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Research Article

Strong Convergence Theorem for Finding a Common Solution of Convex Minimization and Fixed Point Problems in CAT(0) Spaces

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In this paper, we introduce a proximal point algorithm for approximating a common solution of finite family of convex minimization problems and fixed point problems for k-demicontractive mappings in complete CAT(0) spaces. We prove a strong convergence result and obtain other consequence results which generalize and extend some recent results in the literature. We further provide a numerical example to illustrate the convergence behaviour of the sequence generated by our algorithm.

1. Introduction

Let (X, d) be a geodesic metric space and $f : X \longrightarrow (-\infty,\infty]$ be a proper, convex, and lower semicontinuous function. One of the problems arising in optimization theory is to find a minimizer of f, that is, find $z \in X$ such that

$$f(z) = \min_{k\in X} f(k). \quad (1)$$

We denote the set of minimizers of f by $\operatorname{argmin} f$. A strong tool used for solving such problems is known as the proximal point algorithm (PPA) which was first initiated by Martinet [1]. Rockafellar [2] generalized the study to Hilbert spaces H as follows: for $s_1 \in H$,

$$s_{n+1} = \operatorname{argmin}\left(f(p) + \frac{1}{2\sigma_n}\|s_n - p\|^2\right), \quad p \in H, \quad (2)$$

where $\sigma_n > 0$ for all $n \in \mathbb{Z}^+$. In 2013, Bačák [3] extended the PPA to CAT(0) spaces as follows: $s_1 \in X$

$$s_{n+1} = \operatorname{argmin}\left(f(p) + \frac{1}{2\sigma_n} d^2(p, s_n)\right), \quad p \in X, \quad (3)$$

where $\sigma_n > 0$ for all $n \in \mathbb{Z}^+$. It was shown that if f has a minimizer and $\liminf_{n\longrightarrow\infty}\sigma_n > 0$, then the sequence $\{s_n\}\Delta$-converges to its minimizer. Bačák [4] further employed a split version of the PPA for sum of convex operators in CAT(0) spaces. Furthermore, Cholamjiak [5] modified the PPA in CAT(0) space X using the Halpern method as follows: $s_1 \in X$,

$$\begin{cases} s_n = \underset{p\in Z}{\arg\min}\left(f(p) + \dfrac{1}{2\sigma} d^2(p, s_n)\right), \\ s_{n+1} = b_n u \oplus (1 - b_n)s_n, \, n \in \mathbb{Z}^+, \end{cases} \quad (4)$$

where $p \in X, \sigma > 0$, $\lim_{n\longrightarrow\infty} b_n = 0$, and $\sum_{n=1}^{\infty} b_n = \infty$. He showed that the sequence $\{s_n\}$ strongly converges to its minimizer.

On the other hand, one of the most fundamental findings in fixed point theory is the Banach contraction principle [6]. It has been used to prove the existence of the solutions of different nonlinear functional equations. Recently, efforts have been done to obtain a fixed point in partially ordered sets. Ran and Reurings (see [7]) generalized the Banach contraction to ordered metric space. Nieto and Rodriguez-Lopez [8] extended the results of fixed point theory in partially ordered metric space and used them to study the existence of the solution of differential equations. Fixed point theory in CAT(0) spaces was introduced by Kirk [9]; since then, many others have studied fixed point theory for different types of mappings, such as nonexpansive, Lipschitz, quasinonexpansive, k-strictly pseudocontractive, and k-demicontractive together with nonlinear optimization problems (see [10–15]).

Definition 1 (see [16]). Let C be a nonempty subset of a metric space (X, d) and $T : C \longrightarrow C$ be a nonlinear mapping. A point $x \in C$ is called a fixed point of T if $Tx = x$. The set of fixed points of T is denoted by $F(T)$, i.e., $F(T) = \{x \in C : T x = x\}$.

Definition 2 (see [17]). Let X be a Hadamard space and C be a nonempty closed and convex subset of X. A mapping $T : C \longrightarrow C$ is said to be

(i) a contraction, if there exists $\alpha \in (0, 1)$, such that

$$d(Tx, Ty) \leq \alpha d(x, y), \quad \forall x, y \in C, \tag{5}$$

when $\alpha = 1$, then T is said to be nonexpansive.

(ii) quasinonexpansive, if $F(T) \neq \emptyset$ and

$$d(p, Tx) \leq d(p, x), \quad \forall p \in F(T), x \in C. \tag{6}$$

(iii) k-strictly pseudocontractive, if there exists $k \in [0, 1)$ such that

$$d^2(Tx, Ty) \leq d^2(x, y) + k(d(x, Tx) + d(x, Ty))^2, \quad \forall x, y \in C. \tag{7}$$

(iv) k-demicontractive, if $F(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that

$$d^2(Tx, p) \leq d^2(x, p) + kd^2(Tx, x), \quad \forall x \in C, p \in F(T). \tag{8}$$

We note that the class of k-demicontractive mappings contains the class of nonexpansive, quasinonexpansive, and k-strictly pseudocontractive mappings (see [18, 19]).

In 2014, Cholamjiak and Abdou [20] modified (4) in CAT(0) spaces for approximating the minimizers of f and common fixed point of nonexpansive mappings T_1 and T_2 as follows: given $s_1 \in X$,

$$\begin{cases} q_n = \underset{p\in X}{\arg\min} \left(f(p) + \dfrac{1}{2\sigma_n} d(p, s_n) \right), \\ r_n = (1 - a_n)s_n \oplus a_n T_1 q_n, \\ s_{n+1} = (1 - b_n)T_1 s_m \oplus b_n T_2 r_n \forall n \in \mathbb{Z}^+. \end{cases} \tag{9}$$

Also, they established the Δ-convergence for sequence $\{s_n\}$ generated by (9). Motivated by the results of Liu and Chang [17], the authors in [21] introduced a modified PPA for finding a common solution of convex minimization problems and fixed point of nonexpansive mappings in CAT(0) spaces. In particular, the authors proved the Δ-convergence for the following results.

