Introduction and Preliminaries

Fixed point theory is one of the most active research areas of mathematics. The findings are used to solve problems in a wide variety of disciplines, including transportation theory, economics, and biomathematics. The development of functional analysis methods assisted in the broadening of the well-known Banach contraction theory (see [1–21] and references therein). One of them is the extension of the principle of contraction to cover applications and multivalued applications.

Samet et al. [20] presented the definition of $\alpha$-admissible mapping in complete metric spaces and partially ordered metric spaces in 2012 and proved several fixed point theorems under the generalized contraction in both. Following that, Hasanzade et al. [22] introduced the idea of $\alpha_*$-admissible mapping, which is the multivalued edition of $\alpha$-admissible single-valued mapping established in [20], and they provided a fixed point result for multivalued mappings in complete metric spaces satisfying many generalized contractive conditions.

In 2013, Mohammadi et al. [16] developed the notion of an $\alpha_*$-admissible mapping to the class of $\alpha$-admissible mappings.

In 2014, Jleli and Samet [11] expanded the contractive condition by considering the following function $\Theta$:

**Definition 1** ([11]). Let $\Theta$ denote the set of all functions $\theta : (0, \infty) \rightarrow (1, \infty)$ fulfilling the following criteria:

1. $\Theta_1$: $\theta$ is nondecreasing;
2. $\Theta_2$: for every sequence $\{z_n\} \subseteq \mathbb{R}^+$, $\lim_{n \rightarrow \infty} \theta(z_n) = 1$ if and only if $\lim_{n \rightarrow \infty} z_n = 0$;
3. $\Theta_3$: there exists $0 < r < 1$ and $l \in (0, \infty)$ such that $\lim_{z \rightarrow 0^+}(\theta(z) - 1)/z^r = l$;
4. $\Theta_4$: every $\theta \in \Theta$ is continuous.

**Definition 2** ([11]). Let $(\Omega, d)$ be a metric space. A mapping $F : \Omega \rightarrow \Omega$ is called a $\theta$-contraction if $\theta : (0, \infty) \rightarrow (1, \infty)$ fulfills $(\Theta_1) - (\Theta_3)$ and there exists a constant $k \in (0, 1)$ such that for all $x, y \in \Omega$,

$$d(Fx, Fy) \leq \theta(d(x, y))^k. \quad (1)$$

Several researchers extended (1) in various ways and proved fixed point theorems for a single and multivalued contractive mappings (see [7, 12, 17, 18, 23]).

Any Banach contraction is a $\theta$-contraction, although this is not so for the inverse ([7]).

**Definition 3** ([24, 25]). Let $s \geq 1$ be a given real number and $\Omega$ be a nonempty set. A function $d : \Omega \times \Omega \rightarrow \mathbb{R}^+$ is a $b$-metric if the following conditions are fulfilled for all $x, y, z \in \Omega$:
(bM₁): $d(x, x) = 0$ if and only if $x = y$;
(bM₂): $d(x, y) = d(y, x)$;
(bM₃): $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair $(Ω, d)$ is called a $b$-metric space.

It is worth noting that $b$-metric space is a broader category than metric spaces.

The following notations are used in this paper:

$$M(x, y) = \max \left\{ d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2s} \right\}. \quad (2)$$

$$Q(x, y) = \min \left\{ d(x, fx), d(x, fy), d(y, fx), d(y, fy) \right\}, \text{ and } Ψ \text{ is the set of all self mapping on } [0, +∞) \text{ such that } Ψ \text{ is continuous monotone and nondecreasing, and } Ψ(t) = 0 \iff t = 0.$$ 

Definition 4 ([26]). Let $(Ω, d)$ be a $b$-metric space. $X ∈ Ω$, and consider a sequence $\{u_n\}$ in $Ω$. Then,

(i) $\{u_n\}$ is $b$-convergent if there exists $u ∈ Ω$ such that $d(u_n, u) → 0$ as $n → ∞$.

In this case, we write $\lim_{n→∞} u_n = u$.

(ii) $\{u_n\}$ is $b$-Cauchy sequence if $d(u_n, u_m) → 0$ as $n, m → ∞$.

