

## Research Article

# Projections in Moduli Spaces of the Kleinian Groups

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A two-generator Kleinian group  $\langle f, g \rangle$  can be naturally associated with a discrete group  $\langle f, \phi \rangle$  with the generator  $\phi$  of order two and where  $\langle f, \phi f \phi^{-1} \rangle = \langle f, g f g^{-1} \rangle \subset \langle f, g \rangle$ ,  $[\langle f, \phi \rangle : \langle f, g f g^{-1} \rangle] = 2$ . This is useful in studying the geometry of the Kleinian groups since  $\langle f, g \rangle$  will be discrete only if  $\langle f, \phi \rangle$  is, and the moduli space of groups  $\langle f, \phi \rangle$  is one complex dimension less. This gives a necessary condition in a simpler space to determine the discreteness of  $\langle f, g \rangle$ . The dimension reduction here is realised by a projection of principal characters of the two-generator Kleinian groups. In applications, it is important to know that the image of the moduli space of Kleinian groups under this projection is closed and, among other results, we show how this follows from Jørgensen's results on algebraic convergence.

## 1. Introduction

It is shown in [1] that every two-generator Kleinian group  $f, g$  is determined uniquely up to conjugacy by its *principal character*, i.e., the triple of complex parameters

$$(\gamma, \beta, \tilde{\beta}) = (\gamma(f, g), \beta(f), \beta(g)). \quad (1)$$

Here,

$$\gamma = \gamma(f, g) = \text{tr}[f, g] - 2, \beta = \beta(f) = \text{tr}^2(f) - 4, \tilde{\beta} = \beta(g) = \text{tr}^2(g) - 4, \quad (2)$$

where  $[f, g] = f g f^{-1} g^{-1}$  is the multiplicative commutator. Thus, the space of the two-generator Kleinian groups  $f, g$  can be identified as a subspace of the three complex dimensional space  $\mathbb{C}^3$ . We define  $\mathcal{D} = \{(\gamma, \beta, \tilde{\beta}) : (\gamma, \beta, \tilde{\beta}) \text{ is the principal character of a Kleinian group}\}$  via the mapping

$$f, g \mapsto (\gamma(f, g), \beta(f), \beta(g)). \quad (3)$$

Things are set up so the group  $Id \mapsto (0, 0, 0) \in \mathbb{C}^3$ , where  $Id$  is the identity of the group.

Applying [1, Lemma 2.29], given a two-generator Kleinian group  $\langle f, g \rangle$  with principal character  $(\gamma, \beta, \tilde{\beta})$  with  $\gamma \neq \beta$ , there is a two-generator discrete group  $\Gamma_\phi = \langle f, \phi \rangle$  with principal character  $(\gamma, \beta, -4)$ , i.e.,  $\phi$  is elliptic of order two. However, we can waive the condition  $\gamma \neq \beta$  (see Lemma 8 below).

If the group  $\Gamma_\phi$  is Kleinian (i.e., not virtually abelian), then there are a series of inequalities on the entries of the principal character which need to hold. One such is Jørgensen's inequality,  $|\gamma| + |\beta| \geq 1$ . Conversely, it is often easier to establish new discreteness conditions of two-generator groups with one generator of order two since we know of many polynomial trace identities that yield a polynomial semigroup action on the associated moduli space, and the dynamics of this action can be used to investigate discreteness of groups. Let us give one such example leading to Jørgensen's inequality to motivate our subsequent results. We note

$$\text{tr}[f, g f g^{-1}] - 2 = (\text{tr}[f, g] - 2)(\text{tr}[f, g f g^{-1}] + 2 - \text{tr}^2(f)), \quad (4)$$

which we write as  $\gamma(f, g f g^{-1}) = \gamma(\gamma - \beta)$ . Following [2], we write  $p_\beta(z) = z(z - \beta)$ ,  $p_\beta^0(z) = z$ , and  $p_\beta^{(n)}(z) = (p_\beta \circ p_\beta \circ \dots \circ p_\beta)(z)$ .

$\dots \circ p_\beta)(z)$ . Repeated application of this identity followed by projection yields the sequence

$$(\gamma, \beta, -4) \mapsto (p_\beta(\gamma), \beta, -4) \mapsto \dots \mapsto (p_\beta^{(n)}(\gamma), \beta, -4) \mapsto \dots \tag{5}$$

giving a sequence of traces of commutators  $\{p_\beta^{(n)}(\gamma)\}_{n=0}^\infty$  in the original discrete group  $\Gamma_\phi$ , where the series of commutators is

$$[f, g], [f, [f, g]], [f, [f, [f, g]]], \dots \tag{6}$$

It is usually not too difficult to prove such sequences cannot be infinite and contain convergent subsequences. Jørgensen’s inequality is implied by the assertion that  $p_\beta^{(n)}(\gamma) \rightarrow 0$ ; hence,  $|\gamma| > 1 + |\beta|$ . Every such trace identity yields a new inequality and the semigroup structure is clearly seen in general by

$$(\gamma, \beta, -4) \mapsto (p_\beta(\gamma), \beta, -4) \mapsto (q_\beta(p_\beta(\gamma)), \beta, -4), \tag{7}$$

where for example,  $q_\beta(z) = z(1 + \beta - z)^2$  from the trace identity

$$\begin{aligned} \text{tr}[f, gf g^{-1} f g] - 2 &= (\text{tr}[f, g] - 2)(\text{tr}^2(f) - \text{tr}[f, g] - 1)^2, \gamma \\ &\cdot (f, gf g^{-1} f g) = \gamma(1 + \beta - \gamma)^2. \end{aligned} \tag{8}$$

In another direction, the ergodicity of the polynomial semigroup acting in the complement of the Riley slice,  $\mathcal{R} = \text{int}\{(\gamma, 0, -4): \langle f, g \rangle \text{ is discrete and freely generated by } f \text{ and } g\}$  is used to prove the supergroup density and the existence of the unbounded Nielsen classes of generating pairs [3]. Note that this is not the usual definition of the Riley slice but is known to be equivalent.

In order to further these studies, we establish a number of useful results in this article. Dynamically, the elementary subgroups are sinks for the polynomial semigroup action on a slice and are isolated within the subset corresponding to discrete groups (apart from trivial abelian and dihedral examples). Thus, after we observe elementary properties of a two-generator discrete group with principal character  $(\gamma, \beta, -4)$  in §3, we list the possible principal characters in the first table in §4 representing. Then, we prove the two complex dimensional slice space  $\mathcal{D}_2 = \{(\gamma, \beta): \text{there is } \tilde{\beta} \text{ so } (\gamma, \beta, \tilde{\beta}) \text{ is a principal character of a Kleinian group}\}$  is closed in §5. Then,  $\mathcal{D}_2$  is the projection of the three complex dimensional moduli space  $\mathcal{D}$  to  $\mathbb{C}^2$ . Applications of this set being closed are in the generalizations of Jørgensen’s inequality quantifying the isolated nature of the elementary groups in the moduli space of discrete groups [4].

