Hindawi
Abstract and Applied Analysis
Volume 2022, Article ID 4039408, 7 pages
https://doi.org/10.1155/2022/4039408

Research Article

Maximal Injective Real W∗-Subalgebras of Tensor Products of Real W∗-Algebras

A. A. Rakhimov,[1] **Muzaffar Nurillaev**,[2] **and Zabidin Salleh**[3]

[1]*National University of Uzbekistan Named after M. Ulugbek, 700174 Tashkent, VUZ Gorodok, Uzbekistan*
[2]*Tashkent State Pedagogical University Named after Nizami, 100070 Tashkent, St. Bunyadkor, 27, Uzbekistan*
[3]*Special Interest Group on Modelling and Data Analytics, Faculty of Ocean Engineering Technology and Informatics, Universiti Malaysia Terengganu, 21030 Kuala Nerus, Terengganu, Malaysia*

Correspondence should be addressed to Zabidin Salleh; zabidin@umt.edu.my

Received 11 January 2022; Revised 14 March 2022; Accepted 3 April 2022; Published 25 April 2022

Academic Editor: A. Ghareeb

It is known that injective (complex or real) W∗-algebras with particular factors have been studied well enough. In the arbitrary cases, i.e., in noninjective case, to investigate (up to isomorphism) W∗-algebras is hard enough, in particular, there exist continuum pairwise nonisomorphic noninjective factors of type II. Therefore, it seems interesting to study maximal injective W∗-subalgebras and subfactors. On the other hand, the study of maximal injective W∗-subalgebras and subfactors is also related to the well-known von Neumann's bicommutant theorem. In the complex case, such subalgebras were investigated by S. Popa, L. Ge, R. Kadison, J. Fang, and J. Shen. In recent years, studies have also begun in the real case. Let us briefly recall the relevance of considering the real case. It is known that in the works of D. Topping and E. Stormer, it was shown that the study of JW-algebras (nonassociative real analogues of von Neumann algebras) of types II and III is essentially reduced to the study of real W∗-algebras of the corresponding type. It turned out that the structure of real W∗-algebras, generally speaking, differs essentially in the complex case. For example, in the finite-dimensional case, in addition to complex and real matrix algebras, quaternions also arise, i.e., matrix algebras over quaternions. In the infinite-dimensional case, it is proved that there exist, up to isomorphism, two real injective factors of type III_λ $(0<\lambda<1)$, and a countable number of pairwise nonisomorphic real injective factors of type III_0, whose enveloping (complex) W∗-factors are isomorphic, is constructed. It follows from the above that the study of the real analogue of problems in the theory of operator algebras is topical. Moreover, the real analogue is a generalization of the complex case, since the class of real linear operators is much wider than the class of complex linear operators. In this paper, the maximal injective real W∗-subalgebras of real W∗-algebras or real factors are investigated. For real factors $Q\subset R$, it is proven that if $Q+iQ$ is a maximal injective W∗-subalgebra in $R+iR$, then Q also is a maximal injective real W∗-subalgebra in R. The converse is proved in the case "II_1"-factors, that is, it is shown that if R is a real factor of type II_1, then the maximal injectivity of Q implies the maximal injectivity of $Q+iQ$. Moreover, it is proven that a maximal injective real subfactor Q of a real factor R is a maximal injective real W∗-subalgebra in R if and only if Q is irreducible in R, i.e., $Q'\cap R=\mathbb{R}\mathbb{I}$ where $\mathbb{I}$ is the unit. The "splitting theorem" of Ge-Kadison in the real case is also proven, namely, if R_1 is a finite real factor, R_2 is a finite real W∗-algebra, and R is a real W∗-subalgebra of $R_1\bar{\otimes}R_2$ containing $R_1\bar{\otimes}\mathbb{R}\mathbb{I}$, then there is some real W∗ -subalgebra $Q_2\subset R_2$ such that $R=R_1\bar{\otimes}Q_2$. Moreover, it is given some affirmative answers to the question of S. Popa for the real case.

1. Introduction

Injective (complex or real) W∗-algebras with particular factors have been studied well enough. On the other hand, in the arbitrary cases, i.e., in noninjective case, to investigate (up to isomorphism) W∗-algebras is hard enough, in particular, there exist continuum pairwise nonisomorphic noninjective factors of type *II*. Therefore, it seems interesting to study maximal injective W∗-subalgebras and subfactors. On the other hand, it will be described below that the study of maximal injective W∗-subalgebras and subfactors is also related to the well-known von Neumann's bicommutant theorem. In the complex case, such subalgebras were investigated by S. Popa, L. Ge, R. Kadison, J. Fang, J. Shen, S. Stratila, and L. Zsido [1–4], [5–8], [9]. Now, in more detail, we present the main points of these articles. If N is the algebra (more precisely, factor) of all bounded linear operators acting on a complex Hilbert space H, i.e., if $N = B(H)$, and $M \subset N$ is a subfactor, then by the well-known theorem of von Neumann, the bicommutant of M in N is equal to M, i.e., $M'' = M$ (see [8, 9]). But if N is a continuous factor, then in general the bicommutant of M in N is not equal to M (see [4]). If we denote the relative commutants of M in N and $B(H)$ by ${M_N}'$ and M' $(= {M'}_{B(H)})$, respectively, then in general, its relatives are not related, that is, ${M_N}'' \subset {M_{B(H)}}''$ and ${M_{B(H)}}'' \subset {M_N}''$. Actually, it seems that the typical and more interesting type of imbedding M as a subfactor of a continuous factor N is such that the bicommutant of M in N is trivial, i.e., $M' \cap N = \mathbb{C}\mathbb{I}$, and in this case M is called *irreducible* in N. For example, let $\alpha : G \longrightarrow Aut(M)$ be a properly outer action of a discrete group G on M. Then M is naturally imbedded in the crossed product algebra $M \times_\alpha G$, and by the relative commutant theorem, we have $M' \cap (M \times_\alpha G) = \mathbb{C}\mathbb{I}$ ([8, 9]). A sufficient condition for a subfactor M of the factor N to have a trivial relative commutant is that there exists an abelian ∗-subalgebra $A \subset M$ which is maximal abelian in N (see [1–3], [4, 6]). At the Baton Rouge (USA) conference in 1967, R.V. Kadison asked the question whether or not the converse is true, namely:

If M is an irreducible subfactor of a factor N , will some maximal abelian ∗ -subalgebra of M be a maximal abelian ∗ -subalgebra of N?

