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Research Article

Common Fixed Point Theorem for Multivalued Mappings Using Ternary Relation in G-Metric Space with an Application

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This paper proves a common fixed point theorem for single-valued and multivalued mappings by using ternary relation in G-metric spaces. Henceforth, results obtained will be verified with the help of illustrative examples. Also, we demonstrate the results with an application.

1. Introduction

In 1969, Nadler [1] introduced multivalued contraction mappings using the Hausdorff metric and extended Banach's contraction principle [2] from single-valued to multivalued mappings. Since then, several researchers have generalized these results for multivalued mappings in various spaces. As a result, the multivalued mapping theory has many applications in diverse areas, such as control theory, approximation theory, differential equations, and economics.

Definition 1 (see [1]). Let $(\mathscr{X}, d)$ be a metric space. A map $\mathscr{T} : \mathscr{X} \longrightarrow \mathscr{X}$ is said to be a multivalued contraction if there exists $0 \leq \lambda \leq 1$ such that

$$\mathscr{H}(\mathscr{T}x, \mathscr{T}y) \leq \theta d(x, y), \tag{1}$$

for all $x, y \in \mathscr{X}$.

Kaneko and Sessa [3] extended the concept of compatible mappings due to Jungck [4] to include multivalued mappings as well as single-valued mappings. They followed the works of Kubiak [5] and Nadler [1] and proved coincidence and fixed point theorems for hybrid pair of compatible mappings. Pathak [6] extended the concept of compatible hybrid mappings to f-weak compatible hybrid mappings and proved a coincidence theorem, an extension of the results of Kaneko and Sessa [3].

In 2006, Mustafa and Sims [7, 8] gave a generalization of D-metric space to G-metric space soon after identifying some shortcomings concerning the fundamental topological structure on D-metric spaces. They defined several notions, such as continuity, completeness, compactness, convergence, and space product in the G-metric space setting. Abbas and Rhoades [9] established the concept of a common fixed point in G-metric spaces. In addition, Kaewcharoen and Kaewkhao [10] presented some common fixed points for single-valued and multivalued mappings in G-metric spaces. At the same time, Tahat et al. [11] proved common fixed points for single-valued and multivalued maps satisfying a generalized contraction in G-metric spaces. Recently, Mustafa et al. [12] gave the results in common fixed points for multivalued mappings in G-metric spaces with applications using F-contraction concept, and Shoaib and Shahzad [13] proved the common fixed point of multivalued mappings in ordered dislocated quasi G-metric spaces.

On the other hand, Alam and Imdad [14] gave a generalization of the Banach contraction principle in a complete metric space equipped with binary relation. Their results show that the contraction condition holds only for those elements linked with the binary relation, not for every pair of elements. For more results on binary relation, one can see [15–20] and the references therein. Perveen et al. [21] gave the prove in relation theoretic common fixed point results for generalized weak nonlinear contractions with an application. Hossain et al. [22] gave the study of relation-theoretic metrical fixed point theorem for rational type contraction mapping with an application. Further, Gaba et al. [23] extended the works of Alam and Imdad [14] by using the Banach contraction mapping principle in generalized metric spaces with a ternary relation. Furthermore, Radha and Singh [24] proved a novel approach to G-metric spaces by using ternary relations. Badshah et al. [25] proved some common fixed point theorems for contractive mappings in cone 2-metric spaces equipped with a ternary relation.

This paper is aimed at proving a common fixed point theorem for single-valued and multivalued mappings by using the ternary relation concept in the G-metric space setting. Specifically, we improve and extend the works due to Alam and Imdad [14], Ahmadullah et al. [15, 16], Eke et al. [18], and Radha and Singh [24]. In doing so, we will generalize several other works in the literature having the same directions.

2. Preliminaries

In this section, we introduce some definitions, theorems, and preliminary results, which will be helpful in developing the main result.

Motivated by Mustafa and Sims [8], we can formulate the axioms of G-metric spaces as follows:

Definition 2 (see [8]). Let $\mathcal{X}$ be a nonempty set and $G:\mathcal{X}\times\mathcal{X}\times\mathcal{X}\longrightarrow\mathbb{R}_+$ be a function satisfying the following conditions:

$(G1)G(x,y,z)=0$ if $x=y=z$

$(G2)0<G(x,x,y)$, for all $x,y\in\mathcal{X}$ with z=y

$(G3)G(x,x,y)\le G(x,y,z)$ for all $x,y\in\mathcal{X}$ with z=y

$(G4)G(x,y,z)=G(x,z,y)=G(y,z,x)=\cdots$ (symmetry in all three variables)

$(G5)G(x,y,z)\le G(x,a,a)+G(a,y,z)$, for all $x,y,z,a\in\mathcal{X}$ (rectangle inequality)

The function G is called a generalized metric or G-metric, and the pair $(\mathcal{X},G)$ is called a G-metric space.

The following example satisfies the above axioms.

Example 1 (see [8]). Let $\mathcal{X}=\mathbb{R}$ be the set of all real number. Define $G:\mathbb{R}^3\longrightarrow\mathbb{R}^+$ by

$$G(x,y,z)=|x-y|+|y-z|+|x-z|, \tag{2}$$

for all $x,y,z\in\mathcal{X}$. Then, it is clear that $(\mathcal{X},G)$ is a G-metric space with a G-metric on $\mathcal{X}$.

Note that if $G(x,y,z)=0$ then $x=y=z$.

Mustafa and Sims [8] proved the following proposition satisfying a G-metric properties.

Proposition 3 (see [8]). *Let $(\mathcal{X},G)$ be a G-metric space; then, the metric associated with G satisfies the following:*

(i) $G(x,y,z)\le G(x,x,y)+G(x,x,z)$

(ii) $G(x,y,y)\le 2G(y,x,x)$

(iii) $G(x,y,z)\le G(x,a,a)+G(y,a,a)+G(z,a,a)$

for all $x,y,z,a\in\mathcal{X}$.

Inspired by Mustafa and Sims [8], we established some topological properties such as convergence, completeness, and continuity in G-metric spaces as follows:

Definition 4 (see [8]). Let $(\mathcal{X},G)$ be a G-metric space. A sequence $x_n\in\mathcal{X}$ is said to be the following:

(i) G-convergent to $x\in\mathcal{X}$ if for any $\varepsilon>0$, there exists $k\in\mathbb{N}$ such that $G(x,x_n,x_n)<\varepsilon$ for all $n\ge k$

(ii) G-Cauchy if for $\varepsilon>0$, there exists $k\in\mathbb{N}$ such that for all $\{x_n\}$ in $\mathcal{X}$ is called Cauchy sequence if for each $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that $G(x_n,x_m,x_p)<\varepsilon$ for all $n,m,p\ge n_0$, i.e., $G(x_n,x_m,x_p)\longrightarrow 0$ as n,m, $p\longrightarrow\infty$

Definition 5 (see [8]). A G-metric space is said to be G-complete if every G-Cauchy sequence in $\mathcal{X}$ is G-convergent. Every G-metric on $\mathcal{X}$ defines a metric

$$d_G(x,y)=G(x,y,y)+G(y,x,x), \tag{3}$$

for all $x,y\in\mathcal{X}$.

Proposition 6 (see [8]). *Let $(\mathcal{X},G)$ be a G-metric space. Then, the following properties are equivalent:*

(i) (x_n) is G-convergent to x

(ii) $G(x_n,x_m,x_n)\longrightarrow 0$ as $n,m\longrightarrow\infty$

(iii) $G(x_n,x,x)\longrightarrow 0$ as $n\longrightarrow\infty$

(iv) (x_n) is a G-Cauchy sequence

(v) For every $\varepsilon>0$, there exists $k\in\mathbb{N}$ such that $G(x_n,x_m,x_m)<\varepsilon$ for $n,m>k$

Definition 7 (see [8]). Let $(\mathcal{X},G)$ and $(\mathcal{X}',G')$ be two G-metric spaces, and let $\mathcal{T}:(\mathcal{X},G)\longrightarrow(\mathcal{X}',G')$. Then, the map $\mathcal{T}$ is said to be G-continuous at $x\in\mathcal{X}$ if for $\varepsilon>0$, there exists $\delta>0$ such that for all $x,y\in\mathcal{X}$ and $G(a,x,y)<\delta$, we have $G'(\mathcal{T}a,\mathcal{T}x,\mathcal{T}y)<\varepsilon$. The function $\mathcal{T}$ is G-continuous if it is G-continuous for each $a\in\mathcal{X}$.

