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Research Article

Convergence of Mann and Ishikawa Iterative Processes for Some Contractions in Convex Generalized Metric Space

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In this paper, we consider a JS metric space endowed with convexity structure, which will allow us to examine and study convergence of Mann iteration and Ishikawa iteration for Banach type and Chatterjea type contractions defined on JS metric space.

1. Introduction

Convexity structure in normed spaces and vector topological spaces has stimulated mathematicians to enlarge and generalize this important notion to metric space and generalized metric spaces. This work has been achieved by different ways and various directions. In 1970 W. Takahashi [1] opened up a new line of investigation by introducing an abstract convexity structure in metric space and obtained fixed point theorems for nonexpansive mappings; his results generalize fixed point theorems previously proved by Browder [2] and Kirk [3].

In recent years, many papers studied Mann [4] and Ishikawa [5] processes in various ways for single and multivalued contractive mappings. For more details in this direction, we refer the reader to [6–13].

In 2015, Jleli and Samet [14] introduced a new generalization of metric space that also recovers dislocated metric spaces [15], b-metric spaces [16], and modular metric spaces with Fatou property [17, 18]. Since then, fixed point theory has been widely studied in this new framework, and the reader may refer to [19–22].

In [23], the authors have introduced a Mann type iteration method; they also obtained a result about strongly convergence of this iteration method to a fixed point of nonexpansive mappings on Banach spaces. They provide a result about strong convergence of the iteration method to a common fixed point of two mappings on uniform convex Banach spaces via a new iteration method based on Ishikawa iteration idea.

Motivated by above, in this paper, we consider a JS metric space endowed with convexity structure introduced by Takahashi; this will allow us to examine and study convergence of Mann iteration [4] and Ishikawa iteration [5] for Banach type and Chatterjea type contractions defined on JS metric space.

In the next, we recall the basic definitions which are used throughout in this paper. For every $x \in X$, let us define the set

$$C(D, X, x) = \left\{ (x_n) \subset X : \lim_{n \longrightarrow +\infty} D(x_n, x) = 0 \right\}. \tag{1}$$

Definition 1 (see [14]). We say that $D : X \times X \longrightarrow [0,+\infty]$ is a JS metric on X if it satisfies the following conditions:

(D_1) For every $(x, y) \in X \times X$, we have $D(x, y) = 0 \Rightarrow x = y$

(D_2) For every $(x, y) \in X \times X$, we have $D(x, y) = D(y, x)$

(D_3) There exists $C > 0$ such that if $(x, y) \in X \times X$, $(x_n) \in C(D, X, x)$, then

$$(x, y) \leq C \lim_{n \longrightarrow +\infty} \sup D(x_n, y) \tag{2}$$

Also, we say the pair (X, D) is a JS metric space.

We mention that convergent sequences and Cauchy sequences can be introduced in a similar manner as in metric space.

Definition 2 (see [14]). Let (X, D) be a JS metric space. Let (x_n) be a sequence in X and $x \in X$.

(i) We say that (x_n) converges to x, if $\lim_{n\longrightarrow\infty} D(x_n, x) = 0$

(ii) We say that (x_n) is Cauchy sequence if $\lim_{m,n\longrightarrow\infty} D(x_n, x_m) = 0$

(iii) (X, D) is said to be complete, if every Cauchy sequence in X is convergent to some element in X

In the following, we introduce convexity structure defined on JS metric space. Notice that properties and consequences may differ from those arising from convexity structure defined on ordinary metric spaces; this dissimilarity occurs because of absence of triangular inequality in JS metric space.

Definition 3. Let (X, D) be a JS metric space, and $I = [0, 1]$. A mapping $w : X \times X \times I \longrightarrow X$ is said to be convex structure on X, if for any $(x, y, \lambda) \in X \times X \times I$ and $u \in X$, the following inequality holds:

$$D(u, w(x, y, \lambda)) \leq \lambda D(u, x) + (1 - \lambda) D(u, y). \tag{3}$$

If (X, D) is a JS metric space with a convex structure w, then (X, D, w) is called a convex JS metric space.

In the sequel, we are interested to study the convergence of Mann and Ishikawa iteration procedures for k-contraction mappings and Chatterjea mappings in the frame of JS metric space. We recall that

(i) Mann iteration procedure [4] is defined by

$$x_0 \in X, x_{n+1} = w(x_n, f x_n, \alpha_n), n = 0, 1, 2 \cdots \tag{4}$$

(ii) Ishikawa iteration procedure [5] is defined as follows

$$\begin{cases} x_0 \in X, \\ y_n = w(x_n, f x_n, \beta_n), n = 0, 1, 2, \cdots, \\ x_{n+1} = w(x_n, f y_n, \alpha_n), \end{cases} \tag{5}$$

where $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 0}$ satisfy $\alpha_n, \beta_n \in [0, 1]$, for all $n \in \mathbb{N}$.

2. Main Results

2.1. Convergence of Mann and Ishikawa Iterations for k-Contraction. We first need to state a basic lemma that can be found in [24] page 45.

Lemma 4. *Let $(a_n)_n$ and $(b_n)_n$ be two bounded real sequences such that $(b_n)_n$ converges to $b \geq 0$. Then, $\limsup_{n\longrightarrow\infty} (a_n b_n) = b \limsup_{n\longrightarrow\infty} (a_n)$.*

Lemma 5. *Let $(u_n)_n$ and $(\alpha_n)_n$ be two positive sequences such that*

$$\alpha_n \in [0, 1] \text{ and } u_{n+1} \leq \alpha_n u_n, \tag{6}$$

for all $n \in \mathbb{N}$. Then, we have the following:

(1) The sequence $(u_n)_n$ converges to some $r \geq 0$

(2) If $\limsup_{n\longrightarrow\infty} \alpha_n < 1$, then $\lim_{n\longrightarrow\infty} u_n = 0$

Proof.

(1) The sequence $(u_n)_n$ is positive; then, from (6), we get $0 \leq u_{n+1} \leq u_n$, for all $n \in \mathbb{N}$. Hence, $(u_n)_n$ is decreasing and bounded from below. As a result, $(u_n)_n$ converges to some $r \geq 0$

(2) By (1), the sequence $(u_n)_n$ has a limit $r \geq 0$. Now, suppose that $r > 0$, and by using Lemma 4, we get $r \leq r \limsup_{n\longrightarrow\infty} \alpha_n$; it follows that $1 \leq \limsup_{n\longrightarrow\infty} \alpha_n$, which is a contradiction, and we conclude that $\lim_{n\longrightarrow\infty} u_n = 0$.

