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Research Article

Coupled Fixed Point Theorems in Topological Spaces with Size Function Topology

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Regarding the concept of size function topology, this allows us to view the topological space X as a metric-like space. We prove the existence and uniqueness for a coupled fixed point of the map $F : X \times X \longrightarrow X$ satisfying some certain contractive conditions.

1. Introduction

The Banach contraction mapping theorem is the classical fixed point theorem saying that for a self-map $T : X \longrightarrow X$, if (X, d) is a complete metric space and there exists $\lambda \in (0, 1)$ such that

$$d(T(x), T(y)) \leq \lambda d(x, y), \quad \text{for all } x, y \in X, \tag{1}$$

then T admits a unique fixed point. This classical theorem has been applied to many problems in physics, optimization, and economics. There is no surprise why many researchers have attempted to extend the theorem to the general setting. One way is to replace the contraction condition the weaker contractive conditions (see [1–3]). Another approach is to give the alternative metric structure on the space (see [4–6]).

Later, Bhaskar and Lakshmikantham [7] established some coupled fixed point theorems for a map $F : X \times X \longrightarrow X$. They proved that if X is a partially ordered complete metric space and a map F has a mixed monotone property satisfying a certain contractive condition, then F admits a unique coupled fixed point. After this framework, many researchers have attempted to generalize their results (see [8–15]).

In 2017, Robdera [16] introduced the concept of size function topology. This concept allows us to view the topological space as a metric-like space. Our main goal is to establish some coupled fixed point theorems on the topological space equipped with the size function topology. First, we recall some standard notations and definitions for size function topologies given by Robdera.

Definition 1. Let (X, τ) be a topological space. A function $\delta : \tau \longrightarrow [0,+\infty]$ is called a topology size function or simply τ-size function if

(i) $U = \varnothing$, then $\delta(U) = 0$

(ii) $\max\{\delta(U), \delta(V)\} \leq \delta(U \cup V) \leq \delta(U) + \delta(V)$ for all $U, V \in \tau$

His idea behind this definition has arisen from the diameter of an open interval in the set of real numbers.

Example 2. Let $\mathbb{R}$ be the set of real number. For any open set $U \subseteq \mathbb{R}$, U can be uniquely decomposed as a disjoint union of open intervals (a_α, b_α), $U = \coprod_\alpha (a_\alpha, b_\alpha)$. Define

$$\delta(U) = \sup\{\operatorname{diam}(a_\alpha, b_\alpha) | \quad \text{for all } \alpha\}, \tag{2}$$

where $\operatorname{diam}(A) = \sup\{|x - y| | x, y \in A\}$ for any $A \subseteq \mathbb{R}$.

Then, δ is a topology size function on X.

Definition 3. Let (X, τ) be a topological space and δ be a τ-size function. For $a \in X$ and $r > 0$. We define a δ-ball centered at a of radius r to be

$$B_\delta(a;r) = \bigcup\{U \in \tau | a \in U \text{ and } \delta(U) < r\}. \quad (3)$$

We can verify that the collection of all δ-balls forms a basis for a topology on X. The topology generated by this basis is denoted by τ_δ. Clearly, $\tau_\delta \subseteq \tau$.

In the rest of this paper, we always assume that (X, τ) is a topological space, δ is a τ-size function, and $\mathbb{N}$ denote the set of positive integers.

Definition 4.

(i) A τ-size function δ is said to be k-uniform if for all $a \in X$ and $r > 0$, we have that $\delta(B_\delta(a;r)) \le rk$ for some $k > 1$. For example, the topology size function δ, given in Example 2, is a 2-uniform

(ii) A sequence (x_n) in X is said to δ converge to $x \in X$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \ge N$, $x_n \in B_\delta(x;\varepsilon)$. A point x is called a δ-limit of the sequence (x_n) and simply denoted by $\lim_{n\to\infty} x_n = x$

(iii) A sequence (x_n) in X is said to be δ-Cauchy if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n, m \ge N$, $x_n \in B_\delta(x_m;\varepsilon)$

(iv) A subset A of X is said to be δ complete if every δ -convergent sequence in $A\delta$ converges to a δ-limit in A. Denote the set of all δ-limits of A by $\bar{A}$

Remark 5 (see [16]).

(i) Every δ-convergent sequence is a δ-Cauchy sequence

(ii) Clearly, $A \subseteq \bar{A}$, and we obtain that A is δ-complete if and only if $A = \bar{A}$ if and only if every δ-Cauchy sequence δ converges in A

2. Main Results

2.1. Fixed Point Theorem. For a given topological space (X, τ) and a τ-size function δ. Let $T : X \longrightarrow X$ be a self-map.

Definition 6. A point $x \in X$ is said to be a fixed point of T if $T(x) = x$.

We denote the nth iteration of T by

$$\underbrace{T \circ T \circ T \circ \cdots \circ T}_{n-\text{copies}}(x) = T^n(x) \text{ for all } x \in X. \quad (4)$$

A function $\psi : [0,\infty) \longrightarrow [0,\infty)$ is called a contractant if it is increasing and $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t \in [0,\infty)$. Note that if ψ is a contractant, then $\psi(0) = 0$ and $\psi(t) < t$ for all $t \in (0,\infty)$.

For a subset A of X, define $\delta^*(A) = \sup\{\delta(U) | U \in \tau$ and $U \subset A\}$.