(iii) $X$ is closed if and only if for each sequence $\{u_n\}$ of points in $X$ with $d(u_n, u) → 0$ as $n → ∞$, we have $u ∈ X$.

Remark 5 ([26]). The following statements hold in a $b$-metric space $(Ω, d)$.

(1) A $b$-convergent sequence has a unique limit.
(2) Each $b$-convergent sequence is a $b$-Cauchy sequence.
(3) In general, a $b$-metric is not continuous.
(4) $d$ does not generally induce a topology on $Ω$.

Consider $(Ω; d; s)$ be a $b$-metric space and $CL(Ω)$ the class of nonempty closed subsets of $Ω$. On $CL(Ω)$, consider $H(.; .)$ be the generalized Pompeiu-Hausdorff $b$-metric (see [27]); i.e., for all $G; R ∈ CL(Ω), \begin{align*}
H(G; R) &= \max \left\{ \sup_{a∈G} d(a, R), \sup_{b∈R} d(G, b) \right\}, \text{ if the maximum exists,} \\
&= ∞, \text{ otherwise,} \quad (3)
\end{align*}$

where $d(a; R) = \inf \{ d(a; b) : b ∈ R \}$.

For $G; R ∈ CL(Ω)$, we put $δ(G; R) = \sup_{a∈G} d(a; R)$; we note that $δ(G; R) ≤ H(G; R)$.

Definition 6 ([26, 28]). Let $(Ω, d)$ and $(Y, δ)$ be two $b$-metric spaces.

(D₁): The space $(Ω, d)$ is $b$-complete if every $b$-Cauchy sequence in $Ωb$-converges.

(D₂): A function $f : Ω → Y$ is $b$-continuous at a point $u ∈ Ω$ if it is $b$-sequentially continuous at $u$; that is, whenever $\{u_n\}$ is $b$-convergent to $u$, $\{fu_n\}$ is $b$-convergent to $fu$.

Definition 7 ([16, 22]). Let $Ω$ be a nonempty set, $α : Ω × Ω → [0, +∞]$ be a given mapping, and $N(Ω)$ the class of all nonempty subsets of $Ω$.

(1) If the following condition applies, a mapping $f : Ω → N(Ω)$ is called $α$-admissible: for $x, y ∈ Ω, α(x, y) ≥ 1 ⇐⇒ α(f(x), f(y)) ≥ 1$, where $α_f(x, y) = \inf \{ α(a, b) : a ∈ fx, b ∈ fy \}$

(2) For each $x ∈ Ω$ and $y ∈ fx$ with $α(x, y) ≥ 1$, we have $α(y, z) ≥ 1$; for all $z ∈ fy$, a mapping $f : Ω → N(Ω)$ is said to be $α$-admissible.

Hussain et al. [8] in metric spaces used the concept of $α$-complete. We expand and apply it in $b$-metric spaces here.

Definition 8. Let $(Ω, d)$ be a $b$-metric space and $α : Ω × Ω → [0, +∞]$ be a mapping. The $b$-metric space $Ω$ is said to be $α$-complete if and only if every $b$-Cauchy sequence $\{u_n\}$ in $Ω$ with $α(u_n, u_{n+1}) ≥ 1$, for all $n ∈ N$, $b$-converges in $Ω$.

In 2015, Kutbi and Sintunavarat ([13]) introduced the notion of the $α$-continuity for multivalued mappings in metric spaces. We expand and apply it in $b$-metric spaces here.

Definition 9. Let $(Ω, d)$ be a $b$-metric space and $T : Ω → CL(Ω)$ and $α : Ω × Ω → [0, +∞]$ be two given mappings. We say $T$ is an $α$-continuous multivalued mapping on $(CL(Ω), H)$ if

(i) for each sequence $\{u_n\}$ with $u_n → u ∈ Ω$ as $n → ∞$, and

(ii) $α(u_n, u_{n+1}) ≥ 1$, for all $n ∈ N$, we have $Tu_n → Tu$ as $n → ∞$.

Remark 10.

(1) It is easy to see that $α$-admissibility implies $α$-admissibility. But the converse may not be true as shown in example 15 of [15].

(2) If $Ω$ is $b$-complete $b$-metric space, then $Ω$ is also $α$-complete $b$-metric space. But the converse is not true (see Example 1).

