

Research Article

Using a Divergence Regularization Method to Solve an Ill-Posed Cauchy Problem for the Helmholtz Equation

Benedict Barnes ¹, Anthony Y. Aidoo,² and Joseph Ackora-Prah¹

¹*Kwame Nkrumah University of Science and Technology, Mathematics Department, Ghana*

²*Eastern Connecticut State University, Department of Mathematics and Computer Science, Willimantic, CT, USA*

Correspondence should be addressed to Benedict Barnes; ewiekwamina@gmail.com

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The ill-posed Helmholtz equation with inhomogeneous boundary deflection in a Hilbert space is regularized using the divergence regularization method (DRM). The DRM includes a positive integer scaler that homogenizes the inhomogeneous boundary deflection in the Helmholtz equation's Cauchy issue. This guarantees the existence and uniqueness of the equation's solution. To reestablish the stability of the regularized Helmholtz equation and regularized Cauchy boundary conditions, the DRM uses its regularization term $(1 + \alpha^{2m})e^m$, where $\alpha > 0$ is the regularization parameter. As a result, DRM restores all three Hadamard requirements for well-posedness.

1. Introduction

The Helmholtz equation, $\Delta w + k^2 w = 0$, where k is a constant, i.e., the wave number, models reduced wave equation which yields a time-harmonic solution or monofrequency. This time-harmonic Helmholtz equation has numerous applications in seismology, sonar technology, and noise scattering.

In order to make the Helmholtz equation useful in life, some boundary conditions (the values of the solutions) at the end points of the domains must be provided.

The computation of a Helmholtz equation's solution in a suitable functional space is a direct problem. Imposing boundary constraints on a Helmholtz equation may not have a solution in any functional space in the majority of cases. The existence and uniqueness conditions of well-posedness of a Helmholtz equation are all influenced to a great extent by the number of auxiliary conditions put on the equation. These auxiliary criteria can be solely boundary conditions or a combination of boundary and initial conditions. Usually, evolutionary challenges entail a mixture of factors. Despite this, the boundary requirements that were set on the Dirichlet problem, a Neumann issue, or a Robin

problem could be the Helmholtz equation [1]. If the Helmholtz equation is subjected to these auxiliary conditions, they must be enough to guarantee the existence of solutions. The uniqueness and stability of the solution cannot be emphasized in this ongoing discussion so as to make the Helmholtz equation well-posed. Furthermore, proper boundary or starting data aids in the comprehension of the stationary process. Otherwise, there will be no solution to the Helmholtz equation if these auxiliary conditions are too numerous. If the auxiliary requirements are insufficient, the solution will exist but will not be unique [2]. A typical example of an ill-posed problem for the Helmholtz equation that has received a lot of attention in the scientific community dating back to the 20th century is the Cauchy problem. Cauchy data is simply a combination of Dirichlet and Neumann data on some part of the boundary domain of the Helmholtz equation. The Cauchy problem for the Helmholtz equation is highly ill-posed in the sense of Hadamard. The Cauchy condition specifies data from an unknown field as well as its derivative. This auxiliary condition specifies data information only for a portion of the unknown function's domain, not for the complete domain. Imposing such mixed

boundary conditions on the Helmholtz equation as a result does not result in a solution.

The ill-posed Helmholtz equation, according to Hadamard, has no practical use or is physically nonsensical. The current trends in inverse problems, however, have refuted this assertion. The vibrating membrane system and laser beam models, for example, are an ill-posed [3].

Some regularization approaches have been suggested for solving the Cauchy issue of the Helmholtz equation, based on the assumption of the existence of a unique solution. The Tikhonov regularization method (TRM) is based on the existence of a linear bounded operator A that connects one Hilbert space X to another Hilbert space Y , with $x, y \in D(A)$. The TRM makes the Laplace-type operator in the Helmholtz equation regular. The (precise) solution and variations of the data function in the Helmholtz equation are both constrained in this method to prevent the data function from blowing up due to its emitted faults [4]. The quasinversibility regularization method (Q-RRM) was established by Lattes and Lions [5], which assumes that a linear Laplace-type operator in the Helmholtz equation is linear, but the inverse Laplace-type operator is not continuous from its range into a domain. Only the Helmholtz equation is regularized by the Q-RRM by subtracting a product of a square of a regularization parameter α and a mixed fourth-order partial derivative from the Laplace-type operator in the Helmholtz equation, which is of the form:

$$A = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \alpha^2 \frac{\partial^4}{\partial x^2 \partial y^2}. \quad (1)$$

Khoa and Nhan [6] introduced the stabilized operator to complement the effort of the Laplace-like operator occurring in the Helmholtz equations. The introduction of the stabilized operator rather increased the complexity of the problem of regularizing Helmholtz equation, whose weak solution is sought in the Sobolev spaces.

The Cauchy issue of the Helmholtz equation has been solved using wavelet methods. Vani and Avudainayagam [7] solved the problem in the (Meyer) wavelet domain and demonstrated that the regularized solution converges as the Cauchy data perturbations approach zero. Solutions are occasionally sought in a more convenient function space rather than regular Hilbert space. Dou and Fu [8] regularized the problem in a Sobolev space, and when the number of disturbances in the Cauchy data increased, the numerical solution remained stable. Other less effective methods include the energy regularization method introduced by Han et al. [9] for solving the Cauchy issue of the Helmholtz equation. For the identification of a source data for regularizing the Helmholtz equation with Dirichlet boundary conditions, Zhao et al. [10] compared the efficiency and convergence rate of the mollification method, the modified Tikhonov regularization method, and the Fourier regularization method. All of these regularization methods fail to ensure not only the existence of a unique solution to the Helmholtz equation but also the stability of the solution,

especially when the inhomogeneous Cauchy boundary deflection is imposed on the equation.

In [11], the authors employed Fourier truncation regularization to solve the Helmholtz equation with an altered wave number. The regularized solution's error estimation and the exact solution obtained as a regularization parameter are both varied. However, the regularization method has been observed to avoid singularity in the equation. The singular boundary approach, for example, changes the Helmholtz equation into a boundary integral equation for the determination of singularities in the equation and finally produces the regularized solution to the Helmholtz equation, as described in [12]. All of the strategies mentioned in this work fail to guarantee a solution to the Helmholtz equation with the Cauchy boundary conditions but instead restore the problem's stability.

The third condition of well-posedness can be ensured by the bounded inverse theorem. We note that the Helmholtz equation has a solution if the smoothness requirement is satisfied together with the data compatibility conditions, which we give below.