Theorem 3. *Let D be a nonempty closed convex subset of a complete CAT(0) space. Let $\{K_i\}_{i=1}^N$ be a finite family of proper, convex, and lower semicontinuous functions of D into $(-\infty, \infty]$ and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of D into itself. Suppose that $F = \bigcap_{i=1}^N \underset{a\in D}{\arg\min}\, k_i \bigcap \bigcap_{i=1}^N F(T_i)$ is nonempty. For $x_i \in D$, let $\{x_n\}$ be a sequence in D defined by*

$$\begin{cases} y_n^i = \underset{a\in D}{\arg\min} \left(k_i(a) + \dfrac{1}{2\sigma_n^i} d^2(a, x_n) \right), \\ z_n = \beta_n^0 \oplus \beta_n^1 y_n^1 \oplus \beta_n^2 y_n^2 \oplus \oplus \beta_n^N y_n^N, \\ w_n = r_n^0 z_n \oplus r_n^1 T_1 z_n \oplus r_n^2 T_2 z_n \oplus \oplus r_n^N T_N z_n, \\ x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) w_n, \quad \forall n \geq 1, \end{cases} \tag{10}$$

where $\{\alpha_n\}$, $\{\beta_n^i\}$, and $\{r_n^i\}$ are sequences in $(0, 1)$ such that $0 < a \leq \alpha_n, \beta_n^i, r_n^i \leq b < 1$, $\sum_{i=1}^N \beta_n^i = 1$, and $\sum_{i=1}^N r_n^i = 1$ for all $n \leq 1$ and $\{\lambda_n^i\} \subset (0,\infty)$ for all $n \geq 1, i = 1, 2, 3, N$. Then, the sequence $\{x_n\}$ generated by (1.3) Δ-converges to an element $w \in F$.

Recently, Chidume et al. [22] proved the following theorem for approximating the common fixed points of finite family of k-demicontractive mappings in CAT(0) spaces.

Theorem 4 (see [22]). *Let X be a nonempty closed convex subset of a complete CAT(0) space. Let $T_i : X \longrightarrow CB(X), i = 1, 2, \cdots, m$ be a family of demicontractive mappings with constants $k_j \in (0, 1)$, $i = 1, 2, \cdots, m$, such that $\cap_{i=1}^m F(T_i) \neq \emptyset$. Suppose that T_i is Δ-demiclosed at 0 for all $i = 1, 2, \cdots, m$ and $T_i(p) = \{p\}$ for all $p \in \cap_{i=1}^n F(T_i)$ for arbitrary $x_1 \in H$, define a sequence $\{x_n\}$ by*

$$x_{n+1} = \alpha_0 x_n \oplus \alpha_1 y_n^1 \oplus \alpha_2 y_n^2 \oplus \cdots \oplus \alpha_n y^m, \quad n \geq 1, \tag{11}$$

where $y_n^i \in T_i x_n, i = 1, 2, 3 \cdots, m, \alpha_0 \in (k, 1), \alpha_i \in (0, 1), i = 1, 2, 3 \cdots, m$, such that $\sum_{i=0}^m \alpha_i = 1$ and $k = \max\{k_i\}$. Then, $\{x_n\}$ Δ-converges to a point $p \in \cap_{i=1}^m F(T_i)$.

Motivated by the above results, in this paper, we introduce a modified Halpern method together with the PPA method for approximating a common solution of finite family of convex minimization problems and common fixed point of k-demicontractive mappings in complete CAT(0) spaces. We prove a strong convergence result and provide some consequence results which generalize some other related results in the literature. We also give a numerical example to show the convergence behaviour of the sequence generated by our algorithm.

2. Preliminaries

Definition 5 (see [23]). Let (X, d) be a metric space and x, y ∈ X. A geodesic path from x to y is a map $c : [0, 1] \longrightarrow X$ such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, 1]$. Its image is called a geodesic segment. If every pair of points $x, y \in X$ are joined by a geodesic, we call the space (X, d) a geodesic space, and a space (X, d) is said to be a uniquely geodesic if every pair of points of X are joined by exactly one geodesic segment. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) contains three points $x_1, x_2, x_3 \in X$ (vertices of Δ), and the geodesic segment between each pair of vertices (edge of Δ) of a comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3) \in (X, d)$ is a triangle $\bar{\Delta} \coloneqq \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d^2_{\mathbb{R}}(\bar{x}_i, \bar{x}_j) = d(\bar{x}_i, \bar{x}_j)$ for all $i, j \in \{1, 2, 3\}$. A geodesic space is said to be a CAT(0) space if for each geodesic triangle $\Delta(x_1, x_2, x_3)$ in X and its comparison $\bar{\Delta} = \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \mathbb{R}^2$, the CAT(0) space inequality

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}), \tag{12}$$

is satisfied for all $x, y \in \Delta$ and comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$. If y_0 is the midpoint and x, y_1, y_2 are points in a CAT(0) space, then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2} d^2(x, y_1) + \frac{1}{2} d^2(x, y_2) - \frac{1}{4} d(y_1, y_2). \tag{13}$$

Equation (13) is called the CN-inequality of Bruhat and Tits [23]. Moreover, a geodesic space is a CAT(0) space if and only if CN-inequality is satisfied. Examples of CAT(0) spaces are complete, simply connected Riemannian manifold having nonpositive sectional curvature, $\mathbb{R}$-Trees, Euclidean buildings, pre-Hilbert spaces, the complex Hilbert ball, with hyperbolic spaces. A complete CAT(0) space is called the Hadamard space. The normed linear spaces satisfy the CN-inequality if and only if it satisfies the parallelogram identity (see [24]). It is not so unexpected to have an inner product-like notion in Hadamard spaces.

Definition 6 (see [25]). The concept of quasilinearization was introduced by Berg and Nikolaev [25] as follows: let us first denote a pair $(a, b) \in X \times X$ by $\overleftarrow{ab}$ and call it a vector. Then, quasilinearization is defined as a map $\langle .,. \rangle : (X \times X) \times (X \times X) \longrightarrow \mathbb{R}$ defined by

$$\left\langle \overleftarrow{ab}, \overleftarrow{cd} \right\rangle = \frac{1}{2} d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d), \quad \forall a, b, c, d \in X. \tag{14}$$

It can be seen that $\langle \overleftarrow{ab}, \overleftarrow{cd} \rangle = \langle \overleftarrow{cd}, \overleftarrow{ab} \rangle$, $\langle \overleftarrow{ab}, \overleftarrow{cd} \rangle = -\langle \overleftarrow{ba}, \overleftarrow{cd} \rangle$, and $\langle \overleftarrow{ax}, \overleftarrow{cd} \rangle + \langle \overleftarrow{xb}, \overleftarrow{cd} \rangle = \langle \overleftarrow{ab}, \overleftarrow{cd} \rangle$ for all $a, b, c, d, x \in X$.