Proposition 8 (see [8]). *Let $(\mathcal{X},G)$ and $(\mathcal{X}',G')$ be two G-metric spaces, and let $\mathcal{T}:(\mathcal{X},G)\longrightarrow(\mathcal{X}',G')$. Then, the map $\mathcal{T}$ is said to be G-continuous at $x\in\mathcal{T}$ if and only if $\mathcal{T}$ is sequentially continuous, i.e., whenever (x_n) is G-convergent to x, the sequence $\mathcal{T}(x_n)$ is G-convergent to $\mathcal{T}x$.*

Lemma 9 (see [8]). *Let (X,G) be a G-metric space, then the function $G(x,y,z)$is jointly continuous in all three of its variables.*

Later, Kaewcharoen and Kaewkhao [10] established the multivalued notion in G-metric space. Let $\mathcal{X}$ be a G-metric space. Denote $\mathcal{CB}(\mathcal{X})$ be the class of all nonempty, closed, and bounded subsets of $\mathcal{X}$. Let $\mathcal{H}_G(.,.,.)$ be the Haursdorff-G-distance on $\mathcal{CB}(\mathcal{X})$, that is, for $\mathcal{A}$, $\mathcal{B},\mathcal{C}\in(\mathcal{CB}(\mathcal{X}))$, define

$$\mathcal{H}_G(\mathcal{A},\mathcal{B},\mathcal{C})=\max\left\{\sup_{x\in\mathcal{A}}G(x,\mathcal{B},\mathcal{C}),\sup_{x\in\mathcal{B}}G(x,\mathcal{C},\mathcal{A}),\sup_{x\in\mathcal{C}}G(x,\mathcal{A},\mathcal{B})\right\}, \tag{4}$$

where

$$G(x,\mathcal{B},\mathcal{C})=d_G(x,\mathcal{B})+d_G(\mathcal{B},\mathcal{C})+d_G(x,\mathcal{C}), \tag{5}$$

$$d_G(x,\mathcal{B})=\inf\{d_G(x,y),y\in\mathcal{B}\}, \tag{6}$$

$$d_G(\mathcal{A},\mathcal{B})=\inf\{d_G(a,b),a\in\mathcal{A},b\in\mathcal{B}\}, \tag{7}$$

$$d_G(x,y,\mathcal{C})=\inf\{G(x,y,z),z\in\mathcal{C}\}. \tag{8}$$

Lemma 10 (see [10]). *Let $(\mathcal{X},G)$ be a G-metric space and $\mathcal{A},\mathcal{B}\in\mathcal{CB}(\mathcal{X})$. Then, for each $a\in\mathcal{A}$, we have*

$$G(a,\mathcal{B},\mathcal{B})\le\mathcal{H}_G(\mathcal{A},\mathcal{B},\mathcal{B}). \tag{9}$$

Lemma 11 (see [11]). *Let $(\mathcal{X},G)$ be a G-metric space and $\mathcal{A},\mathcal{B}\in\mathcal{CB}(\mathcal{X})$; then, for each $\varepsilon>0$, there exists $b\in\mathcal{B}$ such that*

$$G(a,b,b)\le\mathcal{H}_G(\mathcal{A},\mathcal{B},\mathcal{B})+\theta. \tag{10}$$

Lemma 12 (see [26]). *Let $\{x_n^1\},\{x_n^2\},\cdots,\{x_n^N\}\subseteq\mathcal{X}$ be N-sequence on a G-metric space $(\mathcal{X},G)$ such that*

$$\lim_{n\longrightarrow\infty}G\left(x_n^i,x_{n+1}^i,x_{n+1}^i\right)=0, \tag{11}$$

for all $i\in 1,2,\cdots,N$.

Lemma 13 (see [26]). *If ψ is a (c)-comparison function, for each $t>0$*

$$\lim_{n\longrightarrow+\infty}\psi^n(t)=0\Longrightarrow\psi(t)=t. \tag{12}$$

Definition 14 (see [10]). Let $\mathcal{X}$ be a nonempty set. Assume $\mathcal{S}\mathcal{X}\longrightarrow\mathcal{X}$ and $\mathcal{T}:\mathcal{X}\longrightarrow 2^{\mathcal{X}}$ are two mappings. If $w=\mathcal{S}x\in\mathcal{T}x$ for some $x\in\mathcal{X}$, then x is called a coincidence point of the pair $(\mathcal{T},\mathcal{S})$ and w is a point of coincidence of $\mathcal{S}$ and $\mathcal{T}$. The mapping $\mathcal{S}$ and $\mathcal{T}$ are called weakly compatible if $\mathcal{S}x\in\mathcal{T}x$ for some $x\in\mathcal{X}$ implies $\mathcal{S}\mathcal{T}x\subseteq\mathcal{T}\mathcal{S}x$.

Proposition 15 (see [10]). *Let $\mathcal{X}$ be a nonempty set. Assume $\mathcal{S}:\mathcal{X}\longrightarrow\mathcal{X}$ and $\mathcal{T}:\mathcal{X}\longrightarrow 2^{\mathcal{X}}$ are weakly compatible mappings. If $\mathcal{S}$ and $\mathcal{T}$ have a unique point of coincidence $u=\mathcal{S}x\in\mathcal{T}x$, then u is a unique common fixed point of $\mathcal{S}$ and $\mathcal{T}$.*

Definition 16 (see [27]). Let $(\mathcal{T},\mathcal{S})$ be a pair of self mappings on an ordered metric space $(\mathcal{X},d,\preceq)$ with $\mathcal{T}(\mathcal{X})\subseteq\mathcal{S}(\mathcal{X})$. For every $x_0\in\mathcal{X}$, consider the sequence $\{x_n\}\in\mathcal{X}$ defined by $\mathcal{T}x_n=\mathcal{S}x_{n+1}$, for all $n\in\mathbb{N}_0$. Then, $\{\mathcal{T}x_n\}$ is called $\mathcal{T}$-$\mathcal{S}$ sequence with initial point x_0.

2.1. Ternary Relation-Theoretic in G-Metric Spaces. In this part, we recall some definition, proposition, and preliminary results in G-metric space related to ternary relation. We define $\mathbb{N}_0=\{\{0\}\cup\mathbb{N}\}$, where $\mathbb{N}$ is a set of natural numbers in $\mathcal{X}$.

Motivated by Nová and Novotný [28], NovotnŶ [29], Šlapal [30], and Alam and Imdad [14], we can introduce the property of ternary relation as follows:

Definition 17 (see [30]). Let $(\mathcal{X})$ be a set and $\mathcal{R}\subseteq\mathcal{X}\times\mathcal{X}\times\mathcal{X}$. Then, $\mathcal{R}$ is said to be ternary relation on $\mathcal{X}$. A ternary relation $\mathcal{R}$ defined on a nonempty set G is called

(1) Reflexive if $(x,x,x)\in\mathcal{R}$ for all $x\in\mathcal{X}$

(2) Symmetric if and only if $(x,y,z)\in\mathcal{R}\Longrightarrow(y,z,x)\in\mathcal{R}$ for any $x,y,z\in G$

(3) Transitive if and only if $(x,y,z)\in\mathcal{R}$, $(y,z,u)\in\mathcal{R}$ imply $(x,z,u)\in\mathcal{R}$, for any x,y,z,u in G

(4) Asymmetric if and only if $(x,y,z)\in\mathcal{R}\Longrightarrow(y,z,x)\notin\mathcal{R}$

(5) Irreflexive if $(x,y,z)\in\mathcal{R}\Longrightarrow x\neq y\neq z$

(6) Irreversible if $(x,y,z)\in\mathcal{R}\Longrightarrow(y,x,p)\notin\mathcal{R}$ for any $p\in G$

(7) Feebly regular if $(x,y,p)\in\mathcal{R}\Longrightarrow(y,z,q)\notin\mathcal{R}$ $(x,y,z)\in\mathcal{R}$

(8) Regular if $(x,y,p)\in\mathcal{R}$, $(y,z,q)\in\mathcal{R}\Longrightarrow(x,y,z)\in\mathcal{R}$

(9) Feebly translative if $(x,y,z)\in\mathcal{R}$, $(y,p,q)\in\mathcal{R}\Longrightarrow\exists r\in G:(x,p,r)\notin\mathcal{R}$

(10) Translative if $\mathcal{R}$ is feebly translative and $(x,y,z)\in\mathcal{R},\Longrightarrow\exists r\in G:(x,z,r)\in\mathcal{R}$ or $(y,z,r)\in\mathcal{R}$

(11) Cyclic if $(x,y,z)\in\mathcal{R}\Longrightarrow(y,z,x)\in\mathcal{R}$ for any $x,y,z\in\mathcal{R}$

(12) Complete if $[x,y,z]\in\mathcal{R}\forall x,y,z\in\mathcal{X}$

(13) Weakly complete if $[x,y,z]\in\mathcal{R}$ or $x=y=z$, $\forall x,y,z\in\mathcal{X}$

Definition 18 (see [14]). A relation $\mathcal{R}$ among sets $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ is a subset of the Cartesian product $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, $\mathcal{R}\subseteq\mathcal{A}\times\mathcal{B}\times\mathcal{C}$.

Proposition 19 (see [24]). *For any ternary relation $\mathcal{R}$ defined on a nonempty set $\mathcal{X}$,*

$$(x,y,z)\in\mathcal{R}^s\Longleftrightarrow[x,y,z]\in\mathcal{R}. \tag{13}$$

Definition 20 (see [14]). Let $\mathcal{X}$ be a nonempty set and $\mathcal{T}$ a self- mapping on $\mathscr{X}$. A ternary relation $\mathscr{R}$ defined on $\mathscr{X}$ is called $\mathscr{T}$-closed if for $x, y, z \in \mathscr{X}$,

$$(x, y, z) \in \mathscr{R} \Longrightarrow (\mathscr{T}x, \mathscr{T}y, \mathscr{T}z) \in \mathscr{R}. \tag{14}$$

Motivated by Alam and Imdad [14] and Perveen et al. [21], we introduce the following definitions and proposition for multivalued mapping in the context of G-metric space setting.

Definition 21. Let $\mathscr{T}$ and $\mathscr{S}$ be two multivalued mappings defined on a nonempty set $\mathscr{X}$. Then, a ternary relation $\mathscr{R}$ defined on $\mathscr{X}$ is called $(\mathscr{T}, \mathscr{S})$-closed if $(\mathscr{S}x, \mathscr{S}y, \mathscr{S}z) \in \mathscr{R} \Longrightarrow (\mathscr{T}x, \mathscr{T}y, \mathscr{T}z) \in \mathscr{R}, \forall x, y, z \in \mathscr{X}$.