□

Lemma 6. *Let $(\alpha_n)_n$ be a sequence in $[0, 1]$.*
If $t \in]0, 1]$ and $\sum_{i=0}^{\infty} \alpha_i$ diverges, then $\prod_{i=0}^{\infty} (1 - t\alpha_i) = 0$.

Proof. For all $t \in]0, 1]$ and $x \in [0, 1]$, we have $0 \leq 1 - tx \leq \exp(-tx)$, which yields $0 \leq \prod_{i=0}^{n} 1 - t\alpha_i \leq \exp(-t\sum_{i=0}^{n} \alpha_i)$.

Since $\sum_{i=0}^{\infty} \alpha_i = \infty$, we get $\lim_{n\longrightarrow\infty} \exp(-t\sum_{i=0}^{n} \alpha_i) = 0$.

Hence, it follows that $\prod_{i=0}^{\infty} (1 - t\alpha_i) = 0$. □

Definition 7 (Definition 3.1 in [14]). Let (X, D) be a JS metric space and $f : X \longrightarrow X$ be a mapping. Let $k \in (0, 1)$. We say that f is a k-contraction if

$$D(f(x), f(y)) \leq k D(x, y), \text{for every } (x, y) \in X \times X. \tag{7}$$

Remark 8. Let f be a k-contraction on X.

The ordering in inequality (7) is defined in extended real number system. Note that if we have $D(f(x), f(y)) = \infty$ for some elements $x, y \in X$, then $D(x, y) = \infty$.

For every $x \in X$, let

$$\delta(D, f, x) = \sup \left\{ D\left(f^i x, f^j x\right) \mid i, j \in \mathbb{N} \right\}. \tag{8}$$

Theorem 9 (Theorem 3.3 in [14]). *Suppose that the following conditions hold:*

(i) (X, D) is complete

(ii) f is a k-contraction for some $k \in [0, 1)$

(iii) There exists $x_0 \in X$ such that $\delta(D, f, x_0) < \infty$

Then, $(f^n x_0)$ *converges to* $\omega \in X$*, a fixed point of* f*. Moreover, if* $\omega' \in X$ *is another fixed point of* f *such that* $D(\omega, \omega') < \infty$*, then* $\omega = \omega'$.

Our first essential main result is given as follows:

Theorem 10. *Let* (X, D, w) *be a complete convex JS metric space and* f *a* k*-contraction for some* $k \in [0, 1)$*. Suppose that there exists* $x_0 \in X$ *such that* $\delta(D, f, x_0) < \infty$*. If* $\limsup_{n\longrightarrow\infty} \alpha_n < 1$*, then Mann iterative process defined in (4) converges to a fixed point of* f.

Proof. All conditions of Theorem 9 are satisfied, so $Fix(f) \neq \emptyset$.

Let $\omega \in Fix(f)$ and $(u_n)_n$ the sequence defined by $u_{n+1} = w(u_n, fu_n, \alpha_n)$ and $u_0 = x_0$.

Set $d_n := D(u_n, \omega)$ for all $n \in \mathbb{N}$. The proof goes in two steps.

Step 1: we shall prove that $(d_n)_n$ is bounded.

For every $p \in \mathbb{N}$, we have

$$D(u_0, f^p x_0) \leq \delta(D, f, x_0). \tag{9}$$

Now suppose that for every $p \in \mathbb{N}$, $D(u_n, f^p x_0) \leq \delta(D, f, x_0)$. We have

$$\begin{aligned} D(u_{n+1}, f^p x_0) = D(w(u_n, fu_n, \alpha_n), f^p x_0) &\leq \alpha_n D(u_n, f^p x_0) \\ &+ (1-\alpha_n) D(fu_n, f^p x_0) \leq \alpha_n D(u_n, f^p x_0) \\ &+ k(1-\alpha_n) D(u_n, f^{p-1} x_0) \\ &\leq (\alpha_n + k(1-\alpha_n))\delta(D, f, x_0) \leq \delta(D, f, x_0). \end{aligned} \tag{10}$$

We conclude that

$$D(u_n, f^p x_0) \leq \delta(D, f, x_0), \text{for every } n, p \in \mathbb{N}. \tag{11}$$

By Theorem 9, we have $(f^p x_0) \in C(D, X, \omega)$ and using (D_3), we get

$$d_n = D(u_n, \omega) \leq C \limsup_{p\longrightarrow+\infty} D(u_n, f^p x_0). \tag{12}$$

Hence, by (11), we obtain

$$d_n \leq C\delta(D, f, x_0), \text{for all } n \in \mathbb{N}. \tag{13}$$

Therefore, $(d_n)_n$ is bounded.

Step 2: $(d_n)_n$ converges to 0 (i.e., $(u_n)_n$ converges to ω).

From (3) and (7), we have

$$\begin{aligned} d_{n+1} = D(\omega, u_{n+1}) = D(\omega, w(u_n, fu_n, \alpha_n)) &\leq \alpha_n D(\omega, u_n) \\ + (1-\alpha_n) D(\omega, fu_n) &\leq \alpha_n D(\omega, u_n) + (1-\alpha_n) D(f\omega, fu_n) \\ &\leq \alpha_n D(\omega, u_n) + (1-\alpha_n) k D(\omega, u_n) \\ &\leq [\alpha_n + (1-\alpha_n)k] D(\omega, u_n) \leq [k + (1-k)\alpha_n] d_n. \end{aligned} \tag{14}$$

Denote $\beta_n = k + (1-k)\alpha_n$, and we obtain

$$d_{n+1} \leq \beta_n d_n. \tag{15}$$

Since $\limsup_{n\longrightarrow\infty} \alpha_n < 1$, we get $\limsup_{n\longrightarrow\infty} \beta_n < 1$.

Hence, by Lemma 5, it follows that $\lim_{n\longrightarrow+\infty} d_n = \lim_{n\longrightarrow+\infty} D(\omega, u_n) = 0$.

Therefore, Mann iterative process converges to the fixed point ω. □

Theorem 11. *Let* (X, D, w) *be a complete convex JS metric space and* f *a* k*-contraction for some* $k \in [0, 1)$*. Suppose that there exists* $x_0 \in X$ *such that* $\delta(D, f, x_0) < \infty$*. If* $\sum_{i=0}^{\infty} 1 - \alpha_i$ *diverges, then Mann iterative process defined in (4) converges to a fixed point of* f.

Proof. We proceed as in the proof of Theorem 10. By the first step, the sequence $(d_n)_n$ is bounded.