This problem has been solved in some particular cases. Namely, in 1981, Popa [2] proved that if N is a separable factor, $M \subset N$ is a semifinite irreducible subfactor, and there exists a normal conditional expectation $E : N \longrightarrow M$, then there is an abelian W∗-subalgebra $A \subset M$ which is a maximal abelian W∗-subalgebra in N. Kadison also conjectured that every separable abelian W∗-subalgebra of a factor is contained in some hyperfinite subfactor. However, Kadison's conjecture turned out to be wrong. In 1983, Popa [1] gave a counterexample in which an abelian W∗-algebra can be imbedded in a factor of type II_1 as a maximal injective W∗-subalgebra, i.e., it is not an imbedding as a hyperfinite subfactor. This was also mentioned in [10]. Recall that a W∗-subalgebra $M \subset N$ is called *maximal injective* in N if it is injective and maximal, i.e., there is no other injective W∗-subalgebra containing M. Note that every W∗-algebra N has one maximal injective W∗-subalgebra at least. Indeed, if $\{M_\alpha\}$ is a family of injective W∗-subalgebras of N which is inductively ordered by inclusion, then the weak operator closure of $\bigcup_\alpha M_\alpha$ is an injective W∗-subalgebra of N which contains all M_α. By Zorn's lemma, N has a maximal injective W∗-subalgebra. We also note that the maximality of an injective W∗-subalgebra and the maximality of a hyperfinite subfactor are not connected with each other, i.e., one does not imply the other. In this line, at first, Murray and von Neumann assumed that any separable factor of type II_1 contains a hyperfinite subfactor as a maximal injective W∗-subalgebra. This result was proved by Sorin Popa [1] in 1983. Without the condition of separability, the problem was affirmatively solved by Fang [5].

On the other hand, it turns out that the maximality of an injective W∗ -subalgebra is related to its irreducibility. Namely, in 2007, Fang [5] (see also [6]) showed that if M is a maximal hyperfinite subfactor of a factor N, then M is a maximal injective W∗-subalgebra if and only if it is irreducible, i.e., $M' \cap N = \mathbb{C}\mathbb{I}$. Thus, if we look chronologically, the maximality of an injective W∗-subalgebra is concerned with the bicommutant theorem.

In this paper, we will investigate the real analogues of the considered notions and the results above. Let us explain the topicality of considering the real analogue of problems in the theory of W∗-algebras. In the mid-1960s, in the works of D. Topping and E. Stormer, Jordan (nonassociative real) analogues of von Neumann algebras JW-algebras were first considered, i.e., real linear spaces of self-adjoint algebraic operators $B(H)$, that contain $\mathbb{I}$, are closed under Jordan multiplication $a \circ b = (ab + ba)/2$ and are closed in the weak operator topology. The structure of these algebras turned out to be close to the structure of von Neumann algebras (that is, W∗-algebras), and in the study of JW-algebras, it turned out to be possible to apply ideas similar to those of the theory of W∗-algebras. In these works, an initial classification of JW-algebras by types was obtained, JW-algebras of type I were completely studied, and it was shown that the study of types II and III is essentially reduced to the study of real W∗-algebras of the corresponding type. Since the mid-1980s, the theory of real W∗-algebras has received significant development. It turned out that the structure of real W∗-algebras, generally speaking, differs essentially in the complex case. For example, in the finite-dimensional case, in addition to complex and real matrix algebras, quaternions also arise, i.e., matrix algebras over quaternions. In the infinite-dimensional case, it is proved that there exist, up to isomorphism, two real injective factors of type III_λ $(0 < \lambda < 1)$, and a countable number of pairwise nonisomorphic real injective factors of type III_0, whose enveloping W∗-factors are isomorphic, is constructed. It follows from the above that the study of the real analogue of problems in the theory of operator algebras is topical. Moreover, the real analogue is a generalization of the complex case, since the class of real linear operators is much wider than the class of complex linear operators.

2. Preliminaries

Let A be a Banach ∗-algebra over the field $\mathbb{C}$. The algebra A is called a C∗- *algebra* if $\|aa^*\| = \|a\|^2$ for any $a \in A$. A C∗-algebra M is called a W∗-*algebra* if there exists a Banach space M_*, so-called a *predual* of M such that $(M_*)^* = M$. Let $B(H)$ be the algebra of all bounded linear operators acting on a complex Hilbert space H and let $M \subset B(H)$ be a ∗-subalgebra. The subset $M' = \{a \in B(H) \colon ba = ab, \forall b \in M\}$

is called the *commutant* of M. It is easy to see that

$$M \subset M'' = M^{IV} = M^{VI} = \cdots \text{and}\, M' = M^{III} = M^{V} = \cdots \quad (1)$$

where $M'' = (M')'$. If $M = M''$, then it is called a *von Neumann algebra*. The set $Z(M) = \{a \in M : ba = ab, \forall b \in M\}$ is called the *center* of M. It is obvious that $Z(M) = M \cap M'$. A von Neumann algebra M is called a *factor* if its center is trivial, i.e., $Z(M) = \{\lambda \mathbb{I} : \lambda \in \mathbb{C}\}$, where $\mathbb{I}$ is the unit of M. By von Neumann's bicommutant theorem, any von Neumann algebra is W∗- *algebra*, i.e., M is weakly closed with $\mathbb{I} \in M$. The converse is also true. Therefore, W∗-algebras are also called von Neumann algebras (see [11]).