Definition 7. A map $T : X \longrightarrow X$ is called a δ-contraction if there exists a contractant $\psi : [0,\infty) \longrightarrow [0,\infty)$ such that

$$\delta^*\left(T(\bar{U})\right) < \psi(\delta(U)), \quad (5)$$

for all $U \in \tau$, $\delta(U)$ is finite.

The uniqueness for the fixed point of the map T is relied heavily on the following property of the τ-size function δ.

Definition 8. A τ-size function δ is said to separate points in X or simply X is δ separated if any two distinct points x and y in X, there exist $r_x, r_y > 0$ such that $B_\delta(x;r_x) \cap B_\delta(y;r_y) = \varnothing$.

Example 9. Consider the discrete space $(X, P(X))$ where $P(X)$ is the power set of X. Define a topology size function $\delta : P(X) \longrightarrow [0,\infty]$ by

$$\delta(A) = |A| \text{ for all } A \in P(X), \quad (6)$$

where $|A|$ is the number of elements in A. Clearly, X is δ separated.

Observe that if we consider $(X, P(X))$ as the discrete metric space, then we have that $\operatorname{diam}(\{x\}) = 0$, but $\delta(B_\delta(x;2)) = \delta(\{x\}) = 1$ for all $x \in X$.

Robdera proved the extended Matkowski fixed point theorem as follows.

Theorem 10 (Robdera M. Theorem 3.1). *Let (X, τ) be a topological space and δ be a k-uniform τ-size function. Let $T : X \longrightarrow X$ be a δ-contraction. Assume that X is δ complete, δ separated, and $\delta(X)$ is finite. Then, T admits a unique fixed point.*

Based on the work of Robdera, the fixed point results were obtained by the fact that the size of the domain $\delta(X)$ is finite. In this work, we weaken this condition by introducing the notion of locally finite size as follows.

Definition 11. Let (X, τ) be a topological space and δ be a τ -size function. We say that X is δ locally finite size if for each $x \in X$, there exists $U \in \tau$ such that $x \in U$ and $\delta(U)$ is finite.

We can verify that if $\delta(X)$ is finite, then X is δ locally finite size. The discrete space $(X, P(X))$ in Example 9 is an example of a topological size function space which is locally finite size.

Theorem 12. *Let (X, τ) be a topological space and δ be a k -uniform τ-size function. Let $T : X \longrightarrow X$ be a δ-contraction. Assume that X is δ complete, δ separated, and δ locally finite size. Then, T admits a unique fixed point.*

Proof. Let $\psi : [0,\infty) \longrightarrow [0,\infty)$ be a contractant and x_0 be an arbitrary point in X. Then, we define a sequence (x_n) in X by setting $x_n = T^n(x_0)$.

First, we will show that the sequence (x_n) is δ-Cauchy. Since X is δ locally finite size, there exists $U \in \tau$ such that $x_0, T(x_0) \in U$ and $\delta(U)$ is finite, which says $\delta(U) = \delta_0$.

Hence, we have that $T(x_0), T^2(x_0) \in T(U)$ and $\delta^*(T(U)) \leq \delta^*(T(\bar{U})) < \psi(\delta(U)) = \psi(\delta_0)$. This implies that there exists $U_1 \in \tau$ such that $T(x_0), T^2(x_0) \in U_1 \subseteq T(U)$ and $\delta(U_1) < \psi(\delta_0)$.

Then, $T^2(x_0), T^3(x_0) \in T(U_1) \subseteq T^2(U)$ and $\delta^*(T(U_1)) \leq \delta^*(T(\bar{U}_1)) < \psi^2(\delta(U)) = \psi^2(\delta_0)$.

Continuing in this way, we obtain that for $n \in \mathbb{N}$, $T^n(x_0)$, $T^{n+1}(x_0) \in T^n(U)$ and $\delta^*(T^n(U)) < \psi^n(\delta_0)$.

Given $\varepsilon > 0$, then $\varepsilon > \psi(\varepsilon) > 0$. Since $\psi^n(\delta_0) \longrightarrow 0$ as $n \longrightarrow \infty$, there exists $N \in \mathbb{N}$ such that for $n \geq N$, we have $\delta^*(T^n(U)) < \varepsilon - \psi(\varepsilon)$ and $\psi^n(\delta_0) < \psi(\varepsilon)$.

Hence, for $n, m \geq N$, we have that $T^n(x_0)$ and $T^m(x_0)$ are in $T^n(U) \cup T^m(U)$ and

$$\begin{aligned} \delta^*(T^n(U) \cup T^m(U)) &\leq \delta^*(T^n(U)) + \delta^*(T^m(U)) \\ &\leq \psi^n(\delta_0) + \varepsilon - \psi(\varepsilon) < \varepsilon. \end{aligned} \tag{7}$$

This implies that $T^m(x_0) \in B_\delta(T^n(x_0); \varepsilon)$. Thus, $(x_n) = (T^n(x_0))$ is δ-Cauchy. Since X is δ complete, there exists $x_\infty \in X$ such that $(T^n(x_0))\delta$ converges to x_∞.