Also by (15), we have

$$d_{n+1} \leq \beta_n d_n \text{ for all } n \in \mathbb{N}, \tag{16}$$

with $\beta_n = k + (1-k)\alpha_n$, which can be rewritten as follows:

$$\beta_n = 1 - (1-k)(1-\alpha_n). \tag{17}$$

Hence, inductively, we get

$$d_{n+1} \leq d_0 \prod_0^n [1 - (1-k)(1-\alpha_n)]. \tag{18}$$

Then, since $(\alpha_n)_n \subset [0, 1]$, $0 < 1-k < 1$ and $\sum_{i=0}^{\infty} 1 - \alpha_i = \infty$, it follows by Lemma 6 that $d_n \longrightarrow 0$ as $n \longrightarrow \infty$. So, Mann iterative process converges to the fixed point ω, as desired. □

Remark 12. If $x_0 \in X$ is an element such that $D(x_0, x_0) = 0$ and $\alpha_n = 0$ for all $n \in \mathbb{N}$, then Mann iteration is simply reduced to Picard iteration. Indeed, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} D(u_{n+1}, f^{n+1} x_0) &= D(w(u_n, fu_n, \alpha_n), f^{n+1} x_0) \\ &\leq \alpha_n D(u_n, f^{n+1} x_0) + (1-\alpha_n) D(fu_n, f^{n+1} x_0). \end{aligned} \tag{19}$$

By (11), take $p = n+1$, and we obtain $D(u_n, f^{n+1} x_0) < \infty$; hence,

$$
\begin{aligned}
D\left(u_{n+1}, f^{n+1}x_0\right) &\le kD\left(u_n, f^n x_0\right)\\
&\vdots\\
&\le k^{n+1}D(x_0, x_0) = 0.
\end{aligned}
\tag{20}
$$

Therefore, by (D_2), we obtain $u_{n+1} = f^{n+1}x_0$, for all $n \in \mathbb{N}$. Also, we have $u_0 = x_0$; thus, $u_n = f^n x_0$, for all $n \in \mathbb{N}$.

In [14], Jleli and Samet showed that every b-metric space, with coefficient $s \ge 1$, is a JS metric space for $C = s$, and every modular space with the Fatou property is a JS metric space for $C = 1$. Hence, the following Corollaries are immediate consequences of Theorems 10 and 11.

Corollary 13. *Let (X, D, s, w) be a complete convex b-metric space with constant $s \ge 1$ and f a k-contraction for some $k \in [0, 1)$. If either $\limsup_{n\longrightarrow\infty} \alpha_n < 1$ or $\sum_{i=0}^{\infty} 1 - \alpha_i$ diverges, then Mann iterative process defined in (4) converges to a fixed point of f.*

Remark 14. The condition that there exists $x_0 \in X$ such that $\delta(D, f, x_0) < \infty$ is omitted. Since for each $x_0 \in X$ and for each $i, j \in \mathbb{N}$ with $i \le j$, we have

$$
\begin{aligned}
d\left(f^i x_0, f^j x_0\right) &\le sd\left(f^i x_0, f^{i+1}x_0\right) + \cdots + s^{j-i} d\left(f^{j-1}x_0, f^j x_0\right)\\
&\le sk^i d(x_0, fx_0) + \cdots + s^{j-i}k^{j-1} d(x_0, fx_0)\\
&\le sk^i \frac{\left(1 - (sk)^{j-i}\right)}{(1 - sk)} d(x_0, fx_0) \le \frac{s}{1 - sk} d(x_0, fx_0).
\end{aligned}
\tag{21}
$$

Therefore, for each $x_0 \in X$, we have $\delta(D, f, x_0) < \infty$.

Corollary 15. *Let (X_ρ, ρ) be a complete convex modular space and $f : X \longrightarrow X$ be a mapping. Suppose that the following conditions hold:*

(i) ρ satisfies the Fatou property

(ii) For some $k \in (0, 1)$, we have

$$
\rho(f(x) - f(y)) \le k\rho(x - y), \text{for every } (x, y) \in X_\rho \times X_\rho \tag{22}
$$

(iii) There exists $x_0 \in X_\rho$ such that

$$
\sup\left\{\rho\left(f^i(x_0) - f^j(x_0)\right) : i, j \in \mathbb{N} < \infty\right\} \tag{23}
$$

(iv) $\limsup_{n\longrightarrow\infty} \alpha_n < 1$ or $\sum_{i=0}^{\infty} 1 - \alpha_i$ diverges

Then, Mann iterative process defined in (4) converges to a fixed point of f.

In our next theorem, we give a convergence result of Ishikawa iteration for k-contraction considered in a complete convex JS metric space.

Theorem 16. *Let (X, D, w) be a complete convex JS metric space and f a k-contraction for some $k \in [0, 1)$. Suppose that there exists $x_0 \in X$ such that $\delta(D, f, x_0) < \infty$. If either $\limsup_{n\longrightarrow\infty} \alpha_n < 1$ or $\sum_{i=0}^{\infty} 1 - \alpha_i$ diverges, then Ishikawa iterative process defined in (5) converges to a fixed point of f.*

Proof. By Theorem 9, there exists some $\omega \in Fix(f)$. Let $(x_n)_n$ and $(y_n)_n$ be the sequences defined in (5) and set $d_n := D(x_n, \omega)$ for all $n \in \mathbb{N}$. Now, we complete the proof in the following steps.

Step 1: we shall prove that $(d_n)_n$ is bounded.

For every $p \in \mathbb{N}$, we have $D(x_0, f^p x_0) \le \delta(D, f, x_0)$.

Now suppose that $D(x_n, f^p x_0) \le \delta(D, f, x_0)$ for every $p \in \mathbb{N}$.