Proposition 22. *Let $\mathscr{X}$ be nonempty set, $\mathscr{R}$ a ternary relation on $\mathscr{X}$, and $\mathscr{T}, \mathscr{S}$ two multivalued mappings on $\mathscr{X}$. If $\mathscr{R}$ is $(\mathscr{T}, \mathscr{S})$-closed, then $\mathscr{R}^s$ is $(\mathscr{T}, \mathscr{S})$-closed.*

Definition 23. Let $(\mathscr{X}, G)$ be a G-metric space endowed with a ternary relation $\mathscr{R}$ and $\mathscr{T}, \mathscr{S} : \mathscr{X} \longrightarrow \mathscr{CB}(\mathscr{X})$. $\mathscr{T}$ and $\mathscr{S}$ are said to be $\mathscr{R}$-compatible if for any sequence $\{x_n\}$ such that $\{\mathscr{T}x_n\}$ and $\{\mathscr{S}x_n\}$ are $\{\mathscr{R}\}$-preserving and $\lim_{n\longrightarrow\infty} \mathscr{T}x_n = \lim_{n\longrightarrow\infty} \mathscr{S}x_n$, we have

$$\lim_{n\longrightarrow\infty} G(\mathscr{T}(\mathscr{S}x_n), \mathscr{S}(\mathscr{T}x_n), \mathscr{S}(\mathscr{T}x_n)) = 0. \tag{15}$$

Inspired by Kolman et al. [19], we introduce the following definition in G-metric space setting.

Definition 24. Let $(\mathscr{X}, G)$ be a G-metric space, $\mathscr{R}$ a ternary relation defined on $\mathscr{X}$, and x, y a pair of points in $\mathscr{X}$. Then, a finite sequence $\{z_0, z_1, z_2, \cdots, z_l\} \in \mathscr{X}$ is said to be a path of length l(where $l \in N$) joining x to y in $\mathscr{R}$ if $z_0 = x, z_l = y$ and $[z_i, z_{i+1}, z_{i+1}] \in \mathscr{R}$ for each $i \in \{1, 2, 3, \cdots l-1\}$.

Next, we state some preliminary results which will be helpful to develop our main results.

Ahmadullah et al. [17] proved the results in metric-like space as well as partial metric spaces equipped with a binary relation as follows:

Theorem 25 (see [17]). *Let $(\mathscr{X}, \sigma,)$ be a metric-like spaces equipped with a binary relation $\mathscr{R}$ defined on $\mathscr{X}$ and f a self-mapping on $\mathscr{X}$. Suppose that the following conditions are satisfied:*

(a) There exists a subset $Y \subseteq \mathscr{X}$ with $fX \subseteq Y$ such that (Y, σ) is $\mathscr{R}$-complete

(b) There exists x_0 such that $(x_0, fx_0) \in \mathscr{R}$

(c) $\mathscr{R}$ is f-closed

(d) Either f is $\mathscr{R}$-continuous-like or $\mathscr{R}|_Y$ is σ-self-closed

(e) There exists a constant $k \in [0, 1)$ such that $(\forall x, y \in \mathscr{X}$ with $x, y \in \mathscr{R})$

$$\sigma(fx, fy) \leq k\sigma(x, y). \tag{16}$$

Then, f has a fixed point. Moreover, if

(f) $Y(fx, fy, \mathscr{R}^s)$ is nonempty, for each $x, y \in \mathscr{X}$. Then, f has a unique fixed point

Radha and Singh [24] proved the following theorem in G-metric spaces with ternary relation as follows:

Theorem 26 (see [24]). *Let $(\mathscr{X}, G)$ be a complete G-metric space, $\mathscr{R}$ a ternary relation on $\mathscr{X}$, and $\mathscr{T}$ a self-mapping on $\mathscr{X}$. Suppose that the following conditions hold:*

(i) $\mathscr{X}(\mathscr{T}, \mathscr{R})$ is nonempty

(ii) $\mathscr{R}$ is $\mathscr{T}$-closed

(iii) Either $\mathscr{T}$ is continuous or $\mathscr{R}$ is G-self-closed

(iv) There exists a constant $\theta \in [0, 1)$ such that $(\forall x, y, z \in \mathscr{X}$ with $x, y, z \in \mathscr{R})$

$$G(\mathscr{T}x, \mathscr{T}y, \mathscr{T}z) \leq \theta G(x, y, z). \tag{17}$$

Then, $\mathscr{T}$ has a fixed point. Moreover, if

(v) $\gamma(x, y, z, \mathscr{R}^s)$ is nonempty, for each $x, y, z \in \mathscr{X}$. Then, $\mathscr{T}$ has a unique fixed point

3. Main Results

In this section, we prove the following theorem which is a generalization and improvement of Theorem 26.

Theorem 27. *Let $(\mathscr{X}, G)$ be a complete G-metric space equipped with a ternary relation $\mathscr{R}$ defined on $\mathscr{X}$. Let $S : \mathscr{X} \longrightarrow \mathscr{X}$ and $T : \mathscr{X} \longrightarrow \mathscr{CB}(\mathscr{X})$ be a pair of hybrid multivalued mapping on $\mathscr{X}$. Suppose that the following conditions hold:*

(i) There exists $\mathscr{T}\mathscr{X} \subseteq \mathscr{S}\mathscr{X}$ such that $(\mathscr{X}, G)$ is $\mathscr{R}$-complete

(ii) There exists x_0 such that $(\mathscr{S}x_0, \mathscr{T}x_0, \mathscr{T}x_0) \in \mathscr{R}$

(iii) $\mathscr{X}(\mathscr{T}, \mathscr{S}, \mathscr{R})$ is nonempty

(iv) For n even $\{\mathscr{S}x_{2n}\}$ and $\{\mathscr{T}x_{2n}\}$ converge to a common fixed point

(v) Either (T, S) is $\mathscr{R}$-continuous- or $\mathscr{R}$ is G-self-closed

(vi) There exists a constant $\theta \in [0, 1)$ such that $(\forall x, y, z \in \mathscr{X}$ with $x, y, z \in \mathscr{R})$

$$\mathscr{H}_G(\mathscr{T}x, \mathscr{T}y, \mathscr{T}z) \leq \theta(\mathscr{M}(x, y, z)), \tag{18}$$

where

$$\mathcal{M}(x,y,z) \le \max \left\{ \begin{array}{c} G(\mathcal{S}x,\mathcal{T}y,\mathcal{T}y), G(\mathcal{S}x,\mathcal{T}z,\mathcal{T}z) \\ G(\mathcal{S}y,\mathcal{T}x,\mathcal{T}x), G(\mathcal{S}y,\mathcal{T}z,\mathcal{T}z) \\ G(\mathcal{S}z,\mathcal{T}x,\mathcal{T}x), G(\mathcal{S}z,\mathcal{T}y,\mathcal{T}y) \end{array} \right\}. \tag{19}$$

Then, $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point. Moreover, if

(vii) $\Upsilon_{\mathcal{T}x,\mathcal{T}y,\mathcal{T}z}(\mathcal{T}x,\mathcal{T}y,\mathcal{T}z,\mathcal{R}^s)$ is nonempty, for each $x,y,z\in\mathcal{X}$. Then, $\mathcal{T}$ and $\mathcal{S}$ have a unique common fixed point if z and w are coincidence points of $\mathcal{T}$ and $\mathcal{S}$ such that $d_G(z,w)\le 0$

Proof. Suppose that $\mathcal{T}\mathcal{X}\subseteq\mathcal{S}\mathcal{X}$, $\mathcal{T}\mathcal{X}$ is a G-complete subspace of $\mathcal{X}$ and $(\mathcal{X},G)$ is $\mathcal{R}$-complete, for x_0 with $(\mathcal{S}x_0,\mathcal{T}x_0,\mathcal{T}x_0)\in\mathcal{R}$. We can construct a $\mathcal{T}$-$\mathcal{S}$-sequence such that $\{\mathcal{T}x_n\}\in\mathcal{R}$ with initial point x_0 satisfying

$$(\mathcal{S}x_{2n},\mathcal{T}x_{2n-1},\mathcal{T}x_{n-1}),(\mathcal{S}x_{2n+1},\mathcal{T}x_{2n},\mathcal{T}x_{n}),(\mathcal{S}x_{2n+2},\mathcal{T}x_{2n+1},\mathcal{T}x_{n+1})\cdots, \tag{20}$$

$\forall n\in\mathbb{N}_0$, such that $\{\mathcal{T}x_{2n}\}$, $\{\mathcal{S}x_{2n}\}\in\mathcal{T}(\mathcal{X})$.

By (iii), let x_0 be an arbitrary element of $\mathcal{X}(\mathcal{T},\mathcal{S},\mathcal{R})$, then $(\mathcal{S}x_0,\mathcal{T}x_0,\mathcal{T}x_0)\in\mathcal{R}$. If $Sx_0=\mathcal{T}x_0$, then x_0 is a common fixed point of $\mathcal{T}$ and $\mathcal{S}$ and proof is completed. On contrary, if $\mathcal{T}x_0\ne\mathcal{S}x_0$, then $\mathcal{S}\mathcal{X}\subset\mathcal{T}\mathcal{X}$. Now, we choose $x_1\in\mathcal{X}$ such that $\mathcal{S}x_1=\mathcal{T}x_0$. Again, we can choose $x_2\in\mathcal{X}$ such that $\mathcal{S}x_2=\mathcal{T}x_1$. Proceeding the same way and applying Definition 16, we can construct a sequence $\{x_n\}\subset\mathcal{X}$, such that

$$\begin{aligned} \mathcal{S}x_{2n} &= \mathcal{T}x_{2n-1}, \\ \mathcal{S}x_{2n+1} &= \mathcal{T}x_{2n}, \end{aligned} \tag{21}$$

which is equivalently to

$$(\mathcal{S}x_{2n},\mathcal{S}x_{2n+1})=(\mathcal{T}x_{2n-1},\mathcal{T}x_{2n}). \tag{22}$$