Next, we will show that $T(x_\infty) = x_\infty$. Given $\varepsilon > 0$, since T is continuous (see [11]), there exists $r > 0$ such that $x \in B_\delta(x_\infty; r)$ implies that $T(x) \in B_\delta(T(x_\infty); \varepsilon/2)$. Setting $\eta = \min\{\varepsilon/2, r/2\}$. Since $T^n(x_\infty) \longrightarrow x_\infty$ as $n \longrightarrow \infty$, there exists $M \in \mathbb{N}$ such that for $n \geq M$, $T^n(x_\infty) \in B_\delta(x_\infty; \eta)$.

In particular, $T^M(x_\infty) \in B_\delta(x_\infty; \eta)$ implies that there exists $U' \in \tau$ such that $T^M(x_\infty), x\infty \in U'$ and $\delta(U') < \eta$. By the continuity of T, we have that $T^{M+1}(x_\infty), T(x_\infty) \in T(U')$ and $\delta^*(T(U')) < \varepsilon/2$.

Hence, x_∞, $T(x_\infty) \in U' \cup T(U')$ and

$$\delta^*\left(U' \cup T\left(U'\right)\right) \leq \delta^*\left(U'\right) + \delta^*\left(T\left(U'\right)\right) \leq \eta + \frac{\varepsilon}{2} < \varepsilon. \tag{8}$$

Therefore, $T(x_\infty) \in B_\delta(x_\infty; \varepsilon)$ for every $\varepsilon > 0$. Hence, $T(x_\infty) = x_\infty$.

Finally, we will show that x_∞ is the unique fixed point of T. Assume that there exists $y_\infty \in X$ such that $T(y_\infty) = y_\infty$. Suppose that $x_\infty \neq y_\infty$. Since X is δ separated, there exists $\varepsilon > 0$ such that $B_\delta(x_\infty; \varepsilon) \cap B_\delta(y_\infty; \varepsilon) = \varnothing$. Obviously, $y_\infty \notin B_\delta(x_\infty; \varepsilon)$. Observe that $T(x_\infty), T(y_\infty) \in T(B_\delta(x_\infty; \varepsilon/2k)) \cup T(B_\delta(y_\infty; \varepsilon/2k))$ and

$$\begin{aligned} &\delta^*\left(T\left(B_\delta\left(x_\infty; \frac{\varepsilon}{2k}\right)\right) \cup T\left(B_\delta\left(y_\infty; \frac{\varepsilon}{2k}\right)\right)\right) \\ &\quad \leq \delta^*\left(T\left(B_\delta\left(x_\infty; \frac{\varepsilon}{2k}\right)\right)\right) + \delta^*\left(T\left(B_\delta\left(y_\infty; \frac{\varepsilon}{2k}\right)\right)\right) \\ &\quad < \psi\left(\delta\left(B_\delta\left(x_\infty; \frac{\varepsilon}{2k}\right)\right)\right) + \psi\left(\delta\left(B_\delta\left(y_\infty; \frac{\varepsilon}{2k}\right)\right)\right) \\ &\quad < \psi\left(\frac{\varepsilon}{2}\right) + \psi\left(\frac{\varepsilon}{2}\right) < \varepsilon. \end{aligned} \tag{9}$$

Hence, $y_\infty = T(y_\infty) \in B_\delta(x_\infty; \varepsilon)$ which is a contradiction. Therefore, $x_\infty = y_\infty$. The proof is complete. □

Notice that the map $\psi : [0,\infty) \longrightarrow [0,\infty)$ is defined by $\psi(t) = \alpha t$, where $\alpha \in (0, 1)$ is a contractant. Immediately, we obtain the extension of the Banach contraction mapping theorem as follows.

Corollary 13. *Let (X, τ) be a topological space and δ be a k-uniform τ-size function. Assume that there exists $\alpha \in (0, 1)$ such that*

$$\delta^*\left(T(\bar{U})\right) < \alpha\delta(U), \tag{10}$$

for all $U \in \tau$, $\delta(U)$ is finite. If X is δ complete, δ separated, and δ locally finite size, then T admits a unique fixed point.

2.2. Coupled Fixed Point Theorem. Let (X, τ) be a topological space and δ be a τ-size function. Let $F : X \times X \longrightarrow X$ be a map.

Definition 14.

(i) A point $(x, y) \in X \times X$ is called a coupled fixed point of F if $F(x, y) = x$ and $F(y, x) = y$.

(ii) A point $(x, x) \in X \times X$ is called a fixed point of F if $F(x, x) = x$. Clearly, a fixed point of F is a coupled fixed point of .

Definition 15. A map $F : X \times X \longrightarrow X$ is said to satisfy the product contraction I if there exists a contractant $\psi : [0,\infty) \longrightarrow [0,\infty)$ such that

$$\delta^*\left(F(\overline{U \times V})\right) < \psi(\min\{\delta(U), \delta(V)\}), \tag{11}$$

for all $U, V \in \tau$, $\delta(U)$ and $\delta(V)$ are finite.

Example 16. Let $X = [0,\infty) \subset \mathbb{R}$. Let τ be the topology on X given by

$$\tau = \{\varnothing, X, [0, a) | a > 0\}. \tag{12}$$

Define the topology size function δ by $\delta(\varnothing) = 0$, $\delta(X) = \infty$, and $\delta([0, a)) = a$.

Let $F : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ be defined by

$$F(x, y) = 1, \tag{13}$$

for all $x, y \in [0,\infty)$.