From (3) and (7), we have

$$
\begin{aligned}
D\left(x_{n+1}, f^p x_0\right) = D\left(w(x_n, fy_n, \alpha_n), f^p x_0\right) &\le \alpha_n D\left(x_n, f^p x_0\right)\\
&+ (1 - \alpha_n) D\left(fy_n, f^p x_0\right) \le \alpha_n D\left(x_n, f^p x_0\right)\\
&+ k(1 - \alpha_n) D\left(y_n, f^{p-1}x_0\right).
\end{aligned}
\tag{24}
$$

And similarly,

$$
\begin{aligned}
D\left(y_n, f^{p-1}x_0\right) = D\left(w(x_n, fx_n, \beta_n), f^{p-1}x_0\right) &\le \beta_n D\left(x_n, f^{p-1}x_0\right)\\
&+ (1 - \beta_n) D\left(fx_n, f^{p-1}x_0\right) \le \beta_n D\left(x_n, f^{p-1}x_0\right)\\
&+ k(1 - \beta_n) D\left(x_n, f^{p-2}x_0\right).
\end{aligned}
\tag{25}
$$

Since $D(x_0, f^{p-1}x_0) \le \delta(D, f, x_0)$ and $D(x_0, f^{p-2}x_0) \le \delta(D, f, x_0)$, we get

$$
D\left(y_n, f^{p-1}x_0\right) \le (\alpha_n + k(1 - \alpha_n))\delta(D, f, x_0) \le \delta(D, f, x_0). \tag{26}
$$

It follows that $D(x_{n+1}, f^p x_0) < \delta(D, f, x_0)$. Thus,

$$
D\left(x_n, f^p x_0\right) \le \delta(D, f, x_0), \text{for every } n, p \in \mathbb{N}. \tag{27}
$$

By Theorem 9, we have $(f^p x_0) \in C(D, X, \omega)$, and using (D_3), we get

$$
d_n = D(x_n, \omega) \le C \limsup_{p\longrightarrow+\infty} D\left(x_n, f^p x_0\right). \tag{28}
$$

Hence, by (27), we obtain

$$
d_n \le C\delta(D, f, x_0), \text{for all } n \in \mathbb{N}. \tag{29}
$$

Then, $(d_n)_n$ is bounded as desired.

Step 2: $(d_n)_n$ converges to 0 (i.e., $(x_n)_n$ converges to ω).

From (3) and (7), we have

$$\begin{aligned} d_{n+1} = D(\omega, x_{n+1}) &= D(\omega, w(x_n, fy_n, \alpha_n)) \\ &\le \alpha_n D(\omega, x_n) + (1-\alpha_n) D(\omega, fy_n) \\ &\le \alpha_n D(\omega, x_n) + (1-\alpha_n) D(f\omega, fy_n) \\ &\le \alpha_n D(\omega, x_n) + (1-\alpha_n) k D(\omega, y_n). \end{aligned} \tag{30}$$

We have, for all $n \in \mathbb{N}$,

$$\begin{aligned} D(\omega, y_n) = D(\omega, w(x_n, fx_n, \beta_n)) &\le \beta_n D(\omega, x_n) + (1-\beta_n) D(\omega, fx_n) \\ &\le \beta_n D(\omega, x_n) + k(1-\beta_n) D(\omega, x_n) \\ &\le [\beta_n + (1-\beta_n)k] D(\omega, x_n) \le D(\omega, x_n). \end{aligned} \tag{31}$$

Set

$$\gamma_n \coloneqq k + (1-k)\alpha_n = \alpha_n + (1-\alpha_n)k = 1 - (1-k)(1-\alpha_n). \tag{32}$$

Now, by (30) and (31), we obtain

$$d_{n+1} \le \gamma_n d_n. \tag{33}$$

Therefore,

$$d_{n+1} \le d_0 \prod_0^n [1 - (1-k)(1-\alpha_n)]. \tag{34}$$

Now, if $\limsup_{n\longrightarrow\infty} \alpha_n < 1$, then we get $\limsup_{n\longrightarrow\infty} \gamma_n < 1$; hence, by (33) and Lemma 5, it follows that $\lim_{n\longrightarrow+\infty} d_n = \lim_{n\longrightarrow+\infty} D(\omega, x_n) = 0$.

If $\sum_{i=0}^{\infty} 1 - \alpha_i = \infty$, then the same result follows by (34) and Lemma 6.

Consequently, Ishikawa iterative process converges to the fixed point ω.□ □

We have the following results, which can be deduced from Theorem 16.

Corollary 17. *Let (X, D, s, w) be a complete convex b-metric space with constant $s \ge 1$ and f a k-contraction for some $k \in [0, 1)$. If either $\limsup_{n\longrightarrow\infty} \alpha_n < 1$ or $\sum_{i=0}^{\infty} 1 - \alpha_i$ diverges, then Ishikawa iterative process defined in (4) converges to a fixed point of f.*

Corollary 18. *Let (X_ρ, ρ) be a complete convex modular space and $f : X \longrightarrow X$ be a mapping. Suppose that the following conditions hold:*

(i) ρ satisfies the Fatou property

(ii) For some $k \in (0, 1)$, we have

$$\rho(f(x) - f(y)) \le k\rho(x - y), \text{for every } (x, y) \in X_\rho \times X_\rho \tag{35}$$

(iii) There exists $x_0 \in X_\rho$ such that

$$\sup \left\{ \rho\left(f^i(x_0) - f^j(x_0) \right) : i, j \in \mathbb{N} < \infty \right\} \tag{36}$$

(iv) $\limsup_{n\longrightarrow\infty} \alpha_n < 1$ or $\sum_{i=0}^{\infty} 1 - \alpha_i$ diverges

Then, Ishikawa iterative process defined in (4) converges to a fixed point of f.

Remark 19. If $x_0 \in X$ is an element such that $D(x_0, x_0) = 0$ and $\alpha_n = \beta_n = 0$ for all $n \in \mathbb{N}$, then $(x_n)_n$ in Ishikawa iteration is simply reduced to Picard iteration of f^2. Indeed, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} D\left(x_{n+1}, f^{2(n+1)} x_0\right) = D\left(w(x_n, fy_n, \alpha_n), f^{2n+2} x_0\right) &\le \alpha_n D\left(x_n, f^{2n+2} x_0\right) \\ &+ (1-\alpha_n) D\left(fy_n, f^{2n+2} x_0\right). \end{aligned} \tag{37}$$

By (27), we obtain $D(x_n, f^{2n+2} x_0) < \infty$ and $D(x_n, f^{2n+1} x_0) < \infty$; hence,

$$\begin{aligned} D\left(x_{n+1}, f^{2(n+1)} x_0\right) &= D\left(x_{n+1}, f^{2n+2} x_0\right) \\ &\le k D\left(y_n, f^{2n+1} x_0\right) \\ &\le k D\left(w(x_n, fx_n, \beta_n), f^{2n+1} x_0\right) \\ &\le k\beta_n D\left(x_n, f^{2n+1} x_0\right) + k(1-\beta_n) D\left(fx_n, f^{2n+1} x_0\right) \\ &\le k D\left(fx_n, f^{2n+1} x_0\right) \\ &\le k^2 D\left(x_n, f^{2n} x_0\right) \\ &\vdots \\ &\le k^{2(n+1)} D(x_0, x_0) = 0. \end{aligned} \tag{38}$$

Therefore, by (D_2), we obtain $x_n = f^{2n} x_0$ and $y_n = f^{2n+1} x_0$, for all $n \in \mathbb{N}$.