Now, a real ∗-subalgebra $R \subset B(H)$ with unit $\mathbb{I}$ is called a *real* W∗- *algebra* if it is weakly closed and $R \cap iR = \{0\}$. The smallest (complex) W∗-algebra $U(R)$ containing R coincides with its complexication $R + iR$, i.e., $U(R) = R + iR$. It is known that R generates a natural involutive (i.e., of order 2) ∗-antiautomorphism α_R of $U(R)$, namely, $\alpha_R(x + iy) = x^* + iy^*$, where $x + iy \in U(R)$, $x, y \in R$. In this case $R = \{x \in U(R): \alpha(x) = x^*\}$. Conversely, given a W∗-algebra U and any involutive ∗-antiautomorphism α on U, the set $\{z \in U : \alpha(z) = z^*\}$ is a real W∗-algebra.

Let $R \subset B(H)$ be a real W∗-algebra. It is known that there is a real Hilbert space H_r with

$$H_r + iH_r = H, R \subset B(H_r) \subset B(H_r) + iB(H_r) = B(H). \quad (2)$$

The commutant of a ∗-algebra R is defined similarly to the complex case:

$$R' = \{a \in B(H_r): ab = ba, \forall b \in R\}. \quad (3)$$

It is directly verified that $(R + iR)' = R' + iR'$ (see [11]).

A real Banach ∗-algebra R is called a *real* C∗- *algebra* if $\|aa^*\| = \|a\|^2$ and the element $\mathbb{I} + aa^*$ is invertible for any $a \in R$. Examples of such algebras are the algebra of all linear bounded operators acting on a real Hilbert space H_r, and the algebra of all compact operators on H_r is a real C∗-algebra, but is not a real W∗-algebra. It is known that R is a real C∗-algebra if and only if the norm on R can be extended to the complexification $A = R + iR$ of R so that A is a C∗-algebra (see [12], 5.1.1). Let A be a C∗-algebra. Denote by $M_n(A)$ the algebra of all $n \times n$ matrices over A which is also a C∗-algebra. Recall that a continuous linear mapping φ between two C∗-algebras A and B is called *completely positive* if for any $n \geq 1$, the natural map φ_n from the C∗-algebra $A \otimes M_n$ to the C∗-algebra $B \otimes M_n$ defined by

$$\varphi_n\left(\left(a_{i,j}\right)_{i,j=1}^n\right) = \left(\varphi\left(a_{i,j}\right)\right)_{i,j=1}^n \quad (4)$$

is positive, where M_n is the C∗-algebra of $n \times n$ matrices over $\mathbb{C}$. We say that a W∗-algebra A is *injective* if the following condition is held: for every C∗-algebra B, for every self-adjoint linear subspace S of B containing the identity $\mathbb{I}$, and for every completely positive linear map $\varphi : S \longrightarrow A$, there is a completely positive linear map $\bar{\varphi} : B \longrightarrow A$ such that $\bar{\varphi}|_S = \varphi$ ([13, 14]). All the notions above are defined similarly for real C∗- and W∗-algebras. Recall [14] that a W∗-algebra $A \subset B(H)$ is injective if and only if it has the extension property *property of E in short* (extension property), i.e., there exists a projection $P : B(H) \longrightarrow A$ such that $\|P\| = 1$, $P(\mathbb{I}) = \mathbb{I}$. In this case, the map P is completely positive (see [14]).

3. Maximal Injective Real W∗ -Subalgebras

Let Q and R be real W∗-algebras. Similarly to the complex case, we define: a real W∗-subalgebra $Q \subset R$ is called *maximal injective* in R if it is injective and maximal, i.e. there is no other injective real W∗-subalgebra of R containing Q.

Theorem 1. *Let Q be a real W ∗-subalgebra of a real W ∗-algebra R. If $Q + iQ$ is maximal injective in $R + iR$, then Q is also maximal injective in R.*

Proof. Let $Q + iQ$ be a maximal injective W∗-subalgebra of $R + iR$. Then by Theorem 8 in [15], Q is injective. If there exists an injective real W∗-subalgebra Q_1 of R containing Q, then by Theorem 8 in [15], the algebra $Q_1 + iQ_1$ is injective and $Q + iQ \subset Q_1 + iQ_1 \subset R + iR$ which contradicts the maximality of the injective subalgebra $Q + iQ$. □

In general, the converse of Theorem 1 is still open, but we will prove in a particular case.

Theorem 2. *Let R be a real factor of type II_1. If Q is a maximal injective real subfactor in R, then $Q + iQ$ is also a maximal injective subfactor in $R + iR$.*

Proof. Let $N \subset R + iR$ be an injective subfactor with $Q + iQ \subset N$. Since the class of injective factors of type II_1 is unique, then the subfactors $Q + iQ$ and N are isomorphic, i.e., there is an isomorphism $\theta : N \longrightarrow Q + iQ$. Let α be involutive ∗-antiautomorphism of $Q + iQ$, generating Q, i.e., $Q = \{a : \alpha(a) = a^*\}$. Put $\beta = \theta^{-1}\alpha\theta$. It is clear that β is an involutive ∗-antiautomorphism on N. Put $Q_0 = \{x \in N : \beta(x) = x^*\}$. Then $N = Q_0 + iQ_0$ and Q_0 is a real injective subfactor of N. And, $Q' = \{y : \theta\beta\theta^{-1}(y) = y^*\} \subset R + iR$ is also injective. But, since $Q' \subset Q + iQ$ and Q is maximal injective, then $Q' = Q$, consequently, $N = Q + iQ$. Thus, $Q + iQ$ is a maximal injective subfactor in $R + iR$.

Now, we prove one auxiliary result. □

Lemma 3. *Let R be a real W ∗-algebra and let Q be a maximal injective real W ∗-subalgebra of R. Then Q is singular in R, i.e., for a unitary element $u \in R$ with $uQu^* = Q$, it follows that $u \in Q$. In particular $Q' \cap Q = Q' \cap R = Z$, where Z is the center of Q.*

Proof. Let u be a unitary element in R with $uQu^* = Q$. Then the real W∗-subalgebra of R generated by Q and the element u is injective. Since Q is maximal injective in R, it follows that $u \in Q$. In particular, it is held that $Q' \cap R \subset Q$ and $Z \subset Q' \cap R \subset Q' \cap Q = Z$.

