag{24}$$

and using the monotonicity of θ, we shall have

$$d(x_{n+1},x_{n+2}) \le \mu d(x_n,x_{n+1}), \tag{25}$$

for all $n \in \mathbb{N}$, where $\mu = 1/s \in (0,1)$.

By Lemma 13, we conclude that $\{x_n\}$ is a b-Cauchy sequence.

For $s = 1$, we can use similar arguments as in the proof of Theorem 2.1 of [9] to prove that there exists $n_1 \in \mathbb{N}$ and $r \in (0,1)$ such that

$$d(x_n,x_{n+1}) \le \frac{1}{n^{1/r}}, \tag{26}$$

for all $n \geq n_1$. Then, for $m > n > n_1$, we have

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/r}}. \quad (27)$$

Since $0 < r < 1$, $\sum_{i=n}^{\infty} (1/i^{1/r})$ converges. Therefore, $d(x_n, x_m) \longrightarrow 0$ as $m, n \longrightarrow \infty$. It follows that $\{x_n\}$ is a Cauchy sequence in Ω; since Ω is α-complete b-metric space, there exists $x \in \Omega$ such that

$$\lim_{n\longrightarrow\infty} d(x_n, x) = 0. \quad (28)$$

Since f is α-continuous, we have

$$\lim_{n\longrightarrow\infty} H(fx_n, fx) = 0, \quad (29)$$

and hence,

$$\begin{aligned} d(x, fx) &\leq sd(x, x_{n+1}) + sd(x_{n+1}, fx) \\ &\leq sd(x_{n+1}, x) + sH(fx_n, fx), \end{aligned} \quad (30)$$

which implies that $d(x, fx) = 0$. And since fx is closed, we obtain $x \in fx$.

Assume that f has two fixed points u and v such $d(u, v) \neq 0$ and $\alpha(u, v) \geq 1$.

We have

$$\begin{aligned} 1 &\leq \theta(d(u, v)) \leq \theta(sH(fu, fv)) \\ &\leq \lambda\theta(M(u, v))^k - \varphi(M(u, v)) + LQ(u, v) \\ &\leq \lambda\theta(M(u, v))^k + LQ(u, v) \\ &\leq \theta(M(u, v))^k + LQ(u, v), \end{aligned} \quad (31)$$

where

$$\begin{aligned} M(u, v) &= \max \left\{d(u, v), d(u, fu), d(v, fv), \frac{d(u, fv) + d(v, fu)}{2s}\right\} = d(u, v), \\ Q(u, v) &= \min \{d(v, fv), d(u, fv), d(v, fu)\} = 0. \end{aligned} \quad (32)$$

Then,

$$1 < \theta(d(u, v)) \leq \theta(d(u, v))^k, \quad (33)$$

which is a contradiction since $k \in (0, 1)$; thus, $u = v$, which ends the proof. □

Corollary 17. *Let (Ω, d, s) be a b-complete b-metric space and $f : \Omega \longrightarrow CL(\Omega)$ is weak $(\alpha, k, \theta, \varphi)$-contractive multi-valued mapping. Assume that the following conditions hold:*

(i) f is a α-admissible

(ii) There exist x_0 and $x_1 \in fx_0$ such that $\alpha(x_0, x_1) \geq 1$

(iii) f is a α-continuous

Then, f has a fixed point.

Corollary 18. *Let $(\Omega, d, s \geq 1)$ be a α-complete b-metric space and $f : \Omega \longrightarrow CL(\Omega)$ is weak $(\alpha, k, \theta, \varphi)$-contractive multivalued mapping. Assume that the following conditions are true:*

(i) f is a α-admissible

(ii) There exist x_0 and $x_1 \in fx_0$ such that $\alpha(x_0, x_1) \geq 1$

(iii) f is a continuous

Then, f has a fixed point.

Corollary 19. *Let (Ω, d, s) be a α-complete b-metric space and $f : \Omega \longrightarrow CL(\Omega)$ is weak $(\alpha, k, \theta, \varphi)$-contractive multi-valued mapping. Assume that the following conditions are true:*

(i) f is a α_*-admissible

(ii) There exist x_0 and $x_1 \in fx_0$ such that $\alpha(x_0, x_1) \geq 1$

(iii) f is a continuous

Then, f has a fixed point.

Theorem 20. *Let (Ω, d, s) be a α-complete b-metric space and $f : \Omega \longrightarrow CL(\Omega)$ is a weak $(\alpha, k, \theta, \varphi)$-contractive multivalued mapping. Assume that the following conditions hold:*

(i) f is a α-admissible

(ii) There exists x_0 and $x_1 \in fx_0$ such that $\alpha(x_0, x_1) \geq 1$

(iii) If $\{x_n\}$ is sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \longrightarrow x \in X$ as $n \longrightarrow \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$

Then, f has a fixed point.