Also the Cauchy-Schwartz inequality is satisfied if and only if

$$\left\langle \overleftarrow{ab}, \overleftarrow{cd} \right\rangle \leq d(a, b) d(c, d), \quad \forall a, b, c, d \in X. \tag{15}$$

Lemma 7. *Let X be CAT(0) space, $w, x, y, z \in X$, and $t, s \in [0, 1]$. Then,*

(i) $d^2(tx \oplus (1 - t)y, z) \leq t^2 d^2(x, z) + (1 - t)^2 d^2(y, z) + 2t(1 - t)\langle \overleftarrow{xz}, \overleftarrow{yz} \rangle$ [26]

(ii) $d^2(tx \oplus (1 - t)y, z) \leq t d^2(x, z) + (1 - t) d^2(y, z) - t(1 - t) d^2(x, y)$ [10]

(iii) $d((1 - t)x \oplus ty, z) \leq (1 - t) d(x, z) + t d(y, z)$ [23]

Definition 8 (see [10]; demiclosedness principle). Let C be a nonempty closed and convex subset of a Hadarmard space X and $T : C \longrightarrow C$ be a mapping. T is said to be Δ -demiclosed at zero if there exists a bounded sequence $\{x_n\} \subset X$ such that Δ-converges to p and $\lim_{n\longrightarrow\infty} = 0$, then $p \in F(T)$.

The notion of asymptotic properties was introduced in a general framework of Hadamard space as follows.

Definition 9 (see [10]). Let $\{x_n\}$ be a bounded sequence in Hadamard space (X, d). For any $x \in X$, we put $r\{x, \{x_n\}\} = \lim_{n\longrightarrow\infty} \sup d(x, x_n)$.

(1) The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by $r(\{x_n\}) = \inf \{r(x, x_n) : x \in X\}$.

(2) The asymptotic centre $A(x_n)$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, x_n)\} = r(\{x_n\}). \tag{16}$$

In Hadamard space, $A(\{x_n\})$ consist of exactly one point.

Lemma 10 (see [10]). *If $\{x_n\}$ is a bounded sequence in Hadamard space with $A(\{x_n\}) = \{x\}$, $\{u_n\}$ is the subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Let $f : X \longrightarrow (-\infty, \infty]$ be the proper, convex, and lower semicontinuous function. For all $\lambda > 0$, we define the Moreau-Yosida resolvent of f in Hadamard space X as follows:

$$J_\lambda(x) = \underset{x\in X}{argmin} \left[f(y) + \frac{1}{2\lambda} d^2(y, x) \right] \quad \forall y \in X. \tag{17}$$

It was proven in [24] that the set $F(J_\lambda)$ of a fixed point of the resolvent J_λ associated with f coincides with the set argminf, i.e., the set of minimizers of f. Also, for $\lambda > 0$, the resolvent J_λ of f is nonexpansive (see [27]).

Lemma 11 (see [28]). *Let (X, d) be a Hadamard space and $f : X \longrightarrow (-\infty, \infty]$ be a proper, convex, and lower* *semicontinuous function. Then, for all $x, y \in X$ and $\lambda > 0$, we have*

$$\frac{1}{2\lambda} d^2(J_\lambda x, y) - \frac{1}{2\lambda} d^2(x, y) + \frac{1}{2\lambda} d^2(x, J_\lambda x) + f(J_\lambda x) \le f(y). \quad (18)$$

Lemma 12 (see [29]). *Let (X, d) be a CAT(0) space and $x \in X$. Let $x_0, x_1, x_2, \cdots, x_N \in X$ and $\gamma_0, \gamma_1, \cdots, \gamma_N$, be the real numbers in $[0, 1]$ such that $\sum_{i=0}^{N} \gamma_i = 1$. Then, the following inequality holds:*

$$d\left(x, \underset{i=0}{\overset{N}{\oplus}} \gamma_i x_i\right)^2 \le \sum_{i=0}^{N} \gamma_i d(x, x_i)^2 - \sum_{i,j=0, i\neq j}^{N} \gamma_i \gamma_j d(x_i, x_j)^2. \quad (19)$$

Lemma 13 (see [30]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{u_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} u_n < \infty$, $\{u_n\}$ and $\{t_n\}$ be a sequence of real numbers with $\limsup_{n\longrightarrow\infty} t_n \le 0$. Suppose that*

$$s_n + 1 \le (1 - \alpha_n) s_n + \alpha_n t_n + u_n, \quad (20)$$

for all $\in \mathbb{N}$. Then, $\lim_{n\longrightarrow\infty} s_n = 0$.

Lemma 14 (see [31]). *Let X be a complete CAT(0) space, $\{x_n\}$ a sequence in X, and $x \in X$, then $\{x_n\}$ Δ-convergence to x if and only if $\limsup_{n\longrightarrow\infty} \langle \overleftarrow{x_n y}, \overleftarrow{y_n y} \rangle \le 0$ for all $y \in X$.*

3. Iterative Algorithm

In this section, we present our main result which deals with approximation of the common solution of the finite family of convex minimization problems and common fixed point of k-demicontractive mappings in complete CAT(0) space. First, we give a coincise statement for our algorithm as follows.

Let X be a complete CAT(0) space and C be a nonempty closed convex subset of X. For $i = 1, 2, \cdots, N$, let $f_i : X \longrightarrow \mathbb{R} \bigcup \{+\infty\}$ be a finite family of proper, convex, and lower semicontinuous functions, and for $j = 1, 2, \cdots, M$, let $T_j : X \longrightarrow X$ be a finite family of k_j-demicontractive mappings such that T_j is demiclosed at zero. Suppose the solution set

$$\Omega = \bigcap_{i=1}^{N} \operatorname{argmin} f_i \cap \bigcap_{j=1}^{M} F(T_j) \neq \emptyset. \quad (21)$$

Let $\{\alpha_n\}$ and $\{\gamma_{n,j}\}_{j=0}^{M}$ be sequences in $(0, 1)$ such that

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = +\infty$

(C2) $\sum_{j=0}^{M} \gamma_{n,j} = 1$ and $\liminf_{n\longrightarrow\infty} (\gamma_{n,0} - k) > 0$, where $k = \max \{k_j\}, 1 \le j \le M$

(C3) $\liminf_{n\longrightarrow\infty} \lambda_{n,i} > 0$

Choose $x_0 \in X$ arbitrarily and let $\{x_n\}$ be a sequence generated by the following algorithm.

Algorithm 15.

$$\begin{cases} y_{n,1} = \underset{y\in X}{\operatorname{argmin}} \left(f_1(y) + \dfrac{1}{2\lambda_{n,1}} d^2(y, x_n) \right), \\ y_{n,2} = \underset{y\in X}{\operatorname{argmin}} \left(f_2(y) + \dfrac{1}{2\lambda_{n,2}} d^2(y, y_{n,1}) \right), \\ \vdots \\ y_{n,N-1} = \underset{y\in X}{\operatorname{argmin}} \left(f_{N-1}(y) + \dfrac{1}{2\lambda_{n,N-1}} d^2(y, y_{n,N-2}) \right), \\ y_n = y_{n,N} = \operatorname{argmin}_{y\in X} \left(f_N(y) + \dfrac{1}{2\lambda_{n,N}} d^2(y, y_{n,N-1}) \right), \\ z_n = \gamma_{n,0} y_n \oplus \underset{j=1}{\overset{M}{\oplus}} \gamma_{n,j} T_j y_n, \\ x_{n+1} = \alpha_n x_0 \oplus (1 - \alpha_n) z_n, \quad \forall n \ge 0. \end{cases} \quad (22)$$

We begin the proof of the convergence of Algorithm 15 with the following lemmas.