Below we prove the real analogue of the theorem of J. Fang which is the solution to the problem of Murray and von Neumann for real factors. □

Theorem 4. *Let R be a real factor and let Q be a maximal injective real subfactor of R. Then Q is a maximal injective real $W*$-subalgebra in R if and only if Q is irreducible in R, i.e. $Q' \cap R = \mathbb{R}\mathbb{1}$.*

Proof. If Q is a maximal injective real W∗-subalgebra of R, then by Lemma 3, $Z(Q) = Q' \cap R$. Since Q is a real factor, $Z(Q) = \mathbb{R}\mathbb{1}$, i.e., $Q' \cap R = \mathbb{R}\mathbb{1}$. Conversely, suppose that $Q' \cap R = \mathbb{R}\mathbb{1}$. For any injective real W∗-algebra F with $Q \subset F \subset R$, we have $F' \cap F = Q' \cap R = \mathbb{R}\mathbb{1}$. Therefore, F is an injective real subfactor of R. Since Q is a maximal injective real subfactor of R, $F = Q$. □

Recall that a positive linear mapping from R onto Q is called a *conditional expectation* if $E(abc) = aE(b)c$ for all a, $c \in Q$, $b \in R$.

Lemma 5. *Let R be a real $W*$-algebra and let F, Q be real $W*$-subalgebras of R such that $F \subset Q$. If E is a conditional expectation from R onto Q, then E induces a conditional expectation from $F' \cap R$ onto $F' \cap Q$.*

Proof. Let $x \in F' \cap R$ and $y \in F$. Then $xy = yx$. Apply the conditional expectation E to the both sides of the equality $xy = yx$. Since $\subset QE(x)y = yE(x)$. Thus, $E(x) \in F' \cap Q$. Since $F' \cap Q \subset F' \cap R$, E is a conditional expectation from $F' \cap R$ onto $F' \cap Q$ when E is restricted on $F' \cap R$. □

Lemma 6. *Let F_i be a real $W*$-algebra and let S_i be a real $W*$-subalgebra of $F_i (i = 1, 2)$. If F is a real $W*$-algebra such that $S_1 \bar{\otimes} S_2 \subset F \subset F_1 \bar{\otimes} F_2$ and there is a conditional expectation $E : F_1 \bar{\otimes} F_2 \longrightarrow F$, then E induces a conditional expectation from $(S_{1'} \cap F_1) \bar{\otimes} F_2$ onto $((S_{1'} \cap F_1) \bar{\otimes} F_2) \cap F$.*

Proof. By Lemma 1.7 in [5], we have that

$$\begin{aligned} &(U(F_1) \bar{\otimes} U(F_2)) \cap (U(S_1) \bar{\otimes} \mathbb{C}\mathbb{1})' \\ &= (U(F_1) \bar{\otimes} U(F_2)) \cap \left(U(S_1)' \bar{\otimes} B(K)\right) \\ &= \left(U(S_1)' \cap U(F_1)\right) \bar{\otimes} U(F_2), \end{aligned} \tag{5}$$

where $U(F_i)$ is the enveloping W∗-algebra of the real W∗-algebra F_i. In general, if F_1 and F_2 are real W∗-algebras, then it holds

$$U(F_1 \bar{\otimes} F_2) = U(F_1) \bar{\otimes} U(F_2). \tag{6}$$

Then $(F_1 \bar{\otimes} F_2) \cap (S_1 \bar{\otimes} \mathbb{R}\mathbb{1})' = (S_1' \cap F_1) \bar{\otimes} F_2$. By Lemma 5, E induces a conditional expectation from $(S_1' \cap F_1) \bar{\otimes} F_2$ onto $((S_1' \cap F_1) \bar{\otimes} F_2) \cap F$. □

From Lemma 6, we have the following:

Corollary 7. *Let the condition of Lemma 6 be satisfied and $S_{1'} \cap F_1 = \mathbb{R}\mathbb{1}$. Consider the set $L = \{x \in F_2 : \mathbb{1} \otimes x \in F\}$. Then E induces a conditional expectation from F_2 onto L.*

Using Lemma 6 and Corollary 7, we will prove the real analogue of Ge-Kadison's theorem, so-called "splitting theorem".

Theorem 8. *If R_1 is a finite real factor, R_2 is a finite real $W*$-algebra, and R is a real $W*$-subalgebra of $R_1 \bar{\otimes} R_2$ containing $R_1 \bar{\otimes} \mathbb{R}\mathbb{1}$, then there is some real $W*$-subalgebra $Q_2 \subset R_2$ such that $R = R_1 \bar{\otimes} Q_2$.*

Proof. Let Q be a real W∗-algebra such that $R_1 \bar{\otimes} \mathbb{R}\mathbb{1} \subset Q \subset R_1 \bar{\otimes} R_2$. Since R_1 and R_2 are finite, there is a conditional normal expectation E from $R_1 \bar{\otimes} R_2$ onto Q. By Corollary 7, E induces a conditional expectation from R_2 onto $Q_2 = \{T \in R_2 : \mathbb{1} \otimes T \in Q\}$ which we denote by E_2. Then we have $E(x \otimes y) = x \otimes E_2(y) \in R_1 \bar{\otimes} Q_2$, for any $x \in R_1$, $y \in R_2$. Since E is normal, it follows that $Q = E(R_1 \bar{\otimes} R_2) \subset R_1 \bar{\otimes} Q_2$. Since $Q \supset R_1 \bar{\otimes} Q_2$, $Q = R_1 \bar{\otimes} Q_2$. □

4. Maximal Injective Real W∗-Subalgebras of Tensor Products of Real W∗-Algebras

Let $R \subset B(H)$ be a real W∗-algebra. It is obvious that if Q is a maximal injective real W∗ -subalgebra of R, then Q' is the minimal injective real W∗-subalgebra containing the algebra R'. Therefore, for any such algebra, there always exists as a maximal injective real W∗-subalgebra and such a minimal injective W∗-extension.