2. Preliminary Result

In this section of the paper, the lemma and theorem that will be used in achieving the main result in the next section are provided. Also, throughout this paper, the supremum norm is applied in establishing results.

Lemma 1 (Compatibility of data in Neumann problem). *Let Ω denote a bounded region in R^2 having a smooth boundary $\partial\Omega$. The Neumann problem for the Helmholtz equation*

$$\begin{aligned} (\Delta + k^2)w &= 0 \text{ in } \Omega, \\ \partial\Omega &= g(x) \text{ on } \partial\Omega. \end{aligned} \quad (2)$$

A necessary and sufficient condition for the existence of a solution to the Neumann problem for the Helmholtz (homogeneous) equation is [1]

$$\int_{\partial\Omega} g(s) ds = 0. \quad (3)$$

Proof. Considering the vector identity $\Delta u = \nabla \cdot \nabla u$, applying Green's first identity to the operator equation above, we have

$$\begin{aligned} \int_{\Omega} (\nabla \cdot (\nabla w) + k^2 w) d\Omega &= \int_{\partial\Omega} g(s) ds, \\ \int_{\Omega} (\nabla^2 + k^2) w d\Omega &= \int_{\partial\Omega} g(s) ds, \\ 0 &= \int_{\partial\Omega} g(s) ds. \end{aligned} \quad (4)$$

□

Theorem 2 (Bounded inverse theorem). *Let A be a bounded linear Laplace-type operator in the Helmholtz equation from a subspace Ω in a Hilbert space H into a Hilbert space H .*

Then, A has a continuous inverse operator A^{-1} from its range $R(A)$ into Ω . Conversely, if there is a continuous inverse operator

$$A^{-1} : R(A) \longrightarrow \Omega, \tag{5}$$

then there is a positive constant C such that [13]

$$\|A(w(x, y))\|_H \geq C\|w(x, y)\|, \forall w \in \Omega. \tag{6}$$

In some cases, the Helmholtz equation with the abovementioned boundary conditions does not produce a solution in any functional space. As a result, none of the three well-posedness conditions are satisfied, rendering the Cauchy problem of the Helmholtz equation ill-posed in Hadamard’s meaning.

The following is the Helmholtz equation with Cauchy boundary conditions.

$$\begin{aligned} \frac{\partial^2 w(x, y)}{\partial x^2} + \frac{\partial^2 w(x, y)}{\partial y^2} + k^2 w(x, y) &= 0, 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq 1, \\ \frac{\partial w(x, 0)}{\partial y} &= \frac{1}{n} \sin (nx), 0 \leq x \leq \frac{\pi}{2}, \\ w(x, 0) &= 0, 0 \leq x \leq \frac{\pi}{2}. \end{aligned} \tag{7}$$

We prove that in Hilbert space, the aforementioned Helmholtz equation with Cauchy boundary conditions has no solution.

Furthermore, we can observe from the Helmholtz equation’s initial deflection condition that

$$\int_0^{\frac{\pi}{2}} \frac{1}{n} \sin (nx) dx = \frac{1}{n^2} \left[1 - \cos \left(\frac{n\pi}{2} \right) \right]. \tag{8}$$

Thus,

$$\frac{1}{n^2} \left[1 - \cos \left(\frac{n\pi}{2} \right) \right] = \begin{cases} \frac{1}{n^2}, & \forall n = 1, 3, \dots, \\ \frac{2}{n^2}, & \forall n = 2, 6, \dots, \\ 0, & \forall n = 4, 8, \dots \end{cases} \tag{9}$$

If $n = 1$, we obtain

$$\int_0^{\frac{\pi}{2}} \frac{1}{n} \sin (nx) dx = 1 \neq 0. \tag{10}$$

Thus,

$$\int_0^{\frac{\pi}{2}} \frac{1}{n} \sin (nx) dx \neq 0, \forall n = \text{odd or } n = 2, 6, \dots \tag{11}$$

The data compatibility requirement is not satisfied by the equation. This means that there is no solution to the Helmholtz equation with Cauchy boundary conditions. As

a result, in Hadamard’s understanding, the Helmholtz equation with Cauchy boundary conditions is ill-posed.

How can there be a stable solution without a solution to the Helmholtz equation? Assuming heuristically, this is not unusual among the scientific community. How can there be an estimation of error between the regularized solution and an exact solution when the Helmholtz equation has no solution in any functional space? It takes more information in its construction, the right number of boundary conditions, and appropriate restrictions to make the inhomogeneous boundary deflection in the Cauchy problem of the Helmholtz equation well-posed. Numerical stability is unquestionably possible when the inhomogeneous Cauchy problem of the Helmholtz equation is well-posed, whether addressed with finite precision or with data errors.

The DRM is used to regularize both imposed Cauchy boundary conditions on the Helmholtz equation and equations in a Hilbert space. This method uses a positive integer scalar η to homogenize the inhomogeneous boundary deflection in the Cauchy data of the Helmholtz equation. As a result, the Helmholtz equation has a solution thanks to the positive integer scalar. The DRM employs a regularization term $(1 + \alpha^{2m})e^m$, which restores the stability of solution of the Helmholtz equation. The approach uses Green’s first identity to the Laplace-type operator of $(1 + \alpha^{2m})e^m$ and $w(x, y)$ appearing in the Helmholtz equation to determine the uniqueness of the solution. This results in a piecewise smooth boundary of two disjoint complementary sections $\partial\Omega_1$ and $\partial\Omega_2$ with $w(x, 0)$ and $\partial w(x, l)/\partial y$ on $\partial\Omega_1$ and $w(l, y)$ and $\partial w(0, y)/\partial x$ on $\partial\Omega_2$, respectively. The uniqueness of the solution of the regularized Helmholtz equation with regularized Cauchy boundary conditions is proven by contradiction with the help of DRM. Using the DRM, we demonstrated that the operator $R(A)$ has a closed range, the null space of the operator $N(A)$ is trivial, and the inverse operator

$$A^{-1} : R(A) \longrightarrow \Omega \tag{12}$$

has a continuous range.

3. Main Result

We can see that equation (3) is ill-posed, and that Cauchy data is provided on the arc of the boundary $\partial\Omega$ instead of the whole boundary of the domain Ω . We first extend the arc of the boundary (hypersurface) $\partial\Omega$ to the full boundary of the domain Ω using the DRM to restore the well-posedness of the Helmholtz equation with Cauchy boundary conditions where the boundary deflection is inhomogeneous.