(3) Note that the $b$-continuity implies the $α$-continuity. In general, the converse is not true (see in Example 2).
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Example 1. Let \( \Omega = (0, \infty) \) and \( x, y \in \Omega \), and \( \Lambda \) be a closed subset of \( \Omega \). Define \( \alpha : \Omega \times \Omega \rightarrow [0, \infty) \) by the following:

\[
\alpha(x, y) = \begin{cases} 
(x - y)^2 + 1, & \text{if } x, y \in \Lambda, \\
0, & \text{otherwise}. 
\end{cases}
\]

Clearly, \((\Omega, d)\) is not a \(b\)-complete \(b\)-metric space, but it is an \(a\)-complete \(b\)-metric space.

Example 2. Let \( \Omega = [0, \infty) \) and \( d(x, y) = |x - y|^2 \) for all \( x, y \in \Omega \). Let \( T : \Omega \rightarrow CL(\Omega) \) and \( \alpha : \Omega \times \Omega \rightarrow [0, \infty) \) defined by the following:

\[
T = \begin{cases} 
\{ e^x \}, & \text{if } x \in [0, 1], \\
\{ \cos (\pi x) + 3 \}, & \text{if } x \in (1, \infty), 
\end{cases}
\]

\[
\alpha(x, y) = \begin{cases} 
2^x, & \text{if } x, y \in [0, 1], \\
0, & \text{otherwise}. 
\end{cases}
\]

Clearly, \( T \) is not a \(b\)-continuous multivalued mapping on \((CL(\Omega), d)\). Indeed, if \( \{z_n\} \in \Omega \) defined by \( z_n = 1 + 1/n \) for all \( n \geq 1 \), we see that \( z_n = 1 + 1/n \rightarrow d 1 \) but \( T z_n = \{ \cos (\pi/n) + 3 \} \rightarrow H(1) \neq \{1\} \); however, \( T \) is a \(a\)-continuous multivalued mapping on \((CL(\Omega), d)\). Indeed, if \( z_n \rightarrow z \) as \( n \rightarrow \infty \) with \( \alpha(z_n, z_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \), then \( z_n \rightarrow z \) in \( \|z_{n+1} - z_n\| \) for all \( n \in \mathbb{N} \) and so \( T z_n = \{ z_n \} \rightarrow z \) as \( n \rightarrow \infty \).

Lemma 11 [29]. Let \((\Omega, d, \theta)\) be a \(b\)-metric space, and let \( R \in CL(\Omega) \). If \( \xi \in \Omega \) and \( d(\xi; R) \leq \eta \), then there exists \( \nu \in \Omega \) such that \( d(\xi; \nu) \leq \eta \).

Lemma 12 [1]. Let \((\Omega, d, \theta)\) be a \(b\)-metric space, and let \( R \in CL(\Omega) \) and \( \xi \in \Omega \). Then, we have

\[
d(\xi; R) = 0 \text{ if and only if } \xi \in R = R,
\]

where \( R \) denotes the closure of \( R \).

Lemma 13 [30]. Let \( \{u_n\} \) be a sequence in a \(b\)-metric space \((\Omega, d, \theta)\) with \( s \geq 1 \) such that

\[
d(u_m, u_n) \leq \lambda d(u_{m-1}, u_n),
\]

for some \( \lambda \in (0, 1) \), and each \( n \in \mathbb{N} \). Then, \( \{u_n\} \) is a \(b\)-Cauchy sequence in \((\Omega, d, \theta)\).

Lemma 14 [31]. If \((\Omega, d)\) is a \(b\)-metric space with \( s \geq 1 \) and \( \{u_n\} \) and \( \{y_n\} \) are \(b\)-convergent to \( u \) and \( y \) respectively, we have

\[
\frac{1}{s} d(u, y) \leq \liminf_{n \rightarrow \infty} d(u, y_n) \leq \limsup_{n \rightarrow \infty} d(u, y_n) \leq s^2 d(u, y).
\]

In particular, if \( u = y \), then we have \( \lim_{n \rightarrow \infty} d(u_n, y_n) = 0 \).