Let us prove one auxiliary result.

Lemma 9. *Let R be an abelian real $W*$-algebra, Q be a real $W*$-algebra, and let F_2 be a minimal separable injective real $W*$-algebra containing Q. If F is an injective real $W*$-algebra such that*

$$R \bar{\otimes} Q \subset F \subset R \bar{\otimes} F_2, \tag{7}$$

then $F = R \bar{\otimes} F_2$.

Proof. By formulas (6) and (7), we have

$$U(R) \bar{\otimes} U(Q) \subset U(F) \subset U(R) \bar{\otimes} U(F_2). \tag{8}$$

By Lemma 2.4 from [5], we have $U(F) = U(R) \bar{\otimes} U(F_2)$. Since

$$U(R) \bar{\otimes} U(F_2) = U(R \bar{\otimes} F_2), Q \subset F \subset F_2, \tag{9}$$

then from $U(F) = U(R \bar{\otimes} F_2)$, we get $F = R \bar{\otimes} F_2$. □

The Lemma 9 can be proved in another way. Indeed:

Proof. Let R be an abelian real W∗-algebra and let F_2 be a minimal separable injective real W∗-extension of a real W∗-algebra Q. Then it is directly verified that $U(R)$ is also abelian and $U(F_2)$ is the minimal separable injective

W∗ -extension of $U(Q)$. By (7) we have

$$U(R\bar{\otimes}Q) \subset U(F) \subset U(R\bar{\otimes}F_2). \quad (10)$$

Then by Lemma 2.4 in [5], we have $U(F) = U(R\bar{\otimes}F_2)$. Consequently, since $U(F) = U(R\bar{\otimes}F_2)$ and $F \subset R\bar{\otimes}F_2$, it follows that $F = R\bar{\otimes}F_2$. □

Now, we will prove the main result of this section.

Theorem 10. *Let R_1 be an injective real W ∗-algebra and R_2 be a separable real W ∗-algebra. If Q_2 is a maximal injective real W ∗-subalgebra of R_2, then $R_1\bar{\otimes}Q_2$ is a maximal injective real W ∗-subalgebra of $R_1\bar{\otimes}R_2$.*

Proof. Without loss of generality, we can assume that R_1 and R_2 are real W∗-algebras acting on complex Hilbert spaces H and K, respectively. Then by the hypothesis of the theorem, K is separable. Let A be the center of R_1. Suppose that Q is an injective real W∗-algebra such that $R_1\bar{\otimes}Q_2 \subset Q \subset R_1\bar{\otimes}R_2$. Then we have $R_{1'}\bar{\otimes}Q_{2'} \supset Q' \supset R_{1'}\bar{\otimes}R_{2'}$. Since Q' is an injective real W∗ -subalgebra of $R_{1'}\bar{\otimes}Q_{2'}$, there is a conditional expectation E from $R_{1'}\bar{\otimes}Q_{2'}$ onto Q'. By Lemma 5, E induces a conditional expectation from $A\bar{\otimes}Q_{2'}$ onto $F := (A\bar{\otimes}Q_{2'}) \cap Q'$. Consequently, F is an injective real W∗ -algebra such that $A\bar{\otimes}Q_{2'} \supset F \supset A\bar{\otimes}R_{2'}$. Since Q_2 is a maximal injective real W∗-subalgebra of R_2, $Q_{2'}$ is the minimal injective real W ∗ -extension of $R_{2'}$. By Lemma 9, we get $F = A\bar{\otimes}Q_{2'}$. Thus $\mathbb{R}\mathbb{I}\bar{\otimes}Q_{2'} \subset F \subset Q'$. So, $Q' = R_{1'}\bar{\otimes}Q_{2'}$ and $Q = R_1\bar{\otimes}Q_2$. □

5. Some Particular Cases

Let us consider the following problem which was formulated by Popa (see [1]) in the complex case:

If R_1 and R_2 are real W ∗ -algebras, Q_1 and Q_2 are maximal injective real W ∗ -subalgebras of R_1 and R_2 , respectively, then will the algebra $Q_1\bar{\otimes}Q_2$ be a maximal injective real W ∗ -subalgebra of $R_1\bar{\otimes}R_2$?

In some certain cases, we will give affirmative answers to the question.

(1) The case when Q_1 is a real factor and R_1 is separable

As examples of such algebras, one can take Q_1 as an infinite tensor product of a matrix algebra $M_2(\mathbb{R})$, which is a real factor of type II_1, and R_1 as an arbitrary real W∗ -algebra acting on a separable real Hilbert space.

Theorem 11. *Let $R_i(\subset B(H_i))$ be a real W ∗-algebra and let Q_i be a maximal injective real W ∗-subalgebra of $R_i(i = 1, 2)$. If R_1 is separable and Q_1 is a real factor, then $Q_1\bar{\otimes}Q_2$ is a maximal injective real W ∗-subalgebra of $R_1\bar{\otimes}R_2$.*

Proof. Let Q be an injective real W∗-subalgebra with $Q_1\bar{\otimes}Q_2 \subset Q \subset R_1\bar{\otimes}R_2$. Then we obtain $Q_{1'}\bar{\otimes}Q_{2'} \supset Q' \supset R_{1'}\bar{\otimes}R_{2'}$. Since Q_i is a maximal injective real W∗-subalgebra of R_i, Q' is the minimal injective real W∗-extension of $R_{i'}(i = 1, 2)$. Since Q' is an injective real W∗-subalgebra of the W∗ -subalgebra $Q_{1'}\bar{\otimes}Q_{2'}$, there is a conditional expectation $E : Q_{1'}\bar{\otimes}Q_{2'} \longrightarrow Q'$. □