Lemma 16. *The sequence $\{x_n\}$ generated by Algorithm 15 is bounded.*

Proof. Let $p \in \Omega$, then $f_i(p) \le f_i(y)$ for all $y \in X, i = 1, 2, \cdots, N$ and $T_j p = p$ for all $j = 1, 2, \cdots, M$. Thus, we have

$$f_N(p) + \frac{1}{2\lambda_{n,N}} d(p, p)^2 \le f_N(y) + \frac{1}{2\lambda_{n,N}} d(y, p)^2, \quad \forall y \in X. \quad (23)$$

Hence, $p = J_{\lambda_{n,N}}(p)$ for $n \in \mathbb{N}$. Since $y_{n,N} = J_{\lambda_n,i}(y_{n,N-1})$ and $J_{\lambda_n,i}$ is nonexpansive, then we have

$$d(y_{n,N}, p) = d\left(J_{\lambda_{n,N}}(y_N - 1), J_{\lambda_{n,N}}(p)\right) \le d(y_{n,N-1}, p). \quad (24)$$

Similarly, $p = J_{\lambda_{n,N-1}}(p)$ for all $n \in \mathbb{N}_0$ and $y_{n,N-1} = J_{\lambda_n,i}(y_{n,N-2})$. Thus, we obtain

$$d(y_{n,N-1}, p) = d\left(J_{\lambda_{n,N-1}}(y_{N-2}), J_{\lambda_{n,N-1}}(p)\right) \le d(y_{n,N-2}, p) \vdots \le d(y_{n,1}, p). \quad (25)$$

Moreover, $y_{n,1} = J_{\lambda_{n,1}}(x_n)$, then we obtain

$$d(y_{n,1}, p) \le d\left(J_{\lambda_{n,1}} x_n, J_{\lambda_{n,1}} p\right) \le d(x_n, p). \quad (26)$$

Therefore, from (24), (25), and (26), we have

$$d(y_n, p) = d(y_{n,N}, p) \le d(x_n, p). \quad (27)$$

Furthermore, it follows from Lemma 7 (ii) and Lemma 12 that

$$\begin{aligned} d(z_n,p)^2 &= d\left(\gamma_{n,0}y_n \oplus \bigoplus_{j=1}^{M} \gamma_{n,j}T_jy_n, p\right)^2 \le \gamma_{n,0}d(y_n,p)^2 \\ &\quad + \sum_{j=1}^{M}\gamma_{n,j}d\left(T_jy_n,p\right)^2 \le \gamma_{n,0}d(y_n,p)^2 \\ &\quad + \sum_{j=1}^{M}\gamma_{n,j}d\left(T_jy_n,p\right)^2 - \sum_{j=1}^{M}\gamma_{n,0}\gamma_{n,j}d\left(y_n,T_jy_n\right)^2 \\ &\le \gamma_{n,0}d(y_n,p)^2 + \sum_{j=1}^{M}\gamma_{n,j}\left[d(y_n,p)+k_jd\left(y_n,T_jy_n\right)\right] \\ &\quad - \sum_{j=1}^{M}\gamma_{n,0}\gamma_{n,j}d\left(y_n,T_jy_n\right)^2 = d(y_n,p)^2 \\ &\quad + \sum_{j=1}^{M}\gamma_{n,j}k_jd\left(y_n,T_jy_n\right)^2 - \sum_{j=1}^{M}\gamma_{n,0}\gamma_{n,j}d\left(y_n,T_jy_n\right)^2 \\ &\le d(y_n,p)^2 - \sum_{j=1}^{M}\gamma_{n,j}\left(\gamma_{n,0}-k\right)d\left(y_n,T_jy_n\right)^2. \end{aligned} \tag{28}$$

Thus, from (C2), we obtain

$$d(z_n,p) \le d(y_n,p) \le d(x_n,p). \tag{29}$$

Therefore, from Lemma 7 (v), we get

$$\begin{aligned} d(x_{n+1},p) &= d(\alpha_nx_0 \oplus (1-\alpha_n)z_n,p) \le \alpha_nd(x_0,p) \\ &\quad + (1-\alpha_n)d(z_n,p) \le \alpha_nd(x_0,p) \\ &\quad + (1-\alpha_n)d(x_n,p) \le \max\{d(x_0,p),d(x_n,p)\} \vdots \\ &\le \max\{d(x_0,p),d(x_0,p)\} = d(x_0,p). \end{aligned} \tag{30}$$

This implies that $\{d(x_n,p)\}$ is bounded. Therefore, $\{x_n\},\{y_n\},\{z_n\}$ are bounded. □

Lemma 17. *Let $\{x_{n_k}\}$ be a subsequence of the sequence $\{x_n\}$ generated by Algorithm 15 such that $\liminf_{k\longrightarrow\infty}(d(x_{n_k+1},p) - d(x_{n_k},p)) \ge 0$ for all $p\in\Omega$. Thus, the following conclusions hold:*

(i) $\lim_{k\longrightarrow\infty} d(x_{n_k},p)$ *exists for all* $p\in\Omega$

(ii) $\lim_{k\longrightarrow\infty} d(y_{n_k,i},y_{n_k,i-1}) = 0$ *for* $i=1,2,\cdots,N$

(iii) $\lim_{k\longrightarrow\infty} d(y_{n_k},T_jy_{n_k}) = 0$ *for* $j=1,2,\cdots,M$

Proof. From (30), we have

$$\begin{aligned} 0 &\le \liminf_{k\longrightarrow\infty}\left(d\left(x_{n_k+1},p\right)-d\left(x_{n_k},p\right)\right) \\ &\le \liminf_{k\longrightarrow\infty}\left(\alpha_{n_k}\mathrm{d}(x_0,p)+\left(1-\alpha_{n_k}\right)d\left(x_{n_k},p\right)-d\left(x_{n_k},p\right)\right) \\ &\le \liminf_{k\longrightarrow\infty}\alpha_{n_k}\left(d(x_0,p)-d\left(x_{n_k},p\right)\right)=0. \end{aligned} \tag{31}$$

Hence, $\lim_{k\longrightarrow\infty}d(x_{n_k+1},x_{n_k})=0$. Therefore, $\lim_{k\longrightarrow\infty} d(x_{n_k},p)$ exists.