Then, for $a, b > 0$, we have that for any contractant $\psi : [0,\infty) \longrightarrow [0,\infty)$

$$\begin{aligned} 0 &= \delta^*\left(F(\overline{[0, a) \times [0, b)})\right) \\ &< \psi(\min\{\delta([0, a)), \delta([0, b))\}) \\ &= \psi(\min\{a, b\}). \end{aligned} \tag{14}$$

Therefore, F satisfies the product contraction I. Observe that F admits a unique coupled fixed point $\{(1,1)\}$.

Remark 17. Observe that we can naturally define the product topology on $X \times X$ inherited from X. More precisely, the basis element is of the form $U \times V$ where $U, V \in \tau_\delta$. But it is still questionable to define the size function corresponding to this product topology in term of δ, because the union of two basis elements might not be a product form. Nevertheless, the following lemma can be obtained by the product contraction I.

Lemma 18. *Let (X,τ) be a topological space and δ be a τ-size function. Let $F : X \times X \longrightarrow X$ be a map satisfying the product contraction I. Then, for every $(x,y) \in X \times X$ and $\varepsilon > 0$, there exists $r > 0$ such that for $U, V \in \tau$, if $x \in U, y \in V$ and* min *$\{\delta(U), \delta(V)\} < r$, then $\delta^*(F(U \times V)) < \varepsilon$.*

Proof. Let $(x,y) \in X \times X$ and $\varepsilon > 0$. Choose $r = \varepsilon$. Then, for $U, V \in \tau$, if $x \in U, y \in V$ and $\min\{\delta(U), \delta(V)\} < r$, we have that $F(x,y) \in F(U \times V)$ and

$$\begin{aligned}\delta^*(F(U \times V)) &\le \delta^*\left(F(\overline{U \times V})\right)\\ &< \psi(\min\{\delta(U), \delta(V)\})\\ &< \psi(\varepsilon) < \varepsilon.\end{aligned} \tag{15}$$

□

Theorem 19. *Let (X,τ) be a topological space and δ be a k-uniform τ-size function. Let $F : X \times X \longrightarrow X$ be a map satisfying the product contraction I. Assume that X is δ complete, δ separated, and δ locally finite size. Then, F has a unique coupled fixed point.*

Proof. Let $\psi : [0,\infty) \longrightarrow [0,\infty)$ be a contractant and choose an arbitrary point $(x_0, y_0) \in X \times X$. Since X is δ locally finite size, there exist $U_0, V_0 \in \tau$ such that $x_0 \in U_0, y_0 \in V_0, \delta(U_0)$ and $\delta(V_0)$ are finite. Setting $\delta(U_0) = \alpha_0$ and $\delta(V_0) = \beta_0$.

We inductively define two sequences in X as follows:

$$\begin{gathered}x_1 = F(x_0, y_0), y_1 = F(y_0, x_0),\\ x_n = F(x_{n-1}, y_{n-1}), y_n = F(y_{n-1}, x_{n-1}), \text{for all } n \ge 2.\end{gathered} \tag{16}$$

Next, we will show that both (x_n) and (y_n) are δ-Cauchy. Clearly, $x_1 \in F(U_0 \times V_0)$ and $\delta^*(F(U_0 \times V_0)) < \psi(\alpha_0)$. Similarly, $y_1 \in F(V_0 \times U_0)$ and $\delta^*(F(V_0 \times U_0)) < \psi(\beta_0)$.

Hence, there exist $U_1, V_1 \in \tau$ such that

$$\begin{gathered}x_1 \in U_1 \subseteq F(U_0 \times V_0),\\ \delta(U_1) < \psi(\alpha_0),\\ y_1 \in V_1 \subseteq F(V_0 \times U_0),\\ \delta(V_1) < \psi(\beta_0).\end{gathered} \tag{17}$$

We obtain that $x_2 = F(x_1, y_1) \in F(U_1 \times V_1)$ and

$$\delta^*(F(U_1 \times V_1)) < \psi(\min\{\delta(U_1), \delta(V_1)\}) < \psi(\delta(U_1)) < \psi^2(\alpha_0). \tag{18}$$

Similarly, $y_2 \in F(V_1 \times U_1)$ and $\delta^*(F(V_1 \times U_1)) < \psi^2(\beta_0)$.

Inductively, we obtain that for $n \in \mathbb{N}$ there exist U_{n-1}, $V_{n-1} \in \tau$ such that

$$\begin{gathered}x_n \in F(U_{n-1} \times V_{n-1}), \delta^*(F(U_{n-1} \times V_{n-1})) < \psi^n(\alpha_0),\\ y_n \in F(V_{n-1} \times U_{n-1}),\end{gathered} \tag{19}$$

$\delta^*(F(V_{n-1} \times U_{n-1})) < \psi^n(\beta_0)$.

Given $\varepsilon > 0$, then $\psi(\varepsilon) < \varepsilon$ or $\varepsilon - \psi(\varepsilon) > 0$. Since $\lim_{n\longrightarrow\infty}\psi^n(\alpha_0) = 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that for $n \ge N_1$ implies that $\psi^n(\alpha_0) < \psi(\varepsilon)$ and for $n \ge N_2$ implies that $\delta^*(F(U_{n-1} \times V_{n-1})) < \varepsilon - \psi(\varepsilon)$.