Moreover, for each \( z \in \Omega \), we have

\[
\frac{1}{s} d(u, z) \leq \liminf_{n \rightarrow \infty} d(u_n, z) \leq \limsup_{n \rightarrow \infty} d(u_n, z) \leq s d(u, z).
\]

We will prove some new results of the fixed point in \(a\)-\(b\)-metric spaces, we will add a new kind of contraction for multivalued mappings called \((a, k, \theta, \varphi)\)-contraction multivalued mappings, and we give some examples to illustrate the main results of this paper.

2. Results

Definition 15. Let \((\Omega, d, \theta)\) be a \(b\)-metric space. A mapping \( f : \Omega \rightarrow CL(\Omega) \) called weak \((a, k, \theta, \varphi)\)-contractive multivalued mapping if there exist \( \alpha : \Omega \times \Omega \rightarrow [0, \infty) \), \( k \in (0, 1), L \geq 0, \varphi \in \Psi \) and constant \( \lambda \in (0, 1] \) such that

\[
\alpha(x, y) \geq 1 \implies \theta(sH(fx, fy)) \leq \lambda \theta(M(x, y)) + LQ(x, y),
\]

for all \( x, y \in \Omega \).

Example 3. Let \( \Omega = [12/10, \infty) \), and take the \(b\)-metric \( d(x, y) = |x - y|^2 \) for all \( x, y \in \Omega \). Define \( f : \Omega \rightarrow CL(\Omega), \varphi : [0, \infty) \rightarrow [0, \infty), a : \Omega \times \Omega \rightarrow [0, \infty), \) and \( \theta : [0, \infty) \rightarrow [1, \infty) \) by the following:

\[
f(x) = \left[ \begin{array} {ccc} 12/10 & 12/10 & x \end{array} \right],
\]

\[
\varphi(x) = \left\{ \begin{array} {l} 1/2 \rbrack x, & \text{if } x \in [12/10, y, \infty), \\
0, & \text{otherwise}, 
\end{array} \right.
\]

and \( \theta(t) = e^{dt} \). Note that for all \( t \geq 12/10 \), one has

\[
e^t \leq \frac{3}{10} e^{12t} - \frac{7}{10} t.
\]

Now, for all \( x, y \in \Omega \) such that \( x \in [12/10 + y, \infty) \), we have

\[
H(fx, fy) = |x - y|^2 / 16.
\]

Then,

\[
\theta(sH(fx, fy)) = \left( e^{(x-y)^2/16} - sH(fx, fy) \right)^{1/2} \leq \left( e^{M(x, y)} \right) e^{M(x, y)^s/16} \leq \lambda \left( \theta(M(x, y)) \right)^{1/2} \leq e^{M(x, y)^s/16}
\]

\[
= \frac{3}{10} \left( \theta(M(x, y)) \right)^{1/2} - \frac{7}{10} M(x, y)
\]

\[
= \lambda \left( \theta(M(x, y)) \right)^{1/2} - \varphi(M(x, y))
\]

\[
= \lambda \left( \theta(M(x, y)) \right)^{1/2} - \varphi(M(x, y)).
\]
And thus,
\[
\theta(sH(f, f')) \leq \lambda(\theta(M(x, y)))^k - \varphi(M(x, y)) + LQ(x, y).
\]
(14)

Thus, \( f \) is weak \((a, k, \theta, \varphi)\)-contractive multivalued mapping, with \( \lambda = 3/10 \) and \( k = 1/2 \).

The following is our primary result:

**Theorem 16.** Let \((\Omega, d, s)\) be a \(a\)-complete \(b\)-metric space and \( f : \Omega \rightarrow CL(\Omega) \) is a weak \((a, k, \theta, \varphi)\)-contractive multivalued mapping. Assume that the following conditions hold:

(i) \( f \) is a \(a\)-admissible

(ii) There exists \( x_0, x_1 \in \Omega \) such that \( x_1 \in f x_0 \) and \( a(x_0, x_1) \geq 1 \)

(iii) \( f \) is \(a\)-continuous.

Then, \( f \) has a fixed point. Moreover, if \( u \) and \( v \) are fixed points of \( f \) such that \( a(u, v) \geq 1 \), then \( u = v \).

**Proof.** If \( x_0 = x_1 \) or \( x_0 \neq x_1 \) and \( x_1 \in f x_0 \), then \( x_1 \) is a fixed point of \( f \).

Assume that \( x_0 \neq x_1 \) and \( x_1 \notin f x_0 \).