(ii) Also, from Lemma 7, we have

$$\begin{aligned} d^2\left(x_{n_k+1},p\right) &= d^2\left(\alpha_{n_k}x_0 \oplus \left(1-\alpha_{n_k}\right)z_{n_k},p\right) \le \alpha_{n_k}^2d^2(x_0,p) \\ &\quad + \left(1-\alpha_{n_k}\right)^2d^2\left(z_{n_k},p\right) + 2\alpha_{n_k}\left(1-\alpha_{n_k}\right)\left\langle \overleftarrow{x_0p},\overleftarrow{z_{n_k}p}\right\rangle \\ &\le \alpha_{n_k}d^2(x_0,p)+\left(1-\alpha_{n_k}\right)d^2\left(y_{n_k},p\right) \\ &\quad + 2\alpha_{n_k}\left(1-\alpha_{n_k}\right)\left\langle \overleftarrow{x_0p},\overleftarrow{z_{n_k}p}\right\rangle. \end{aligned} \tag{32}$$

Also from Lemma 11, we obtain

$$\begin{aligned} &\frac{1}{2\lambda_{n_k}}d\left(y_{n_k,1},p\right)^2 - \frac{1}{2\lambda_{n_k}}d\left(x_{n_k},p\right)^2 + \frac{1}{2\lambda_{n_k}}d\left(x_{n_k},y_{n_k,1}\right)^2 \\ &\quad \le f_1(p)-f_1\left(y_{n_k,1}\right). \end{aligned} \tag{33}$$

Thus, from (32), we get

$$\begin{aligned} d^2\left(x_{n_k},y_{n_k,1}\right) &\le d^2\left(x_{n_k},p\right)-d^2\left(y_{n_k,1},p\right)=d^2\left(x_{n_k},p\right) \\ &\quad - d^2\left(x_{n_k+1},p\right)+d^2\left(x_{n_k+1},p\right)-d^2\left(y_{n_k,1},p\right) \\ &\le d^2\left(x_{n_k},p\right)-d^2\left(x_{n_k+1},p\right)+\alpha_{n_k}d^2(x_0,p) \\ &\quad + \left(1-\alpha_{n_k}\right)d^2\left(y_{n_k},p\right)+2\alpha_{n_k}\left(1-\alpha_{n_k}\right)\left\langle \overleftarrow{x_0p},\overleftarrow{z_{n_k}p}\right\rangle \\ &\quad - d^2\left(y_{n_k,1},p\right)=d^2\left(x_{n_k},p\right)-d^2\left(x_{n_k+1},p\right) \\ &\quad + \alpha_{n_k}\left[d^2(x_0,p)-d\left(y_{n_k,1},p\right)+2\left(1-\alpha_{n_k}\right)\left\langle \overleftarrow{x_0p},\overleftarrow{z_{n_k}p}\right\rangle\right]. \end{aligned} \tag{34}$$

Therefore,

$$\lim_{k\longrightarrow\infty} d\left(x_{n_k},y_{n_k,1}\right)=0. \tag{35}$$

Now, we show that $d(y_{n_k,1},y_{n_k,2})=0$. From the fact that $J_{\lambda_{n_k},1}(x_{n_k})=y_{n_k,1}$ and $J_{\lambda_{n_k},1}(y_{n_k,1})=y_{n_k,2}$, we get

$$d\left(y_{n_k,1},y_{n_k,2}\right)=d\left(J_{\lambda_{n_k},1}\left(x_{n_k}\right),J_{\lambda_{n_k},1}\left(y_{n_k,1}\right)\right)\le d\left(x_{n_k},y_{n_k,1}\right). \tag{36}$$

Hence,

$$\lim_{k\longrightarrow\infty} d\left(y_{n_k,1}, y_{n_k,2}\right) = 0. \quad (37)$$

Continuing in this manner, we get

$$d\left(y_{n_k,N-1}, y_{n_k,N}\right) = d\left(J_{\lambda_{n_k},N-1}\left(y_{n_k-2}\right), J_{\lambda_{n_k},N-1}\left(y_{n_k,N-1}\right)\right) \leq d\left(y_{n_k,N-2}, y_{n_k,N-1}\right). \quad (38)$$

Hence, by induction,

$$\lim_{k\longrightarrow\infty} d\left(y_{n_k,i-1}, y_{n_k,i}\right) = 0, \quad \text{for } i = 1, 2, \cdots, N. \quad (39)$$

In addition, using the triangle inequality with (35), (37), and (39), we get the following:

$$d\left(x_{n_k}, y_{n_k}\right) = d\left(x_{n_k}, y_{n_k,N}\right) \leq d\left(x_{n_k}, y_{n_k,1}\right) + d\left(y_{n_k,1}, y_{n_k,2}\right) + \cdots + d\left(y_{n_k,N-1}, y_{n_k,N}\right). \quad (40)$$

Therefore,

$$\lim_{k\longrightarrow\infty} d\left(x_{n_k}, y_{n_k}\right) = 0. \quad (41)$$

(iii) Now, we show that

$$\lim_{n\longrightarrow\infty} d\left(y_{n_k}, T_j y_{n_k,i-1}\right) = 0. \quad (42)$$

From (28), we have
$d^2(z_{n_k}, p) \leq d^2(y_n, p) - \sum_{j=1}^{M} \gamma_{n,j}(\gamma_{n_k,0} - k_j) d(y_{n_k}, T_j y_{n_k}).$
(30)
Thus,

$$\sum_{j=1}^{M} \gamma_j\left(\gamma_{n_k,0} - k_j\right) d\left(y_{n_k}, T_j y_n\right) \leq d^2\left(y_{n_k}, p\right) - d^2\left(z_{n_k}, p\right) \leq d^2\left(x_{n_k}, p\right) - d^2\left(z_{n_k}, p\right) = d^2\left(x_{n_k}, p\right) - d^2\left(x_{n_k+1}, p\right) + d^2\left(x_{n_k+1}, p\right) - d^2\left(z_{n_k}, p\right). \quad (43)$$

From (32), we have

$$\sum_{j=1}^{M} \gamma_j\left(\gamma_{n_k,0} - k_j\right) d\left(y_{n_k}, T_j y_{n_k}\right) \leq d^2\left(x_{n_k}, p\right) - d^2\left(x_{n_k+1}, p\right) + \alpha_{n_k} d^2(x_0, p) + \left(1 - \alpha_{n_k}\right) d^2\left(z_{n_k}, p\right) + 2\alpha_{n_k}\left(1 - \alpha_{n_k}\right)\left\langle \overleftarrow{x_0 p}, \overleftarrow{z_{n_k} p}\right\rangle - d\left(z_{n_k}, p\right)^2 = d^2\left(x_{n_k}, p\right) - d^2\left(x_{n_k+1}, p\right) + \alpha_{n_k}\left[d^2(x_0, p) - d\left(z_{n_k}, p\right) + 2\left(1 - \alpha_{n_k}\right)\left\langle \overleftarrow{x_0 p}, \overleftarrow{z_{n_k} p}\right\rangle\right]. \quad (44)$$

Hence,

$$\lim_{n\longrightarrow\infty} \sum_{j=1}^{M} \gamma_j\left(\gamma_{n_k,0} - k_j\right) d\left(y_n, T_j y_{n_k}\right) = 0. \quad (45)$$

This implies that

$$\lim_{n\longrightarrow\infty} d\left(y_{n_k}, T_j y_{n_k,i-1}\right) = 0. \quad (46)$$

□

Now, we prove the strong convergence of the iterative scheme.