Let $L = \{x \in Q_{2'} : \mathbb{I}\otimes x \in Q'\}$. Then $L \subset Q_{2'}$. By Lemma 3 we have

$$(R_{1'})' \cap Q_{1'} = Q_1 \cap R_1 = Q_1 \cap Q_1 = \mathbb{R}\mathbb{I}. \quad (11)$$

By Corollary 7, the mapping E induces a conditional expectation from $Q_{2'}$ onto L. Thus, L is injective. Since $R_{2'} \subset L \subset Q_{2'}$ and $Q_{2'}$ is the minimal injective real W∗-extension of $R_{2'}$, $L = Q_{2'}$. So, $\mathbb{R}\mathbb{I}\bar{\otimes}Q_{2'} \subset Q'$. This implies that $Q_{1'}\bar{\otimes}Q_{2'} \supset Q' \supset R_{1'}\bar{\otimes}Q_{2'}$ and hence, $Q_1\bar{\otimes}Q_2 \subset Q \subset R_1\bar{\otimes}Q_2$. By Theorem 10, we have $Q = Q_1\bar{\otimes}Q_2$.

(2) The case when R_1 is separable and the center of Q_1 is atomic

Recall that a real or complex von Neumann algebra A is called *atomic* if every nonzero projection of A majorizes a nonzero minimal projection (atom), i.e., under each nonzero projection, there is an atom.

As examples of such algebras, one can take R_1 as an arbitrary real W∗-algebra acting on a separable real Hilbert space, and Q_1 as an arbitrary real W∗-algebra whose center is a commutative algebra.

Let us prove one auxiliary result which in itself is interesting.

Theorem 12. *Let Q be a real W ∗-algebra and R be the minimal injective real W ∗-algebra containing Q. If θ is a ∗ -automorphism of R such that $\theta(x) = x$, $\forall x \in Q$, then $\theta(y) = y$. That is, if a ∗-automorphism is identical to a real W ∗ algebra, then it is also identical to its minimal injective real W ∗-extension.*

Proof. Let R act on a Hilbert space H, i.e., $R \subset B(H)$. $\bar{\theta}$ denotes a linear extension of the ∗-automorphism θ on the W∗-algebra $M = R + iR$ that is defined as $\bar{\theta}(a + ib) = \theta(a) + i\theta(b)$. As is known, every ∗-automorphism of $B(H)$ is inner, i.e., there exists some unitary element $u \in B(H)$ with $\bar{\theta}(x) = uxu^* = Adu(x)$, $x \in M$. By the condition of the theorem, we have $uxu^* = x$ for all $x \in Q$, consequently, for all $x \in N = Q + iQ$. Hence, we can get: $u \in N'$. Put $\bar{\theta}'(y)\bar{\theta}'(y') = uy'u^*$, for $y' \in M'$. It is easy to see that $\bar{\theta}'(y')\bar{\theta}(y) = \bar{\theta}(y)\bar{\theta}'(y')$, for all $y \in M$, $y' \in M'$. Since $\bar{\theta}(M) = M$, $\bar{\theta}'(y') \in M'$ for all $y' \in M'$, i.e., $uM'u^* \subset M'$. Note that $\bar{\theta}^{-1}(y) = u^*yu$ is also a ∗-automorphism on M. Similar reasoning for the ∗-automorphism $\bar{\theta}^{-1}$, we can obtain $u^*M'u \subset M'$. So, $uM'u^* = M'$. It implies that $u \in N'$ is a normalizer of M'. By Lemma 1.1 in [5], $u \in M'$. Consequently, $\bar{\theta}(y) = uyu^* = y$ for all $y \in M$, i.e., $\bar{\theta}|_M = id$. In particular, $\theta|_R = id$.

Now, using Theorem 12, we prove the following auxiliary results. □

Lemma 13. *If R is the minimal injective real $W*$-extension of a real $W*$-algebra Q and p, q are nonzero central orthogonal projections in R, then there is not a $*$-isomorphism $\varphi : R_p \longrightarrow R_q$ with $\varphi(px) = qx$, $\forall x \in Q$.*

Proof. Suppose that there is a $*$-isomorphism $\varphi : R_p \longrightarrow R_q$ with $\varphi(px) = qx$, $\forall x \in Q$. Then $y = pq + qy + (\mathbb{1} - p - q)y$ for all $y \in R$. Define the mapping $\theta : R \longrightarrow R$ as

$$\theta(y) = \varphi(py) + \varphi^{-1}(qy) + (\mathbb{1} - p - q)y. \tag{12}$$

Since p, q are mutually orthogonal central projections and the mapping $\varphi : R_p \longrightarrow R_q$ is a $*$-isomorphism, it is easy to see that the mapping θ is a $*$-automorphism of R. Moreover, for any $x \in Q$, we have also

$$\theta(x) = \varphi(px) + \varphi^{-1}(qx) + (1 - p - q)x = qx + px + (1 - p - q)x = x. \tag{13}$$

That is, the $*$-automorphism θ acts identically on Q. Then by Theorem 12, we have $\theta(y) = y$, for any $y \in R$. That is, the $*$-automorphism θ is identical to R. Consequently, $p = \theta(p) = \varphi(p) = q$. Hence, we obtain that $p = pq = 0$. It contradicts to the assumption that $p \neq 0$. □

Lemma 14. *Let R be the minimal injective real $W*$-extension of a real $W*$-algebra Q and let p, q be nonzero central projections in R. If there is a $*$-isomorphism $\varphi : R_p \longrightarrow R_q$ with $\varphi(px) = qx$, $\forall x \in Q$, then $p = q$ and $\varphi(py) = py$ for any $y \in R$.*