By substituting $x=x_{2n-1}$ and $y=x_{2n}$ in (18), we have

$$\mathcal{H}_G(\mathcal{T}x_{2n-1},\mathcal{T}x_{2n},\mathcal{T}z)\le\theta(\mathcal{M}(x_{2n-1},x_{2n},z)), \tag{23}$$

where

$$\mathcal{M}(x_{2n-1},x_{2n},z)\le\max\left\{\begin{array}{c} G(\mathcal{S}x_{2n-1},\mathcal{T}x_{2n},\mathcal{T}x_{2n}) \\ G(\mathcal{S}x_{2n-1},\mathcal{T}z,\mathcal{T}z) \\ G(\mathcal{S}x_{2n},\mathcal{T}x_{2n-1},\mathcal{T}x_{2n-1}) \\ G(\mathcal{S}x_{2n},\mathcal{T}z,\mathcal{T}z) \\ G(\mathcal{S}z,\mathcal{T}x_{2n-1},\mathcal{T}x_{2n-1}) \\ G(\mathcal{S}z,\mathcal{T}x_{2n},\mathcal{T}x_{2n}) \end{array}\right\}. \tag{24}$$

Applying $y=z=x_{2n}$ in (24), we obtain

$$\begin{aligned} \mathcal{M}(x_{2n-1},x_{2n},x_{2n}) &\le \max\left\{\begin{array}{c} G(\mathcal{S}x_{2n-1},\mathcal{T}x_{2n},\mathcal{T}x_{2n}) \\ G(\mathcal{S}x_{2n-1},\mathcal{T}x_{2n},\mathcal{T}x_{2n}) \\ G(\mathcal{S}x_{2n},\mathcal{T}x_{2n-1},\mathcal{T}x_{2n-1}) \\ G(\mathcal{S}x_{2n},\mathcal{T}x_{2n},\mathcal{T}x_{2n}) \\ G(\mathcal{S}x_{2n},\mathcal{T}x_{2n-1},\mathcal{T}x_{2n-1}) \\ G(\mathcal{S}x_{2n},\mathcal{T}x_{2n},\mathcal{T}x_{2n}) \end{array}\right\}, \\ &\le \max\left\{\begin{array}{c} G(\mathcal{S}x_{2n-1},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1}) \\ G(\mathcal{S}x_{2n-1},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1}) \\ G(\mathcal{S}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n}) \\ G(\mathcal{S}x_{2n},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1}) \\ G(\mathcal{S}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n}) \\ G(\mathcal{S}x_{2n},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1}) \end{array}\right\}. \end{aligned} \tag{25}$$

By $(G5)$, we have

$$\begin{aligned} G(\mathcal{S}x_{2n-1},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1}) \le{}& G(\mathcal{S}x_{2n-1},\mathcal{S}x_{2n},\mathcal{S}x_{2n}) \\ &+ G(\mathcal{S}x_{2n},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1}). \end{aligned} \tag{26}$$

Using (26) in (25), we get

$$\mathcal{M}(x_{2n-1},x_{2n},x_{2n})\le\max\left\{\begin{array}{c} G(\mathcal{S}x_{2n-1},\mathcal{S}x_{2n},\mathcal{S}x_{2n})+ \\ G(\mathcal{S}x_{2n},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1}) \\ G(\mathcal{S}x_{2n-1},\mathcal{S}x_{2n},\mathcal{S}x_{2n})+ \\ G(\mathcal{S}x_{2n},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1}) \\ G(\mathcal{S}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n}) \\ G(\mathcal{S}x_{2n},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1}) \\ G(\mathcal{S}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n}) \\ G(\mathcal{S}x_{2n},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1}) \end{array}\right\}. \tag{27}$$

$$\begin{aligned} \mathcal{M}(x_{2n-1},x_{2n},x_{2n}) ={}& G(\mathcal{S}x_{2n-1},\mathcal{S}x_{2n},\mathcal{S}x_{2n}) \\ &+ G(\mathcal{S}x_{2n},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1}). \end{aligned} \tag{28}$$

From (23), (27), and Lemma 11, we get

$$\begin{aligned} G(\mathcal{S}x_{2n},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1}) \le{}& \theta G(\mathcal{S}x_{2n-1},\mathcal{S}x_{2n},\mathcal{S}x_{2n}) \\ &+ \theta G(\mathcal{S}x_{2n},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1})+\theta, \end{aligned}$$

$$(1-\theta)G(\mathcal{S}x_{2n},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1})\le\theta G(\mathcal{S}x_{2n-1},\mathcal{S}x_{2n},\mathcal{S}x_{2n})+\theta,$$
$$G(\mathcal{S}x_{2n},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1})\le\frac{\theta}{(1-\theta)}G(\mathcal{S}x_{2n-1},\mathcal{S}x_{2n},\mathcal{S}x_{2n})+\frac{\theta}{(1-\theta)}. \tag{29}$$

Similarly, for $\mathcal{S}x_{2n+1}\in\mathcal{T}x_{2n}$, there exists $\mathcal{S}x_{2n+2}\in\mathcal{T}x_{2n+1}$ such that

$$G(\mathcal{S}x_{2n},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1})\le\mathcal{H}_G(\mathcal{T}x_{2n},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1})+\theta^2, \le\frac{\theta^2}{(1-\theta)^2}G(\mathcal{S}x_{2n-1},\mathcal{S}x_{2n},\mathcal{S}x_{2n})+\frac{2\theta^2}{(1-\theta)^2}. \tag{30}$$

Letting $\psi=\theta/(1-\theta)$ and (30), we have

$$G(\mathcal{S}x_{2n},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1})\le\psi^2G(\mathcal{S}x_{2n-1},\mathcal{S}x_{2n},\mathcal{S}x_{2n})+2\psi^2. \tag{31}$$

Continuing this process, we obtain by mathematical induction a sequence $\{x_n\}$ with $\mathcal{S}x_{2n+1}\in\mathcal{T}x_{2n}$ such that

$$G(\mathcal{S}x_{2n},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1})\le\psi^nG(\mathcal{S}x_{2n-1},\mathcal{S}x_{2n},\mathcal{S}x_{2n})+n\psi^n. \tag{32}$$

It follows that

$$\sum_{n=0}^{\infty}G(\mathcal{S}x_{2n},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1})\le\sum_{n=0}^{\infty}\psi^nG(\mathcal{S}x_{2n-1},\mathcal{S}x_{2n},\mathcal{S}x_{2n})+\sum_{n=0}^{\infty}n\psi^n<\infty. \tag{33}$$

By taking limits on both sides in (33), we obtain

$$\lim_{n\longrightarrow\infty}G(\mathcal{S}x_{2n},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1})=G(\mathcal{S}x_0,\mathcal{S}x_1,\mathcal{S}x_1)=0. \tag{34}$$

Using (32) and (G4)-symmetric properties, for all n, $m\in\mathbb{N}_0$ with $m>n$, we obtain

$$\begin{aligned}G(\mathcal{S}x_{2n},\mathcal{S}x_{2n+m},\mathcal{S}x_{2n+m})&\le G(\mathcal{S}x_{2n},\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+1})\\&+G(\mathcal{S}x_{2n+1},\mathcal{S}x_{2n+2},\mathcal{S}x_{2n+2})+\cdots+G(\mathcal{S}x_{2n+m-2},\mathcal{S}x_{2n+m-1},\mathcal{S}x_{2n+m-1})\\&+G(\mathcal{S}x_{2n+m-1},\mathcal{S}x_{2n+m},\mathcal{S}x_{2n+m}),\\&\le(\psi^n+\psi^{n+1}+\psi^{n+2}+\cdots+\psi^{n+m-1})(G(\mathcal{S}x_{2n-1},\mathcal{S}x_{2n},\mathcal{S}x_{2n})+n),\\&\le\psi^n(1+\psi^n+\psi^{n+1}+\psi^{n+2}+\cdots+\psi^{n+m-1})\sigma,\\&\le\frac{\psi^n}{1-\psi}\sigma,\longrightarrow 0\text{ as }n\longrightarrow\infty.\end{aligned} \tag{35}$$

where $\sigma=G(\mathcal{S}x_{2n-1},\mathcal{S}x_{2n},\mathcal{S}x_{2n})+n$ and $0<\psi<1$. This shows that the sequence $\{\mathcal{S}x_{2n}\}$ is a G-Cauchy sequence in the complete subspace $\mathcal{S}\mathcal{X}$.

The results obey a G-Cauchy sequence properties of completeness. Hence, $\{\mathcal{S}x_{2n}\}$ is $\mathcal{R}$-preserving Cauchy sequence. If the pair $(\mathcal{T},\mathcal{S})$ is closed and $\mathcal{R}$-compatible, using Definition 23, we have

$$\begin{aligned}\mathcal{T}\mathcal{S}x_{2n}&=\mathcal{S}\mathcal{T}x_{2n},\\\mathcal{T}x_{2n}&=\mathcal{S}x_{2n},\\\mathcal{S}\mathcal{T}x_{2n}&=\mathcal{S}\mathcal{S}x_{2n},\\\mathcal{T}\mathcal{S}x_{2n}&=\mathcal{T}\mathcal{T}x_{2n}.\end{aligned} \tag{36}$$