As a result, the domain of Cauchy data’s boundary is quadratured to a piecewise smooth boundary comprising two disjoint complementary parts.

Theorem 3 (Divergence regularization method). *Let Cauchy boundary conditions be imposed on Helmholtz equation where the boundary deflection is inhomogeneous; then, the Cauchy problem of the Helmholtz equation is given by*

$$\begin{aligned} \frac{\partial^2 w(x, y)}{\partial x^2} + \frac{\partial^2 w(x, y)}{\partial y^2} + k^2 w(x, y) &= 0 \text{ in } \Omega, \\ \frac{\partial w(x, 0)}{\partial y} &= h(x) \text{ on } \partial\Omega_1, \\ w(x, 0) &= 0 \text{ on } \partial\Omega_2, \end{aligned} \tag{13}$$

where

$$\int_{\partial\Omega} h(x) dx \neq 0 \text{ and } \partial\Omega = \partial\Omega_1 \cup \partial\Omega_2. \tag{14}$$

The corresponding regularized Cauchy problem of the Helmholtz equation is

$$\frac{\partial^2 w_{\eta,0}(x, y)}{\partial x^2} + \frac{\partial^2 w_{\eta,0}(x, y)}{\partial y^2} + (1 + \alpha^{2m})^{-1} e^{-m} k^2 w_{\eta,0}(x, y) = 0, \text{ in } \Omega, \tag{15}$$

$$\frac{\partial w_{\eta,0}(x, l)}{\partial y} = (1 + \alpha^{2m})^{-1} e^{-m} h(\eta x), \text{ on } \partial\Omega_1, \tag{16}$$

$$w_{\eta}(x, 0) = 0, \text{ on } \partial\Omega_1, \tag{17}$$

$$\frac{\partial w_{\eta,0}(0, y)}{\partial x} = 0, \text{ on } \partial\Omega_2, \tag{18}$$

$$w_{\eta,0}(l, y) = (1 + \alpha^{2m})^{-1} e^{-m} h(y), \text{ on } \partial\Omega_2, \tag{19}$$

where

$$\int_{\partial\Omega_1} h(\eta x) dx = 0. \tag{20}$$

$h(x) \neq 0, h(y) \neq 0, \alpha \in (-\infty, -1) \cup (1, \infty)$ is the regularization parameter, $m \in \mathbb{Z}^+$ is a positive integer, k is the wave number, $[0, l]$ is the square domain with l is a radian number, and η is any even positive integer.

This Theorem 3 ensures the existence, uniqueness, and stability of the Cauchy problem for the Helmholtz equation.

Proof. We write the homogeneous Helmholtz equation as

$$\nabla \cdot (v \nabla w(x, y)) + k^2 w(x, y) = 0. \tag{21}$$

Applying the dot product, product rule, and integrating both sides over Ω , we obtain

$$\begin{aligned} \iint_{\Omega} \nabla \cdot (v \nabla w(x, y)) dx dy + \iint_{\Omega} k^2 w(x, y) dx dy \\ = \iint_{\Omega} \nabla w(x, y) \cdot \nabla v dx dy + \iint_{\Omega} v \Delta w(x, y) dx dy \\ + \iint_{\Omega} k^2 w(x, y) dx dy = 0. \end{aligned} \tag{22}$$

The x - coordinate of the unknown function $w(x, y)$ in equation (22) is then scaled by a factor η . The trigonometric

function of the Helmholtz equation at inhomogeneous boundary deflection becomes zero as a result of this scalar, and when integrated over the domain boundary, we get

$$\begin{aligned} \iint_{\Omega} \nabla \cdot (v \nabla w_{\eta,0}(x, y)) dx dy + \iint_{\Omega} k^2 w_{\eta,0}(x, y) dx dy \\ = \iint_{\Omega} \nabla w_{\eta,0}(x, y) \cdot \nabla v dx dy + \iint_{\Omega} v \Delta w_{\eta,0}(x, y) dx dy \\ + \iint_{\Omega} k^2 w_{\eta,0}(x, y) dx dy = 0. \end{aligned} \tag{23}$$

Substituting $v = (1 + \alpha^{2m})e^m$ into equation (3), the right hand side becomes

$$\iint_{\Omega} (1 + \alpha^{2m})e^m \Delta w_{\eta,0}(x, y) dx dy + \iint_{\Omega} k^2 w_{\eta,0}(x, y) dx dy = 0. \tag{24}$$

Applying Green's first identity to the first term of above equation yields

$$\begin{aligned} \iint_{\Omega} (1 + \alpha^{2m})e^m \Delta w_{\eta,0}(x, y) dx dy + k^2 \iint_{\Omega} w_{\eta,0}(x, y) dx dy \\ = \int_{\partial\Omega_1} (1 + \alpha^{2m})e^m \left(\frac{\partial w_{\eta,0}(x, l)}{\partial y} - w_{\eta,0}(x, 0) \right) dx \\ + \int_{\partial\Omega_2} (1 + \alpha^{2m})e^m \left(w_{\eta,0}(l, y) - \frac{\partial w_{\eta,0}(0, y)}{\partial x} \right) dy. \end{aligned} \tag{25}$$

A regularized Helmholtz equation with regularized Cauchy boundary conditions is obtained as shown above.

We show that our regularized Helmholtz equation, when combined with Cauchy boundary conditions, meets all three well-posedness requirements. We begin by demonstrating that equations (15)–(19) has a solution in the Hilbert space. \square

3.1. Well-Posedness of Regularized Helmholtz Equation with Regularized Cauchy Boundary Conditions. We now show that the regularized Helmholtz equation with regularized Cauchy boundary conditions is well-posed.