For $n, m \ge N = \max\{N_1, N_2\}$, we have that $x_n, x_m \in F(U_{n-1} \times V_{n-1}) \cup F(U_{m-1} \times V_{m-1})$ and

$$\begin{aligned}&\delta^*(F(U_{n-1} \times V_{n-1}) \cup F(U_{m-1} \times V_{m-1}))\\ &\quad \le \delta^*(F(U_{n-1} \times V_{n-1})) + \delta^*(F(U_{m-1} \times V_{m-1}))\\ &\quad < \varepsilon - \psi(\varepsilon) + \psi^m(\alpha_0) < \varepsilon.\end{aligned} \tag{20}$$

This means that $x_n \in B_\delta(x_m; \varepsilon)$, so (x_n) is δ-Cauchy. By the analogous argument, we can show that (y_n) is also δ-Cauchy.

By the δ-completeness of X, there exist x_∞ and y_∞ in X such that $\lim_{n\longrightarrow\infty}x_n = x_\infty$ and $\lim_{n\longrightarrow\infty}y_n = y_\infty$.

Next, we will show that (x_∞, y_∞) is a coupled fixed point or $F(x_\infty, y_\infty) = x_\infty$ and $F(y_\infty, x_\infty) = y_\infty$.

Given $\varepsilon > 0$. By Lemma 18, there is $r > 0$ such that for $U, V \in \tau$ if $(x_\infty, y_\infty) \in U \times V$ and $\min\{\delta(U), \delta(V)\} < r$, then $\delta^*(F(U \times V)) \le \varepsilon/2$.

Since $\lim_{n\longrightarrow\infty}x_n = x_\infty$ and $\lim_{n\longrightarrow\infty}y_n = y_\infty$, there exist $M_1, M_2 \in \mathbb{N}$ such that

$$\begin{gathered}n > M_1 \text{ implies that } x_n \in B_\delta(x_\infty; \eta),\\ n > M_2 \text{ implies that } y_n \in B_\delta(y_\infty; \eta),\end{gathered} \tag{21}$$

where $\eta = \min\{\varepsilon/2, r/2\}$.

In particular, there exist $U', V' \in \tau$ such that $x_{M_1}, x_\infty \in U'$, $y_{M_2}, y_\infty \in V'$, and $\delta(U'), \delta(V') < \eta \le r/2$. Then, $\min\{\delta(U'), \delta(V')\} < r$.

Hence, we have that $F(x_\infty, y_\infty) \in F(U' \times V')$ and $\delta^*(F(U' \times V')) < \varepsilon/2$.

Thus, $x_\infty, F(x_\infty, y_\infty) \in U' \cup F(U' \times V')$ and

$$\delta^*\left(U' \cup F\left(U' \times V'\right)\right) \le \delta^*\left(U'\right) + \delta^*\left(F\left(U' \times V'\right)\right) < \eta + \frac{\varepsilon}{2} < \varepsilon. \tag{22}$$

That is $F(x_\infty, y_\infty) \in B_\delta(x_\infty; \varepsilon)$ for all $\varepsilon > 0$. Hence, $F(x_\infty, y_\infty) = x_\infty$. Similarly, we can show that $F(y_\infty, x_\infty) = y_\infty$.

Finally, we will show that (x_∞, y_∞) is the unique coupled fixed point of F. Suppose that there is $(x'_\infty, y'_\infty) \in X \times X$ such that $F(x'_\infty, y'_\infty) = x'_\infty$ and $F(y'_\infty, x'_\infty) = y'_\infty$. Suppose that $(x_\infty, y_\infty) \neq (x'_\infty, y'_\infty)$. Then, $x_\infty \neq x'_\infty$ or $y_\infty \neq y'_\infty$.

Assume that $x_\infty \neq x'_\infty$. Since X is δ separated and k-uniform, there exists $\varepsilon > 0$ such that $B_\delta(x_\infty; \varepsilon) \cap B_\delta(x'_\infty; \varepsilon) = \varnothing$. Clearly, $x'_\infty \notin B_\delta(x_\infty; \varepsilon)$. Since X is δ locally finite size, there exist $V, V' \in \tau$ such that $y_\infty \in V, y'_\infty \in V'$ and $\delta(V), \delta(V')$ are finite. We have that $F(x_\infty, y_\infty), F(x'_\infty, y'_\infty) \in F(B_\delta(x_\infty; \varepsilon/2k) \times V) \cup F(B_\delta(x'_\infty; \varepsilon/2k) \times V')$ and

$$\begin{aligned} &\delta^*\left(F\left(B_\delta\left(x_\infty; \frac{\varepsilon}{2k}\right) \times V\right) \cup F\left(B_\delta\left(x'_\infty; \frac{\varepsilon}{2k}\right) \times V'\right)\right) \\ &\quad \leq \delta^*\left(F\left(B_\delta\left(x_\infty; \frac{\varepsilon}{2k}\right) \times V\right)\right) + \delta^*\left(F\left(B_\delta\left(x'_\infty; \frac{\varepsilon}{2k}\right) \times V'\right)\right) \\ &\quad < \psi\left(\frac{\varepsilon}{2}\right) + \psi\left(\frac{\varepsilon}{2}\right) < \varepsilon. \end{aligned} \tag{23}$$

This implies that $x'_\infty = F(x'_\infty, y'_\infty) \in B_\delta(x_\infty; \varepsilon)$ which is a contradiction.

