Hindawi
Abstract and Applied Analysis
Volume 2022, Article ID 6385401, 6 pages
https://doi.org/10.1155/2022/6385401

Research Article

A Study on the Impact of Nonlinear Source Term in Black-Scholes Option Pricing Model

P. Sozhaeswari,[1] R. Sowrirajan,[1] K. Loganathan ,[2] and Sonam Gyeltshen [3]

[1] *Department of Mathematics, Dr.N.G.P. Arts and Science College, Coimbatore, 641035 Tamilnadu, India*
[2] *Department of Mathematics and Statistics, Manipal University Jaipur, Jaipur, 303007 Rajasthan, India*
[3] *Department of Humanities and Management, Jigme Namgyel Engineering College, Royal University of Bhutan, Bhutan*

Correspondence should be addressed to K. Loganathan; loganathankaruppusamy304@gmail.com and Sonam Gyeltshen; sonamgyeltshen@jnec.edu.bt

Received 9 June 2022; Revised 27 October 2022; Accepted 15 November 2022; Published 28 November 2022

Academic Editor: Paul Eloe

In this work, we study the effect of nonlinear source term in Black-Scholes model by finding the solution of it. We use the mathematical concepts of existence and uniqueness to arrive the conclusion. The transformation of the nonlinear equation into heat equation leads to the existence of solution through fixed-point theorems, semigroup theory, and certain regularity conditions imposed on variables.

1. Introduction

The Black-Scholes equation is a mathematical model that plays an important role in dynamics of financial and economic identity. In order to increase the efficiency of Black-Scholes model, the condition on the parameters is made uncertain, and their behaviours are studied in [1–4]. Agyeman and Oduro in [5]studied the existence with nonlinear force term depending on this deviation of the option price.

Based on this theory, let us study the solvability of the following nonlinear Black-Scholes type equation.

$$\tilde{V}_t + \frac{1}{2}\sigma^2 S^2 \tilde{V}_{SS} + rS\tilde{V}_S - r\tilde{V} = \tilde{\varphi}\left(\int_0^T f\left(S, t, \tilde{V}_S\right) dt\right)\tilde{V}, \quad (1)$$

where $S > 0$, $t \in (0, T)$, and $\Omega^* \subset \mathfrak{R}^n$.

Here, $\tilde{V}$ is the option price; σ is the volatility constant; S is the stock price; r is the risk-free interest rate; $\tilde{\varphi}$ is the potential function; time $T > 0$ and the nonlinear term $f(S, t, \tilde{V}_S)$dt represent the various price effects that happens during the period. Assume that the force effect happen as a multiple of the option price. In [1], the function $f : \Omega^* \times [0, T] \times \Omega^* \longrightarrow \Omega^*$ is assumed to satisfy the Caratheodory conditions that, $f(.,.,\tilde{V})$ is measurable over all $\tilde{V} \in \Omega^*$. In addition, the function $f(.,t,.)$ and its derivatives are bounded and continuous throughout its entire region for all time t. More details on the function spaces are explained in [6].

The nonlinear term $\tilde{\Phi}(\cdot)\tilde{V}$ on the right hand side represents the monetary assistance provided to the investor. Pilant and Rundell in [7] studied the existence and uniqueness of the partial differential equations (PDEs) with nonlinear term using fixed-point technique. Shu and Shi in [8] studied the mild solution of impulsive fractional evolution equations through fixed-point technique. Following this, Nanda and Das [9] extended the aforementioned method for solving various forms heat conduction problems involving nonlinear source terms. The similar problem is studied by one of the authors in [10], where fixed-point technique is applied but for its inverse theory. In [11, 12], the authors adopted fixed-point technique in the discussion on the existence of various forms of solution of an almost periodic stochastic differential equations. The works in [13–15] studied the existence of solutions for PDE under nonlocal conditions. Moreover, the works in [16–20] also deals with the different forms of Black-Scholes equation and studied the existence theory. In [21], the author used the semigroup theory and operators to study the Black-Scholes model. Hence, considering the mentioned points, it is clear that the discussion on Black-Scholes model using fixed-point theory is of great interest. The problem under consideration is a special one as it depends on the forcing term that reflects today's economic outlook.

The novelty of this problem lies in the form under investigation. This is one of the most difficult forms of the Black-Scholes equation. Physically, this problem is interpreted as follows: find the value for option "$\tilde{V}$" that make up for any losses incurred while exercising the option. The integral term is defined to check the cash flow required for offsetting. The loss must be within the specified range. This paper is organized as follows: the existence part is discussed is Section 2, the uniqueness property is established in Section 3, and the conclusion in Section 4. This discussion is based on the technique adapted in [22].