Theorem 4 (Existence of solution of the regularized Helmholtz equation). *In Theorem 3, the regularized Cauchy boundary conditions on the regularized Helmholtz equation have two boundary deflections, homogeneous*

$$\frac{\partial w_{n,0}(0, y)}{\partial x} = 0 \text{ on } \partial\Omega_2 \tag{26}$$

and inhomogeneous

$$\frac{\partial w_{n,0}(x, l)}{\partial y} = (1 + \alpha^{2m})^{-1} e^{-m} h(\eta x) \text{ on } \partial\Omega_1. \tag{27}$$

It is clear that the homogeneous boundary deflection satisfies the data compatibility condition. We demonstrate that the inhomogeneous boundary deflection also satisfies the data compatibility condition as follows:

$$\int_0^l (1 + \alpha^{2m})^{-1} e^{-m} h(\eta x) dx = (1 + \alpha^{2m})^{-1} e^{-m} \int_0^l h(\eta x) dx. \tag{28}$$

Since $h(\eta x)$ is a periodic function such as $\sin(x)$ or $\cos(x)$ the integral of the periodic function of a scalar multiplier of the spatial variable over the boundary always yields zero, then

$$\int_0^l (1 + \alpha^{2m})^{-1} e^{-m} h(\eta x) dx = (1 + \alpha^{2m})^{-1} e^{-m} (0) = 0. \tag{29}$$

This result implies that the regularized Helmholtz equation with regularized Cauchy boundary conditions has a solution in the Hilbert space.

We prove that the DRM provides a unique solution of regularized Helmholtz equation together with regularized Cauchy boundary conditions as follows.

Theorem 5 (Uniqueness). *Suppose that Ω denotes a rectangular domain whose boundary consists of two disjoint, complementary parts $\partial\Omega_1$ and $\partial\Omega_2$. Let $h(\eta x)$ and $h(y)$ denote given data functions, and then equations (15), (16), (17), (18), (19) has at most one smooth solution.*

Proof. Suppose that equations (15), (16), (17), (18), (19) has two different smooth solutions denoted by $u_{\eta,0}(x, y)$ and $v_{\eta,0}(x, y)$, then

$$w_{\eta,0}(x, y) = u_{\eta,0}(x, y) - v_{\eta,0}(x, y) \tag{30}$$

is a solution of equations (15), (16), (17), (18), (19).

$$\begin{aligned} w_{\eta,0}(x, 0) &= u_{\eta,0}(x, 0) - v_{\eta,0}(x, 0), \\ w_{\eta,0}(x, 0) &= 0, \\ \frac{\partial w_{\eta,0}(x, l)}{\partial y} &= \frac{\partial u_{\eta,0}(x, l)}{\partial y} - \frac{\partial v_{\eta,0}(x, l)}{\partial y}, \\ w_{\eta,0}(l, y) &= u_{\eta,0}(l, y) - v_{\eta,0}(l, y), \\ w_{\eta,0}(l, y) &= (1 + \alpha^{2m})^{-1} e^{-m} (h_1(y) - h_2(y)), \\ \frac{\partial w_{\eta,0}(0, y)}{\partial x} &= \frac{\partial u_{\eta,0}(0, y)}{\partial x} - \frac{\partial v_{\eta,0}(0, y)}{\partial x}, \\ \frac{\partial w_{\eta,0}(0, y)}{\partial x} &= 0. \end{aligned} \tag{31}$$

□

Multiplying both sides of equations (15), (16), (17), (18), (19) by $w_{\eta,0}(x, y)$ and integrating over a domain Ω , we obtain

$$\begin{aligned} &\iint_{\Omega} w_{\eta,0}(x, y) \frac{\partial^2 w_{\eta,0}(x, y)}{\partial x^2} dx dy \\ &+ \iint_{\Omega} w_{\eta,0}(x, y) \frac{\partial^2 w_{\eta,0}(x, y)}{\partial y^2} dx dy \\ &+ \iint_{\Omega} (1 + \alpha^{2m})^{-1} e^{-m} k^2 |w_{\eta,0}(x, y)|^2 dx dy = 0. \end{aligned} \tag{32}$$

Applying Green's first identity to the first two terms on the left hand side of equation (32), we obtain

$$\begin{aligned} &\iint_{\Omega} w_{\eta,0}(x, y) \Delta w_{\eta,0}(x, y) dx dy \\ &= \int_{\partial\Omega_1} w_{\eta,0}(x, 0) \frac{\partial w_{\eta,0}(x, l)}{\partial y} dx \\ &+ \int_{\partial\Omega_2} w_{\eta,0}(l, y) \frac{\partial w_{\eta,0}(0, y)}{\partial x} dy \\ &- \int_{\Omega} |\nabla w_{\eta,0}(x, y)|^2 dx dy, \end{aligned} \tag{33}$$

$$\begin{aligned} &\iint_{\Omega} w_{\eta,0}(x, y) \Delta w_{\eta,0}(x, y) dx dy \\ &= \int_{\partial\Omega_1} 0 \times (1 + \alpha^{2m})^{-1} e^{-m} h(\eta x) dx \\ &+ \int_{\partial\Omega_2} (1 + \alpha^{2m})^{-1} e^{-m} h(y) \times 0 dy \\ &- \iint_{\Omega} |\nabla w_{\eta,0}(x, y)|^2 dx dy, \end{aligned} \tag{34}$$

$$\iint_{\Omega} w_{\eta,0}(x, y) \Delta w_{\eta,0}(x, y) dx dy = 0 - \iint_{\Omega} |\nabla w_{\eta,0}(x, y)|^2 dx dy, \tag{35}$$

$$\iint_{\Omega} w_{\eta,0}(x, y) \Delta w_{\eta,0}(x, y) dx dy = - \iint_{\Omega} |\nabla w_{\eta,0}(x, y)|^2 dx dy. \tag{36}$$

Substituting equation (34) into equation (32) yields

$$\begin{aligned} &\iint_{\Omega} w_{\eta,0}(x, y) \frac{\partial^2 w_{\eta,0}(x, y)}{\partial x^2} dx dy \\ &+ \iint_{\Omega} w_{\eta,0}(x, y) \frac{\partial^2 w_{\eta,0}(x, y)}{\partial y^2} dx dy \\ &+ \iint_{\Omega} (1 + \alpha^{2m})^{-1} e^{-m} k^2 |w_{\eta,0}(x, y)|^2 dx dy \\ &= - \iint_{\Omega} |\nabla w_{\eta,0}(x, y)|^2 dx dy \\ &+ \iint_{\Omega} (1 + \alpha^{2m})^{-1} e^{-m} k^2 |w_{\eta,0}(x, y)|^2 dx dy = 0 \\ &= - \iint_{\Omega} |\nabla w_{\eta,0}(x, y)|^2 dx dy \\ &+ \iint_{\Omega} (1 + \alpha^{2m})^{-1} e^{-m} k^2 |w_{\eta,0}(x, y)|^2 dx dy. \end{aligned} \tag{37}$$