Proof. Suppose that $p \neq q$. Let $e = pq$. Without loss of generality, we can assume that $p_1 = p - e > 0$. Let $q_1 = \theta(p_1) \leq q$. Then p_1, q_1 are nonzero central projections in R and $p_1 q_1 = q_1 p_1 = 0$. It is easy to see that the $*$-isomorphism $\varphi : R_p \longrightarrow R_q$ induces a $*$-isomorphism $\psi : R_{p_1} \longrightarrow R_{q_1}$ with $\psi(p_1 y) = \varphi(p_1 y)$ for all $y \in R$. Moreover, for any $x \in Q$, we have:

$$\psi(p_1 x) = \varphi(p_1 x) = \varphi(p_1)\varphi(px) = q_1 qx = q_1 x. \tag{14}$$

As we see that it contradicts Lemma 13. Thus $p = q$. Now, define a mapping θ on R as:

$$\theta(y) = \phi(py) + (1 - p)y. \tag{15}$$

It is directly shown that θ is $*$-automorphism of R and $\theta(x) = x$, for all $x \in Q$. Then by Theorem 12, we have: $\theta(y) = y$ for all $y \in R$. Hence, $py = \theta(py) = \varphi(py)$ for all $y \in R$.

Now, let us prove the main result of the section of the paper. □

Theorem 15. *Let $R_i(\subset B(H_i))$ be a real $W*$-algebra and let Q_i be a maximal injective real $W*$-subalgebra of R_i $(i = 1, 2)$. If R_1 is separable and the center of Q_1 is atomic, then $Q_1 \bar{\otimes} Q_2$ is maximal injective in $R_1 \bar{\otimes} R_2$.*

Proof. Let Q be an injective real W$*$-subalgebra with

$$Q_1 \bar{\otimes} Q_2 \subset Q \subset R_1 \bar{\otimes} R_2. \tag{16}$$

To prove the theorem, we have to show that $Q = Q_1 \bar{\otimes} Q_2$. By Theorem 10, it is sufficient to prove that $Q \subseteq R_1 \bar{\otimes} Q_2$. For that, it is enough to prove that $\mathbb{R}\mathbb{1} \bar{\otimes} Q_{2'} \subset Q'$. Put $A = Q_1 \cap Q_{1'} = R_1 \cap Q_{1'}$, i.e., let A be the center of Q_1 which is atomic according to the condition. Since the algebras $Q_1 \subset R_1$ are separable, then the set of minimal projections is countable, which we denote by $\{p_n : n = 1, 2, \cdots\}$. Put $F = Q' \cap (A \bar{\otimes} Q_{2'}) = Q' \cap (R_{1'} \bar{\otimes} \mathbb{R}\mathbb{1})'$. As in the proof of Theorem 10, it is similarly shown that F is injective and $\mathbb{R}\mathbb{1} \bar{\otimes} R_{2'} \subseteq F \subseteq A \bar{\otimes} Q_{2'}$. □

Note that $p_n \otimes \mathbb{1} \in F'$ for every n. Since $Q_{2'}$ is the minimal injective real W$*$-extension of R_2, we have $F(p_n \otimes \mathbb{1}) = p_n \otimes Q_{2'}$ for every n. We denote by z_n the smallest projection in $Z(F)$ satisfying the condition: $p_n \otimes \mathbb{1} \leq z_n$. Then we can get a $*$-isomorphism $\theta_n : Q_{2'} \longrightarrow Fz_n$ uniquely determined by the formula

$$\theta_n(y)(p_n \otimes \mathbb{1}) = p_n \otimes y \text{ for all } y \in Q_{2'}. \tag{17}$$

Since $\mathbb{R}\mathbb{1} \bar{\otimes} R_{2'} \subseteq F$, it follows that $\theta_n(x) = (\mathbb{1} \otimes x)z_n$ for all $x \in R_{2'}$.

Thus for all n, m and for all $x \in R_{2'}$, we have $\theta_n(x)z_m = \theta_m(x)z_n$. Since $Q_{2'}$ is the minimal injective real W$*$-extension of $R_{2'}$, then by Lemma 14, the last equation is satisfied for all $y \in Q_{2'}$ and all n, m. Moreover, hence, for every $y \in Q_{2'}$ there is an element $a \in F$ such that $az_n = \theta_n(y)$ for all n. In particular $a(p_n \otimes \mathbb{1}) = p_n \otimes y$, i.e., $a = \mathbb{1} \otimes y$. Therefore, $\mathbb{R}\mathbb{1} \bar{\otimes} Q_{2'} \subseteq F$.

(3) The case when Q_i, R_i $(i = 1, 2)$ are real factors

Theorem 16. *Let $R_i(\subset B(H_i))$ be a real factor and Q_i be a maximal injective real subfactor of R_i, $(i = 1, 2)$. If $Q_{1'} \cap R_1 \cong \mathbb{R}^N (1 \leq N \leq \infty)$ and $Q_{2'} \cap R_2 = \mathbb{R}\mathbb{1}$, then $Q_1 \bar{\otimes} Q_2$ is a maximal injective real subfactor of $R_1 \bar{\otimes} R_2$.*

Proof. Let Q be an injective real W$*$-algebra with

$$Q_1 \bar{\otimes} Q_2 \subset Q \subset R_1 \bar{\otimes} R_2. \tag{18}$$

Then we have $Q_{1'} \bar{\otimes} Q_{2'} \supset Q' \supset R_{1'} \bar{\otimes} R_{2'}$. By Theorem 4, Q_2 is a maximal injective real W$*$-subalgebra of R_2, and consequently, $Q_{2'}$ is the minimal injective real W$*$-extension of $R_{2'}$. By assumption, $Q_{2'} \cap R_2 = Q_{2'} \cap Q_2 = \mathbb{R}\mathbb{1}$. Let $E : Q_1 \bar{\otimes} Q_2 \longrightarrow Q'$ be a conditional expectation and $F = Q_{1'} \cap R_1 \cong \mathbb{R}^N$. By Lemma 6, E induces a conditional expectation from $F \bar{\otimes} Q_{2'}$ onto B, where $B = (F \bar{\otimes} Q_{2'}) \cap Q'$. Therefore, B is an injective real W$*$-algebra such that

$$Q' \supseteq B \supseteq (F \bar{\otimes} Q_{2'}) \cap (R_{1'} \bar{\otimes} R_{2'}) = \mathbb{R}\mathbb{1} \bar{\otimes} R_{2'}. \tag{19}$$