Similarly, if we assume that $y_\infty \neq y'_\infty$, then it will also lead to the contradiction. Therefore, $(x_\infty, y_\infty) = (x'_\infty, y'_\infty)$. The proof is complete. □

Remark 20. Observe that in the proof of Theorem 19, we can replace the product contraction I by the weaker condition:

$$\delta^*\left(F(\bar{U} \times V)\right) < \frac{1}{2}[\psi(\delta(U)) + \psi(\delta(V))] \text{ for all } U, V \in \tau, \tag{24}$$

if the contractant ψ is assumed to be a linear function. By the Cauchy functional equation, this implies that $\psi(t) = \lambda t$ for some $0 < \lambda < 1$.

Definition 21. A map $F : X \times X \longrightarrow X$ is said to satisfy the product contraction II if there exists $\lambda \in (0, 1)$ such that

$$\delta^*\left(F(\bar{U} \times V)\right) \leq \frac{\lambda}{2}(\delta(U) + \delta(V)), \tag{25}$$

for all $U, V \in \tau$, $\delta(U)$ and $\delta(V)$ are finite.

Example 22. Let $X = [0,\infty) \subset \mathbb{R}$. Let τ be the topology on X given by

$$\tau = \{\varnothing, X, [0, a) | a > 0\}. \tag{26}$$

Define the topology size function δ by $\delta(\varnothing) = 0, \delta(X) = \infty$ and $\delta([0, a)) = a$.

Let $F : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ be defined by

$$F(x, y) = \frac{x + y}{6}, \tag{27}$$

for all $x, y \in [0,\infty)$.

Then, for $a, b > 0$,

$$\frac{a + b}{6} = \delta^*\left(F(\overline{[0, a)} \times [0, b))\right) \leq \frac{\lambda}{2}(\delta([0, a)) + \delta([0, b))) = \frac{\lambda}{2}(a + b), \tag{28}$$

where $\lambda = 1/3$.

Therefore, F satisfies the product contraction II. Observe that F has a unique coupled fixed point $\{(0, 0)\}$.

Before we prove the next main theorem, the following lemma is required.

Lemma 23. *Let (X, τ) be a topological space and δ be a τ-size function. Let $F : X \times X \longrightarrow X$ be a map satisfying the product contraction II. Then, for every $(x, y) \in X \times X$ and $\varepsilon > 0$, there exists $r > 0$ such that for $U, V \in \tau$, if $x \in U, y \in V$ and $\delta(U) + \delta(V) < r$, then $\delta^*(F(U \times V)) < \varepsilon$.*

Proof. Let $(x, y) \in X \times X$ and $\varepsilon > 0$. Choose $r = 2\varepsilon/\lambda$. Then, for $U, V \in \tau$, if $x \in U, y \in V$ and $\delta(U) + \delta(V) < r$, we have that $F(x, y) \in F(U \times V)$ and

$$\delta^*(F(U \times V)) \leq \delta^*\left(F(\bar{U} \times V)\right) < \frac{\lambda}{2}(\delta(U) + \delta(V)) < \frac{\lambda}{2}\frac{2\varepsilon}{\lambda} = \varepsilon. \tag{29}$$

□

Theorem 24. *Let (X, τ) be a topological space and δ be a k-uniform τ-size function. Let $F : X \times X \longrightarrow X$ be a map satisfying the product contraction II. If X is δ complete, δ separated, and δ locally finite size. Then, F admits a unique coupled fixed point.*

Proof. Let $(x_0, y_0) \in X \times X$ be an arbitrary point. Since X is δ locally finite size, there exist $U_0, V_0 \in \tau$ such that $x_0 \in U_0$, $y_0 \in V_0, \delta(U_0)$ and $\delta(V_0)$ are finite. Setting $\delta(U_0) = \alpha_0$ and $\delta(V_0) = \beta_0$ and let $\alpha_0 + \beta_0 = \gamma_0$.

We inductively define two sequences in X as follows:

$$\begin{gathered} x_1 = F(x_0, y_0), y_1 = F(y_0, x_0), \\ x_n = F(x_{n-1}, y_{n-1}), y_n = F(y_{n-1}, x_{n-1}) \text{ for all } n \geq 2. \end{gathered} \tag{30}$$

Next, we will show that both (x_n) and (y_n) are δ-Cauchy. Clearly, $x_1 \in F(U_0 \times V_0)$ and $\delta^*(F(U_0 \times V_0)) < \lambda/2(\gamma_0)$. Similarly, $y_1 \in F(V_0 \times U_0)$ and $\delta^*(F(V_0 \times U_0)) < \lambda/2(\gamma_0)$.

Hence, there exist $U_1, V_1 \in \tau$ such that

$$\begin{aligned} x_1 &\in U_1 \subseteq F(U_0 \times V_0), \\ \delta(U_1) &< \frac{\lambda}{2}(\gamma_0), \\ y_1 &\in V_1 \subseteq F(V_0 \times U_0), \\ \delta(V_1) &< \frac{\lambda}{2}(\gamma_0). \end{aligned} \tag{31}$$

We obtain that $x_2 = F(x_1, y_1) \in F(U_1 \times V_1)$ and

$$\delta^*(F(U_1 \times V_1)) \leq \delta^*\left(F(\overline{U_1 \times V_1})\right) = (\lambda/2)^2\gamma_0. \tag{32}$$

Similarly, $y_2 \in F(V_1 \times U_1)$ and $\delta^*(F(V_1 \times U_1)) < (\lambda/2)^2\gamma_0$.