## 2. Preliminaries

Let  $\text{Möb}^+(\bar{\mathbb{C}})$  be the Möbius group of the normalized orientation preserving Möbius transformations on the Riemann sphere  $\bar{\mathbb{C}}$ :

$$\text{Möb}^+(\bar{\mathbb{C}}) = \left\{ \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{C} \text{ and } ad - bc = 1 \right\}, \tag{9}$$

and let  $\text{PSL}(2, \mathbb{C})$  be the projective special linear topological quotient group:

$$\text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C}) / \{\pm Id\}, \tag{10}$$

where  $\text{SL}(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C} \text{ and } ad - bc = 1 \right\}$ .

Then, these two groups are topologically isomorphic,  $\text{Möb}^+(\bar{\mathbb{C}}) \cong \text{PSL}(2, \mathbb{C})$  and so there are at least two different ways of thinking about groups throughout this paper, as either subgroups of  $\text{Möb}^+(\bar{\mathbb{C}})$  or subgroups of  $\text{PSL}(2, \mathbb{C})$ . Each of these groups has its own topology; however, the topological isomorphism shows us that the concept of discreteness is the same.

A subgroup  $\Gamma$  of  $\text{Möb}^+(\bar{\mathbb{C}})$  is called a *Kleinian group* if it is a nonelementary discrete group (see [5, 6]). Here, a group is *elementary* if it is virtually abelian, i.e., a finite extension of an abelian group. The elementary groups are classified [5, Section 5.1]. A subgroup of  $\text{Möb}^+(\bar{\mathbb{C}})$  is *nonelementary* if the limit set,  $L(\Gamma) = \text{the set of accumulation points of a generic orbit } \Gamma(z) = \{f(z): f \in \Gamma\}$ , contains infinitely many points. Notice from [7] that a group  $\Gamma$  is Kleinian if and only if every two-generator subgroup of  $\Gamma$  is discrete. It is this characterization of the Kleinian groups that motivates a focus on two-generator groups.

The principal character of a representation of the two-generator group  $f, g$  conveniently encodes various other geometric quantities. One of the most important is the *complex hyperbolic distance*  $\delta + i\theta$  between two axes  $\text{ax}(f)$  and  $\text{ax}(g)$  of the two elliptic or loxodromic Möbius transformations  $f$  and  $g$ , where  $\text{ax}(f)$  is the hyperbolic line in the hyperbolic 3-space  $\mathbb{H}^3$  joining the fixed points of  $f$  and  $g$  in  $\bar{\mathbb{C}}$ ,  $\delta$  is the hyperbolic distance between those two axes, and  $\theta$  is the holonomy of the transformation whose axis contains the common perpendicular between those two axes and moves  $\text{ax}(f)$  to  $\text{ax}(g)$ . Thus,

$$\sinh^2(\delta + i\theta) = \frac{4\gamma}{\beta\tilde{\beta}}. \tag{11}$$

The parameter  $\beta(f)$  describes the conjugacy class of the Möbius transformation in which  $f$  lies.

**Lemma 1.** *If an element  $f$  of  $\text{Möb}^+(\bar{\mathbb{C}})$  is not the identity, then*

- (a)  *$f$  is parabolic if and only if  $\beta(f) = 0$ .*
- (b)  *$f$  is elliptic if and only if  $\beta(f) \in [-4, 0)$ .*
- (c)  *$f$  is loxodromic if and only if  $\beta(f) \in \mathbb{C} \setminus [-4, 0]$ .*

In addition, the parameter  $\beta(f)$  gives the order of each elliptic element.

**Lemma 2.** *If an element  $f$  of  $M\ddot{o}b^+(\mathbb{C})$  is not the identity, then  $f$  is elliptic of order  $p$  if and only if*

$$\beta(f) = -4 \sin^2\left(\frac{k\pi}{p}\right), \text{ for } 1 \leq k < p \text{ and } (k, p) = 1. \quad (12)$$

We use the following easy lemma: a group is Kleinian if and only if it is *virtually Kleinian*, i.e., has a Kleinian subgroup of finite index.

**Lemma 3.** *Let  $H$  be a finite index subgroup of  $\Gamma$ . Then,  $\Gamma$  is Kleinian if and only if  $H$  is Kleinian.*

The proof follows directly from the fact that if  $H$  has finite index in  $\Gamma$ , then it has a coset decomposition

$$\Gamma = \phi_1 H \cup \phi_2 H \cup \dots \cup \phi_n H, \quad (13)$$

where  $\phi_1 = Id, \phi_2, \dots, \phi_n \in \Gamma$  for some  $n$ . Thus, any sequence  $\{g_n\}$  contains a subsequence  $\{g_{n_k}\}$  lying in one coset.

### 3. Slicing Spaces of Two-Generator Discrete Groups

Suppose that  $f, g$  is two-generator discrete subgroup of  $M\ddot{o}b^+(\mathbb{C})$  with  $\gamma = \gamma(f, g) \neq 0$  and  $\gamma(f, g) = \gamma \neq \beta = \beta(f)$ . Then, Gehring and Martin [1] state that there exist elliptics  $\phi$  and  $\psi$  of order two such that  $\Gamma_\phi = \langle f, \phi \rangle$  and  $\Gamma_\psi = \langle f, \psi \rangle$  are discrete with the related two-generator groups principal characters  $(\gamma, \beta, -4)$  and  $(\beta - \gamma, \beta, -4)$ , respectively.

Notice that  $\gamma(f, g) = 0$  implies  $f$  and  $g$  have a common fixed point on  $\mathbb{C}$  and hence  $\langle f, g \rangle$  is elementary (see [5], Theorem 4.3.5 and [1], Identity (1.5)). Thus, the condition  $\gamma \neq 0$  is necessary for Kleinian groups. We now consider the other restriction  $\gamma \neq \beta$ .

We start with our observation of the properties of a two-generator discrete group with the principal character  $(\gamma, \beta, -4)$ , which has one generator of order two.

**Lemma 4.** *Let  $\Gamma = f, g$  be a discrete group with principal character  $(\gamma, \beta, -4)$ . If  $\gamma = \beta$ , then  $\Gamma$  is an elementary discrete group, which is isomorphic to a dihedral group  $D_p$ , for some  $p \in \{1, 2, \dots, \infty\}$ .*

*Proof.* There are two cases to consider. If  $f$  is parabolic, then  $\gamma = \beta = 0$  by Lemma 1, and hence,  $f$  and  $g$  have a common fixed point which we may assume by conjugacy is  $\infty \in \mathbb{C}$ . Then,  $\Gamma$  is a subgroup of the upper-triangular matrices, where

$$f = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ and } g = \begin{pmatrix} \pm i & y \\ 0 & \mp i \end{pmatrix}, \quad (14)$$

and the result follows. We may now assume  $f$  and  $g$  have

the form

$$f = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \text{ and } g = \begin{pmatrix} \alpha & \mu \\ \delta & \nu \end{pmatrix}, \text{ where } \lambda \neq 0, \pm 1, \alpha\nu - \mu\delta = 1. \quad (15)$$