We can show that for n, even $\mathcal{T}x_{2n}$ and $\mathcal{S}x_{2n}$ converge to a common fixed point of $\mathcal{T}$ and $\mathcal{S}$. To see this, let $x=\mathcal{T}x_{2n}$, $y=\mathcal{S}x_{2n}$, using (18) which gives

$$\mathcal{H}_G(\mathcal{T}\mathcal{S}x_{2n},\mathcal{T}\mathcal{T}x_{2n},\mathcal{T}\mathcal{T}x_{2n})\le\theta(\mathcal{M}(\mathcal{S}x_{2n},\mathcal{T}x_{2n},\mathcal{T}x_{2n})), \tag{37}$$

where

$$\mathcal{M}(\mathcal{S}x_{2n},\mathcal{T}x_{2n},\mathcal{T}x_{2n})\le\max\left\{\begin{array}{c}G(\mathcal{S}\mathcal{S}x_{2n},\mathcal{T}\mathcal{T}x_{2n},\mathcal{T}\mathcal{T}x_{2n})\\G(\mathcal{S}\mathcal{S}x_{2n},\mathcal{T}\mathcal{T}x_{2n},\mathcal{T}\mathcal{T}x_{2n})\\G(\mathcal{S}\mathcal{T}x_{2n},\mathcal{T}\mathcal{S}x_{2n},\mathcal{T}\mathcal{S}x_{2n})\\G(\mathcal{S}\mathcal{T}x_{2n},\mathcal{T}\mathcal{T}x_{2n},\mathcal{T}\mathcal{T}x_{2n})\\G(\mathcal{S}\mathcal{T}x_{2n},\mathcal{T}\mathcal{S}x_{2n},\mathcal{T}\mathcal{S}x_{2n})\\G(\mathcal{S}\mathcal{T}x_{2n},\mathcal{T}\mathcal{T}x_{2n},\mathcal{T}\mathcal{T}x_{2n})\end{array}\right\},$$
$$\mathcal{M}(\mathcal{S}x_{2n},\mathcal{T}x_{2n},\mathcal{T}x_{2n})=G(\mathcal{S}\mathcal{T}x_{2n},\mathcal{T}\mathcal{S}x_{2n},\mathcal{T}\mathcal{S}x_{2n}), \tag{38}$$

which implies that

$$\mathcal{H}_G(\mathcal{T}\mathcal{S}x_{2n},\mathcal{T}\mathcal{T}x_{2n},\mathcal{T}\mathcal{T}x_{2n})\le\theta(G(\mathcal{S}\mathcal{T}x_{2n},\mathcal{T}\mathcal{S}x_{2n},\mathcal{T}\mathcal{S}x_{2n})). \tag{39}$$

By using Lemma 10 and (39), we obtain

$$G(\mathcal{S}x_{2n},\mathcal{T}x_{2n},\mathcal{T}x_{2n})\le\theta(G(\mathcal{S}\mathcal{T}x_{2n},\mathcal{T}\mathcal{S}x_{2n},\mathcal{T}\mathcal{S}x_{2n})). \tag{40}$$

Since $\mathcal{X}$ is a complete G-metric space and $(\mathcal{T},\mathcal{S})$ is closed and $\mathcal{R}$-compatible, from Definition 23, we get

$$\lim_{n\longrightarrow\infty}G(\mathcal{S}\mathcal{T}x_{2n},\mathcal{T}\mathcal{S}x_{2n},\mathcal{T}\mathcal{S}x_{2n})=0. \tag{41}$$

Assume that z is a common fixed point of $\mathcal{T}$ and $\mathcal{S}$ such that $\lim_{n\longrightarrow\infty}x_{2n}=z$. Thus, $\mathcal{S}\mathcal{S}z=\mathcal{S}\mathcal{S}z=\mathcal{S}z$. So $\mathcal{S}z$ is a common fixed point of $\mathcal{T}$ and $\mathcal{S}$.

Now we show that z is the unique common fixed point of $\mathcal{T}$ and $\mathcal{S}$. For the uniqueness, take $z=\mathcal{S}z$ as a common fixed point of $\mathcal{T}$ and $\mathcal{S}$. Assume that $w = \mathcal{T}w$ and $z \neq w$, and using $x = z, y = w$ in (3.1), we get

$$\mathcal{H}_G(\mathcal{T}z, \mathcal{T}w, \mathcal{T}w) \leq \theta(\mathcal{M}(z, w, w)), \tag{42}$$

where

$$\begin{aligned}\mathcal{M}(z, w, w) &\leq \max\left\{\begin{matrix} G(\mathcal{S}z, \mathcal{T}w, \mathcal{T}w), G(\mathcal{S}z, \mathcal{T}w, \mathcal{T}w) \\ G(\mathcal{S}w, \mathcal{T}z, \mathcal{T}z), G(\mathcal{S}w, \mathcal{T}w, \mathcal{T}w) \\ G(\mathcal{S}w, \mathcal{T}z, \mathcal{T}z), G(\mathcal{S}w, \mathcal{T}w, \mathcal{T}w) \end{matrix}\right\} \\ &= G(\mathcal{S}z, \mathcal{T}w, \mathcal{T}w).\end{aligned} \tag{43}$$

Consequently,

$$\mathcal{H}_G(\mathcal{T}z, \mathcal{T}w, \mathcal{T}w) \leq \theta(G(\mathcal{S}z, \mathcal{T}w, \mathcal{T}w)). \tag{44}$$

Using Lemma 10 and (44), we get

$$G(\mathcal{S}z, \mathcal{T}w, \mathcal{T}w) \leq \theta(G(\mathcal{S}z, \mathcal{T}w, \mathcal{T}w)). \tag{45}$$

Hence, we get

$$G(\mathcal{S}z, \mathcal{T}w, \mathcal{T}w) \leq 0, \Longrightarrow G(\mathcal{S}z, \mathcal{T}w, \mathcal{T}w) = 0, \tag{46}$$

and we conclude that $\mathcal{S}z = \mathcal{T}w$, which is a contradiction. Therefore, z is a unique common fixed point of $\mathcal{T}$ and $\mathcal{S}$.

Using the condition considered in Theorem 27, we prove assertion (vii) as follows: we observe that $\mathcal{C}(\mathcal{T}, \mathcal{S})$ is nonempty, so let us take a pair of elements say (a, b) in $\mathcal{C}(\mathcal{T}, \mathcal{S})$ such that

$$\begin{aligned}\mathcal{T}a &= \mathcal{S}a = \bar{a}, \\ \mathcal{T}b &= \mathcal{S}b = \bar{b}.\end{aligned} \tag{47}$$

Next, we are required to show that $\bar{a} = \bar{b}$. By observing the above assertion, there exists an S-path (say, $z_0, z_1, z_2, \cdots, z_l$) of length l in $\mathcal{R}^s$ from $\mathcal{T}a$ to $\mathcal{T}b$, with

$$\mathcal{S}z_0 = \mathcal{T}a, \tag{48}$$

$$\mathcal{S}z_l = \mathcal{T}b, \tag{49}$$

such that

$$[\mathcal{S}z_{2i}, \mathcal{S}z_{2i+1}, \mathcal{S}z_{2i+1}] \in \mathcal{R}^s \subseteq \mathcal{R}, \tag{50}$$

for all $i \in 0, 1, 2, 3, \cdots l-1$.

And

$$[\mathcal{S}z_{2i}, \mathcal{T}z_{2i}, \mathcal{T}z_{2i}] \in \mathcal{R}^s \subseteq \mathcal{R}, \tag{51}$$

for every $i \in 0, 1, 2, 3, \cdots l-1$.

Define two constant sequences such that

$$\begin{aligned}z_{2n}^0 &= a, \\ z_{2n}^l &= b.\end{aligned} \tag{52}$$

By using (49), for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\mathcal{T}z_{2n}^0 &= Ta = \bar{a}, \\ \mathcal{T}z_{2n}^l &= \mathcal{T}b = \bar{b}.\end{aligned} \tag{53}$$

By usual substitution for $z_0^i = z_i$ for each $i \in 0, 1, 2, \cdots l$, that is,

$$\begin{aligned}z_0^1 &= z_1, \\ z_0^2 &= z_2, \\ z_0^3 &= z_3, \\ z_0^4 &= z_4, \\ z_0^{l-1} &= z_{l-1}.\end{aligned} \tag{54}$$

Recall that $\mathcal{T}\mathcal{X} \subseteq S\mathcal{X}$ and Lemma 12. Thus, we construct a sequence

$$\{z_{2n}^1\}, \{z_{2n}^2\}, \{z_{2n}^3\}, \cdots \{z_{2n}^i\} \in \mathcal{X}. \tag{55}$$

In general, $\{z_n^1\} \in \mathcal{X}$

$$\begin{aligned}\mathcal{S}z_{2n+1}^1 &= \mathcal{T}z_{2n}^1, \\ \mathcal{S}z_{2n+1}^2 &= \mathcal{T}z_{2n}^2, \\ \mathcal{S}z_{2n+1}^3 &= \mathcal{T}z_{2n}^3, \\ \mathcal{S}z_{2n+1}^4 &= \mathcal{T}z_{2n}^4, \\ \mathcal{S}z_{2n+1}^{l-1} &= \mathcal{T}z_{2n}^{l-1}, \forall n \in \mathbb{N}.\end{aligned} \tag{56}$$

We obtain

$$Sz_{2n+1}^i = Tz_{2n}^i, \tag{57}$$

for all $i \in [0, l-1]$. Corresponding to each z_i, we have $[Sz_0^i, Sz_1^i] \in \mathcal{R}$ from (49), (50), and $(\mathcal{T}, \mathcal{S})$-compactness of $\mathcal{R}$, and we get

$$\lim_{n \longrightarrow \infty} G(Sz_{2n}^i, Sz_{2n+1}^i, Sz_{2n+1}^i) = 0, \tag{58}$$

for each $i \in 1, 2, 3, \cdots l-1$.

Thus, $\mathcal{R}$ is $(\mathcal{T}, \mathcal{S})$-closed, and we conclude that $[\mathcal{T}z_{2n}^i, \mathcal{T}z_{2n}^{i+1}, \mathcal{T}z_{2n}^{i+1}] \in \mathcal{R}$, for each $i \in 0, 1, 2, 3, \cdots l-1$ and for all $n \in \mathbb{N}$.

Otherwise, $[\mathcal{S}z_{2n}^i, \mathcal{S}z_{2n}^{i+1}, \mathcal{S}z_{2n}^{i+1}] \in \mathcal{R}$, for each $i \in 0, 1, 2, 3, \cdots l-1$ and for all $n \in \mathbb{N}$.