Since \( f \) is weak \((a, k, \theta, \varphi)\)-contractive multivalued mapping, we obtain

\[
1 < \theta(d(x_1, f x_1)) \leq \theta(sH(f x_0, f x_1)) \\
\leq \lambda \theta(M(x_0, x_1))^k - \varphi(M(x_0, x_1)) + LQ(x_0, x_1) \\
\leq \theta(M(x_0, x_1))^k + LQ(x_0, x_1),
\]

with

\[
M(x_0, x_1) = \max \left\{ d(x_0, x_1), d(x_0, f x_0), d(x_1, f x_0), \frac{d(x_0, f x_1) + d(x_1, f x_0)}{2s}, \max \{d(x_0, x_1), d(x_1, f x_1)\}, \min \{d(x_1, f x_1), d(x_1, f x_0), d(x_1, f x_0)\} = 0 \right\}.
\]

Assume that \( M(x_0, x_1) = d(x_1, f x_1) \), and from (15), we get

\[
1 < \theta(d(x_1, f x_1)) \leq \theta(sH(f x_0, f x_1)) \leq \theta(d(x_1, f x_1))^k, \tag{17}
\]

which is a contradiction; it follows that \( M(x_0, x_1) = d(x_0, x_1) \), and (15) becomes

\[
1 < \theta(d(x_1, f x_1)) \leq \theta(sH(f x_0, f x_1)) \leq \theta(d(x_0, x_1))^k. \tag{18}
\]

By Lemma 11, there exists \( x_2 \in f x_1 \) such that \( d(x_1, x_2) \leq d(x_0, x_1) \), and by the monotonicity of \( \theta \), we obtain

\[
1 < \theta(d(x_1, x_2)) \leq \theta(d(x_0, x_1))^k. \tag{19}
\]

Since \( f \) is \((a, k, \theta, \varphi)\)-admissible and \( a(x_0, x_1) \geq 1 \), we have \( a(x_1, x_2) \geq 1 \).

If \( x_2 \in f x_2 \), this ends the proof; if \( x_2 \notin f x_2 \), we have \( d(x_2, f x_2) > 0 \).

From (10), we have

\[
1 < \theta(d(x_2, f x_2)) \leq \theta(sH(f x_1, f x_2)) \\
\leq \lambda \theta(M(x_1, x_2))^k - \varphi(M(x_1, x_2)) + LQ(x_1, x_2) \\
\leq \theta(M(x_1, x_2))^k + LQ(x_1, x_2), \tag{20}
\]

As above, we obtain

\[
1 < \theta(d(x_2, f x_2)) \leq \theta(sH(f x_1, f x_2)) \leq \theta(d(x_1, x_2))^k, \tag{21}
\]

and then, there exists \( x_3 \in f x_2 \) such that

\[
1 < \theta(d(x_2, x_3)) \leq \theta(d(x_1, x_2))^k \leq \theta(d(x_0, x_1))^k. \tag{22}
\]

We will assume that there exists the sequence \( \{x_n\} \) in \( X \) such that \( x_n \neq x_{n+1} \in f x_n \), \( a(x_n, x_{n+1}) \geq 1 \), using an inductive method.

And

\[
1 < \theta(d(x_{n+1}, x_{n+2})) \leq \theta(d(x_0, x_1))^k, \tag{23}
\]

for all \( n \in \mathbb{N} \). This shows that \( \lim_{n \to \infty} \theta(d(x_n, x_{n+1})) = 1 \) and so
\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = 0.
\]

Prove that \( \{x_n\} \) is a \(b\)-Cauchy sequence.

For \( s > 1 \), we have

\[
1 < \theta(d(x_{n+1}, x_{n+2})) \leq \theta(sH(x_n, x_{n+1})) \\
\leq \theta(sH(f x_n, f x_{n+1})) \leq \theta(d(x_n, x_{n+1}))^k \\
\leq \theta(d(x_n, x_{n+1})), \tag{24}
\]

and using the monotonicity of \( \theta \), we shall have

\[
d(x_{n+1}, x_{n+2}) \leq \mu d(x_n, x_{n+1}), \tag{25}
\]

for all \( n \in \mathbb{N} \), where \( \mu = 1/s \in (0, 1) \).