Example 1. Let $X = [0, 1]$ and let D be the JS metric defined by the following:

$$\begin{cases} D(y, x) = D(x, y) = \infty, \text{if } x = 1 \text{ or } y = 1, \\ D(x, y) = x + y, \text{ if } x \ne 1 \text{ and } y \ne 1. \end{cases} \tag{39}$$

Define $w : X \times X \times [0, 1] \longrightarrow X$ by $w(x, y, \lambda) = \lambda x + (1-\lambda) y$; then, (X, D) is a complete convex JS metric space.

Now, consider the mapping $T : [0, 1] \longrightarrow [0, 1]$ given by

$$\begin{cases} T(1) = 1, \\ T(x) = \dfrac{6x}{7}, \text{if} x \neq 1. \end{cases} \tag{40}$$

The mapping T is a k-contraction with $k = 6/7$, and for every $x \in [0, 1[$, we have $Tx \leq x$. In addition, $D(x, y) < 1$ for all $x, y \in [0, 1[$; consequently, $\delta(D, f, x) < \infty$.

Hence, by Theorems 9, 11, and 16, T has a fixed point $\omega \in X$ that can be approximated by Picard iteration, Mann iteration with $\alpha = 1/n + 1$, and Ishikawa iteration with $\alpha_n = \beta_n = 1/n + 1$, and we take for initial value $x_0 = 0.5$.

The convergence behaviors of the three iterative schemes are displayed in Table 1 and graphically in Figure 1.

2.2. Convergence of Mann and Ishikawa Iterations for Chatterjea Contraction

Definition 20. Let (X, D) be a JS metric space, $f : X \longrightarrow X$ be a mapping, and $k \in [0, 1/2[$. We say that f is Chatterjea contraction if

$$D(fx, fy) \leq k[D(y, fx) + D(x, fy)], \text{for every } (x, y) \in X \times X. \tag{41}$$

In the next, we need the following result proved by El Kouch and Marhrani, in [19].

Theorem 21 (Theorem 3.9 in [19]). *Let (X, D) be a complete generalized metric space, $k \in [0, 1/2[$, and let $f : X \longrightarrow X$ be a Chatterjea mapping.*

If there exists a point $x_0 \in X$ such that $\delta(D, f, x_0) < \infty$, then the sequence $(f^n x_0)_n$ converges to some $\omega \in X$. Moreover, if $D(x_0, f\omega) < \infty$, then ω is a fixed point of f, and for any fixed point ω' of f such that $D(\omega, \omega') < \infty$, we have $\omega = \omega'$.

Now, we give a convergence result of Mann iterative process for Chatterjea mapping in complete convex JS metric space.

Theorem 22. *Let (X, D, w) be a complete convex JS metric space and $f : X \longrightarrow X$ be a Chatterjea mapping. Let $(x_n)_n$ be the sequence defined by $x_{n+1} = w(x_n, fx_n, \alpha_n)$, $x_0 \in X$. Suppose that the following conditions hold:*

(i) $\delta(D, f, x_0) < \infty$

(ii) For all $n, m \in \mathbb{N}$, $D(fx_n, f^m x_0) < \infty$

(iii) $(f^n x_0)_n$ converges to some $\omega \in X$ satisfying $D(x_0, f\omega) < \infty$

(iv) $\limsup_{n\longrightarrow\infty} \alpha_n < 1$ or $\sum_{i=0}^{\infty} 1 - \alpha_i$ diverges

Then, Mann iterative process defined in (4) converges to the fixed point ω.

Table 1: Convergence behavior of the three iterative processes to the fixed point.

No. of iterations	Picard iteration	Mann iteration	Ishikawa iteration
0	0.5000000000	0.5000000000	0.5000000000
1	0.4285714285	0.4642857142	0.4489795918
2	0.3673469387	0.4200680272	0.3817853672
3	0.3148688046	0.3750607385	0.3145833510
4	0.2698875468	0.3321966541	0.2539779054
5	0.2313321830	0.2926494334	0.2021456798
6	0.1982847283	0.2568148089	0.1592076137
7	0.1699583385	0.2247129578	0.1243809482
8	0.1456785758	0.1961779790	0.0965527088
9	0.1248673507	0.1709550960	0.0745623368
⋮	⋮	⋮	⋮
45	0.0004856959	0.0008512181	0.0000002419
46	0.0004163108	0.0007322028	0.0000001797
47	0.0003568378	0.0006297815	0.0000001335
48	0.0003058609	0.0005416488	0.0000009914
49	0.0002621665	0.0004658180	0.0000007360
50	0.0002247142	0.0004005774	0.0000005463

Proof. All conditions of Theorem 21 are satisfied and $\omega \in Fix(f)$. Set $d_n := D(x_n, \omega)$ for all $n \in \mathbb{N}$. The proof goes in three steps.

Step 1: set $\delta = \delta(D, f, x_0)$, and we will prove that

$$\begin{aligned} &\text{For every } n \in \mathbb{N}, \text{there exists } p_n \in \mathbb{N} \text{ such that} \\ &\quad m \geq p_n \text{ implies } D(x_n, f^m x_0) \leq \delta. \end{aligned} \tag{42}$$

We will prove the statement (42) by induction on n.

For $n = 0$, we have $D(x_0, f^m x_0) \leq \delta$ for all $m \in \mathbb{N}$, so we take $p_0 = 0$.

Suppose that (42) is true for some $n \in \mathbb{N}$, that is,

$$\text{There exists } p_n \in \mathbb{N} \text{ such that } m \geq p_n \text{ implies } D(x_n, f^m x_0) \leq \delta. \tag{43}$$

Now, we will prove that it is true again for $n + 1$.