Theorem 18. *The sequence $\{x_n\}$ generated by Algorithm 15 converges strongly to an element $p \in \Omega$, where $\Omega = \bigcap_{i=1}^{N} \operatorname{argmin} f_i \cap \bigcap_{j=1}^{M} F(T_j)$.*

Proof. Let $p \in \Omega$. From Lemma 7 (i), we obtain

$$d^2(x_{n+1}, p) = d^2(\alpha_n x_0 \oplus (1 - \alpha_n) z_n, p) \leq \alpha_n^2 d^2(x_0, p) + (1 - \alpha_n)^2 d^2(z_n, p) + 2\alpha_n(1 - \alpha_n)\left\langle \overleftarrow{x_0 p}, \overleftarrow{z_n p}\right\rangle. \quad (47)$$

Since

$$d(z_n, p) \leq d(x_n, p), \quad (48)$$

so from (47), we obtain

$$d^2(x_{n+1}, p) \leq \alpha_n^2 d^2(x_0, p) + (1 - \alpha_n)^2 d^2(x_n, p) + 2\alpha_n(1 - \alpha_n)\left\langle \overleftarrow{x_0 p}, \overleftarrow{z_n p}\right\rangle. \quad (49)$$

Moreover, since $\alpha_n \in (0, 1)$, we have

$$d^2(x_{n+1}, p) \leq (1 - \alpha_n) d^2(x_n, p) + \alpha_n\left(\alpha_n d^2(x_0, p) + 2(1 - \alpha_n)\left\langle \overleftarrow{x_0 p}, \overleftarrow{z_n p}\right\rangle\right). \quad (50)$$

Hence, from (50), we have

$$S_{n+1} \leq (1-\alpha_n)S_n + \alpha_n b_n + c_n, \tag{51}$$

where

$$\begin{aligned} S_n &= d^2(x_{n+1}, p), \\ b_n &= \alpha_n d^2(x_0, p) + 2(1-\alpha_n)\left\langle \overleftarrow{x_0 p}, \overleftarrow{z_n p} \right\rangle \\ c_n &= 0. \end{aligned} \tag{52}$$

Obviously, $d^2(x_n, p) \geq 0$, and from condition (C1), we have $\sum_{n=0}^{\infty}\alpha_n = +\infty$. Also, since $\alpha_n \longrightarrow 0$, we obtain $\alpha_n d^2(x_n, p) \longrightarrow 0$, and from Lemma 14, we have $\limsup_{n\longrightarrow\infty}\langle \overleftarrow{x_0 p}, \overleftarrow{z_n p} \rangle \leq 0$. Thus, $\limsup_{n\longrightarrow\infty} b_n \leq 0$. Clearly, $\sum_{n=0}^{\infty} c_n < \infty$. Therefore, from Lemma 13, we obtain that $\lim_{n\longrightarrow\infty} S_n = 0$. This implies that $\lim_{n\longrightarrow\infty} d(x_n, p) = 0$. Therefore, $\{x_n\}$ converges strongly to $p \in \Omega$. □

The following can be obtained as consequence results from our main theorem.

(i) If $N = M = 1$, then we have the following result for approximating a common solution of minimization and fixed point problem for a demicontractive mapping T.

Corollary 19. *Let X be a complete CAT(0) space and C be a nonempty closed convex subset of X. Let $f : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a proper, convex, and lower semicontinuous function, and let $T : C \longrightarrow C$ be k-demicontractive mappings such that $I - T$ is demiclosed at zero. Suppose $\Gamma = argminf \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\gamma_n\}_{j=0}^{M}$ be sequences in $(0,1)$ such that*

(C1) $\lim_{n\longrightarrow\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = +\infty$

(C2) $\liminf_{n\longrightarrow\infty}(\gamma_n - k) > 0$

(C3) $\liminf_{n\longrightarrow\infty}\lambda_n > 0$

Choose $x_0 \in X$ arbitrarily and let $\{x_n\}$ be a sequence generated by the following algorithm.

Algorithm 20.

$$\begin{cases} y_n = \underset{y\in X}{\operatorname{argmin}}\left(f(y) + \dfrac{1}{2\lambda_n} d^2(y, x_n)\right), \\ z_n = \gamma_n y_n \oplus (1-\gamma_n) T y_n, \\ x_{n+1} = \alpha_n x_0 \oplus (1-\alpha_n) z_n, \quad \forall n \geq 1. \end{cases} \tag{53}$$

Then, the sequence $\{x_n\}$ converges strongly to a solution $p \in \Gamma$.

Also, changing T_j from k_j demicontractive into quasinonexpansive mapping, we have the following result.

Corollary 21. *Let X be a complete CAT(0) space and C be a nonempty closed convex subset of X. For $i = 1, 2, \cdots, N$, let $f_i : X \longrightarrow \mathbb{R}\bigcup\{+\infty\}$ be a finite family of proper, convex, and lower semicontinuous functions, and for $j = 1, 2, \cdots, M$, let $T_j : C \longrightarrow C$ be a finite family of quasinonexpansive mappings such that $I - T_j$ is demiclosed at zero. Suppose*

$$\Omega = \bigcap_{i=1}^{N} argminf_i \cap \bigcap_{j=1}^{M} F(T_j) \neq \emptyset. \tag{54}$$

Let $\{\alpha_n\}$ and $\{\gamma_{n,j}\}_{j=0}^{M}$ be sequences in $(0,1)$ such that

(C1) $\lim_{n\longrightarrow\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = +\infty$

(C2) $\sum_{j=0}^{M}\gamma_{n,j} = 1$ *and* $\liminf_{n\longrightarrow\infty}(\gamma_{n,i}) > 0$

(C3) $\liminf_{n\longrightarrow\infty}\lambda_{n,i} > 0$

Choose $x_0 \in X$ arbitrarily and let $\{x_n\}$ be a sequence generated by the following algorithm.