Define $G_{2n}^i = G(\mathcal{S}z_{2n}^i, \mathcal{S}z_{2n}^{i+1}, \mathcal{S}z_{2n}^{i+1})$, for each $i \in 0, 1, 2, 3, \cdots l-1$ and for all $n \in \mathbb{N}$. We assert that $\lim_{n \longrightarrow \infty} G_{2n}^i > 0$. Assume that $\lim_{n \longrightarrow \infty} G_{2n}^i = G > 0$.

Since $[\mathcal{S}z_{2n}^{i}, \mathcal{S}z_{2n}^{i+1}, \mathcal{S}z_{2n}^{i+1}] \in \mathcal{R}$, either $[\mathcal{S}z_{2n}^{i}, \mathcal{S}z_{2n}^{i+1}, \mathcal{S}z_{2n}^{i+1}] \in \mathcal{R}$ or $[\mathcal{S}z_{2n}^{i+1}, \mathcal{S}z_{2n}^{i}, \mathcal{S}z_{2n}^{i}] \in \mathcal{R}$.

If $[\mathcal{S}z_{2n}^{i}, \mathcal{S}z_{2n}^{i+1}, \mathcal{S}z_{2n}^{i+1}] \in \mathcal{R}$, for $x = z_{2n}^{i}$ and $y = z_{2n}^{i+1}$, then applying the condition (vi), we have

$$\mathcal{H}_G\left(\mathcal{T}z_{2n}^{i}, \mathcal{T}z_{2n}^{i+1}, \mathcal{T}z_{2n}^{i+1}\right) \leq \theta\left(\mathcal{M}\left(z_{2n}^{i}, z_{2n}^{i+1}, z_{2n}^{i+1}\right)\right), \quad (59)$$

where

$$\begin{aligned}
\mathcal{M}\left(z_{2n}^{i}, z_{2n}^{i+1}, z_{2n}^{i+1}\right) &\leq \max \left\{ \begin{array}{c} G\left(\mathcal{S}z_{2n}^{i}, \mathcal{T}z_{2n}^{i+1}, \mathcal{T}z_{2n}^{i+1}\right) \\ G\left(\mathcal{S}z_{2n}^{i}, \mathcal{T}z_{2n}^{i+1}, \mathcal{T}z_{2n}^{i+1}\right) \\ G\left(\mathcal{S}z_{2n}^{i+1}, \mathcal{T}z_{2n}^{i}, \mathcal{T}z_{2n}^{i}\right) \\ G\left(\mathcal{S}z_{2n}^{i+1}, \mathcal{T}z_{2n}^{i+1}, \mathcal{T}z_{2n}^{i+1}\right) \\ G\left(\mathcal{S}z_{2n}^{i+1}, \mathcal{T}z_{2n}^{i}, \mathcal{T}z_{2n}^{i}\right) \\ G\left(\mathcal{S}z_{2n}^{i+1}, \mathcal{T}z_{2n}^{i+1}, \mathcal{T}z_{2n}^{i+1}\right) \end{array} \right\} \\
&\leq \max \left\{ \begin{array}{c} G\left(\mathcal{S}z_{2n}^{i}, \mathcal{S}z_{2n+1}^{i+1}, \mathcal{S}z_{2n+1}^{i+1}\right) \\ G\left(\mathcal{S}z_{2n}^{i}, \mathcal{S}z_{2n+1}^{i+1}, \mathcal{S}z_{2n+1}^{i+1}\right) \\ G\left(\mathcal{S}z_{2n}^{i+1}, \mathcal{S}z_{2n+1}^{i}, \mathcal{S}z_{2n+1}^{i}\right) \\ G\left(\mathcal{S}z_{2n}^{i+1}, \mathcal{S}z_{2n+1}^{i+1}, \mathcal{S}z_{2n+1}^{i+1}\right) \\ G\left(\mathcal{S}z_{2n}^{i+1}, \mathcal{S}z_{2n+1}^{i}, \mathcal{S}z_{2n+1}^{i}\right) \\ G\left(\mathcal{S}z_{2n}^{i+1}, \mathcal{S}z_{2n+1}^{i+1}, \mathcal{S}z_{2n+1}^{i+1}\right) \end{array} \right\} \\
&= G\left(\mathcal{S}z_{2n}^{i}, \mathcal{S}z_{2n+1}^{i+1}, \mathcal{S}z_{2n+1}^{i+1}\right).
\end{aligned} \quad (60)$$

By Lemma 10 and (60), we get

$$\begin{aligned}
G\left(\mathcal{S}z_{2n}^{i}, \mathcal{T}z_{2n}^{i+1}, \mathcal{T}z_{2n}^{i+1}\right) &\leq \theta\left(G\left(\mathcal{S}z_{2n}^{i}, \mathcal{S}z_{2n+1}^{i+1}, \mathcal{S}z_{2n+1}^{i+1}\right)\right), \\
G\left(\mathcal{S}z_{2n}^{i}, \mathcal{S}z_{2n+1}^{i+1}, \mathcal{S}z_{2n+1}^{i+1}\right) &\leq \theta\left(G\left(\mathcal{S}z_{2n}^{i}, \mathcal{S}z_{2n+1}^{i+1}, \mathcal{S}z_{2n+1}^{i+1}\right)\right).
\end{aligned} \quad (61)$$

Taking lim as $n \longrightarrow \infty$ and using $\lim_{n\longrightarrow\infty} G_{2n}^{i} = G$, we get

$$G\left(\mathcal{S}z_{2n}^{i}, \mathcal{S}z_{2n+1}^{i+1}, \mathcal{S}z_{2n+1}^{i+1}\right) \leq 0, \quad (62)$$

which is contradiction, and hence,

$$\lim_{n\longrightarrow\infty} G_{2n}^{i} = G = 0. \quad (63)$$

The same for symmetric property (G4), if $(Sz_{2n}^{i+1}, Sz_{2n+1}^{i}, Sz_{2n+1}^{i}) \in \mathcal{R}$, we have

$$\lim_{n\longrightarrow\infty} G_{2n}^{i} = \lim_{n\longrightarrow\infty} G\left(Sz_{2n}^{i+1}, Sz_{2n+1}^{i}, Sz_{2n+1}^{i}\right) = 0, \quad (64)$$

for $i \in 0, 1, 2, \cdots l-1$.

Using (50), $\lim_{n\longrightarrow\infty} G_{2n}^{i} = 0$, and (G4), we have

$$\begin{aligned}
G\left(\bar{a}, \bar{b}, \bar{b}\right) &= G\left(Sz_{2n}^{0}, Sz_{2n}^{i}, Sz_{2n}^{i}\right) \\
&\leq \sum_{i=0}^{l-1} G\left(Sz_{2n}^{i}, Sz_{2n}^{i+1}, Sz_{2n}^{i+1}\right) \\
&\leq \sum_{i=0}^{l-1} G_{2n}^{i}, \longrightarrow 0 \, as \, n \longrightarrow \infty,
\end{aligned} \quad (65)$$

so that

$$\begin{aligned}
G\left(\bar{a}, \bar{b}, \bar{b}\right) &= 0, \\
\bar{a} &= \bar{b}.
\end{aligned} \quad (66)$$

Therefore,

$$\mathcal{S}z = \mathcal{S}w. \quad (67)$$

Now, suppose that $z = \bar{a}$ and $w = \bar{b}$, we claim that for the existence of unique fixed point $d_G(z, w) > 0$. To see this, we have

$$d_G(z, w) = G(z, w, w) + G(w, z, z). \quad (68)$$

From

$$\begin{aligned}
G(z, w, w) &= |z - w| + |w - w| + |z - w| = 2|z - w|, \\
G(w, z, z) &= |w - z| + |z - z| + |w - z| = 2|z - w|,
\end{aligned} \quad (69)$$

we have

$$d_G(z, w) = 2|z - w| + 2|z - w| = 4|z - w|, \quad (70)$$

which is a contradiction. Since $\bar{a} = \bar{b} z = w$, thus z is a unique common fixed point of $\mathcal{T}$ and $\mathcal{S}$, and we have $z = b$. Thus, the proof is completed. □

From Theorem 27, we can deduce the corollary as follows:

(i) $\mathcal{S}\mathcal{X}$ is closed in $(\mathcal{X}, G)$

(ii) There exists a continuous comparison function $\psi \in \Psi$ such that

$$\mathcal{H}_G(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) \leq \psi(\mathcal{M}(x, y, z)), \quad (71)$$

where

$$\mathcal{M}(x, y, z) \leq \left\{ \begin{array}{c} G(\mathcal{S}x, \mathcal{S}y, \mathcal{S}z), G(\mathcal{S}x, \mathcal{T}x, \mathcal{S}z) \\ G(\mathcal{S}z, \mathcal{T}z, \mathcal{S}z), G(\mathcal{S}y, \mathcal{S}y, \mathcal{S}z) \end{array} \right\}. \quad (72)$$

Corollary 28. *Let $(\mathcal{X}, G)$ be a G-metric space and let $\mathcal{S} : \mathcal{X} \longrightarrow \mathcal{X}$ and $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{CB}(\mathcal{X})$ be pair of hybrid mapping such that the following conditions are hold:*

Then, $\mathcal{T}$ and $\mathcal{S}$ have the unique common fixed point.

Proof. The proof of the above corollary follows similar steps of Theorem 27. Therefore, the proof is completed. □

The following example is motivated from [31] for ternary relation.