From (3), we obtain, for all $m \geq p_n$,

$$\begin{aligned} D(x_{n+1}, f^m x_0) = D(w(x_n, fx_n, \alpha_n), f^m x_0) \leq \alpha_n D(x_n, f^m x_0) \\ + (1 - \alpha_n) D(fx_n, f^m x_0). \end{aligned} \tag{44}$$

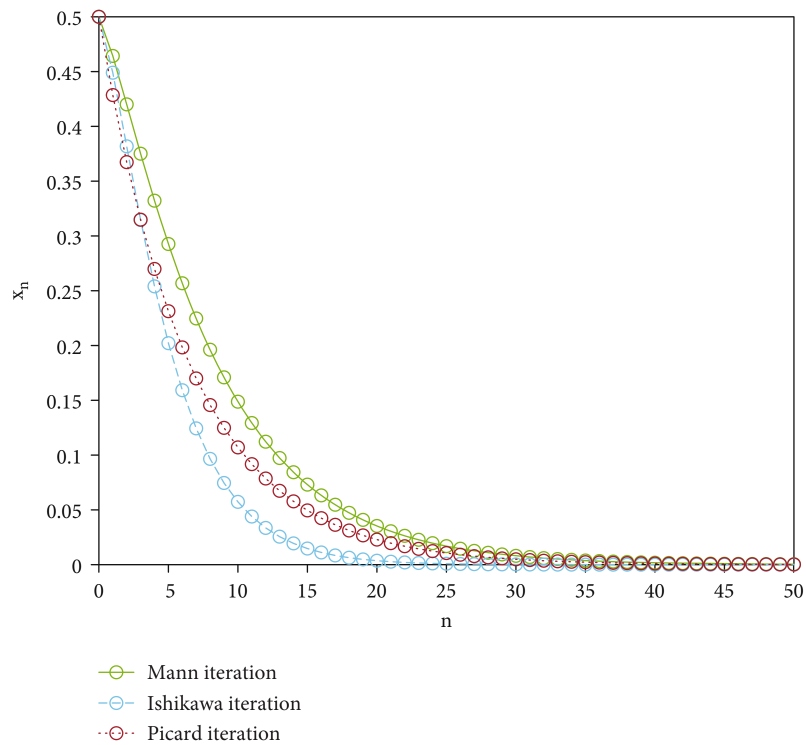

Figure 1: Convergence behavior of the three iterative processes to the fixed point.

By successive application of the inequality (41), we have

$$\begin{aligned} D(fx_n, f^m x_0) \le{}& k\left[D(x_n, f^m x_0) + D(fx_n, f^{m-1}x_0)\right] \\ &\le kD(x_n, f^m x_0) + kD(fx_n, f^{m-1}x_0)\big] \\ &\le kD(x_n, f^m x_0) + k^2 D(x_n, f^{m-1}x_0) \\ &+ k^2 D(fx_n, f^{m-2}x_0) \le kD(x_n, f^m x_0) \\ &+ k^2 D(x_n, f^{m-1}x_0) + \cdots + k^{m-p_n} D(x_n, f^{p_n+1}x_0) \\ &+ k^{m-p_n+1} D(x_n, f^{p_n}x_0) + k^{m-p_n+1} D(fx_n, f^{p_n-1}x_0). \end{aligned} \tag{45}$$

Since for all $m \ge p_n$, we have $D(x_n, f^m x_0) \le \delta$. Thus, from (45), we get

$$\begin{aligned} D(fx_n, f^m x_0) &\le k\delta\left(1 + k + \cdots + k^{m-p_n}\right) + k^{m-p_n+1} D(fx_n, f^{p_n-1}x_0) \\ &\le k\delta\left(\frac{1 - k^{m-p_n+1}}{1-k}\right) + k^{m-p_n+1} D(fx_n, f^{p_n-1}x_0) \\ &\le \frac{k\delta}{1-k} + k^{m-p_n+1}\left(D(fx_n, f^{p_n-1}x_0) - \frac{k\delta}{1-k}\right). \end{aligned} \tag{46}$$

We have $k \in [0, 1/2[$ and $D(fx_n, f^{p_n-1}x_0) < \infty$; hence,

$$\lim_{m\longrightarrow+\infty} k^{m-p_n+1}\left(D(fx_n, f^{p_n-1}x_0) - \frac{k\delta}{1-k}\right) = 0. \tag{47}$$

Using (47) and for $\delta(1-2k)/1-k > 0$, there exists $N \in \mathbb{N}$ such that for all $m \ge N$, we have

$$k^{m-p_n+1}\left(D(fx_n, f^{p_n-1}x_0) - \frac{k\delta}{1-k}\right) < \frac{\delta(1-2k)}{1-k}. \tag{48}$$

We take $p_{n+1} = \max\ (p_n, N)$; thus, from (48) and (46), we obtain, for all $m \ge p_{n+1}$,

$$D(fx_n, f^m x_0) \le \frac{k\delta}{1-k} + \frac{\delta(1-2k)}{1-k} = \delta. \tag{49}$$

From (49) and (43), we have, for all $m \ge p_{n+1}$, $D(x_n, f^m x_0) \le \delta$ and $D(fx_n, f^m x_0) \le \delta$.

Let us now return to (44), and we obtain, for all $m \ge p_{n+1}$, $D(x_{n+1}, f^m x_0) \le \delta$.

Hence, we conclude that the statement (42) is true.

Step 2: we shall prove that $(d_n)_n$ is bounded.

By step 1, we have, for all $n \in \mathbb{N}$ and every $m \ge p_n$,

$$D(x_n, f^m x_0) \le \delta. \tag{50}$$

Using Theorem 21, we have $(f^m x_0)$ converges to ω and by (D_3), and we obtain

$$d_n = D(x_n, \omega) \le C\lim\sup_{m\longrightarrow+\infty} D(x_n, f^m x_0). \tag{51}$$

Hence, by (50), we obtain

$$d_n \leq C\delta, \text{forall } n \in \mathbb{N}. \tag{52}$$

Therefore, $(d_n)_n$ is bounded.

Step 3: $(d_n)_n$ converges to 0 (i.e., $(x_n)_n$ converges to ω). From (3), we have

$$\begin{aligned} d_{n+1} = D(\omega, x_{n+1}) &= D(\omega, w(x_n, fx_n, \alpha_n)) \\ &\leq \alpha_n D(\omega, x_n) + (1-\alpha_n) D(\omega, fx_n). \end{aligned} \tag{53}$$

It follows from (41) that

$$\begin{aligned} D(\omega, fx_n) = D(f\omega, fx_n) &\leq k[D(\omega, fx_n) + D(f\omega, x_n)] \\ &\leq kD(\omega, fx_n) + kD(\omega, x_n). \end{aligned} \tag{54}$$

Hence, we get

$$D(\omega, fx_n) \leq \frac{k}{1-k} D(\omega, x_n). \tag{55}$$

Set $\lambda = k/1-k \in [0,1[$, and we obtain from (53) and (55)

$$d_{n+1} \leq \gamma_n d_n, \tag{56}$$

with $\gamma_n = \lambda + (1-\lambda)\alpha_n = 1-(1-\lambda)(1-\alpha_n)$.

Therefore,

$$d_{n+1} \leq d_0 \prod_0^n [1-(1-\lambda)(1-\alpha_n)]. \tag{57}$$

Now, if $\limsup_{n \longrightarrow \infty} \alpha_n < 1$, then we get $\limsup_{n \longrightarrow \infty} \gamma_n < 1$.