In the above equation, it follows that

$$(1 + \alpha^{2m})^{-1} e^{-m} k^2 \iint_{\Omega} |w_{\eta,0}(x, y)|^2 dx dy = 0 \Rightarrow w_{\eta,0}(x, y) = 0 \text{ in } \Omega,$$

$$\iint_{\Omega} |\nabla w_{\eta,0}(x, y)|^2 dx dy = 0 \Rightarrow \nabla w_{\eta,0}(x, y) = 0 \Rightarrow w_{\eta,0}(x, y) = \text{constant} = 0 \text{ in } \Omega. \tag{38}$$

Also, we observe that

$$|\nabla w_{\eta,0}(x, y)| = 0 \text{ on } \partial\Omega_2 \Rightarrow w_{\eta,0}(x, y) = \text{constant} = 0 \text{ on } \partial\Omega_2, \tag{39}$$

$$h(\eta x) = 0 \text{ on } \partial\Omega_1,$$

$$|\nabla w_{\eta,0}(x, y)| = 0 \Rightarrow w_{\eta,0}(x, y) = \text{constant} = 0 \text{ on } \partial\Omega_1. \tag{40}$$

Thus, $w_{\eta,0}(x, y)$ is smooth and zero in the domain Ω and its boundary $\partial\Omega$. This implies that

$$u_{\eta,0}(x, y) = v_{\eta,0}(x, y). \tag{41}$$

Hence, equation (22) has only one solution in Hilbert Space.

Theorem 6 (Stability). *In demonstrating that the regularized Helmholtz equation is stable to the small changes in the regularized Cauchy boundary conditions, the spatial variable y is perturbed from ε to δ , where $\delta > \varepsilon$. Then,*

$$\begin{aligned} w_{\eta,0}(l, \varepsilon) &= (1 + \alpha^{2m})^{-1} e^{-m} h(\eta\varepsilon), \\ w_{\eta,0}(l, \delta) &= (1 + \alpha^{2m})^{-1} e^{-m} h(\eta\delta), \end{aligned} \tag{42}$$

and the corresponding solutions are

$$\begin{aligned} w_1(x, y) &= \sum_{n=1,3,\dots}^{\infty} \frac{h(\eta\varepsilon) \cosh\left(\sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]} x\right) h(\eta y)}{\pi n (1 + \alpha^{2m}) e^m \cosh\left(2\pi \sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]}\right)}, \\ w_2(x, y) &= \sum_{n=1,3,\dots}^{\infty} \frac{h(\eta\delta) \cosh\left(\sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]} x\right) h(\eta y)}{\pi n (1 + \alpha^{2m}) e^m \cosh\left(2\pi \sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]}\right)}. \end{aligned} \tag{43}$$

The change in the regularized boundary condition is

$$\begin{aligned} &\lim_{m,n \rightarrow \infty} |w_{4,0}(2\pi, \varepsilon) - w_{4,0}(2\pi, \delta)| \\ &= \lim_{m,n \rightarrow \infty} |(1 + \alpha^{2m})^{-1} e^{-m} \cdot h(\eta\varepsilon) - (1 + \alpha^{2m})^{-1} e^{-m} h(\eta\delta)|, \\ &\lim_{m,n \rightarrow \infty} |w_{4,0}(2\pi, \varepsilon) - w_{4,0}(2\pi, \delta)| \leq 2 \lim_{m,n \rightarrow \infty} (1 + \alpha^{2m})^{-1} e^{-m}, \\ &\lim_{m,n \rightarrow \infty} |w_{4,0}(2\pi, \varepsilon) - w_{4,0}(2\pi, \delta)| \longrightarrow 0 \text{ as } m, n \longrightarrow \infty, \end{aligned} \tag{44}$$

on the grounds that $h(\eta x) \leq 1$ and $h(\delta x) \leq 1$ are a periodic function which is $\sin(\eta x)$ or $\cos(\eta x)$.

This implies that there is a small change in the boundary condition. Moreover, we observe the corresponding change in the solution $w(x, y)$ as

$$\begin{aligned} &\lim_{m,n \rightarrow \infty} |w_1(x, y) - w_2(x, y)| \\ &= \lim_{m,n \rightarrow \infty} \left| \sum_{n=1,3,\dots}^{\infty} \frac{4h(\eta\varepsilon) \cosh\left(\sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]} x\right) h(\eta y)}{\pi (1 + \alpha^{2m}) e^m \cosh\left(2\pi \sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]}\right)} \right. \\ &\quad \left. - \sum_{n=1,3,\dots}^{\infty} \frac{4h(\eta\delta) \cosh\left(\sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]} x\right) h(\eta y)}{\pi (1 + \alpha^{2m}) e^m \cosh\left(2\pi \sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]}\right)} \right| \lim_{m,n \rightarrow \infty} \\ &\cdot |w_1(x, y) - w_2(x, y)| \leq \lim_{m,n \rightarrow \infty} \frac{8}{\pi (1 + \alpha^{2m}) e^m} \cdot \frac{e^{\sqrt{(n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2)} \cdot x}}{e^{2\pi \sqrt{(n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2)}}. \end{aligned} \tag{45}$$

Thus,

$$\lim_{m,n \rightarrow \infty} |w_1(x, y) - w_2(x, y)| \leq \lim_{m \rightarrow \infty} \frac{8}{\pi (1 + \alpha^{2m}) e^m}. \tag{46}$$

We can see that

$$\begin{aligned} &(1 + \alpha^{2m})^{-1} e^{-m} \longrightarrow 0 \text{ as } m \longrightarrow \infty \\ &\Rightarrow \lim_{m,n \rightarrow \infty} |w_1(x, y) - w_2(x, y)| \longrightarrow 0 \text{ as } m, n \longrightarrow \infty. \end{aligned} \tag{47}$$

This implies that a small change in the regularized Cauchy boundary condition from $x_1 = \varepsilon$ to $x_2 = \delta$ results in a small change in solution

$$\sum_{n=1,3,\dots}^{\infty} \frac{\cosh\left(\sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]} x\right) h(\eta y)}{(1 + \alpha^{2m}) e^m \cosh\left(2\pi \sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]}\right)}. \tag{48}$$

Thus, the regularized Helmholtz equation is stable. Hence, the regularized Cauchy problem for the regularized Helmholtz equation is well-posed.

4. Applications

The DRM is used in this part to find solutions to the Helmholtz equation with Cauchy boundary conditions in the upper half-plane and the Helmholtz equation with Neumann boundary conditions in the lower half-plane.