Example 2. Consider $\mathcal{X} = [0, 1, 2, 3, \cdots 14, 15]$, $\mathcal{R}$ a ternary relation on $\mathcal{X}$ defined by the equation

$$x^2 + 5y = z, \tag{73}$$

endowed with complete G-metric space, defined by G-metric $G(x, y, z) = |x - y| + |y - z| + |x - z|$ in $\mathbb{R}^3$ with ternary relation. Since $x^2 > 15$ for $x > 3$, we need only to find solutions for y and z when $x = 1, 2, 3$. This yields the following:

$$\mathcal{R} = \{(1, 1, 6), (1, 2, 11), (2, 1, 9), (2, 2, 14), (3, 1, 14)\} \text{ on } \mathcal{X}. \tag{74}$$

We claim that $\mathcal{X}$ is either G-complete or $\mathcal{R}$-complete. To observe this, define $\mathcal{S} : \mathcal{X} \longrightarrow \mathcal{X}$ by

$$\mathcal{S}x = \frac{x}{2}, \forall x \in \mathcal{X}, \tag{75}$$

and $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{CB}(\mathcal{X})$ by

$$\mathcal{T}x = \left[0, \sqrt{x}\right], \forall x \in \mathcal{X}. \tag{76}$$

Then,

(i) $\mathcal{T}\mathcal{X}$ is a G-complete subspace in $\mathcal{X}$

(ii) There exists a constant $\theta \in [0, 1)$ such that $(\forall x, y, z \in \mathcal{X}$ with $x, y, z \in \mathcal{R})$

$$\mathcal{H}_G(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) \leq \theta(\mathcal{M}(x, y, z)), \tag{77}$$

where

$$\mathcal{M}(x, y, z) \leq \max \left\{ \begin{array}{l} G(\mathcal{S}x, \mathcal{T}y, \mathcal{T}y), G(\mathcal{S}x, \mathcal{T}z, \mathcal{T}z) \\ G(\mathcal{S}y, \mathcal{T}x, \mathcal{T}x), G(\mathcal{S}y, \mathcal{T}z, \mathcal{T}z) \\ G(\mathcal{S}z, \mathcal{T}x, \mathcal{T}x), G(\mathcal{S}z, \mathcal{T}y, \mathcal{T}y) \end{array} \right\}. \tag{78}$$

By using (i), we show that $\mathcal{T}\mathcal{X}$ is a G-complete subspace in $\mathcal{X}$. By Definition 5, we have

$$G(x, y, y) \leq |x - y| + |y - y| + |x - y| = 2|x - y|, \tag{79}$$

$$G(y, x, x) \leq |y - x| + |x - x| + |y - x|, = 2|y - x|. \tag{80}$$

Using (79) and (79) in (3), we get

$$d_G(x, y) \leq 2|x - y| + 2|y - x| = 4|x - y|. \tag{81}$$

To prove (ii), let $x, y, z \in \mathcal{X}$. If $x = y = z = 0$, then $\mathcal{T}x = \mathcal{T}y = \mathcal{T}z = 0$ and $\mathcal{H}_G(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) = 0$. The proof is completed. On contrary, suppose that the values of x, y, z are not all zero.

For $x \leq y \leq z$, we have

$$\mathcal{H}_G(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) = \mathcal{H}_G\left(\left[0, \sqrt{x}\right], \left[0, \sqrt{y}\right], \left[0, \sqrt{z}\right]\right). \tag{82}$$

By (4), we have

$$\Longrightarrow \max \left\{ \begin{array}{l} \sup_{0 \leq a \leq \sqrt{x}} G\left(a, \left[0, \sqrt{y}\right], \left[0, \sqrt{z}\right]\right) \\ \sup_{0 \leq b \leq \sqrt{y}} G\left(b, \left[0, \sqrt{x}\right], \left[0, \sqrt{z}\right]\right) \\ \sup_{0 \leq c \leq \sqrt{z}} G\left(c, \left[0, \sqrt{x}\right], \left[0, \sqrt{y}\right]\right) \end{array} \right\}. \tag{83}$$

Since $x \leq y \leq z$, then $[0, \sqrt{x}] \subseteq [0, \sqrt{y}] \subseteq [0, \sqrt{z}]$, using (5), and as a result,

$$\begin{aligned} d_G\left(\left[0, \sqrt{x}\right], \left[0, \sqrt{y}\right]\right) &= 0, \\ d_G\left(\left[0, \sqrt{y}\right], \left[0, \sqrt{z}\right]\right) &= 0, \\ d_G\left(\left[0, \sqrt{x}\right], \left[0, \sqrt{z}\right]\right) &= 0. \end{aligned} \tag{84}$$

Now for each $0 \leq a \leq \sqrt{x}$ and $d_G(x, y) = 4|x - y|$ in (5) and (6), we have

$$\begin{aligned} G\left(a, \left[0, \sqrt{y}\right], \left[0, \sqrt{z}\right]\right) &= d_G\left(a, \left[0, \sqrt{y}\right]\right) + d_G\left(\left[0, \sqrt{y}\right], \left[0, \sqrt{y}\right]\right) \\ &\quad + d_G\left(a, \left[0, \sqrt{z}\right]\right), \\ &\leq \left|4a - 4\sqrt{y} + 0 + 4a - 4\sqrt{z}\right|, \\ &= \left|8a - 4\sqrt{y} - 4\sqrt{z}\right|, \\ &\leq \left|8\sqrt{x} - 4\sqrt{y} - 4\sqrt{z}\right|, \\ &= \left|8\sqrt{x} - 4\sqrt{y} - 4\sqrt{z}\right|. \end{aligned} \tag{85}$$

Next, for each $0 \leq b \leq \sqrt{y}$ and $d_G(x, y) = 4|x - y|$ in (5) and (6), we get

$$\begin{aligned} G\left(b, \left[0, \sqrt{x}\right], \left[0, \sqrt{z}\right]\right) &= d_G\left(b, \left[0, \sqrt{x}\right]\right) \\ &\quad + d_G\left(\left[0, \sqrt{x}\right], \left[0, \sqrt{z}\right]\right) + d_G\left(b, \left[0, \sqrt{z}\right]\right), \\ &\leq \left|4b - 4\sqrt{x} + 0 + 4b - 4\sqrt{z}\right|, \\ &= \left|8b - 4\sqrt{x} - 4\sqrt{z}\right|, \\ &= \left|8\sqrt{y} - 4\sqrt{x} - 4\sqrt{z}\right|, \\ &= \left|8\sqrt{y} - 4\left(\sqrt{x} + \sqrt{z}\right)\right|. \end{aligned} \tag{86}$$

Likewise, for each $0 \leq \sqrt{z}$ and (5) and (6), it gives

$$\begin{aligned} G\left(c,\left[0,\sqrt{x}\right],\left[0,\sqrt{y}\right]\right) &= d_G\left(c,\left[0,\sqrt{x}\right]\right) + d_G\left(\left[\sqrt{x}\right],\left[0,\sqrt{y}\right]\right) \\ &\quad + d_G\left(c,\left[0,\sqrt{y}\right]\right), \\ &\leq \left|4c - 4\sqrt{x} + 0 + 4c - 4\sqrt{y}\right|, \\ &= \left|8c - 4\sqrt{x} - 4\sqrt{y}\right|, \\ &= \left|8\sqrt{z} - 4\left(\sqrt{x} + \sqrt{y}\right)\right|. \end{aligned} \quad (87)$$

Consequently,

$$\begin{aligned} \mathscr{H}_G(\mathscr{T}x,\mathscr{T}y,\mathscr{T}z) &\leq \max \left\{ \begin{array}{c} \sup\limits_{0\leq a\leq\sqrt{x}} \left|8\sqrt{x} - 4\left(\sqrt{y} + \sqrt{z}\right)\right| \\ \sup\limits_{0\leq b\leq\sqrt{y}} \left|8\sqrt{y} - 4\left(\sqrt{z} + \sqrt{x}\right)\right| \\ \sup\limits_{0\leq c\leq\sqrt{z}} \left|8\sqrt{z} - 4\left(\sqrt{x} + \sqrt{y}\right)\right| \end{array} \right\} \\ &= \left|8\sqrt{z} - 4\left(\sqrt{x} + \sqrt{y}\right)\right|. \end{aligned} \quad (88)$$

On the other hand, we formulate the following G-metrics. By (5) and (6), this yields to

$$\begin{aligned} G(\mathscr{S}x,\mathscr{T}y,\mathscr{T}y) &= G\left(\frac{x}{2},[0,\sqrt{y}],[0,\sqrt{y}]\right), \\ &= d_G\left(\frac{x}{2},[0,\sqrt{y}]\right) + d_G([0,\sqrt{y}],[0,\sqrt{y}]) + d_G\left(\frac{x}{2},[0,\sqrt{y}]\right), \\ &\leq \inf\left\{d_G\left(\frac{x}{2},0\right), d_G\left(\frac{x}{2},\sqrt{y}\right)\right\} \\ &\quad + 0 + \inf\left\{d_G\left(\frac{x}{2},0\right), d_G\left(\frac{x}{2},\sqrt{y}\right)\right\}, \\ &= 4\left|\frac{x}{2} - \sqrt{y}\right| + 4\left|\frac{x}{2} - \sqrt{y}\right|, = 4|x - 2\sqrt{y}|. \end{aligned} \quad (89)$$

In similar manner, we can calculate the following G-metrics.

$$\begin{aligned} G(\mathscr{S}y,\mathscr{T}z,\mathscr{T}z) &= G\left(\frac{y}{2},\left[0,\sqrt{z}\right],\left[0,\sqrt{z}\right]\right) = 4\left|y - 2\sqrt{z}\right|, \\ G(\mathscr{S}z,\mathscr{T}x,\mathscr{T}x) &= G\left(\frac{z}{2},\left[0,\sqrt{x}\right],\left[0,\sqrt{x}\right]\right) = 4\left|z - 2\sqrt{x}\right|, \\ G(\mathscr{S}x,\mathscr{T}z,\mathscr{T}z) &= G\left(\frac{x}{2},\left[0,\sqrt{z}\right],\left[0,\sqrt{z}\right]\right) = 4\left|x - 2\sqrt{z}\right|, \\ G(\mathscr{S}y,\mathscr{T}x,\mathscr{T}x) &= G\left(\frac{y}{2},\left[0,\sqrt{x}\right],\left[0,\sqrt{x}\right]\right) = 4\left|y - 2\sqrt{x}\right|, \\ G(\mathscr{S}z,\mathscr{T}y,\mathscr{T}y) &= G\left(\frac{z}{2},[0,\sqrt{y}],[0,\sqrt{y}]\right) = 4|z - 2\sqrt{y}|. \end{aligned} \quad (90)$$