Similar arguments as the proof of Theorem 15 shows that $B \supseteq \mathbb{R}\mathbb{I} \bar{\otimes} Q_{2'}$. So, $Q' \supseteq \mathbb{R}\mathbb{I} \bar{\otimes} Q_{2'}$. By Theorem 8 (the real analogue of Ge-Kadison's splitting theorem), for some real W∗ -subalgebra $N' \subseteq Q_{1'}$ we have: $Q' = N' \bar{\otimes} Q_{2'}$. Therefore, $Q = N \bar{\otimes} Q_2$. Since Q is injective, it follows that N is also injective and $Q_1 \subseteq N \subseteq R_1$. Since Q_1 is a maximal injective real subfactor, then $N = Q_1$. Consequently, $Q = Q_1 \bar{\otimes} Q_2$. □

Corollary 17. *Let R_i be a real factor, Q_i be an irreducible, maximal injective real subfactor of $R_i (i = 1, 2)$. Then $Q_1 \bar{\otimes} Q_2$ is an irreducible, maximal injective real subfactor of $R_1 \bar{\otimes} R_2$. Moreover, by Theorem 4, $Q_1 \bar{\otimes} Q_2$ is a maximal injective real W∗-subalgebra of the factor $R_1 \bar{\otimes} R_2$.*

6. Conclusion

In the article, the maximal injective real W∗-subalgebras of real W∗-algebras or real factors are investigated. For real factors $Q \subset R$, it is proven that if $Q + iQ$ is a maximal injective W∗-subalgebra in $R + iR$, then Q also is a maximal injective real W∗-subalgebra in R. The converse may not be true. It is assumed that using the crossed product of a finite real W∗-algebra on an action, one can construct a maximal injective real subfactor of type III, for which the enveloping complex subfactor is not the maximal injective. Now the authors of the article are preparing a continuation of this article, where this example will be given. Nevertheless, in this article, the validity of the converse statement for finite factors is proved. Exactly, the converse is proved in the case "II_1"-factors, that is, it is shown that if R is a real factor of type II_1, then the maximal injectivity of Q implies the maximal injectivity of $Q + iQ$. Moreover, it is proven that a maximal injective real subfactor Q of a real factor R is a maximal injective real W∗-subalgebra in R if and only if Q is irreducible in R. The above show how the real case differs from the complex case, and this is the contribution of this article. Moreover, the "splitting theorem" of Ge-Kadison in the real case is also proven, namely, if R_1 is a finite real factor, R_2 is a finite real W∗-algebra, and R is a real W∗-subalgebra of $R_1 \bar{\otimes} R_2$ containing $R_1 \bar{\otimes} \mathbb{R}\mathbb{I}$, then there is some real W∗-subalgebra $Q_2 \subset R_2$ such that $R = R_1 \bar{\otimes} Q_2$. And also, it is given some affirmative answers to the question of S. Popa for the real case.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors jointly worked on the results and they read and approved the final manuscript.

Acknowledgments

The authors gratefully acknowledge the Research Management Office, Universiti Malaysia Terengganu that this research was partially supported.

References

[1] S. Popa, "Maximal injective subalgebras in factors associated with free groups," *Advances in Mathematics*, vol. 50, no. 1, pp. 27–48, 1983.

[2] S. Popa, "On a problem of R.V. Kadison on maximal abelian ∗ -subalgebras in factors," *Inventiones Mathematicae*, vol. 65, no. 2, pp. 269–281, 1981.

[3] L. Ge, "On maximal injective subalgebras of factors," *Advances in Mathematics*, vol. 118, no. 1, pp. 34–70, 1996.

[4] B. Fuglede and R. V. Kadison, "On a conjecture of Murray and von Neumann," *Proceedings. National Academy of Sciences United States of America*, vol. 37, no. 7, pp. 420–425, 1951.

[5] J. Fang, "On maximal injective subalgebras of tensor products of von Neumann algebras," *Journal of Functional Analysis*, vol. 244, no. 1, pp. 277–288, 2007.

[6] J. Shen, "Maximal injective subalgebras of tensor products of free group factors," *Journal of Functional Analysis*, vol. 240, no. 2, pp. 334–348, 2006.

[7] M. Gao, "On maximal injective subalgebras," *Proceedings of the American Mathematical Society*, vol. 138, no. 6, pp. 2065–2070, 2010.

[8] S. Stratila and L. Zsido, *Lectures on von Neumann Algebras*, Editura Academiei; Tunbridge Wells; Abacus Press, Bucuresti, 1979.

[9] S. Stratila and L. Zsido, "The commutation theorem for tensor products over von Neumann algebras," *Journal of Functional Analysis*, vol. 165, no. 2, pp. 293–346, 1999.

[10] S. Popa, "Tight decomposition of factors and the single generation problem," *Journal Operator Theory*, vol. 85, pp. 277–301, 2021.

[11] S. A. Ayupov, A. A. Rakhimov, and S. M. Usmanov, *Jordan, Real and Lie Structures in Operator Algebras*, vol. 418, Kluw. Acad. Pub., MAIA, 1997.

[12] B. R. Li, *Real operator algebras*, World Scientific Publishing Co. Pte. Ltd., 2003.

[13] W. B. Arveson, "Subalgebras of C∗-algebras," *Acta Mathematica*, vol. 123, pp. 141–224, 1969.

[14] R. I. Loebl, "Injective von Neumann algebras," *American Mathematical Society*, vol. 44, no. 1, pp. 46–48, 1974.

[15] A. A. Rakhimov and M. E. Nurillaev, "On property of injectivity for real W∗-algebras and JW-algebras," *Positivity*, vol. 22, no. 5, pp. 1345–1354, 2018.