4.1. Helmholtz Equation with Cauchy Boundary Conditions where the Boundary Deflection Is Inhomogeneous. In this subsection of the paper, the DRM is applied to regularize

the Cauchy problem of the Helmholtz equation in a domain $[0, (\pi/2)] \times [0, 2\pi]$. The problem is as follows:

$$\frac{\partial^2 w(x, y)}{\partial x^2} + \frac{\partial^2 w(x, y)}{\partial y^2} + k^2 w(x, y) = 0, 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq 2\pi, \tag{49}$$

$$\frac{\partial w(x, 0)}{\partial y} = \frac{1}{n} \sin (nx), 0 \leq x \leq \frac{\pi}{2}, \tag{50}$$

$$w(x, 0) = 0, 0 \leq x \leq \frac{\pi}{2}. \tag{51}$$

We show that when Cauchy boundary conditions are applied on a homogeneous Helmholtz equation with a non-zero boundary deflection, we get the following result using the method of separation of variables

$$w(x, y) = \sum_{n=1}^{\infty} \frac{1}{n \sqrt{(n^2 - k^2)}} \sin (nx) \sinh \left(\sqrt{(n^2 - k^2)} y \right). \tag{52}$$

For the above function $w(x, y)$ in Equation (52) to be called a solution to Equation (52) together with Cauchy boundary conditions, it must satisfy the smoothness requirement condition as well as the data compatibility condition. The integral of the boundary deflection $\partial w(x, 0)/\partial y$ over $[0, (\pi/2)]$ is

$$\int_0^{\frac{\pi}{2}} \frac{1}{n} \sin (nx) dx = \frac{1}{n^2} \left[1 - \cos \left(\frac{n\pi}{2} \right) \right]. \tag{53}$$

Now,

$$\frac{1}{n^2} \left[1 - \cos \left(\frac{n\pi}{2} \right) \right] = \begin{cases} \frac{1}{n^2}, & \forall n = 1, 3, 5, \dots, \\ \frac{2}{n^2}, & \forall n = 2, 6, 10, \dots, \\ 0, & \forall n = 4, 8, 12, \dots, \end{cases} \tag{54}$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{n} \sin (nx) dx \neq 0, \forall n = \text{odd or } n = 2, 6, \dots.$$

Equation (52) does not satisfy the data compatibility condition. This implies that the function

$$w(x, y) = \sum_{n=1}^{\infty} \frac{1}{n \sqrt{n^2 - k^2}} \sin (nx) \sinh \left(\sqrt{n^2 - k^2} y \right), \forall n \in \mathbb{N} \tag{55}$$

is not a solution of equations (49), (50), (51). Hence, the equation is ill-posed in the sense of Hadamard.

To regularize this equation, we choose $\eta = 4$ in the Theorem 3, and we obtain

$$\begin{aligned} \frac{\partial^2 w_{4,0}(x, y)}{\partial x^2} + \frac{\partial^2 w_{4,0}(x, y)}{\partial y^2} + (1 + \alpha^{2m})^{-1} e^{-m} k^2 w_{4,0}(x, y) \\ = 0, 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq 2\pi, \end{aligned} \tag{56}$$

$$w_{4,0}(x, 0) = 0, 0 \leq x \leq \frac{\pi}{2}, \tag{57}$$

$$\frac{\partial w_{4,0}(x, (\pi/2))}{\partial y} = (1 + \alpha^{2m})^{-1} e^{-m} \frac{1}{n} \sin (4nx), 0 \leq x \leq \frac{\pi}{2}, \tag{58}$$

$$\frac{\partial w_{4,0}(0, y)}{\partial x} = 0, 0 \leq y \leq 2\pi, \tag{59}$$

$$w_{4,0}(2\pi, y) = (1 + \alpha^{2m})^{-1} e^{-m} \frac{1}{n} \sin (ny), 0 \leq y \leq 2\pi. \tag{60}$$

By the method of separation of variables, we obtain

$$w_{4,0}(x, y) = \sum_{n=1,3,\dots}^{\infty} \frac{\cosh \left(\sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]} x \right) \sin (ny)}{n(1 + \alpha^{2m}) e^m \cosh \left(2\pi \sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]} \right)}. \tag{61}$$

It can easily be shown that the regularized equations (56), (57), (58), (59), (60) together with the regularized boundary conditions is well-posed since

$$\int_0^{\frac{\pi}{2}} \frac{1}{n(1 + \alpha^{2m}) e^m} \sin (4nx) dx = 0. \tag{62}$$

Applying the uniqueness of the DRM in Theorem 3, the solution in equation (61) is unique.

Finally, we demonstrate that the solution of the regularized Helmholtz equation is stable to small changes in the boundary condition. In equations (56), (57), (58), (59), (60), we choose $y = \varepsilon$ in the boundary condition $w_{4,0}(2\pi, y) = (1 + \alpha^{2m})^{-1} e^{-m} (1/n) \sin (ny)$, where $0 < \varepsilon \leq \pi/36$. We obtain the regularized Helmholtz equations (56), (57), (58), (59), (60) together with new boundary condition as given below:

$$w_{4,0}(2\pi, \varepsilon) = (1 + \alpha^{2m})^{-1} e^{-m} \frac{1}{n} \sin (n\varepsilon), \forall 0 \leq y \leq 2\pi, \tag{63}$$

and the corresponding solution is as below:

$$w_1(x, y) = \sum_{n=1,3,\dots}^{\infty} \frac{4 \sin (n\varepsilon) \cosh \left(\sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]} x \right) \sin (ny)}{\pi n(1 + \alpha^{2m}) e^m \cosh \left(2\pi \sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]} \right)}. \tag{64}$$

We perturb from

$$w_{4,0}(2\pi, \varepsilon) = (1 + \alpha^{2m})^{-1} e^{-m} \frac{1}{n} \sin(n\varepsilon), 0 \leq y \leq 2\pi \quad (65)$$

to

$$w_{4,0}(2\pi, \delta) = (1 + \alpha^{2m})^{-1} e^{-m} \frac{1}{n} \sin(n\delta), 0 \leq y \leq 2\pi, \quad (66)$$

where $0 < \delta \ll \pi/36, \delta > \varepsilon$ with the corresponding solution as:

$$w_2(x, y) = \sum_{n=1,3,\dots}^{\infty} \frac{4 \sin(n\delta) \cosh\left(\sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]} x\right) \sin(ny)}{\pi n (1 + \alpha^{2m}) e^m \cosh\left(2\pi \sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]}\right)}. \quad (67)$$