Applying the above equality, we obtain

$$\mathscr{M}(x,y,z) \leq \max \left\{ \begin{array}{c} 4\left|x - 2\sqrt{y}\right|, 4\left|x - 2\sqrt{z}\right| \\ 4\left|y - 2\sqrt{x}\right|, 4\left|y - 2\sqrt{z}\right| \\ 4\left|z - 2\sqrt{x}\right|, 4\left|z - 2\sqrt{y}\right|, = 4\left|z - 2\sqrt{x}\right| \end{array} \right\}. \quad (91)$$

From (88) and (91), it follows that

$$|8\sqrt{z} - 4\left(\sqrt{x} + \sqrt{y}\right)| \leq \theta\left(4\,|\,z - 2\sqrt{x}\,|\right). \quad (92)$$

This shows that all $x, y, z \in \mathscr{R}$ and conditions imposed in Theorem 27 are satisfied. Hence, a pair of hybrid mapping $\mathscr{S}$ and $\mathscr{T}$ in G-metric space has a unique common fixed point. Thus, $x = 0$ is a unique common fixed point of $\mathscr{S}$ and $\mathscr{T}$. Clearly, $\mathscr{R}$ is $(\mathscr{T},\mathscr{S})$-closed, and $x_0 = 0$, $(\mathscr{S}0,\mathscr{T}0,\mathscr{T}0) \in \mathscr{R}$.

4. An Application to SODE in G-Metric Space

Next, to validate the result of Theorem 27, we apply the second-order differential equation (SODE) by transforming it to the system of integral equations. The fixed point theory is involved in physics applications, specifically in the solution of electric equations, and this type of work is motivated from [22, 32, 33]. It is well known that an electric circuit can be represented by ternary relation, R a resistor, L an inductor, and C a capacitor on E an electromotive force in series. On the other hand, they can be presented as ternary relations with X_L an inductive reactance, X_C a capacitive reactance, and Z an impedance. If the rate of charge q in condenser with respect to time t is denoted by current I, that is amount to say $I = dq/dt$. We get the following ternary relations:

$$\begin{aligned} V &= IR, \\ V &= \frac{q}{C}, \\ V &= L\frac{dI}{dt}. \end{aligned} \quad (93)$$

By Kirchhoff's law, the sum of voltage drops across the circuit is equal to the supplied voltage. We consider the $R - L - C$-circuit, and the differential equation for the charge q on the condenser is

$$IR + \frac{q}{C} + L\frac{dI}{dt} = V(t), \quad (94)$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V(t), \quad (95)$$

where $V(t)$ is the applied voltage at time t. If the initial conditions are assumed to be

$$q(0) = 0, \frac{dq(0)}{dt} = 0, \quad R^2 > \frac{4L}{C}, \quad \tau = \frac{R}{2L}. \tag{96}$$

The differential equation (95) can be represented using the Green function given by

$$G(t,s) = \begin{cases} -se^{\tau(s-t)}, & 0 \le s \le t \le 1, \\ -te^{\tau(s-t)}, & 0 \le t \le s \le 1. \end{cases} \tag{97}$$

Consider the following systems of equations motivated by [21].

$$\begin{cases} z(t) = \int_0^T G(t,s)\mathcal{K}(t,s,z(s))ds + g(t), \\ w(t) = \int_0^T G(t,s)\mathcal{K}(t,s,w(s))ds + g(t), \end{cases} \tag{98}$$

for all $t, s \in \mathcal{X} = [0,T], T > 0$, where $\mathcal{K} : \mathcal{X} \times \mathcal{X} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $g : \mathcal{X} \longrightarrow \mathbb{R}^n$. Let $\mathcal{R}$ be a ternary relation on $\mathbb{R}^n$ and $\mathcal{X} = C(\mathcal{X}, \mathbb{R}^n)$ on $\mathcal{X}$, set of all continuous mappings from $\mathcal{X} \longrightarrow \mathbb{R}^n$. Consider the ternary relation $\mathcal{R}$ on $\mathcal{X}$ as

$$(z,w,w) \in \mathcal{R} \Longleftrightarrow (z(t), w(t), w(t)) \in \mathcal{R}, \tag{99}$$

for all $t \in \mathcal{X}$. Now we define

$$\mathcal{T}z(t) = \int_0^T G(t,s)\mathcal{K}(t,s,z(s))ds + g(t), \quad \mathcal{S}w(t) = \int_0^T G(t,s)\mathcal{K}(t,s,w(s))ds + g(t), \tag{100}$$

for all $t, s \in \mathcal{X}$.

Let $\mathcal{X} = C([0,T])$ be the set of all continuous function defined on $[0,T]$. Define a G-metric on $\mathcal{X}$, by $G : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}^+$ and

$$G(z,w,w) \le \sup_{t\in[0,T]} |z-w| + \sup_{t\in[0,T]} |w-w| + \sup_{t\in[0,T]} |z-w|, = 2 \sup_{t\in[0,T]} |z-w|. \tag{101}$$

Then, $(\mathcal{X}, G)$ is a G-complete metric space.

Theorem 29. *Suppose the following conditions holds:*

(i) *$\mathcal{K} : [0,T] \times [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous*

(ii) *There exists some $t, s \in [0,T]$ and $z \in \mathcal{X}$ such that*

$$\left(\int_0^T G(t,s)\mathcal{K}(t,s,z(s))ds + g(t), \int_0^T G(t,s)\mathcal{K}(t,s,w(s))ds + g(t), \int_0^T G(t,s)\mathcal{K}(t,s,w(s))ds + g(t)\right) \in \mathcal{R}, \tag{102}$$

for all $t, s \in \mathcal{X}$.

(iii) *There exists some a function $G : [0,T] \times [0,T] \times [0,T] \longrightarrow [0, \mathbb{R}]$, for each $z, w \in \mathcal{R}$ or $z, w \in \mathcal{R}^s$ and $t, s \in [0,T]$ with $\theta \in [0,1]$ such that*

$$|\mathcal{K}(t,s,z(s)) - \mathcal{K}(t,s,w(s))| \le G(t,s)\mathcal{M}(z,w,w), \tag{103}$$

where $\mathcal{M}(z,w,w) = 2 \mid \mathcal{S}z(t) - \mathcal{S}w(t)$ and

$$\mathcal{M}(z,w,w) = \max\left\{ \begin{array}{l} G(\mathcal{S}z, \mathcal{T}w, \mathcal{T}w), G(\mathcal{S}z, \mathcal{T}w, \mathcal{T}w) \\ G(\mathcal{S}w, \mathcal{T}z, \mathcal{T}z), G(\mathcal{S}w, \mathcal{T}w, \mathcal{T}w) \\ G(\mathcal{S}w, \mathcal{T}z, \mathcal{T}z), G(\mathcal{S}w, \mathcal{T}w, \mathcal{T}w) \end{array} \right\}, \tag{104}$$

(iv) *There exists $\theta \in [0,1)$ such that*

$$G(t,s) = \frac{1 - e^{-\tau t} - \tau t e^{\tau(1-t)}}{\tau^2} \le \theta, \tag{105}$$

where $\tau = R/2L$.

Then, the system of equations (95) has a common solution $z(t)$ which is a solution of the integral equation (98).

Proof. For $z, w \in C([0,T])$ with $z < w$, we claim that $\mathcal{H}_G(\mathcal{T}z, \mathcal{T}w, \mathcal{T}w) \nleq \theta(\mathcal{M}(z,w,w))$. Then, we have

$$\begin{aligned}
\mathcal{H}_G(\mathcal{T}z, \mathcal{T}w, \mathcal{T}w) &\le 2 \sup_{0\in[0,T]} |\mathcal{S}z(t) - \mathcal{S}w(t)|, \\
&\le 2 \sup_{0\in[0,T]} \left| \int_0^T G(t,s)\mathcal{K}(t,s,z(s))ds + g(t) - \int_0^T G(t,s)\mathcal{K}(t,s,w(s))ds + g(t) \right|, \\
&\le 2 \sup_{0\in[0,T]} \left| \int_0^T G(t,s)(\mathcal{K}(t,s,z(s)) - \mathcal{K}(t,s,w(s)))ds \right|, \\
&\le 2 \sup_{0\in[0,T]} |\mathcal{S}z(t) - \mathcal{S}w(t)| \int_0^T G(t,s)ds \\
&\le 2 \sup_{0\in[0,T]} |\mathcal{S}z(t) - \mathcal{S}w(t)| \left[\int_0^t -se^{\tau(s-t)}ds - \int_0^1 te^{\tau(s-t)}ds\right], \\
&\le 2 \sup_{0\in[0,T]} |\mathcal{S}z(t) - \mathcal{S}w(t)| \cdot \left[\frac{1 - e^{-\tau t} - \tau t e^{\tau(1-t)}}{\tau^2}\right], \mathcal{H}_G(\mathcal{T}z, \mathcal{T}w, \mathcal{T}w) \\
&\le \theta(\mathcal{M}(z,w,w)),
\end{aligned} \tag{106}$$

which is a contradiction. Hence, z is a common fixed of $\mathcal{T}$ and $\mathcal{S}$ and also a solution to integral equation (98) and a second-order differential equation (95). Hence, we can conclude that all the conditions imposed in Theorems 27 and 29 are satisfied. Therefore, the proof is completed. □

Data Availability

No data were used to support this study.

Disclosure

Conflicts of Interest

The author declares no competing interests.

Authors' Contributions

The author contributed fully and significantly in writing this article. The author read and approved the final manuscript.

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