Hence, by (56) and Lemma 5, it follows that

$$\lim_{n \longrightarrow +\infty} d_n = \lim_{n \longrightarrow +\infty} D(\omega, u_n) = 0. \tag{58}$$

If $\sum_{i=0}^{\infty} 1-\alpha_i = \infty$, then by (57) and Lemma 6, we have the same result.

Consequently, Mann iterative process converges to the fixed point ω.□ □

Theorem 23. *Let (X, D, w) be a complete convex JS metric space and $f : X \longrightarrow X$ be a Chatterjea mapping. Let $(x_n)_n$ and $(y_n)_n$ be the sequences defined in (5), $x_0 \in X$. Suppose that the following conditions hold:*

(i) $\delta(D, f, x_0) < \infty$

(ii) For all $n, m \in \mathbb{N}$, $D(fx_n, f^m x_0) < \infty$ and $D(fy_n, f^m x_0) < \infty$

(iii) $(f^n x_0)_n$ converges to some $\omega \in X$ satisfying $D(x_0, f\omega) < \infty$

(iv) $\limsup_{n \longrightarrow \infty} \alpha_n < 1$ or $\sum_{i=0}^{\infty} 1-\alpha_i$ diverges

Then, Ishikawa iterative process defined in (4) converges to the fixed point ω.

Proof. By Theorem 21, we have $\omega \in Fix(f)$. Let $(x_n)_n$ and $(y_n)_n$ be the sequences defined in (5) and set $d_n := D(x_n, \omega)$ for all $n \in \mathbb{N}$. Now, we complete the proof in the following steps.

Step 1: we prove that

$$\begin{aligned} &\text{For every } n \in \mathbb{N}, \text{there exists } p_n \in \mathbb{N} \text{ such that} \\ &m \geq p_n \text{ implies } D(x_n, f^m x_0) \leq \delta. \end{aligned} \tag{59}$$

We will prove the statement (59) by induction on n.

For $n = 0$, we have $D(x_0, f^m x_0) \leq \delta$ for all $m \in \mathbb{N}$, so we take $p_0 = 0$.

Suppose that (59) is true for some $n \in \mathbb{N}$, that is,

$$\text{There exists } p_n \in \mathbb{N} \text{ such that } m \geq p_n \text{ implies } D(x_n, f^m x_0) \leq \delta. \tag{60}$$

From (3), we obtain, for all $m \geq p_n$,

$$\begin{aligned} D(x_{n+1}, f^m x_0) &= D(w(x_n, fy_n, \alpha_n), f^m x_0) \\ &\leq \alpha_n D(x_n, f^m x_0) + (1-\alpha_n) D(fy_n, f^m x_0). \end{aligned} \tag{61}$$

By (41), we get, for all $m \geq p_n$,

$$D(fy_n, f^m x_0) \leq k\left[D(y_n, f^m x_0) + D\left(fy_n, f^{m-1} x_0\right)\right]. \tag{62}$$

On the other hand,

$$\begin{aligned} D(y_n, f^m x_0) &= D(w(x_n, fx_n, \alpha_n), f^m x_0) \\ &\leq \alpha_n D(x_n, f^m x_0) + (1-\alpha_n) D(fx_n, f^m x_0). \end{aligned} \tag{63}$$

According to (49), there exists $s_n = \max(p_n, N)$ such that for all $m \geq s_n$,

$$D(fx_n, f^m x_0) \leq \delta. \tag{64}$$

Then, from (60), (63), and (64), we have, for all $m \geq s_n$,

$$D(y_n, f^m x_0) \leq \delta. \tag{65}$$

Hence, from (65) and (62), we get, for all $m \geq s_n$,

$$\begin{aligned} D(fy_n, f^m x_0) &\leq kD(y_n, f^m x_0) \\ &\quad + k^2 D\left(y_n, f^{m-1} x_0\right) + \cdots + k^{m-s_n} D\left(y_n, f^{s_n+1} x_0\right) \\ &\quad + k^{m-s_n+1} D\left(y_n, f^{s_n} x_0\right) + k^{m-s_n+1} D\left(fy_n, f^{s_n-1} x_0\right) \\ &\leq k\delta\left(1 + k + \cdots + k^{m-s_n}\right) + k^{m-s_n+1} D\left(fy_n, f^{s_n-1} x_0\right) \\ &\leq \frac{k\delta}{1-k} + k^{m-s_n+1}\left(D\left(fy_n, f^{s_n-1} x_0\right) - \frac{k\delta}{1-k}\right). \end{aligned} \tag{66}$$

We have $k \in [0, 1/2[$ and $D(fy_n, f^{s_n-1}x_0) < \infty$; hence,

$$\lim_{m \longrightarrow +\infty} k^{m-s_n+1}\left(D\left(fx_n, f^{s_n-1}x_0\right) - \frac{k\delta}{1-k}\right) = 0. \quad (67)$$

Using (67), and for $\delta(1-2k)/1-k > 0$, there exists $N' \in \mathbb{N}$ such that for all $m \geq N'$, we have

$$k^{m-s_n+1}\left(D\left(fx_n, f^{s_n-1}x_0\right) - \frac{k\delta}{1-k}\right) < \frac{\delta(1-2k)}{1-k}. \quad (68)$$

If we take $p_{n+1} = \max(s_n, N')$, then from (68) and (66), we obtain, for all $m \geq p_{n+1}$,

$$D(fy_n, f^m x_0) \leq \delta. \quad (69)$$

Now, by (61), (60), and (69), we obtain, for all $m \geq p_{n+1}$,

$$D(x_{n+1}, f^m x_0) \leq \delta. \quad (70)$$

It follows that the statement (59) is true.

Step 2: we shall show that $(d_n)_n$ is bounded.

We have for every $n \in \mathbb{N}$, there exists $p_n \in \mathbb{N}$ such that for all $m \geq p_n$,

$$D(x_n, f^m x_0) \leq \delta. \quad (71)$$

Using Theorem 21, $(f^m x_0)$ converges to ω, and by (D_3), we obtain

$$d_n = D(x_n, \omega) \leq C\limsup_{m \longrightarrow +\infty} D(x_n, f^m x_0) \leq C\delta. \quad (72)$$

Then, $(d_n)_n$ is bounded.

Step 3: $(d_n)_n$ converges to 0 (i.e., $(x_n)_n$ converges to ω).