Proof. As in the proof of Theorem 16, we obtain a b-Cauchy sequence $\{x_n\}$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_n d(x_n, x) = 0$. Assume that $d(x, fx) > 0$.

Since f is weak $(\alpha, k, \theta, \varphi)$-contractive multivalued mapping, we have

$$\begin{aligned} \alpha(x_n, x) \geq 1 \Longrightarrow \theta(sd(x_{n+1}, fx)) &\leq \theta(sH(fx_n, fx)) \\ &\leq \lambda\theta(M(x_n, x))^k - \varphi(M(x_n, x)) + LQ(x_n, x) \\ &\leq \theta(M(x_n, x))^k + LQ(x_n, x), \end{aligned} \quad (34)$$

where

$$M(x_n, x) = \max \left\{d(x_n, x), d(x_n, fx_n), d(x, fx), \frac{d(x_n, fx) + d(x, fx_n)}{2s}\right\}$$

$$= \max\left\{d(x_n, x), d(x_n, x_{n+1}), d(x, fx), \frac{d(x_n, fx) + d(x, x_{n+1})}{2s}\right\}$$
$$\le \max\left\{d(x_n, x), d(x_n, x_{n+1}), d(x, fx), \frac{d(x_n, x) + d(x, fx)}{2} + \frac{d(x, x_{n+1})}{2s}\right\},$$
$$Q(x_n, x) = \min\{d(x, fx), d(x_n, fx), d(x, fx_n)\} = \min\{d(x, fx), d(x_n, fx), d(x, x_{n+1})\}. \quad (35)$$

If $n \longrightarrow \infty$, we obtain that

$$\limsup_{n\longrightarrow\infty} M(x_n, x) = d(x, fx), \qquad \limsup_{n\longrightarrow\infty} Q(x_n, x) = 0. \quad (36)$$

Since $\theta \in \Theta$, we obtain that

$$\begin{aligned}1 < \theta(d(x, fx)) &= \theta\left(d\left(s\frac{1}{s}d(x, fx)\right)\right) \le \theta\left(\limsup_{n\longrightarrow\infty} sd(x_{n+1}, fx)\right) \le \theta\left(\limsup_{n\longrightarrow\infty} sH(fx_n, fx)\right) \\ &\le \theta\left(\limsup_{n\longrightarrow\infty} M(x_n, x)\right)^k - \varphi\left(\liminf_{n\longrightarrow\infty} M(x_n, x)\right) \\ &\quad + L\limsup_{n\longrightarrow\infty} Q(x_n, x) = \theta\left(\limsup_{n\longrightarrow\infty} M(x_n, x)\right)^k \\ &\quad + L\limsup_{n\longrightarrow\infty} Q(x_n, x) = \theta(d(x, fx))^k,\end{aligned} \quad (37)$$

which is a contradiction since $k \in (0, 1)$. Thus, x is a fixed point of f. □

3. Examples

Example 4. Let $\Omega = [0, \infty)$, and take the b-metric $d(x, y) = |x - y|^2$ for all $x, y \in \Omega$. Define $f : \Omega \longrightarrow CL(\Omega)$, $\varphi : \Omega \longrightarrow \Omega$, and $\alpha : \Omega \times \Omega \longrightarrow [0, \infty)$ by the following:

$$fx = \left[1, 1 + \frac{x}{4}\right], \quad \alpha(x, y) = \begin{cases} e^{xy}, & \text{if } x \in [1 + y, \infty), \\ 0, & \text{otherwise}, \end{cases} \quad \varphi(x) = \frac{7}{10}x, \quad (38)$$

and $\theta(t) = e^{te^t}$. Note that for all $t \ge 1$,

$$e^{t/8e^{t/8}} \le \frac{476}{1000}e^{t/2e^t} - \frac{7}{10}t. \quad (39)$$

Now, for all $x, y \in X$ and $x \in [1 + y, \infty)$, we have $H(fx, fy) = |x - y|^2/16$.

Then,

$$\begin{aligned}\theta(sH(fx, fy)) &= e^{|x-y|^2/8e^{|x-y|^2/8}} = e^{d(x,y)/8e^{d(x,y)/8}} \le e^{M(x,y)/8e^{M(x,y)/8}} \\ &\le \frac{476}{1000}e^{M(x,y)/2e^{M(x,y)}} - \frac{7}{10}M(x, y) \\ &= \frac{476}{1000}(\theta(M(x, y)))^{1/2} - \varphi(M(x, y)) \\ &= \lambda(\theta(M(x, y)))^k - \varphi(M(x, y)).\end{aligned} \quad (40)$$

Then,

$$\theta(sH(fx, fy)) \le \lambda(\theta(M(x, y)))^k - \varphi(M(x, y)) + LQ(x, y). \quad (41)$$

Thus, f is weak $(\alpha, k, \theta, \varphi)$-contractive multivalued mapping, with $\lambda = 476/1000$ and $k = 1/2$.