Algorithm 22.

$$\begin{cases} y_{n,1} = \underset{y\in X}{\operatorname{argmin}}\left(f_1(y) + \dfrac{1}{2\lambda_{n,1}} d^2(y, x_n)\right), \\ y_{n,2} = \operatorname{argmin}_{y\in X}\left(f_2(y) + \dfrac{1}{2\lambda_{n,2}} d^2(y, y_{n,1})\right), \\ \vdots \\ y_{n,N-1} = \operatorname{argmin}_{y\in X}\left(f_{N-1}(y) + \dfrac{1}{2\lambda_{n,N-1}} d^2(y, y_{n,N-2})\right), \\ y_n = y_{n,N} = \operatorname{argmin}_{y\in X}\left(f_N(y) + \dfrac{1}{2\lambda_{n,N}} d^2(y, y_{n,N-1})\right), \\ z_n = \gamma_{n,0} y_n \oplus \overset{M}{\underset{j=1}{\oplus}} \gamma_{n,j} T_j y_n, \\ x_{n+1} = \alpha_n x_0 \oplus (1-\alpha_n) z_n, \quad \forall n \geq 1. \end{cases} \tag{55}$$

Then, the sequence $\{x_n\}$ converges strongly to a solution $p \in \Omega$.

4. Numerical Example

In this section, we present a computational example to illustrate the convergence behaviour of the proposed iterative scheme. We perform the numerical experiment using MATLAB programming.

Example 23. Let $X=\mathbb{R}^m$ (Euclidean m-dimensional space) with the Euclidean norm defined by

$$\|x\|=\left(\sum_{i=1}^{m}|x_i|^2\right)^{1/2}, \tag{56}$$

and inner product

$$\langle x,y\rangle=\sum_{i=1}^{m}x_iy_i \quad \forall x,y\in\mathbb{R}^m. \tag{57}$$

It is clear that X is a complete CAT(0) space. Let

$$C=\{x=(x_1,x_2,\cdots,x_m)\in\mathbb{R}^m : -5\le x_i\le 5, i=1,2,\cdots,m\}. \tag{58}$$

Also, for $i=1,2,3,\cdots,N$, let $f_i : X\longrightarrow\mathbb{R}\cup\{+\infty\}$ be defined by

$$f_i(x)=\frac{1}{2}\|A_ix-b_i\|^2, \tag{59}$$

where A_i is a $m\times m$ matrix and b_i is a $m\times 1$ vector for all $i=1,2,\cdots,N$. It is easy to see that f_i is proper, convex, and lower semicontinuous.

Also, for $j=1,2,\cdots,M$, let $T_j:\mathbb{R}^m\longrightarrow\mathbb{R}^m$ be defined by

$$T_jx=\frac{-(j+1)}{2}x,\quad \forall x\in\mathbb{R}^m, j=1,2,\cdots,M. \tag{60}$$

We need to show that T_j is a k_j-demicontractive mapping, for all $j=1,2,3,\cdots,M$.

Recall that T_j is k_j-demicontactive if

$$F(T_j)\neq\varnothing,$$
$$d^2(T_jx,p)\le d^2(x,p)+k_jd^2(x,T_jx)\forall \quad x\in\mathbb{R}, p\in F(T_j), j=1,2,3,\cdots,M. \tag{61}$$

Clearly,

$$0=(0,0,\cdots,0_m)\in F(T_j)\quad \text{for all } j=1,2,3,\cdots,M. \tag{62}$$

Also,

$$d^2(T_jx,p)=\left\|\frac{-(j+1)x}{2}-0\right\|^2=\left\|\frac{-(j+1)x}{2}\right\|^2=\frac{(j+1)^2}{4}\|x\|^2. \tag{63}$$

Table 1: Computation result for Example 23.

	Algorithm 15	Algorithm (3)
Case I		
Iter.	7	12
Time (s)	0.1281	0.1708
Case II		
Iter.	15	24
Time (s)	2.2554	5.8941
Case III		
Iter.	14	22
Time (s)	1.5914	4.8935
Case IV		
Iter.	16	26
Time (s)	1.6134	4.0589

Then,

$$\begin{aligned}
d^2(x,p)&=\|x-0\|^2=\|x\|^2,\\
d^2(x,T_jx)&=\left\|x-\left(-\frac{(j+1)x}{2}\right)\right\|^2=\left\|x+\frac{(j+1)x}{2}\right\|^2\\
&=\left\|\frac{2x+(j+1)x}{2}\right\|^2=\left\|\frac{3x+jx}{2}\right\|^2=\left\|\frac{(3+j)x}{2}\right\|^2\\
&=\frac{(3+j)^2}{4}\|x\|^2.
\end{aligned} \tag{64}$$

Now, since

$$\begin{aligned}
d^2(T_jx,p)&=\frac{(j+1)^2}{4}\|x\|^2=\|x\|^2+\frac{(j^2+2j-3)}{4}\|x\|^2\\
&=\|x\|^2+\frac{(j^2+2j-3)}{(9+6j+j^2)}\times\frac{(3+j)^2}{4}\|x\|^2\\
&=\|x\|^2+\frac{j-1}{j+3}\|x-T_jx\|^2.
\end{aligned} \tag{65}$$

Therefore,

$$d^2(T_jx,p)=d^2(x,p)+k_jd^2(x,T_jx), \tag{66}$$

where $k_j=(j-1)/(j+3)$. Hence, T_j is k_j-demicontractive, where $k_j=(j-1)/(j+3)$.

Since $f_j(0)\le f_j(x)$; for all $x\in\mathbb{R}^m$, and $j=1,2,\cdots,m$, then $0\in\operatorname{argmin}f_j$. Hence,

$$0\in\Omega=\bigcap_{i=0}^{N}\operatorname{argmin}f_j\cap\bigcap_{j=0}^{M}F(T_j)\neq\varnothing. \tag{67}$$