Computing the commutator,

$$[f, g] = \begin{pmatrix} \alpha\nu - \lambda^2\mu\delta & \alpha\lambda^2\mu - \alpha\mu \\ \frac{1}{\lambda^2}\delta\nu - \delta\nu & \alpha\nu - \frac{1}{\lambda^2}\mu\delta \end{pmatrix}, \quad (16)$$

and hence,  $\gamma = \text{tr}[f, g] - 2 = -\mu\delta(\lambda - 1/\lambda)^2$ , and  $\beta = \text{tr}^2(f) - 4 = (\lambda - 1/\lambda)^2$ . The assumption  $\gamma = \beta$  now implies that  $\mu\delta = -1$ . Since  $\alpha\nu - \mu\delta = 1, \alpha\nu = 0$ . Now,  $\beta(g) = -4$ , so  $\text{tr}(g) = 0$ , which gives  $\alpha + \nu = 0$ . Then,  $\alpha = \nu = 0$  and

$$g = \begin{pmatrix} 0 & \mu \\ -\frac{1}{\mu} & 0 \end{pmatrix}. \quad (17)$$

Thus,  $g(z) = -\mu^2/z$  interchanges the fixed points  $0$  and  $\infty$  of  $f$ . So  $\Gamma$  has a finite orbit  $\{0, \infty\}$ , and hence, it is an elementary discrete group, [5]. Computation shows the conjugate  $gf g^{-1} = f^{-1}$ :

$$gf g^{-1} = \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{pmatrix} = f^{-1}. \quad (18)$$

Thus,  $\Gamma$  is a dihedral group  $D_p$ , for some  $p \in \mathbb{N} \cup \{\infty\}$  from Theorem 9.  $\square$

We recall that the Riemann sphere  $\mathbb{C} = \mathbb{C} \cup \{\infty\}$  is the boundary of the hyperbolic 3-space  $\mathbb{H}^3$ .

**Lemma 5.** *Let  $\Gamma = f, g$  be a discrete group with the principal character  $(\gamma, -4, -4)$ , then  $\Gamma$  is isomorphic to one of the following elementary discrete groups:*

- (a) *The dihedral group  $D_\infty$ , if  $ax(f) \cap ax(g) = \emptyset$  or  $ax(f) \cap ax(g) \neq \emptyset$  in  $\mathbb{C}$ .*
- (b) *A dihedral group  $D_p$  for some  $p \in \mathbb{N}$ , if  $ax(f) \cap ax(g) \neq \emptyset$  in  $\mathbb{H}^3$ .*
- (c) *A parabolic group, if  $ax(f) \cap ax(g) = \{\infty\}$ .*

*Proof.*

- (1)  $ax(f) \cap ax(g) = \emptyset$ : Let  $\alpha$  be the common perpendicular between two axes  $ax(f)$  and  $ax(g)$ . Since  $f$  and  $g$  are elliptic of order two, they are the rotations of order two about their axes  $ax(f)$  and  $ax(g)$  in  $\mathbb{H}^3$ , respectively. And each of  $f$  and  $g$  interchanges the

ending points of  $\alpha$  and fixes  $\alpha$  setwise; hence,  $\Gamma$  is elementary. Furthermore, the product  $fg$  fixes the ending points of  $\alpha$  and fixes  $\alpha$  setwise, so the axis  $\text{ax}(fg) = \alpha$ . It follows that  $fg$  is loxodromic

Next, from the Fricke identity (see [8]), we have

$$\gamma(f, g) = \beta(f) + \beta(g) + \beta(fg) - \text{tr}(f)\text{tr}(g)\text{tr}(fg) + 8. \tag{19}$$

Since  $\beta(f) = \beta(g) = -4, \text{tr}(f) = 0$ , and  $f^2 = Id$ , the previous identity (19) gives  $\gamma(f, fg) = \beta(fg)$ , and by Lemma 4,  $\langle f, fg \rangle$  is a dihedral group. Notice that  $fg$  is loxodromic with infinite order; hence,  $f, g = \langle f, fg \rangle \cong D_\infty$ .

- (2)  $\text{ax}(f) \cap \text{ax}(g) \neq \emptyset$  in  $\mathbb{H}^3$  : Since  $f$  and  $g$  have a common fixed point (say  $x_0$ ),  $\Gamma$  is elementary. Since  $\Gamma$  is discrete, the axes of elliptics both of order two must intersect at  $x_0$  an angle  $\theta = k\pi/n$  for some  $k$  and  $n \geq 2$ , the product  $fg$  is a rotation about  $\text{ax}(fg)$  with the oriented angle  $2\theta = 2k\pi/p$  for any  $n \geq 2$  and some  $p \in \mathbb{N}$ , where  $\text{ax}(fg)$  is perpendicular to  $\text{ax}(f)$  and  $\text{ax}(g)$  and passing through  $x_0$ . Therefore,  $fg$  is elliptic of order  $p$ . Notice that  $f$  and  $g$  are rotations of order two and their rotation axes  $\text{ax}(f)$  and  $\text{ax}(g)$  meet in  $\mathbb{H}^3$ , we have  $gfg^{-1} = f^{-1}$ . It follows that  $f, g \cong D_p$
- (3)  $\text{ax}(f) \cap \text{ax}(g) \neq \emptyset$  in  $\bar{\mathbb{C}}$  : We may assume by conjugacy that  $\infty$  is the common fixed point of  $f$  and  $g$  in  $\bar{\mathbb{C}}$ . Then,  $f$  and  $g$  fix  $\infty$ , and hence,  $\text{ax}(f)$  and  $\text{ax}(g)$  are vertical hyperbolic lines. Notice that both  $f$  and  $g$  are rotations with the angles  $\pi$  at distinct centers. Since the sum of two rotation angles is  $2\pi$ , the product  $fg$  is a translation, and hence,  $fg$  is parabolic fixing  $\infty$ . Thus, the classification of elementary groups [5, Section 5.1],  $f, g$  is a group of the Euclidean similarities of  $\mathbb{C}$

We next note the following elementary result. □

**Lemma 6.** *Let  $\Gamma = \langle f, g \rangle$  be a two-generator group with principal character  $(\gamma, \beta, -4)$ , then  $\Gamma^g = \langle f, gfg^{-1} \rangle$  is an index two subgroup of  $f, g$ :  $\Gamma = \Gamma^g \cup g\Gamma^g$ .*

Then, we have the following.

**Lemma 7.** *Suppose that  $f, g$  is a two-generator Kleinian group with principal character  $(\gamma, \beta, \tilde{\beta})$ . If  $\tilde{\beta} \neq -4$ , then*

$$\text{Fix}(f) \cap \text{Fix}(gfg^{-1}) = \emptyset. \tag{20}$$

*Proof.* Since  $f, g$  is a two-generator Kleinian group,  $\gamma \neq 0$ , and hence,  $f$  and  $g$  cannot share any fixed points, so  $\text{Fix}(f) \cap \text{Fix}(g) = \emptyset$ . Since  $g(\text{Fix}(f)) = \text{Fix}(gfg^{-1})$ , there are two cases for us to consider. □

If  $\text{Fix}(f)$  has one element, say  $\text{Fix}(f) = \{z_0\}$ , then  $\text{Fix}(gfg^{-1}) = \{g(z_0)\}$ . But  $g(z_0) \neq z_0$ . Thus,  $\text{Fix}(f) \cap \text{Fix}(gfg^{-1}) = \emptyset$ .