So, all conditions of Theorem 16 are fulfilled, which implies that f has a fixed point, which is $x_0 = 1, 1742$ (in particular if $L = 0$).

Example 5. Let $\Omega = [0, \infty)$, and take the b-metric $d(x, y) = |x - y|^2$ for all $x, y \in \Omega$. Define $f : \Omega \longrightarrow CL(\Omega)$, $\varphi : \Omega \longrightarrow \Omega$, $\alpha : \Omega \times \Omega \longrightarrow [0, \infty)$, and $\theta : [0, \infty) \longrightarrow [1, \infty)$ by the following:

$$fx = \left[0, \frac{x}{4}\right], \quad \varphi(x) = \frac{1}{2}x, \quad \alpha(x, y) = \begin{cases} e^{x-y}, & \text{if } x \ge y + \frac{18}{10}, \\ 0, & \text{otherwise}, \end{cases} \quad (42)$$

and $\theta(t) = e^{te^t}$. Note that for all $t \ge 18/10$, one has

$$e^{t/8e^{t/8}} \le e^{\sqrt{t}/2e^{\sqrt{t}}} - \frac{1}{2}t. \quad (43)$$

Now, for all $x, y \in \Omega$ and $x \ge y + 18/10$, we have $H(fx, fy) = |x - y|^2/16$.

Then,

$$\begin{aligned}\theta(sH(fx, fy)) &= e^{|x-y|^2/8e^{|x-y|^2/8}} = e^{d(x,y)/8e^{d(x,y)/8}} \le e^{M(x,y)/8e^{M(x,y)/8}} \\ &\le e^{\sqrt{M(x,y)}/2e^{\sqrt{M(x,y)}}} - \frac{1}{2}M(x, y) \\ &= (\theta(M(x, y)))^{1/2} - \varphi(M(x, y)) \\ &= \lambda(\theta(M(x, y)))^k - \varphi(M(x, y)).\end{aligned} \quad (44)$$

Then,

$$\theta(sH(fx, fy)) \leq \lambda(\theta(M(x, y)))^k - \varphi(M(x, y)) + LQ(x, y). \tag{45}$$

Thus, f is weak $(\alpha, k, \theta, \varphi)$-contractive multivalued mapping, with $\lambda = 1$ and $k = 1/2$.

So, all conditions of Theorem 16 are verified, which implies that f has a fixed point, which is $x_0 = 0.7586$ (in particular if $L = 0$).

Example 6. Let $\Omega = [0, \infty)$, and take the b-metric $d(x, y) = |x - y|^2$ for all $x, y \in \Omega$. Define $f : \Omega \longrightarrow CL(\Omega)$, $\varphi : \Omega \longrightarrow \Omega$, $\alpha : \Omega \times \Omega \longrightarrow [0, \infty)$, and $\theta : [0, \infty) \longrightarrow [1, \infty)$ by the following:

$$\begin{aligned} fx &= \left[0, \frac{x}{4}\right], \\ \varphi(x) &= \frac{1}{7}x^2, \\ \alpha(x, y) &= \begin{cases} x - y, & \text{if } x \geq 2 + y, \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \tag{46}$$

and $\theta(t) = e^{\sqrt{te^t}}$. Note that for all $t \geq 2$, one has

$$e^{\sqrt{t/8e^{t/8}}} \leq \frac{7}{10}e^{\sqrt{te^t}/3} - \frac{1}{7}t^2. \tag{47}$$

Now, for all $x, y \in \Omega$ and $x \geq 2 + y$, we have $H(fx, fy) = |x - y|^2/16$.

Then,

$$\begin{aligned} \theta(sH(fx, fy)) &= e^{\sqrt{|x-y|^2/8e^{|x-y|^2/8}}} = e^{\sqrt{d(x,y)/8e^{d(x,y)/8}}} \leq e^{\sqrt{\frac{M(x,y)}{8}e^{M(x,y)/8}}} \\ &\leq \frac{7}{10}e^{\sqrt{M(x,y)e^{M(x,y)}}/3} - \frac{1}{7}(M(x, y))^2 \\ &= \frac{7}{10}(\theta(M(x, y)))^{1/3} - \varphi(M(x, y)) \\ &= \lambda(\theta(M(x, y)))^k - \varphi(M(x, y)). \end{aligned} \tag{48}$$

Then,

$$\theta(sH(fx, fy)) \leq \lambda(\theta(M(x, y)))^k - \varphi(M(x, y)) + LQ(x, y). \tag{49}$$

Thus, f is weak $(\alpha, k, \theta, \varphi)$-contractive multivalued mapping, with $\lambda = 7/10$ and $k = 1/3$.

So, all conditions of Theorem 16 are verified, which implies that f has a fixed point, which is $x_0 = 2, 69$ (in particular if $L = 0$).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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