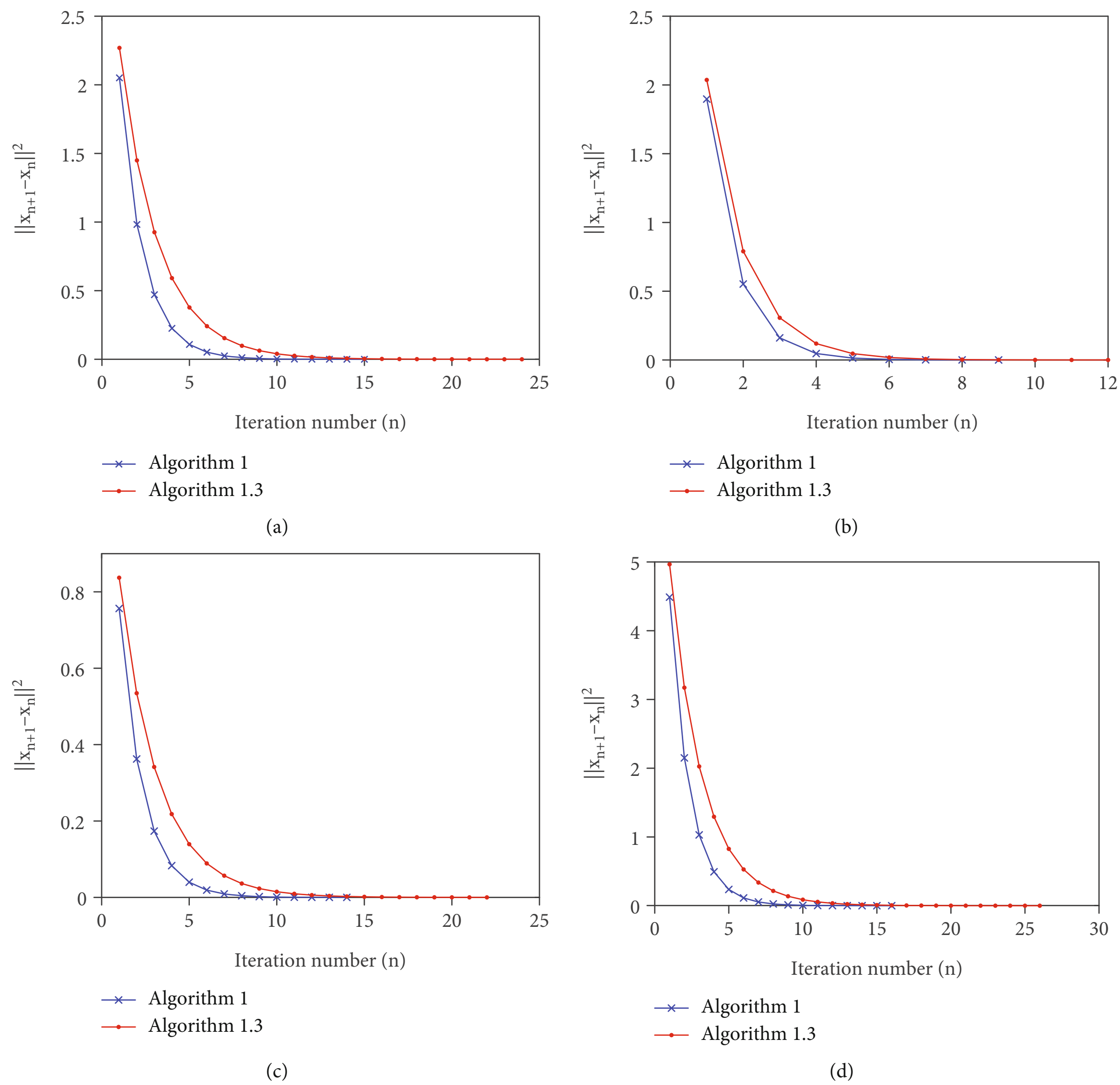

FIGURE 1: Example 23: (a) Case I; (b) Case II; (c) Case III; (d) Case IV.

We choose $\alpha_n = 1/n$, $\gamma_{n,j} = 1/(M+1)$; clearly,

$$\lim_{n\longrightarrow\infty} \alpha_n = 0, \ \sum_{i=0}^{\infty} \alpha_n = +\infty, \ \sum_{j=0}^{M} \gamma_{n,j} = 1. \tag{68}$$

Since $k = \max\{k_j\} = (M-1)/(M+3)$, then

$$\liminf_{n\longrightarrow\infty} (\gamma_{n,0} - k) = \liminf_{n\longrightarrow\infty} \left(\frac{1}{M+1} - \frac{M-1}{M+3} \right) > 0, \tag{69}$$

for $M > 2$. Also, $\lambda_{n,i} = (n+i)/(5n+9)$, $\lim_{n\longrightarrow\infty} \alpha_{n,i} > 0$. Hence, conditions (1)–(3) are satisfied. In this case, our algorithm becomes

$$\begin{cases} y_{n,1} = \operatorname*{argmin}_{y\in X} \left(f_1(y) + \dfrac{5n+9}{2(n+1)} d^2(y, x_n) \right), \\ y_{n,2} = \operatorname{argmin}_{y\in X} \left(f_2(y) + \dfrac{5n+9}{2(n+2)} d^2(y, y_{n,1}) \right), \\ \vdots \\ y_{n,N-1} = \operatorname{argmin}_{y\in X} \left(f_{N-1}(y) + \dfrac{5n+9}{2(n+N-1)} d^2(y, y_{n,N-2}) \right), \\ y_n = y_{n,N} = \operatorname{argmin}_{y\in X} \left(f_N(y) + \dfrac{5n+9}{2(n+N)} d^2(y, y_{n,N-1}) \right), \\ z_n = \dfrac{1}{M+1} y_n + \displaystyle\sum_{j=1}^{M} \dfrac{1}{M+1} \left(-\dfrac{(j+1)}{2} y_n \right), \\ x_{n+1} = \dfrac{1}{n} x_0 + \left(\dfrac{n-1}{n} \right) z_n, \quad \forall n \geq 1. \end{cases} \tag{70}$$

In our computation, the initial values are generated randomly in $\mathbb{R}^m$ where the values of m, N, and M are as follows:

(i) Case I: $m = 10, N = 10, M = 30$

(ii) Case II: $m = 15, N = 10, M = 20$

(iii) Case III: $m = 20, N = 5, M = 10$

(iv) Case IV: $m = 50, N = 25, M = 10$

We compare the performance of Algorithm 15 with Algorithm (3) (which refers to Iterative scheme 10 in Theorem 3), choosing $T_i(x) = x/2i, \alpha_n = 1/(n+1), \beta_n^i = 1/(N+1), r_n^i = 1/2i$, and $\lambda_{n,i} = (n+i)/(5n+9)$. We also use $\|x_{n+1} - x_n\| < 10^{-4}$ as the tolerance level for our computation. The numerical results are shown in Table 1 and Figure 1.

5. Conclusions

In this paper, we introduced a modified Halpern-type algorithm for approximating the common solution of the finite family of convex minimization problems and fixed point problems for k-demicontractive mappings in CAT(0) spaces. A strong result is proven under a mild condition, and a numerical example is provided to illustrate the convergence behaviour of the sequence generated by our iterative scheme. This result improves and generalizes many other recent results such as [32] from Hilbert to CAT(0) spaces. We are hopeful that this result can be extend to the split minimization, split inclusion, and the split fixed problems (see, e.g., [32–34]) in CAT(0) spaces. We wish to consider this in our future research.

Data Availability

No data is need for this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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