If  $\text{Fix}(f)$  has two elements, say  $\text{Fix}(f) = \{z_1, z_2\}$ , then  $\text{Fix}(gfg^{-1}) = \{g(z_1), g(z_2)\}$  and

$$g(z_1) \neq z_1, g(z_2) \neq z_2. \tag{21}$$

If  $\text{Fix}(f) \cap \text{Fix}(gfg^{-1}) \neq \emptyset$ , then by [5, Theorem 5.1.2]  $f$  and  $gfg^{-1}$  cannot share exactly one fixed point, and hence,  $\text{Fix}(f) = \text{Fix}(gfg^{-1}) = g(\text{Fix}(f))$ . Thus, the only possibility is that  $g$  interchange the fixed points of  $f$ , so  $g(z_1) = z_2$  and  $g(z_2) = z_1$ , which implies that  $g^2$  fixes  $z_1$  and  $z_2$  but they are not fixed by  $g$ . We conclude that  $g^2$  has at least three fixed points and this implies that  $g^2$  is the identity,  $g$  has order two and  $\tilde{\beta} = -4$ .

In [1, Remark 2.6] it is shown that if  $f, g$  is a two-generator discrete group, then there are two elliptic elements  $\phi$  and  $\psi$  of order two such that

$$\phi f \phi^{-1} = gfg^{-1} \text{ and } \psi f \psi^{-1} = gf^{-1}g^{-1}. \tag{22}$$

In particular, if  $f$  is elliptic or loxodromic, then the axes  $\text{ax}(\phi)$  and  $\text{ax}(\psi)$  meet at right angles to one and other, and they bisect the common perpendicular between the axes  $\text{ax}(f)$  and  $\text{ax}(gfg^{-1})$ .

**Lemma 8.** *Let  $f, g$  be a Kleinian group with principal character  $(\gamma, \beta, \tilde{\beta})$ . Then, there are two elliptic elements  $\phi$  and  $\psi$  of order two such that  $\Gamma_\phi = \langle f, \phi \rangle$  and  $\Gamma_\psi = \langle f, \psi \rangle$  are discrete  $\mathbb{Z}_2$ -extension of  $\Gamma^g$  with principal characters  $(\gamma, \beta, -4)$  and  $(\beta - \gamma, \beta, -4)$ , respectively.*

*Proof.* Note  $\Gamma^g = \langle f, gfg^{-1} \rangle$  is a subgroup of both  $\Gamma_\phi = \langle f, \phi \rangle$  and  $\Gamma_\psi = \langle f, \psi \rangle$ . Suppose that the principal characters for  $\Gamma_\phi$  and  $\Gamma_\psi$  are  $(\gamma_1, \beta, -4)$  and  $(\gamma_2, \beta, -4)$ , respectively. Using [1, Lemma 2.1]

$$\begin{aligned} \gamma(\gamma - \beta) &= \gamma(f, gfg^{-1}) = \gamma(f, \phi f \phi^{-1}) = \gamma_1(\gamma_1 - \beta), \\ \gamma(\gamma - \beta) &= \gamma(f, gfg^{-1}) = \gamma(f, \psi f \psi^{-1}) = \gamma_2(\gamma_2 - \beta). \end{aligned} \tag{23}$$

We have two quadratic equations

$$\begin{aligned} \gamma_1^2 - \beta\gamma_1 - \gamma(\gamma - \beta) &= 0, \\ \gamma_2^2 - \beta\gamma_2 - \gamma(\gamma - \beta) &= 0. \end{aligned} \tag{24}$$

Solving these two quadratic equations,  $\gamma_1 = \gamma$ , or  $\beta - \gamma$  and  $\gamma_2 = \gamma$ , or  $\beta - \gamma$ . As [9, Identity (6.9)] shows both possibilities occur, so

$$\{\gamma_1, \gamma_2\} = \{\gamma, \beta - \gamma\}. \tag{25}$$

Thus, after relabeling, the principal characters for  $\Gamma_\phi$  and

$\Gamma_\psi$  are  $(\gamma, \beta, -4)$  and  $(\beta - \gamma, \beta, -4)$ , respectively. This completes the proof.  $\square$

### 4. Exceptional Set of Principal Characters

Now, the natural question is to determine whether two discrete groups  $\Gamma_\phi = \langle f, \phi \rangle$  and  $\Gamma_\psi = \langle f, \psi \rangle$  in Lemma 8 are actually Kleinian. Since both  $\Gamma_\phi$  and  $\Gamma_\psi$  are  $\mathbb{Z}_2$ -extension of  $\Gamma^g$ , by Lemma 3, we only need to decide if  $\Gamma^g$  is a Kleinian subgroup. First, we list the possible principal characters in Tables 1–3 representing two-generator elementary discrete groups. The set of principal characters occurring in these three tables is called the *exceptional set of principal characters*.

**Theorem 9.** *Let  $G$  be an elementary discrete group of  $\text{Möb}^+(\bar{\mathbb{C}})$ , then  $G$  is isomorphic to one of the following groups, where  $p \in \{1, 2, \dots, \infty\}$ .*

A cyclic group  $\mathbb{Z}_p$  :

$$\langle a : a^p = Id \rangle. \tag{26}$$

A dihedral group  $D_p \cong \mathbb{Z}_p \rtimes \mathbb{Z}_2$  :

$$\langle a, b : a^p = b^2 = Id, bab^{-1} = a^{-1} \rangle. \tag{27}$$

The group  $(\mathbb{Z}_p \times \mathbb{Z}) \rtimes \mathbb{Z}_2$  or  $\mathbb{Z}_p \times \mathbb{Z}$  :

$$\langle a, b, c : aba^{-1}b^{-1} = b^p = c^2 = Id, cac^{-1} = a^{-1}, cbc^{-1} = b^{-1} \rangle. \tag{28}$$

A Euclidean translation group  $\mathbb{Z} \times \mathbb{Z}$ .

A Euclidean triangle group  $\Delta(2, 3, 6)$ , or  $\Delta(2, 4, 4)$ , or  $\Delta(3, 3, 3)$ :

$$\left\langle a, b : a^p = b^q = (ab)^r = Id, \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \right\rangle. \tag{29}$$

A finite spherical triangle group  $A_4 = \Delta(2, 3, 3)$ , or  $S_4 = \Delta(2, 3, 4)$ , or  $A_5 = \Delta(2, 3, 5)$ :

$$\left\langle a, b : a^p = b^q = (ab)^r = Id, \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1 \right\rangle. \tag{30}$$

Let  $f$  and  $g$  be elliptic of order  $p \in \{2, 3, 4, 5\}$  in  $\text{Möb}^+(\bar{\mathbb{C}})$ , and let  $\theta$  be the angle subtended at the origin between  $\text{ax}(f)$  and  $\text{ax}(g)$ , and hence, the hyperbolic distance between those two axes is  $\delta = 0$ . Then,  $\sin^2(\theta)$  can be computed by using [9, Lemmas 6.19, 6.20, and 6.21], the parameters  $\beta(g)$  and  $\beta(f) \in \{-3, -4, \sqrt{5} - 5/2, -\sqrt{5} + 5/2\}$ , and the commutator parameter is  $\gamma(f, g) = ((-\beta(f)\beta(g))/4) \sin^2(\theta)$ , see [1]. With these elementary observations, some spherical trigonometry, and Theorem 9 about the classification of the elementary discrete groups, we obtain the following Tables 1 and 2, which list the possible principal characters (including the cases in the thesis [10]) of two-generator elementary discrete groups with nonzero commutator parameters  $\gamma(f, g)$

TABLE 1: Principal Characters: 2,  $p$ .