2. Existence Result

In this segment, transform the left hand side of (1) in the form of heat equation and thereby the direct application of the notations and procedure of [22], we obtain the necessary result. Also, the integrodifferential forcing term can be neutralized through the method discussed in [14]. Let

$$S = e^x, t = T - \frac{2\tau}{\sigma^2} \tilde{V}(S,t) = \tilde{v}(x,\tau) = \tilde{v}\left(\ln(S), \frac{\sigma^2}{2}(T-t)\right). \tag{2}$$

The derivatives of transformation are

$$\frac{\partial \tilde{V}}{\partial t} = \frac{-\sigma^2}{2}\tilde{v}_\tau, \frac{\partial \tilde{V}}{\partial S} = \frac{1}{S}\tilde{v}_x, \frac{\partial^2 \tilde{V}}{\partial S^2} = \frac{1}{S^2}(\tilde{v}_{xx} - \tilde{v}_x). \tag{3}$$

Substituting the derivatives of Black-Scholes equation, we get

$$\tilde{V}_t + \frac{1}{2}\sigma^2 S^2 \tilde{V}_{SS} + rS\tilde{V}_S - r\tilde{V} = -\tilde{v}_\tau + \tilde{v}_{xx} - \tilde{v}_x + \frac{2r}{\sigma^2}\tilde{v}_x - \frac{2r}{\sigma^2}\tilde{v}. \tag{4}$$

Setting $k = 2r/\sigma^2$, the equation becomes

$$\tilde{V}_t + \frac{1}{2}\sigma^2 S^2 \tilde{V}_{SS} + rS\tilde{V}_S - r\tilde{V} = -\tilde{v}_\tau + \tilde{v}_{xx} + (k-1)\tilde{v}_x - k\tilde{v}, \tag{5}$$

for $-\infty < x < \infty, 0 \le \tau \le \sigma^2/2T$. As more modifications on variables are required to make the unnecessary terms be removed, let

$$\tilde{v}(x,\tau) = e^{\alpha x + \beta\tau}\tilde{u}(x,\tau) = \phi\tilde{u}. \tag{6}$$

Computing the partial derivatives of $\tilde{v}$ and using this in (5), we get

$$\begin{aligned}\tilde{V}_t + \frac{1}{2}\sigma^2 S^2 \tilde{V}_{SS} + rS\tilde{V}_S - r\tilde{V} = -\tilde{u}_\tau + \tilde{u}_{xx} + [2\alpha + (k-1)]\tilde{u}_x \\ + \left[\alpha^2 + (k-1)\alpha - k - \beta\right]\tilde{u},\end{aligned} \tag{7}$$

since α and β are arbitrary constants, introduced to eliminate the $\tilde{u}_x$ and $\tilde{u}$ terms

$$\alpha = \frac{-(k-1)}{2}, \beta = \alpha^2 + (k-1)\alpha - k = \frac{-(k+1)^2}{4}. \tag{8}$$

Then, α and β form the coefficients of 0, which eliminate $\tilde{u}_x$ and $\tilde{u}$ terms, respectively. The equation is then reduced to a one-dimensional heat equation, we can attain that

$$\tilde{V}_t + \frac{1}{2}\sigma^2 S^2 \tilde{V}_{SS} + rS\tilde{V}_S - r\tilde{V} = -\tilde{u}_\tau + \tilde{u}_{xx}\#. \tag{9}$$

Substituting (9) in (1) and replacing the variables on the R.H.S of (1) $S = x$, $t = \tau$, and $\tilde{V} = \tilde{u}$. Thence, we have

$$\tilde{u}_\tau - \tilde{u}_{xx} + \tilde{\varphi}\left(\int_0^T f(x,\tau,\tilde{u}_x)d\tau\right)\tilde{u} = 0, \tag{10}$$

with conditions

$$\tilde{u}|_{\tau=0} = \tilde{u}^0, \tilde{u}|_{\partial\Omega^*} = 0, (\tau, x) \in (0,T] \times \Omega^*, \tag{11}$$

where Ω^* lies in a multidimensional bounded domain $\mathfrak{C}^2$. Obviously, the significance of the nonlinear term (9) is its dependence on

$$\tilde{u}_T := \int_0^T f(x,\tau,\tilde{u}_x)d\tau. \tag{12}$$

That is, while the existence on $[0,T]$ is unknown at whole interval, $T > 0$ is given a precursor.

From [23, 24], the existence of the solution of (1) should be nonnegative, nondecreasing potential $\tilde{\varphi}$ for which $x \longmapsto \tilde{\varphi}(x)x$ is derivable. We present a brief demonstration on its solvability based on generic assumptions about potential $\tilde{\varphi}$.