By (3) and (7), we have

$$\begin{aligned} d_{n+1} = D(\omega, x_{n+1}) &= D(\omega, w(x_n, fy_n, \alpha_n)) \\ &\leq \alpha_n D(\omega, x_n) + (1-\alpha_n) D(\omega, fy_n). \end{aligned} \quad (73)$$

It follows from (41) that

$$\begin{aligned} D(\omega, fy_n) = D(f\omega, fy_n) &\leq k[D(\omega, fy_n) + D(f\omega, y_n)] \\ &\leq kD(\omega, fy_n) + kD(\omega, y_n). \end{aligned} \quad (74)$$

Hence, we get

$$D(\omega, fy_n) \leq \frac{k}{1-k} D(\omega, y_n). \leq \lambda D(\omega, y_n), \quad (75)$$

with $\lambda = k/1-k \in [0, 1[$.

Similarly, we have

$$D(\omega, fx_n) \leq \lambda D(\omega, x_n). \quad (76)$$

We have, for all $n \in \mathbb{N}$,

$$\begin{aligned} D(\omega, y_n) = D(\omega, w(x_n, fx_n, \beta_n)) &\leq \beta_n D(\omega, x_n) \\ &+ (1-\beta_n) D(\omega, fx_n) \leq \beta_n D(\omega, x_n) \\ &+ \lambda(1-\beta_n) D(\omega, x_n) \leq [\beta_n + (1-\beta_n)\lambda] D(\omega, x_n) \\ &\leq D(\omega, x_n). \end{aligned} \quad (77)$$

Set

$$\gamma_n := \lambda + (1-\lambda)\alpha_n = \alpha_n + (1-\alpha_n)\lambda = 1 - (1-\lambda)(1-\alpha_n). \quad (78)$$

Now, by (73), (75), and (77), we obtain

$$d_{n+1} \leq \gamma_n d_n. \quad (79)$$

Therefore,

$$d_{n+1} \leq d_0 \prod_0^n [1 - (1-\lambda)(1-\alpha_n)]. \quad (80)$$

Now, if $\limsup_{n \longrightarrow \infty} \alpha_n < 1$, then we get $\limsup_{n \longrightarrow \infty} \gamma_n < 1$; hence, by (79) and Lemma 6, it follows that $\lim_{n \longrightarrow +\infty} d_n = \lim_{n \longrightarrow +\infty} D(\omega, x_n) = 0$.

If $\sum_{i=0}^{\infty} 1 - \alpha_i = \infty$, then the same result follows by (80) and Lemma 6.

Consequently, Ishikawa iterative process converges to the fixed point ω.□ □

3. Applications

In this section, we apply Corollaries 13 and 17 in order to show the existence of solution of the Fredholm integral equation:

$$f(t) = \int_0^T K(t, \tau, f(\tau)) d\tau + g(t), \quad (81)$$

where $T > 0$, $K : [0, T] \times [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g : [0, T] \longrightarrow \mathbb{R}$ is continuous function.

Remark 24. The convexity of the function $f(x) = |x|^p$ $(x \in \mathbb{R}, 1 < p < \infty)$ implies the followings:

(i) For all $a, b \in \mathbb{R}$ and $\alpha \in [0, 1]$, we have $|\alpha a + (1-\alpha) b|^p \leq \alpha |a|^p + (1-\alpha)|b|^p$

(ii) Particularly for $\alpha = 1/2$, we have $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$

Let $X = \mathscr{C}[0, T]$ and let $D : X \times X \longrightarrow [0, +\infty)$ be the mapping defined by

$$D(x, y) = \max_{0 \leq t \leq T} |x(t) - y(t)|^p. \quad (82)$$

Using Remark 24 above and by considering $w : X \times X \times [0, 1] \longrightarrow X$ defined by $w(x, y, \lambda) = \lambda x + (1 - \lambda)y$, we can easily show that (X, D, w) is a complete convex b-metric space with $C = 2^{p-1}$.

Theorem 25. *Suppose that the following hypotheses hold:*

(i) There exists a continuous function $\zeta : [0, T] \times [0, T] \longrightarrow \mathbb{R}^+$ *such that for all* $t, s \in [0, T]$ *and for all* $x, y \in \mathscr{C}([0, T])$, *we have*

$$|K(t, s, x(s)) - K(t, s, y(s))| \le \zeta(t, s)|x(s) - y(s)|. \tag{83}$$

(ii) $\max_{t\in[0,T]} \int_0^T \zeta(t, s)ds < 1$.

Then, the integral Equation (81) has a unique solution on $\mathscr{C}[0, T]$. *Moreover, Mann and Ishikawa iteration processes converge to this solution for any sequences* $(\alpha_n)_n$, $(\beta_n)_n \in [0, 1]$ *such that* $\limsup_{n\longrightarrow\infty} \alpha_n < 1$ *or* $\sum_{i=0}^{\infty} 1 - \alpha_i$ *diverges.*

Proof. Define a self-map S on X by

$$Su(t) = \int_0^T K(t, \tau, u(\tau))d\tau + g(t), \text{for all } u \in \mathscr{C}[0, T].$$

$$\begin{aligned} |Su(t) - Sv(t)|^p &= \left|\int_0^T K(t, s, u(s)) - K(t, s, v(s))ds\right|^p \le \left(\int_0^T |K(t, s, u(s)) - K(t, s, v(s))|ds\right)^p \\ &\le \left(\int_0^T \zeta(t, s)|u(s) - v(s)|ds\right)^p \le \left(\max_{t\in[0,T]} \int_0^T \zeta(t, s)ds\right)^p \max_{0\le\tau\le T} |u(\tau) - v(\tau)|^p \le \left(\max_{t\in[0,T]} \int_0^T \zeta(t, s)ds\right)^p D(u, v). \end{aligned} \tag{84}$$

Therefore,

$$D(Su, Sv) \le \left(\max_{t\in[0,T]} \int_0^T \zeta(t, s)ds\right)^p D(u, v). \tag{85}$$

Hence, S is a k-contraction mapping on X with $k = (\max_{t\in[0,T]} \int_0^T \zeta(t, s)ds)^p < 1$. All conditions of Corollaries 13 and 17 are satisfied. Hence, S has a unique fixed point. That is to say, there exists a unique solution to the integral Equation (81). Moreover, this solution could be approximated by Mann and Ishikawa iterative processes.□ □

4. Conclusion

In this paper, we consider a JS metric space endowed with Takahashi convexity structure, and we establish convergence theorems for Mann iteration and Ishikawa iteration for Banach contraction and Chatterjea contraction. Also, we provide an example to validate the established theorems and give an application of our result to Fredholm integral equation. Finally, it remains an open problem to establish the Convergence of Mann iteration and Ishikawa iteration to fixed point for Kannan contraction.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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