$p$	$\sin^2(\theta)$	$\gamma$	Groups	Principal characters
3	$\frac{2}{3}$	-2	$A_4$	$(-2, -3, -4)$
3	$\frac{1}{3}$	-1	$S_4$	$(-1, -3, -4)$
3	$\frac{3 - \sqrt{5}}{6}$	$\sqrt{5} - \frac{3}{2}$	$A_5$	$\left(\frac{\sqrt{5} - 3}{2}, -3, -4\right)$
3	$\frac{3 + \sqrt{5}}{6}$	$-3 + \frac{\sqrt{5}}{2}$	$A_5$	$\left(-\frac{3 + \sqrt{5}}{2}, -3, -4\right)$
3	1	-3	$D_3$	$(-3, -3, -4)$
4	$\frac{1}{2}$	-1	$S_4$	$(-1, -2, -4)$
4	1	-2	$D_4$	$(-2, -2, -4)$
5	$\frac{5 - \sqrt{5}}{10}$	$\sqrt{5} - \frac{3}{2}$	$A_5$	$\left(\frac{\sqrt{5} - 3}{2}, \frac{\sqrt{5} - 5}{2}, -4\right)$
5	$5 - \sqrt{5}$	-1	$A_5$	$\left(-1, -\frac{5 + \sqrt{5}}{2}, -4\right)$
5	$\frac{5 + \sqrt{5}}{10}$	-1	$A_5$	$\left(-1, \frac{\sqrt{5} - 5}{2}, -4\right)$
5	$\frac{5 + \sqrt{5}}{10}$	$-\frac{3 + \sqrt{5}}{2}$	$A_5$	$\left(-\frac{3 + \sqrt{5}}{2}, -\frac{5 + \sqrt{5}}{2}, -4\right)$
5	1	$\frac{\sqrt{5} - 5}{2}$	$D_5$	$\left(\frac{\sqrt{5} - 5}{2}, \frac{\sqrt{5} - 5}{2}, -4\right)$
5	1	$-\frac{5 + \sqrt{5}}{2}$	$D_5$	$\left(-\frac{5 + \sqrt{5}}{2}, -\frac{5 + \sqrt{5}}{2}, -4\right)$

TABLE 2: Principal Characters: 3,  $p$ .

$p$	$\sin^2(\theta)$	$\gamma$	Groups	Principal characters
3	$\frac{4}{9}$	-1	$A_5$	$(-1, -3, -3)$
3	$\frac{8}{9}$	-2	$A_4$	$(-2, -3, -3)$
4	$\frac{2}{3}$	-1	$S_4$	$(-1, -2, -3)$
5	$\frac{10 - 2\sqrt{5}}{15}$	$\frac{\sqrt{5} - 3}{2}$	$A_5$	$\left(\frac{\sqrt{5} - 3}{2}, \frac{\sqrt{5} - 5}{2}, -3\right)$
5	$\frac{10 - 2\sqrt{5}}{15}$	-1	$A_5$	$\left(-1, -\frac{5 + \sqrt{5}}{2}, -3\right)$
5	$\frac{10 + 2\sqrt{5}}{15}$	-1	$A_5$	$\left(-1, \frac{\sqrt{5} - 5}{2}, -3\right)$
5	$\frac{10 + 2\sqrt{5}}{15}$	$-\frac{3 + \sqrt{5}}{2}$	$A_5$	$\left(-\frac{3 + \sqrt{5}}{2}, -\frac{5 + \sqrt{5}}{2}, -3\right)$

TABLE 3: Principal Characters: 4,4 and 5,5.

$p, P$	$\sin^2(\theta)$	$\gamma$	Group	Parameters
4,4	1	-1	$S_4$	$(-1, -2, -2)$
5,5	$\frac{4}{5}$	$\sqrt{5} - 3/2$	$A_5$	$((\sqrt{5} - 3/2), (\sqrt{5} - 5/2), (\sqrt{5} - 5/2))$
5,5	$\frac{4}{5}$	-1	$A_5$	$(-1, -5 + \sqrt{5}/2, (\sqrt{5} - 5/2))$
5,5	$\frac{4}{5}$	$-3 + \sqrt{5}/2$	$A_5$	$(-3 + \sqrt{5}/2, -5 + \sqrt{5}/2, -5 + \sqrt{5}/2)$

$\neq 0$ . They are either the dihedral groups (if  $\gamma(f, g) = \beta(f)$  and  $\beta(g) = -4$ ) or the finite spherical triangle groups  $A_4 = \Delta(2, 3, 3)$ ,  $S_4 = \Delta(2, 3, 4)$ , and  $A_5 = \Delta(2, 3, 5)$ .

Notice that the angle between intersecting axes of elliptics of order 4 in a discrete group is always either 0 when they meet on the Riemann sphere  $\bar{\mathbb{C}}$  or  $\pi/2$ . This yields the additional parameter  $(-1, -2, -2)$  for the elementary group  $S_4$  when generated by two elements of order 4. Furthermore, the angle between intersecting axes of elliptics of order 5 in a discrete group is either  $\arcsin 2/\sqrt{5}$  or its complement  $\arcsin -2/\sqrt{5}$ . After possibly taking powers of the generator of order 5, the three additional principal characters can be obtained in the following Table 3.

The classification of the elementary discrete groups (Theorem 9) plays an important role in listing the tables above. We make the following additional remarks.

- (i) The axes of elliptics both of order two can intersect at an angle  $k\pi/p$  for any  $k$  and  $p \geq 2$  giving the dihedral group  $D_p$  with principal character  $(-4 \sin^2((k\pi)/p), -4, -4)$  (see Lemma 5).
- (ii) The axes of elliptics elements of orders 2 and  $p, p \geq 3$ , can meet at right angles. In this case, the principal character of dihedral group  $D_p$  is  $(-4 \sin^2(\pi/p), -4 \sin^2(\pi/p), -4)$ .
- (iii) The axes of elliptics of order  $p$  and  $q$  ( $p \leq q$ ) in a discrete group meet on  $\bar{\mathbb{C}}$ , i.e., meeting with angle 0, if and only if  $(p, q) \in \{(2, 2), (2, 3), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)\}$ .

For all of these groups  $\gamma(f, g) = 0$ . In particular, the Euclidean triangle groups  $\Delta(2, 3, 6)$ ,  $\Delta(3, 3, 3)$ ,  $\Delta(2, 4, 4)$ , and cyclic groups have  $\gamma(f, g) = 0$ . Also, the Euclidean translation groups have  $\gamma = 0$ .