We observe the following inequalities:

$$\|\sin(n\varepsilon)\| \leq 1, \|\sin(n\delta)\| \leq 1 \text{ and } \|\cosh(y)\| \leq e^y. \quad (68)$$

The change in the boundary condition is

$$\begin{aligned} & \lim_{m,n \rightarrow \infty} |w_{4,0}(2\pi, \varepsilon) - w_{4,0}(2\pi, \delta)| \\ &= \lim_{m,n \rightarrow \infty} \left| (1 + \alpha^{2m})^{-1} e^{-m} \frac{1}{n} \sin(n\varepsilon) \right. \\ & \quad \left. - (1 + \alpha^{2m})^{-1} e^{-m} \frac{1}{n} \sin(n\delta) \right| \\ &\leq \lim_{m,n \rightarrow \infty} (1 + \alpha^{2m})^{-1} e^{-m} \\ & \quad \cdot \frac{1}{n} (\|\sin(n\varepsilon)\| + \|\sin(n\delta)\|) \lim_{m,n \rightarrow \infty} |w_{4,0}(2\pi, \varepsilon) \\ & \quad - w_{4,0}(2\pi, \delta)| \leq 2 \lim_{m,n \rightarrow \infty} (1 + \alpha^{2m})^{-1} e^{-m} \cdot \frac{1}{n} \end{aligned}$$

$$\lim_{m,n \rightarrow \infty} |w_{4,0}(2\pi, \varepsilon) - w_{4,0}(2\pi, \delta)| \longrightarrow 0 \text{ as } m, n \longrightarrow \infty. \quad (69)$$

This implies that there is a small change in the boundary condition. Moreover, we observe the corresponding change in the solution $w(x, y)$ as

$$\begin{aligned} & \lim_{m,n \rightarrow \infty} |w_1(x, y) - w_2(x, y)| \\ &= \lim_{m,n \rightarrow \infty} \left| \sum_{n=1,3,\dots}^{\infty} \frac{4 \sin(n\varepsilon) \cosh\left(\sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]} x\right) \sin(ny)}{\pi n (1 + \alpha^{2m}) e^m \cosh\left(2\pi \sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]}\right)} \right. \end{aligned}$$

$$\begin{aligned} & \left. - \sum_{n=1,3,\dots}^{\infty} \frac{4 \sin(n\delta) \cosh\left(\sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]} x\right) \sin(ny)}{\pi n (1 + \alpha^{2m}) e^m \cosh\left(2\pi \sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]}\right)} \right| \\ &\leq \lim_{m,n \rightarrow \infty} \frac{4}{\pi (1 + \alpha^{2m}) e^m} \left\| \sum_{n=1,3,\dots}^{\infty} \frac{\cosh\left(\sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]} x\right)}{\cosh\left(2\pi \sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]}\right)} \right\| \\ & \quad \times (\|\sin(n\varepsilon)\| \|\sin(ny)\| + \|\sin(n\delta)\| \|\sin(ny)\|) \lim_{m,n \rightarrow \infty} |w_1(x, y) \\ & \quad - w_2(x, y)| \leq \lim_{m,n \rightarrow \infty} \frac{8}{\pi (1 + \alpha^{2m}) e^m} \cdot \frac{e^{\sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]} x}}{e^{2\pi \sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]}}}. \quad (70) \end{aligned}$$

Then,

$$\lim_{m,n \rightarrow \infty} |w_1(x, y) - w_2(x, y)| \leq \lim_{m \rightarrow \infty} \frac{8}{\pi (1 + \alpha^{2m}) e^m}. \quad (71)$$

We can see that

$$\begin{aligned} & (1 + \alpha^{2m})^{-1} e^{-m} \longrightarrow 0 \text{ as } m \longrightarrow \infty \\ & \Rightarrow \lim_{m,n \rightarrow \infty} |w_1(x, y) - w_2(x, y)| \longrightarrow 0 \text{ as } m, n \longrightarrow \infty. \quad (72) \end{aligned}$$

This implies that a small change in the boundary condition $w_{4,0}(\pi/2, y)$ from $x_1 = \varepsilon$ to $x_2 = \delta$ results in a small change in solution as shown below:

$$\sum_{n=1,3,\dots}^{\infty} \frac{\cosh\left(\sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]} x\right) \sin(ny)}{n (1 + \alpha^{2m}) e^m \cosh\left(2\pi \sqrt{[n^2 - (1 + \alpha^{2m})^{-1} e^{-m} k^2]}\right)}. \quad (73)$$

Thus, the regularized equations (56), (57), (58), (59), (60) is stable. Hence, the regularized Cauchy problem for the regularized Helmholtz equation is well-posed.

Another type of the Cauchy problem for the Helmholtz equation is if the first boundary condition in equations (49), (50), (51) is replaced by $(1/n) \cos(nx)$. Thus,

$$\begin{aligned} & \frac{\partial^2 w(x, y)}{\partial x^2} + \frac{\partial^2 w(x, y)}{\partial y^2} + k^2 w(x, y) \\ &= 0, 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \frac{\pi}{2}, \\ & \frac{\partial w(x, 0)}{\partial y} = \frac{1}{n} \cos(nx), 0 \leq x \leq \frac{\pi}{2}, \\ & w(x, 0) = 0, 0 \leq x \leq \frac{\pi}{2}. \quad (74) \end{aligned}$$

The regularized Helmholtz equation is produced using a process similar to the one described before, with the

exception that we use the scaling factor $\eta = 2$. Because we can see from the above that the nonzero endpoint of the domain's boundary is $\pi/2$, we choose $\eta = 2$, that is, the denominator of the nonzero endpoint of the domain's boundary. The following is the regularized equation:

$$\frac{\partial^2 w_{2,0}(x, y)}{\partial x^2} + \frac{\partial^2 w_{2,0}(x, y)}{\partial y^2} + (1 + \alpha^{2m})^{-1} e^{-m} k^2 w_{2,0}(x, y) = 0, 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \frac{\pi}{2},$$

$$w_{2,0}(x, 0) = 0, 0 \leq x \leq \frac{\pi}{2},$$

$$\frac{\partial w_{2,0}(x, (\pi/2))}{\partial y} = (1 + \alpha^{2m})^{-1} e^{-m} \frac{1}{n} \cos(2nx), 0 \leq x \leq \frac{\pi}{2},$$

$$\frac{\partial w_{2,0}(0, y)}{\partial x} = 0, 0 \leq y \leq \frac{\pi}{2},$$

$$w_{2,0}(\pi, y) = (1 + \alpha^{2m})^{-1} e^{-m} \frac{1}{n} \cos(2ny), 0 \leq y \leq \frac{\pi}{2}. \quad (75)$$

But if, $(1/n) \sin(nx)$, is imposed at boundary deflection in the Cauchy problem of the Helmholtz equation, then we choose $\eta = 4$ instead of 2.