Continuing in this manner, we obtain that for $n \in \mathbb{N}$ there exist $U_{n-1}, V_{n-1} \in \tau$ such that

$$\begin{aligned} x_n &\in F(U_{n-1} \times V_{n-1}), \delta^*(F(U_{n-1} \times V_{n-1})) < \left(\frac{\lambda}{2}\right)^n \gamma_0, \\ y_n &\in F(V_{n-1} \times U_{n-1}), \\ \delta^*&(F(V_{n-1} \times U_{n-1})) < \left(\frac{\lambda}{2}\right)^n \gamma_0. \end{aligned} \tag{33}$$

Given $\varepsilon > 0$, then $\lambda\varepsilon < \varepsilon$ or $\varepsilon(1-\lambda) > 0$. Since $\lim_{n\longrightarrow\infty}(\lambda/2)^n\gamma_0 = 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that for $n \geq N_1$ implies that $(\lambda/2)^n\gamma_0 < \lambda\varepsilon$ and for $n \geq N_2$ implies that $\delta^*(F(U_{n-1} \times V_{n-1})) < \varepsilon(1-\lambda)$.

For $n, m \geq N = \max\{N_1, N_2\}$, we have that $x_n, x_m \in F(U_{n-1} \times V_{n-1}) \cup F(U_{m-1} \times V_{m-1})$ and

$$\begin{aligned} &\delta^*(F(U_{n-1} \times V_{n-1}) \cup F(U_{m-1} \times V_{m-1})) \\ &\quad \leq \delta^*(F(U_{n-1} \times V_{n-1})) + \delta^*(F(U_{m-1} \times V_{m-1})) \\ &\quad < \varepsilon(1-\lambda) + \left(\frac{\lambda}{2}\right)^m \gamma_0 < \varepsilon. \end{aligned} \tag{34}$$

This means $x_n \in B_\delta(x_m; \varepsilon)$, so (x_n) is δ-Cauchy. By the similar argument, we can show that (y_n) is also δ-Cauchy.

By the δ-completeness of X, there exist x_∞ and y_∞ in X such that $\lim_{n\longrightarrow\infty} x_n = x_\infty$ and $\lim_{n\longrightarrow\infty} y_n = y_\infty$.

Next, we will show that (x_∞, y_∞) is a coupled fixed point or $F(x_\infty, y_\infty) = x_\infty$ and $F(y_\infty, x_\infty) = y_\infty$.

Given $\varepsilon > 0$, by Lemma 23, there is $r > 0$ such that for $U, V \in \tau$ if $(x_\infty, y_\infty) \in U \times V$ and $\delta(U) + \delta(V) < r$, then $\delta^*(F(U \times V)) < \varepsilon/2$.

Since $\lim_{n\longrightarrow\infty} x_n = x_\infty$ and $\lim_{n\longrightarrow\infty} y_n = y_\infty$, there exist $M_1, M_2 \in \mathbb{N}$ such that

$$\begin{aligned} n > M_1 &\text{ implies that } x_n \in B_\delta(x_\infty; \eta), \\ n > M_2 &\text{ implies that } y_n \in B_\delta(y_\infty; \eta), \end{aligned} \tag{35}$$

where $\eta = \min\{\varepsilon/2, r/2\}$.

In particular, there exist $U', V' \in \tau$ such that $x_{M_1}, x_\infty \in U'$, $y_{M_2}, y_\infty \in V'$, and $\delta(U'), \delta(V') < \eta \leq r/2$. Then, $\delta(U') + \delta(V') < r$.

Hence, we have that $F(x_\infty, y_\infty) \in F(U' \times V')$ and $\delta^*(F(U' \times V')) < \varepsilon/2$.

Thus, $x_\infty, F(x_\infty, y_\infty) \in U' \cup F(U' \times V')$ and

$$\delta^*\left(U' \cup F\left(U' \times V'\right)\right) \leq \delta^*\left(U'\right) + \delta^*\left(F\left(U' \times V'\right)\right) < \eta + \frac{\varepsilon}{2} < \varepsilon. \tag{36}$$

That is $F(x_\infty, y_\infty) \in B_\delta(x_\infty; \varepsilon)$ for all $\varepsilon > 0$. Hence, $F(x_\infty, y_\infty) = x_\infty$. Similarly, we can show that $F(y_\infty, x_\infty) = y_\infty$.

Finally, we will show that (x_∞, y_∞) is the unique coupled fixed point of F. Suppose that there is $(x'_\infty, y'_\infty) \in X \times X$ such that $F(x'_\infty, y'_\infty) = x'_\infty$ and $F(y'_\infty, x'_\infty) = y'_\infty$. Suppose that $(x_\infty, y_\infty) \neq (x'_\infty, y'_\infty)$. Then, $x_\infty \neq x'_\infty$ or $y_\infty \neq y'_\infty$.