**Lemma 10.** *Let  $f, g$  be a Kleinian group with the principal character  $(\gamma, \beta, -4)$  that is not one of those exceptional groups listed in Table 1. Then, the subgroup  $\Gamma^g = \langle f, gfg^{-1} \rangle$  is Kleinian.*

*Proof.* We need to show that the discrete subgroup  $\Gamma^g$  of the Kleinian group  $f, g$  can not be elementary. Let  $\gamma = \gamma(f, g)$  and  $\beta = \beta(f)$ . By Lemma 8 there is an elliptic  $h$  of order two such that  $\Gamma_h = \langle f, h \rangle$  is a discrete group containing  $\Gamma^g$

with index two and the principal character for  $\Gamma_h$  is  $(\gamma, \beta, -4)$ . By the hypothesis,  $(\gamma, \beta, -4)$  is not one of those exceptional groups listed in Table 1; then,  $\Gamma_h$  is not a finite spherical triangle group  $A_4, S_4$ , and  $A_5$ . Thus,  $\Gamma^g$  can only be elementary if  $\Gamma^g = \gamma(\gamma - \beta) = 0$ , i.e.,  $\gamma = 0$  or  $\gamma = \beta$ . Since  $f, g$  is Kleinian,  $\gamma \neq 0$ . So it can only be  $\gamma = \beta$ , and hence, it is the dihedral group. In this case, since  $f$  is not of order two,  $gfg^{-1} = f^{\pm 1}$ . In the case  $gfg^{-1} = f$ , it gives  $g(\text{Fix}(f)) = \text{Fix}(gfg^{-1}) = \text{Fix}(f)$ , and hence,  $g$  is elliptic of order two (Lemma 7) fixing or interchanging the fixed points of  $f$  in  $\bar{\mathbb{C}}$ . In the case  $gfg^{-1} = f^{-1}$ ,  $g$  might be a power of  $f$ . In either case  $f, g$  is not Kleinian, a contradiction.  $\square$

The following theorem guarantees that  $\Gamma^g$  is a Kleinian group if  $f$  is not elliptic of order  $p \leq 6$ .

**Theorem 11.** *Suppose that  $f, g$  is a Kleinian group. If  $f$  is not elliptic of order  $p \in \{2, 3, 4, 6\}$ , then the subgroup  $\Gamma^g = \langle f, gfg^{-1} \rangle$  is Kleinian.*

*Proof.* (a) Suppose that  $g$  is elliptic of order two. Then,  $\Gamma^g$  is a subgroup of index two in  $\Gamma$  and Lemma 3 ensures that  $\Gamma^g$  is also Kleinian. (b) Suppose that the order of  $g$  is not 2. We need to show that  $\Gamma^g$  is nonelementary. If  $\Gamma^g$  is elementary, then (see [5], Section 5.1).

- (i)  $\Gamma^g$  is elementary of type I; each nontrivial element is elliptic. Since the order of  $f$  is not  $p \leq 6$  or  $p = 5$ , neither is the order of  $gfg^{-1}$ . If  $\gamma(f, gfg^{-1}) \neq 0$ , then from Tables 1–3,  $\Gamma^g$  is neither the finite spherical triangle groups  $A_4, S_4$ , and  $A_5$ , nor the dihedral groups  $D_3, D_4$ , and  $D_5$ . If  $\gamma(f, gfg^{-1}) = 0$ , then  $\gamma = \beta$ , and  $\text{Fix}(f) \cap \text{Fix}(gfg^{-1}) \neq \emptyset$ . On the other hand, by Lemma 7,  $\text{Fix}(f) \cap \text{Fix}(gfg^{-1}) = \emptyset$ , a contradiction.
- (ii) Suppose  $\Gamma^g$  is an elementary group of type II; then,  $\Gamma^g$  is conjugate to a subgroup of  $\text{Möb}(\bar{\mathbb{C}})$  fixing  $\infty$  whose every element is parabolic of the form  $z \mapsto z + b, b \in \mathbb{C}$ . Thus, the group  $\Gamma^g$  is abelian, and hence,  $g(\text{Fix}(f)) = \text{Fix}(gfg^{-1}) = \text{Fix}(f)$ , a contradiction to  $\Gamma$  being Kleinian.
- (iii) Suppose  $\Gamma^g$  is an elementary group of type III; then, both  $f$  and  $gfg^{-1}$  are elliptic or both are

loxodromic. In either case  $\Gamma^g$  is abelian and as above this is a contradiction.

We have shown that  $\Gamma^g$  cannot be elementary if  $g$  does not have order two. Hence, in all two cases,  $\Gamma^g$  is a Kleinian group.  $\square$

Notice that a Kleinian group  $\langle f, g \rangle$  with  $f$  elliptic of order  $p = 5$  or  $p \geq 7$  and  $\beta(g) \neq -4$  cannot have  $\gamma = \beta$ . If so, then  $f$  and  $gf g^{-1}$  have a common fixed point. But this discrete group cannot be a Euclidean triangle group as they do not have elliptics of order 5 or greater than 6. Thus,  $f$  and  $gf g^{-1}$  have two common fixed points and the same trace. Hence,  $gf g^{-1} = f^{\pm 1}$  and  $g$  must fix or interchange the fixed points of  $f$ . The first is not possible, and the second shows  $g^2 = Id$ , the identity.

### 5. Projections and Principal Characters

Our methods to establish new inequalities rely on a fundamental result concerning spaces of finitely generated Kleinian groups. Namely they are closed in the topology of algebraic convergence—a result originally due to Jørgensen [7]. We state this now.

*Definition 12.* A sequence of  $n$ -generator subgroups  $\Gamma_j = \langle f_{j,1}, f_{j,2}, \dots, f_{j,n} \rangle$  is said to be convergent algebraically to a  $n$ -generator subgroup  $\Gamma = \langle f_1, f_2, \dots, f_n \rangle$  in  $\text{Möb}(\mathbb{C})$  if for each  $k = 1, 2, \dots, n$ , the sequence of the corresponding generators  $\{f_{j,k}\}$  converges uniformly to  $f_k$  in the spherical metric of  $\mathbb{C}$ .

**Theorem 13.** (Jørgensen).

That the map back is an isomorphism was proved by Jørgensen in [7]. There is a higher dimensional analogue to this Jørgensen’s theorem, but additional hypotheses are necessary—for instance giving a uniform bound on the torsion in a sequence or asking that the limit set be in geometric position (see [11], Proposition 5.8).

Now, we show the slice  $\mathcal{D}_2$  is closed. Notice that we are free to normalize a sequence of two-generator groups by conjugation and to pass to subsequences. First show a preliminary result.

**Theorem 14.** *The slice*

$\mathcal{D}^* = \{(\gamma, \beta, -4) : (\gamma, \beta, -4) \text{ is the principal character of a Kleinian group}\}$  is closed in the two complex dimensional space  $\mathbb{C}^2$  in the usual topology.

*Proof.* Suppose that  $\{(\gamma_j, \beta_j, -4)\}$  is a sequence of principal characters of Kleinian groups, say  $\{f_j, g_j\}$ . We proceed by considering two cases: up to subsequence,  $f_i$  is parabolic for all  $i$  or not.