Also, if $(1/n) \sin(nx)$ is imposed at the boundary deflection in the Cauchy problem of the Helmholtz equation, then we choose $\eta = 2m$, whereas when $(1/n) \cos(nx)$ is imposed, then we choose $\eta = m$, where m is the denominator of the nonzero endpoint of the boundary of the domain in the Cauchy boundary conditions. The DRM regularizes the ill-posed Helmholtz equation with a periodic function such as $\sin(nx)$ and $\cos(nx)$ imposed at boundary deflection in the Cauchy problem.

In regularizing Neumann problem of Helmholtz equation, we make use of a shifting operator $\tau_{1,0}$ of x coordinate of unknown function $w(x, y)$, regularization term $(1 + \alpha^{2m}) e^m$ and then apply Green's first identity. That is, the DRM solves

$$Aw(x, y) = f, \quad (76)$$

where the null space

$$N(A) = \{w : Aw(x, y) = 0\} \quad (77)$$

is not trivial.

Unlike other methods of regularization discussed in the literature, the new DRM regularizes the Cauchy problem of the Helmholtz equation where the boundary deflection is inhomogeneous.

Thus, equations (56), (57), (58), (59), (60) has at least a solution. The other properties follow.

4.2. Helmholtz Equation with Neumann Boundary Conditions in the Upper Half-Plane. We now show that the Neumann problem on the upper half-plane for the Helmholtz equation has a solution but is not unique. We provide the rigorous proof for our claim. We impose Neumann

boundary conditions on homogeneous Helmholtz equation on the upper half-plane as follows:

$$\frac{\partial^2 w(x, y)}{\partial x^2} + \frac{\partial^2 w(x, y)}{\partial y^2} + k^2 w(x, y) = 0, -1 \leq x \leq 1, 0 \leq y \leq 1,$$

$$\frac{\partial w(-1, y)}{\partial x} = \frac{\partial w(1, y)}{\partial x} = 0, 0 \leq y \leq 1,$$

$$\frac{\partial w(x, 0)}{\partial y} = 0 - 1 \leq x \leq 1,$$

$$\frac{\partial w(x, 1)}{\partial y} = \cos(2\pi x) - 1 \leq x \leq 1. \quad (78)$$

The solution is given by

$$w(x, y) = \begin{cases} \sum_{n=1,5,9,\dots}^{\infty} \frac{2n\pi \cdot \cosh\left(\sqrt{((n\pi/2)^2 - k^2)}y\right) (\cos(n\pi x/2) + \sin(n\pi x/2))}{(n^2 - 16)\pi^2 \cdot \sqrt{((n\pi/2)^2 - k^2)} \sinh\left(\sqrt{((n\pi/2)^2 - k^2)}\right)}, \\ \sum_{n=3,7,11,\dots}^{\infty} \frac{-2n\pi \cdot \cosh\left(\sqrt{((n\pi/2)^2 - k^2)}y\right) (\cos(n\pi x/2) + \sin(n\pi x/2))}{(n^2 - 16)\pi^2 \cdot \sqrt{((n\pi/2)^2 - k^2)} \sinh\left(\sqrt{((n\pi/2)^2 - k^2)}\right)}, \\ \sum_{n=1}^{\infty} \frac{\cosh\left(\sqrt{((n\pi)^2 - k^2)}y\right) (\cos(n\pi x) + \sin(n\pi x))}{2\sqrt{((n\pi)^2 - k^2)} \cdot \sinh\left(\sqrt{((n\pi)^2 - k^2)}\right)}. \end{cases} \quad (79)$$

We can see that all the coefficients of partial derivatives appearing in equation (80) are continuously differentiable, and the expression on the right hand side of the second equation in equation (80) is zero, which is continuous. Thus, equation (80) meets the smoothness requirement condition. In addition, we observe that all the boundary conditions are zero except $\partial w(x, 1)/\partial y = \cos(2\pi x)$. We can see that

$$\int_{-1}^1 \cos(2\pi x) dx = 0. \quad (80)$$

This implies that the function that appears in equation $w(x, y)$ is a solution of equation (80) However, it can easily be shown that the equation has more than one solution in $L^2([-1, 1] \times [0, 1])$. Hence, the equation is ill-posed in the sense of Hadamard. Using a similar technique as demonstrated above, the DRM can be used to regularize this equation.

5. Conclusion

The DRM, which regularizes both the Helmholtz equation and the imposed boundary conditions, was devised to solve the ill-posed Cauchy issue of the Helmholtz equation where the boundary deflection is inhomogeneous, as well as the Neumann problem. In the Cauchy boundary conditions, the DRM uses a positive integer scaler in the x spatial variable of the unknown function $w(x, y)$ to homogenize the inhomogeneous boundary deflection. The integer scaler, η , assures that the Helmholtz equation has a solution. In the Cauchy problem of the Helmholtz equation, the value of the scaler is

determined by the periodic function prescribed as well as the scalar multiplier of spatial variable imposed at inhomogeneous boundary deflection. The regularization term restores the Helmholtz equation's Cauchy problem's stability.

When $(1/n) \sin(nx)$ is imposed at boundary deflection in the Cauchy problem of the Helmholtz equation, we choose $\eta = 2n$, whereas when $(1/n) \cos(nx)$ is imposed, we choose $\eta = n$, where n is the denominator of the nonzero endpoint of the domain boundary in the Cauchy boundary conditions. In the Cauchy problem, the DRM regularizes ill-posed Helmholtz equations with periodic functions such as $\sin(nx)$ and $\cos(nx)$ imposed at boundary deflection.

In the Cauchy problem of the Helmholtz equation, the value of η is determined by periodic function functions such as $\sin(nx)$, $\cos(nx)$, and the scalar multiplier of the spatial variable (angle) imposed at inhomogeneous border deflection. The proposed method, DRM, depends on the domain of the imposed function on the inhomogeneous boundary deflection on the Helmholtz equation. The DRM regularizes Cauchy problem of the Helmholtz equation.

Data Availability

No available data for this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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