By Lemma 13, we conclude that \( \{x_n\} \) is a \(b\)-Cauchy sequence.

For \( s = 1 \), we can use similar arguments as in the proof of Theorem 2.1 of [9] to prove that there exists \( n_1 \in \mathbb{N} \) and \( r \in (0, 1) \), such that

\[
d(x_n, x_{n+1}) \leq \frac{1}{n^{1/r}}, \tag{26}
\]

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for all \( n \geq n_1 \). Then, for \( m > n > n_1 \), we have

\[
d(x_m, x_n) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \frac{1}{i^{1/r}}.
\]

(27)

Since \( 0 < r < 1 \), \( \sum_{i=n}^{\infty} (1/i^{1/r}) \) converges. Therefore, \( d(x_m, x_n) \to 0 \) as \( m, n \to \infty \). It follows that \( \{x_n\} \) is a Cauchy sequence in \( \Omega \); since \( \Omega \) is \( \alpha \)-complete \( b \)-metric space, there exists \( x \in \Omega \) such that

\[
\lim_{n \to \infty} d(x_n, x) = 0.
\]

(28)

Since \( f \) is \( \alpha \)-continuous, we have

\[
\lim_{n \to \infty} H(fx_n, fx) = 0,
\]

(29)

and hence,

\[
d(x, fx) \leq sd(x, x_{n+1}) + sd(x_{n+1}, fx) \\
\leq sd(x_{n+1}, x) + sH(fx_n, fx),
\]

(30)

which implies that \( d(x, fx) = 0 \). And since \( fx \) is closed, we obtain \( x \in fx \).

Assume that \( f \) has two fixed points \( u \) and \( v \) such that \( d(u,v) \neq 0 \) and \( d(u,v) \geq 1 \).

We have

\[
1 \leq \theta(d(u, v)) \leq \theta(sH(fu, fv)) \\
\leq \lambda \theta(M(u, v))^k - \varphi(M(u, v)) + LQ(u, v) \\
\leq \lambda \theta(M(u, v))^k + LQ(u, v) \\
\leq \theta(M(u, v))^k + LQ(u, v),
\]

where

\[
M(u, v) = \max\{d(u, v), d(u, fu), d(v, fv), \\
\frac{d(u, fu) + d(v, fu)}{2s} \} = d(u, v),
\]

(32)

\[
Q(u, v) = \min\{d(v, fu), d(u, fu), d(v, fu)\} = 0.
\]

Then,

\[
1 < \theta(d(u, v)) \leq \theta(d(u, v))^k,
\]

(33)

which is a contradiction since \( k \in (0, 1) \); thus, \( u = v \), which ends the proof.

Corollary 17. Let \( (\Omega, d, s) \) be a \( b \)-complete \( b \)-metric space and \( f : \Omega \to \text{CL}(\Omega) \) is weak \( (\alpha, k, \theta, \varphi) \)-contractive multivalued mapping. Assume that the following conditions hold:

(i) \( f \) is a \( \alpha \)-admissible
(ii) There exist \( x_0 \) and \( x_1 \in fx_0 \) such that \( \alpha(x_0, x_1) \geq 1 \)
(iii) \( f \) is a \( \alpha \)-continuous

Then, \( f \) has a fixed point.

Corollary 18. Let \( (\Omega, d, s) \) be a \( \alpha \)-complete \( b \)-metric space and \( f : \Omega \to \text{CL}(\Omega) \) is weak \( (\alpha, k, \theta, \varphi) \)-contractive multivalued mapping. Assume that the following conditions are true:

(i) \( f \) is a \( \alpha \)-admissible
(ii) There exist \( x_0 \) and \( x_1 \in fx_0 \) such that \( \alpha(x_0, x_1) \geq 1 \)
(iii) \( f \) is a continuous

Then, \( f \) has a fixed point.

Corollary 19. Let \( (\Omega, d, s) \) be a \( \alpha \)-complete \( b \)-metric space and \( f : \Omega \to \text{CL}(\Omega) \) is weak \( (\alpha, k, \theta, \varphi) \)-contractive multivalued mapping. Assume that the following conditions are true:

(i) \( f \) is a \( \alpha \)-admissible
(ii) There exist \( x_0 \) and \( x_1 \in fx_0 \) such that \( \alpha(x_0, x_1) \geq 1 \)
(iii) \( f \) is a continuous

Then, \( f \) has a fixed point.