*Case (a).* Suppose  $f_j$  is parabolic for all  $j$ . Conjugate each  $\Gamma_j$  so that the first generator is now represented by the matrix  $f_j = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . Then,  $f_j$  is constant and it remains

to show that the sequence  $\{g_j\}$  also converges. Since  $\text{tr}(g_j) = 0$ , we suppose the matrix for second generator is

$$g_j = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix}. \tag{31}$$

We calculate that  $\gamma_j = c_j^2$ , and hence,  $c_j \rightarrow c$ . Jørgensen’s inequality gives  $|\gamma_j| \geq 1$  for all  $j$ , and hence,  $c_j \rightarrow c \neq 0$ .

Set

$$h_j = \begin{pmatrix} 1 & \frac{-a_j + i\sqrt{1+c_j}}{c_j} \\ 0 & 1 \end{pmatrix} \text{ and } h_j g_j h_j^{-1} = \begin{pmatrix} i\sqrt{1+c_j} & 1 \\ c_i & -i\sqrt{1+c_j} \end{pmatrix}. \tag{32}$$

Since  $f_j$  commutes with  $h_j$ ,  $f_j$  is left unchanged under conjugation. The result follows by Theorem 13.

*Case (b).* We now suppose that  $f_j$  is not parabolic for all  $j$ . By conjugation, we may assume that  $f_j$  is represented by the matrix

$$f_j = \begin{pmatrix} \lambda_j & 0 \\ 0 & \frac{1}{\lambda_j} \end{pmatrix}, \tag{33}$$

where  $\lambda_j \neq 0, \pm 1$ . Thus, the parameter  $\beta_j = (\lambda_j - 1/\lambda_j)^2$ , which gives the quadratic equation

$$\lambda_j^2 - \sqrt{\beta_j} \lambda_j - 1 = 0, \tag{34}$$

and hence,  $\lambda_j = \sqrt{\beta_j} \pm \sqrt{\beta_j + 4}/2$ . Since  $\beta_j \rightarrow \beta, \lambda_i$  converges to  $\lambda = \sqrt{\beta} \pm \sqrt{\beta + 4}/2$ . Noting that  $\lambda \neq 0, \pm 1$  for either choice of  $\lambda$ , we conclude that  $f_j$  converges to the element  $f$  represented by the matrix

$$f_j \rightarrow f = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \text{ and } \beta_j \rightarrow \beta = \left(\lambda - \frac{1}{\lambda}\right)^2. \tag{35}$$

It remains to show that the order two elements  $g_j$  also converge. Note first that  $\gamma_j = -\beta_j b_j c_j$  and so we may assume that  $b_j c_j \neq 0$ . As in the previous case, the matrix for  $g_j$  can be written as

$$g_j = \begin{pmatrix} a_j & -\frac{1+a_j^2}{c_j} \\ c_j & -a_j \end{pmatrix}, \tag{36}$$

so that  $a_j \neq \pm i$  and  $1 + a_j^2 \neq 0$ . Consider the conjugacy of the

group  $\langle f_j, g_j \rangle$  by the following diagonal matrix,

$$\phi_j = \begin{pmatrix} i\sqrt{\frac{c_j}{1+a_j^2}} & 0 \\ 0 & -i\sqrt{\frac{1+a_j^2}{c_j}} \end{pmatrix}, \tag{37}$$

and then,  $g_j$  (replaced by the conjugacy  $\phi_j g_j \phi_j^{-1}$ ) has the form

$$g_j = \begin{pmatrix} a_j & 1 \\ -(1+a_j^2) & -a_j \end{pmatrix}. \tag{38}$$

Since  $f_j$  commutes with  $\phi_j$ ,  $f_j$  is left unchanged under the conjugation.

The commutator is now

$$[f_j, g_j] = f_j g_j f_j^{-1} g_j^{-1} = \begin{pmatrix} -a_j^2 + \lambda_j^2(1+a_j^2) & -a_j + \lambda_j^2 a_j \\ \frac{a_j(1+a_j^2)}{\lambda_j^2} - a_j(1+a_j^2) & \frac{1+a_j^2}{\lambda_j^2} - a_j^2 \end{pmatrix}. \tag{39}$$

Hence, the parameter  $\gamma_j$  for the group  $f_j, g_j$  is

$$\gamma_j = \frac{a_j^2(\lambda_j^2 - 1)^2 + (\lambda_j^2 - 1)^2}{\lambda_j^2}. \tag{40}$$

At this point we would like to show  $\lambda_j \rightarrow 1$ , but this can actually happen in limits of Dehn surgeries—see the following example. Thus, we have two subcases:

Case (b) (i) Suppose  $\lambda_j \rightarrow 1$

Solving for  $a_j$  yields

$$a_j^2 = \frac{\gamma_j \lambda_j^2 - (\lambda_j^2 - 1)^2}{(\lambda_j^2 - 1)^2} \rightarrow \frac{\gamma \lambda^2 - (\lambda^2 - 1)^2}{(\lambda^2 - 1)^2}. \tag{41}$$

Let  $a^2 = \gamma \lambda^2 - (\lambda^2 - 1)^2 / (\lambda^2 - 1)^2$ , then  $g_j$  converges to the element  $g$  of  $\text{PSL}(2, \mathbb{C})$  represented by the matrix

$$g = \begin{pmatrix} a & 1 \\ -(1+a^2) & -a \end{pmatrix}, \tag{42}$$

and  $\gamma_j \rightarrow \gamma(f, g) = \gamma$ . Applying Theorem 13, the limit group  $\Gamma = \langle f, g \rangle$  is Kleinian.

Case (b) (ii) Suppose  $\lambda_j \rightarrow 1$ . This is equivalent to  $\beta_j \rightarrow 0$ . For each  $j$ , we may conjugate  $f_j$  and  $g_j$  so that  $f_j$  has a fixed point at  $\infty$  and  $g_j$  has a fixed point at 0. Thus,

$f_j$  is upper triangular and  $g_j$  is lower triangular. We may further conjugate by a diagonal matrix to achieve the upper entry in the matrix representing  $f_j$  is 1. With this normalization we have

$$f_j = \begin{pmatrix} \lambda_j & 1 \\ 0 & \frac{1}{\lambda_j} \end{pmatrix} \text{ and } g_j = \begin{pmatrix} i & 0 \\ c_j & -i \end{pmatrix}, \tag{43}$$

and  $\lambda_j \rightarrow 1$ . We calculate that

$$\gamma_j = \gamma(f_j, g_j) = c_j \left( c_j + \frac{2i(\lambda_j^2 - 1)}{\lambda_j} \right). \tag{44}$$

It follows that  $c_j \rightarrow \pm\sqrt{\gamma}$ , for one choice of sign. Thus,  $f_j \rightarrow f$  and  $g_j \rightarrow g$

$$f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } g = \begin{pmatrix} i & 0 \\ \sqrt{\gamma} & -i \end{pmatrix}. \tag{45}$$

This group is nonelementary by Theorem 13 so  $\gamma \neq 0$ . One can see this since as soon as  $|\beta_j| < 1/2$ , Jørgensen's inequality gives  $|\gamma_j| > 1/2$ . This completes the proof.  $\square$