Assume that $x_\infty \neq x'_\infty$. Since X is δ separated and k-uniform, there exists $\varepsilon_1 > 0$ such that $B_\delta(x_\infty; \varepsilon_1) \cap B_\delta(x'_\infty; \varepsilon_1) = \emptyset$. Clearly, $x'_\infty \notin B_\delta(x_\infty; \varepsilon_1)$. □

Case 1. If $y_\infty \neq y'_\infty$, then there exists $\varepsilon_2 > 0$ such that $B_\delta(y_\infty; \varepsilon_2) \cap B_\delta(y'_\infty; \varepsilon_2) = \emptyset$. Setting $\varepsilon_3 = \min\{\varepsilon_1, \varepsilon_2\}$. We have that $F(x_\infty, y_\infty), F(x'_\infty, y'_\infty) \in F(B_\delta(x_\infty; \varepsilon_3/4k) \times B_\delta(y_\infty; \varepsilon_3/4k)) \cup F(B_\delta(x'_\infty; \varepsilon_3/4k) \times B_\delta(y'_\infty; \varepsilon_3/4k))$ and

$$\begin{aligned} &\delta^*\left(F\left(B_\delta\left(x_\infty; \frac{\varepsilon_3}{4k}\right) \times B_\delta\left(y_\infty; \frac{\varepsilon_3}{4k}\right)\right) \cup F\left(B_\delta\left(x'_\infty; \frac{\varepsilon_3}{2k}\right)\right) \times B_\delta\left(y'_\infty; \frac{\varepsilon_3}{4k}\right)\right)\Big) \\ &\leq \delta^*\left(F\left(B_\delta\left(x_\infty; \frac{\varepsilon_3}{4k}\right)\right) \times B_\delta\left(y_\infty; \frac{\varepsilon_3}{4k}\right)\right)\Big) \\ &\quad + \delta^*\left(F\left(B_\delta\left(x'_\infty; \frac{\varepsilon_3}{4k}\right)\right) \times B_\delta\left(y'_\infty; \frac{\varepsilon_3}{4k}\right)\right)\Big) \\ &< \frac{\lambda}{2}\left(\delta\left(B_\delta\left(x_\infty; \frac{\varepsilon_3}{4k}\right)\right) + \delta\left(B_\delta\left(y_\infty; \frac{\varepsilon_3}{4k}\right)\right)\right) \\ &\quad + \frac{\lambda}{2}\left(\delta\left(B_\delta\left(x'_\infty; \frac{\varepsilon_3}{4k}\right)\right) + \delta\left(B_\delta\left(y'_\infty; \frac{\varepsilon_3}{4k}\right)\right)\right) \\ &< \frac{\lambda}{2}\left[\frac{\varepsilon_3}{2} + \frac{\varepsilon_3}{2}\right] < \varepsilon_3. \end{aligned} \tag{37}$$

This implies that $x'_\infty = F(x'_\infty, y'_\infty) \in B_\delta(x_\infty; \varepsilon_3) \subset B_\delta(x_\infty; \varepsilon_1)$ which is a contradiction.

Case 2. If $y_\infty = y'_\infty$, then there exists $V \in \tau$ such that $y_\infty = y'_\infty \in V$ and $\delta(V)$ is finite. Setting $\varepsilon_4 = \min\{\varepsilon_1, \delta(V)\}$. We have that $F(x_\infty, y_\infty), F(x'_\infty, y'_\infty) \in F(B_\delta(x_\infty; \varepsilon_4/4k) \times B_\delta(y_\infty; \varepsilon_4/4k)) \cup F(B_\delta(x'_\infty; \varepsilon_4/4k) \times B_\delta(y'_\infty; \varepsilon_4/4k))$ and we can similarly show that

$$\begin{aligned} &\delta^*\left(F\left(B_\delta\left(x_\infty; \frac{\varepsilon_4}{4k}\right) \times B_\delta\left(y_\infty; \frac{\varepsilon_4}{4k}\right)\right) \cup F\left(B_\delta\left(x'_\infty; \frac{\varepsilon_4}{4k}\right)\right)\right. \\ &\quad \left.\times B_\delta\left(y'_\infty; \frac{\varepsilon_4}{4k}\right)\right)\Big) < \varepsilon_4. \end{aligned} \tag{38}$$

This implies that $x'_\infty = F(x'_\infty, y'_\infty) \in B_\delta(x_\infty; \varepsilon_4) \subset B_\delta(x_\infty; \varepsilon_1)$ which is a contradiction.

Similarly, if we assume that $y_\infty \neq y'_\infty$, then it will also lead to the contradiction. Hence, $(x_\infty, y_\infty) = (x'_\infty, y'_\infty)$. The proof is complete.

If space X failed to be separated by the topology size function δ, then the map F might not have a unique coupled fixed point.

Example 25. Consider the closed interval $[0, 1]$. Let τ be the topology on $[0, 1]$ given by

$$\tau = \{\varnothing, [0, 1], [0, a) \mid a < 1\}. \tag{39}$$

Define the topology size function δ by $\delta(\varnothing) = 0$, $\delta([0, 1]) = 1$, and $\delta([0, a)) = a$. Clearly, δ does not separate the points on $[0, 1]$.

Let $F : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ be defined by

$$F(x, y) = x^2 + y^2, \tag{40}$$

for all $x, y \in [0, 1]$. We obtain that F satisfies the product contraction II. The coupled fixed points of F are $(0, 0)$ and $(1/2, 1/2)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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