Theorem 20. Let \( (\Omega, d, s) \) be a \( \alpha \)-complete \( b \)-metric space and \( f : \Omega \to \text{CL}(\Omega) \) is a weak \( (\alpha, k, \theta, \varphi) \)-contractive multivalued mapping. Assume that the following conditions hold:

(i) \( f \) is a \( \alpha \)-admissible
(ii) There exists \( x_0 \) and \( x_1 \in fx_0 \) such that \( \alpha(x_0, x_1) \geq 1 \)
(iii) If \( \{x_n\} \) is sequence in \( X \) such that \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \) and \( x_n \to x \in X \) as \( n \to \infty \), then \( \alpha(x_n, x) \geq 1 \) for all \( n \in \mathbb{N} \)

Then, \( f \) has a fixed point.

Proof. As in the proof of Theorem 16, we obtain a \( b \)-Cauchy sequence \( \{x_n\} \) such that \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} d(x_n, x) = 0 \). Assume that \( d(x, fx) > 0 \).

Since \( f \) is weak \( (\alpha, k, \theta, \varphi) \)-contractive multivalued mapping, we have

\[
\alpha(x_n, x) \geq 1 \implies \theta(sd(x_{n+1}, fx)) \leq \theta(sH(fx_n, fx)) \\
\leq \lambda \theta(M(x_n, x))^k - \varphi(M(x_n, x)) + LQ(x_n, x) \\
\leq \lambda \theta(M(x_n, x))^k + LQ(x_n, x),
\]

(34)

where

\[
M(x_n, x) = \max\{d(x_n, x), d(x_n, fx_n), d(x, fx) \\
\frac{d(x_n, fx_n) + d(x, fx)}{2s} \}
\]
\[ f(x, y) = |x - y|^2 / 16. \]

Then,

\[ \theta(sH(f, f)) = e^{(y^2 - y^2 - T^2) / 8e} \leq e^{M(x, y) / 8e_2} \leq e^{M(x, y) / 8e_2}. \]

Thus, \( f \) is weak \((\alpha, k, \theta, \varphi)\)-contractive multivalued mapping, with \( \lambda = 476/1000 \) and \( k = 1/2 \).

So, all conditions of Theorem 16 are fulfilled, which implies that \( f \) has a fixed point, which is \( x_0 = 1, 1742 \) (in particular if \( L = 0 \)).

**Example 5.** Let \( \Omega = [0, \infty) \), and take the \( b \)-metric \( d(x, y) = |x - y|^2 \) for all \( x, y \in \Omega \). Define \( f : \Omega \rightarrow \text{CL}(\Omega), \varphi : \Omega \rightarrow \Omega, \) and \( \alpha : \Omega \times \Omega \rightarrow [0, \infty) \) by the following:

\[ f(x) = \begin{cases} 0, & x < \frac{X}{4}, \\ \frac{X}{4}, & x \geq \frac{X}{4}. \end{cases} \]

\[ \varphi(x) = \begin{cases} \frac{x}{2}, & x \leq \frac{X}{4}, \\ 0, & x > \frac{X}{4}. \end{cases} \]

\[ \alpha(x, y) = \begin{cases} e^{x^2 - y^2}, & x \geq y + \frac{18}{10}, \\ 0, & \text{otherwise}. \end{cases} \]

and \( \theta(t) = e^{t^2} \). Note that for all \( t \geq 18/10 \), one has

\[ e^{t^2} \leq e^{t^2} - \frac{1}{2} t. \]

Now, for all \( x, y \in \Omega \) and \( x \geq y + 18/10 \), we have \( H(f, f) = |x - y|^2 / 16. \)

Then,

\[ \theta(sH(f, f)) = e^{(y^2 - y^2 - T^2) / 8e} \leq e^{M(x, y) / 8e_2} \leq e^{M(x, y) / 8e_2}. \]

Thus, \( f \) is weak \((\alpha, k, \theta, \varphi)\)-contractive multivalued mapping, with \( \lambda = 476/1000 \) and \( k = 1/2 \).