Example 1. We next give an example to show the last case can occur. That is  $\beta_j \rightarrow 0$  and  $\gamma_j \rightarrow \gamma$  (and  $|\gamma| = 1$ ). Let

$$f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } h = \begin{pmatrix} 1 & 0 \\ a_\infty & 1 \end{pmatrix}. \tag{46}$$

Then,  $\langle f, h \rangle$  is (a representation of) the two bridge figure of eight knot group if  $a_\infty = (1 + i\sqrt{3})/2$ . The relator in this group is

$$hfh^{-1}fh = fhf^{-1}hf. \tag{47}$$

If we perform  $(p, 0)$  Dehn surgery on the complement of this knot, we obtain a Kleinian group generated by two elliptics of order  $p$ ,  $f_p$ , and  $h_p$ . Then, up to conjugacy  $f_p$  and  $h_p$  have the form

$$f_p = \begin{pmatrix} e^{i\pi/p} & 1 \\ 0 & e^{-i\pi/p} \end{pmatrix} \text{ and } h_p = \begin{pmatrix} e^{i\pi/p} & 0 \\ a_p & e^{-i\pi/p} \end{pmatrix}. \tag{48}$$

Then, the relator (47) holds if and only if

$$a_p = \frac{1}{2} \left( 3 - 2 \cos \frac{2\pi}{p} - \sqrt{-1 - 4 \cos \frac{2\pi}{p} + 2 \cos \frac{4\pi}{p}} \right) \tag{49}$$

or its conjugate. Thus,  $\langle f_p, h_p \rangle$  is the group obtained from



this orbifold Dehn surgery. As such it is discrete and non-elementary. Then, as  $p \rightarrow \infty$ ,

$$\begin{aligned} \beta(f_p) &= \beta(h_p) = -4 \sin^2 \frac{\pi}{p} \rightarrow 0 = \beta(f), \\ \gamma(f_p, h_p) &= a_p \left( -2 + a_p + 2 \cos \frac{2\pi}{p} \right) \rightarrow a_\infty^2 = \gamma(f, g). \end{aligned} \tag{50}$$

As  $\beta(f_p) = \beta(h_p)$ , we find an involution  $g_p$  such that  $h_p = g_p f_p g_p^{-1}$  with  $\gamma(f_p, g_p) = \gamma(f_p, h_p)$ .

We next have the following lemma.

**Lemma 15.** *Let  $(\gamma_j, \beta_j, \tilde{\beta}_j)$  be a sequence of principal characters of the two-generator Kleinian groups  $f_j, g_j$ , and let  $(\gamma, \beta, \tilde{\beta})$  be the principal character of two-generator group  $f, g$ . Suppose that  $(\gamma_j, \beta_j)$  converges to  $(\gamma, \beta)$  and  $f$  is not elliptic of order  $p \in \{2, 3, 4, 6\}$ . Then,  $\gamma \neq 0$  and  $\gamma \neq \beta$ .*

*Proof.* Since  $f$  is not elliptic of order  $p \in \{2, 3, 4, 6\}$ ,  $f_j$  is likewise not elliptic of these orders for all but finitely many  $j$ . Applying Theorem 11,  $\{f_j, g_j f_j g_j^{-1}\}$  is a sequence of Kleinian groups with corresponding principal characters  $(\gamma_j(\gamma_j - \beta_j), \beta_j, \tilde{\beta}_j)$  which converge to  $(\gamma(\gamma - \beta), \beta, \tilde{\beta})$ . Now,  $f_j, g_j f_j g_j^{-1}$  is a Kleinian group generated by two elements of the same trace. Thus,  $\gamma_j(\gamma_j - \beta_j) \neq 0$ , and hence,  $\gamma_j \neq 0$  and  $\gamma_j \neq \beta_j$ .

Now we may apply a result by Cao [12, Theorem 5.1] which gives a universal lower bound on the commutator parameter for such groups.

$$\gamma_j(\gamma_j - \beta_j) \geq 0.198. \tag{51}$$

Thus,  $\lim_{j \rightarrow \infty} \gamma_j(\gamma_j - \beta_j) = \gamma(\gamma - \beta) \geq 0.198$ . It follows that  $\gamma \neq 0$  and  $\gamma \neq \beta$ .  $\square$

**Theorem 16.** *The two complex dimensional space*

$\mathcal{D}_2 = \{(\gamma, \beta) : (\gamma, \beta, \tilde{\beta}) \text{ is the principal character of a Kleinian group for some } \tilde{\beta}\}$  *is closed in the two complex dimensional space  $\mathbb{C}^2$  in the usual topology.*

*Proof.* Suppose that  $\{(\gamma_j, \beta_j)\}$  is a sequence in  $\mathcal{D}_2$  with limit  $(\gamma, \beta)$  in  $\mathbb{C}^2$ . By the definition of  $\mathcal{D}_2$ , there is a sequence  $\{\tilde{\beta}_j\}$  so that  $(\gamma_j, \beta_j, \tilde{\beta}_j)$  is the principal character of a Kleinian group. Then, we project to the principal character  $(\gamma_j, \beta_j, -4)$  of  $\langle f_j, \phi_j \rangle$ . If infinitely many of these groups are Kleinian, the result we seek follows from Theorem 14. After passing to a subsequence, we are left with the case  $\langle f_j, \phi_j \rangle$  is elementary for every  $j$ .

*Case (i).* Suppose  $f_j$  is not elliptic of order  $p \in \{2, 3, 4, 6\}$  for infinitely many  $j$ . Then, Lemma 15 tells us that  $\gamma_j \notin \{0, \beta_j\}$ , and hence,  $\langle f_j, \phi_j \rangle$  is Kleinian, so this case cannot occur

*Case (ii).* We may now suppose, after passing to a subsequence, that each  $f_j$  is elliptic of order  $p \in \{2, 3, 4, 5, 6\}$ ,  $\beta(f_j) = -4 \sin^2((k\pi)/p) = \beta$ ,  $(k, p) = 1$ .  $\langle f_j, \phi_j \rangle$  is a discrete elementary group generated by elliptics of orders  $p$  and 2. The order of any such group is less than that of  $A_5$ . If infinitely many of these groups are finite, then in fact  $\langle f_j, \phi_j \rangle$  is elliptic of order 1, 2, 3, 4, 5, or 6. Thus,  $\gamma_j$  lies in a finite set of values.

Again, after a subsequence, we have  $\gamma_j = \gamma$ . Then,  $\{(\gamma_j, \beta_j, \tilde{\beta}_j)\} = \{(\gamma, \beta, \tilde{\beta}_j)\}$  and the latter group is Kleinian by hypothesis. Thus,  $(\gamma, \beta) \in D_2$

*Case (iii).* We are left with the situation that  $\langle f_j, \phi_j \rangle$  is generated by elliptics of orders  $p$  and 2, infinite and elementary for each  $j$ . The classification tells us this only happens for the Euclidean triangle groups and any such group has a common fixed point. Then,  $\gamma(f_j, \phi_j) = 0 = \gamma(f_j, g_j)$  which is not possible as  $\langle f_j, g_j \rangle$  is a Kleinian group.  $\square$

### Data Availability

The manuscript includes the underlying data supporting the results of our study.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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