So, all conditions of Theorem 16 are fulfilled, which implies that \( f \) has a fixed point, which is \( x_0 = 1, 1742 \) (in particular if \( L = 0 \)).

**Example 4.** Let \( \Omega = [0, \infty) \), and take the \( b \)-metric \( d(x, y) = |x - y|^2 \) for all \( x, y \in \Omega \). Define \( f : \Omega \rightarrow \text{CL}(\Omega), \varphi : \Omega \rightarrow \Omega, \) and \( \alpha : \Omega \times \Omega \rightarrow [0, \infty) \) by the following:

\[ f(x) = \begin{cases} 0, & x < \frac{X}{4}, \\ \frac{X}{4}, & x \geq \frac{X}{4}. \end{cases} \]

\[ \varphi(x) = \begin{cases} \frac{x}{2}, & x \leq \frac{X}{4}, \\ 0, & x > \frac{X}{4}. \end{cases} \]

\[ \alpha(x, y) = \begin{cases} e^{x^2 - y^2}, & x \geq y + \frac{18}{10}, \\ 0, & \text{otherwise}. \end{cases} \]

and \( \theta(t) = e^{t^2} \). Note that for all \( t \geq 18/10 \), one has

\[ e^{t^2} \leq e^{t^2} - \frac{1}{2} t. \]
Then,
\[ \theta(sH(f, g)) \leq \lambda(\theta(M(x, y)))^k - \phi(M(x, y)) + LQ(x, y). \]
(45)

Thus, \( f \) is weak \((\alpha, k, \theta, \phi)\)-contractive multivalued mapping, with \( \lambda = 1 \) and \( k = 1/2 \).

So, all conditions of Theorem 16 are verified, which implies that \( f \) has a fixed point, which is \( x_0 = 0.7586 \) (in particular if \( L = 0 \)).

Example 6. Let \( \Omega = [0, \infty) \), and take the \( b \)-metric \( d(x, y) = |x - y|^2 \) for all \( x, y \in \Omega \). Define \( f : \Omega \rightarrow CL(\Omega), \phi : \Omega \rightarrow \Omega, \alpha : \Omega \times \Omega \rightarrow [0, \infty) \), and \( \theta : [0, \infty) \rightarrow [1, \infty) \) by the following:
\[ f(x) = \begin{cases} 0, & \text{if } x \leq \frac{1}{4} \frac{\sqrt{x}}{2}, \\ \frac{1}{2} x^2, & \text{if } x \geq 2 + y, \\ 0, & \text{otherwise,} \end{cases} \]
and \( \theta(t) = \frac{\sqrt{t}}{\sqrt{t}} \). Note that for all \( t \geq 2 \), one has
\[ e^{\sqrt{\frac{\sqrt{t}}{\sqrt{t}}}} \leq 7 \frac{7/10}{1} t^2. \]
(47)

Now, for all \( x, y \in \Omega \) and \( x \geq 2 + y \), we have \( H(f, g) = |x - y|^2/16 \).

Then,
\[ \theta(sH(f, g)) = e^{\sqrt{|x-y|^2/16}} = e^{\sqrt{\frac{\sqrt{t}}{\sqrt{t}}}} \leq e^{\sqrt{\frac{\sqrt{t}}{\sqrt{t}}}} \]
\[ \leq 7 \frac{7/10}{1} (\theta(M(x, y)))^1/3 - \phi(M(x, y)) \]
\[ = 7 \frac{7/10}{1} (\theta(M(x, y)))^1/3 - \phi(M(x, y)) \]
\[ = \lambda(\theta(M(x, y)))^k - \phi(M(x, y)). \]
(48)

Then,
\[ \theta(sH(f, g)) \leq \lambda(\theta(M(x, y)))^k - \phi(M(x, y)) + LQ(x, y). \]
(49)

Thus, \( f \) is weak \((\alpha, k, \theta, \phi)\)-contractive multivalued mapping, with \( \lambda = 7/10 \) and \( k = 1/3 \).

So, all conditions of Theorem 16 are verified, which implies that \( f \) has a fixed point, which is \( x_0 = 2.69 \) (in particular if \( L = 0 \)).

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**References**