The proof depends on fixed-point theorem due to Schauder and the fact that under acceptable presumptions, $\mathscr{B}(\tilde{u}_T) := -\Delta_D + \tilde{\varphi}(\tilde{u}_T)$ from a semigroup $\{e^{-\tau\mathscr{B}(\tilde{u}_T)}; \tau \ge 0\}$ on $\mathscr{L}_p(\Omega^*)$, where $-\Delta_D$ stands for Laplacian, which is subject to Dirichlet boundary conditions. The expected form of solution (1)

$$\tilde{u}(\tau) = e^{-\tau\mathscr{B}(\tilde{u}_T)}\tilde{u}^0, \tau \in [0,T], \tag{13}$$

and, as a result, $\tilde{u}_T$ is described by the relation

$$\tilde{u}_T = \int_0^T e^{-\tau\mathscr{B}(\tilde{u}_T)}\tilde{u}^0 d\tau. \tag{14}$$

Theorem 1. *Let $\tilde{\varphi} \in \mathfrak{C}(\mathfrak{R}, \mathfrak{R})$ be nonnegative and $n < 2p$. If one of the relations*

$$\tilde{u}^0 \in \mathscr{L}_\infty(\Omega^*), \tag{15}$$

or

(i) $\tilde{\varphi}(x) \le a(1 + |x|), x \in \mathfrak{R}$ holds, then for some $a > 0$ and $\tilde{u}^0 \in \mathscr{L}_p(\Omega^*)$.

There exists at least one possible solution

$$\tilde{u} \in \mathfrak{C}([0, T], \mathscr{L}_p(\Omega^*)) \cap \mathfrak{C}^1((0, T], \mathscr{L}_p(\Omega^*)) \cap \mathfrak{C}\left((0, T], \mathscr{W}_p^2(\Omega^*)\right), \tag{16}$$

to (1). Moreover, $\|\tilde{u}(\tau)\|_p \le \|\tilde{u}^0\|_p$ for $\tau \in [0, T]$. If $0 \le \tilde{u}^0$, then $0 \le \tilde{u}(\tau)$ for $\tau \in [0, T]$.

Proof. Part (i).

Take a nonnegative function $\tilde{\varphi} \in \mathfrak{C}(\mathfrak{R}, \mathfrak{R})$. Based on the boundedness of $\tilde{\varphi}$ and by defining $\tilde{u}^0 \in \mathscr{L}_\infty(\Omega^*)$, we arrive at

$$\tilde{\varphi} \in \mathfrak{C}(\mathscr{L}_\infty(\Omega^*), \mathscr{L}_\infty(\Omega^*)), \tag{17}$$

is bounded.

Set $\mathscr{R}_0 \coloneqq T\|\tilde{u}^0\|_\infty$ and let

$$\mathfrak{X}_T \coloneqq \bar{\mathbb{B}}_{\mathscr{L}_\infty(\Omega^*)}(0, \mathscr{R}_0), \tag{18}$$

indicate a closed ball whose cent is at origin and radius $\mathscr{R}_0$. Take $2p \in (n,\infty)$ with the fact that, for any $\tilde{u}_T \in \mathfrak{X}_T$, the mapping $\tilde{\varphi}(\tilde{u}_T) \coloneqq [\Omega^* \mapsto \tilde{\varphi}(\tilde{u}_T)\Omega^*]$ satisfies without a doubt,

$$\|\tilde{\varphi}(\tilde{u}_T)\|_{\mathfrak{L}(\mathscr{L}_p(\Omega^*))} \le \|\tilde{\varphi}(\tilde{u}_T)\|_\infty, \le \max_{[-\mathscr{R}_0,\mathscr{R}_0]} \tilde{\varphi}, \tilde{u}_T \in \mathfrak{X}_T. \tag{19}$$

Combining the results and the fact $-\Delta_D \in \mathfrak{H}(\mathscr{W}_{p,D}^2(\Omega^*), \mathscr{L}_p(\Omega^*))$

$$\mathscr{B}(\tilde{u}_T) \coloneqq -\Delta_D + \tilde{\varphi}(\tilde{u}_T) \in \mathfrak{H}\left(\mathscr{W}_{p,D}^2(\Omega^*), \mathscr{L}_p(\Omega^*) ; \kappa, \omega(\mathscr{R}_0)\right), \tag{20}$$

for some positive $\omega(\mathscr{R}_0)$ and $1 \le \kappa$. Moreover, for a nonnegative $\tilde{\varphi}$, the operator $-\mathscr{B}(\tilde{u}_T)$ generates a positive contraction semigroup $(e^{-\tau\mathscr{B}(\tilde{u}_T)})_{\tau\ge0}$ on each $\mathscr{L}_j(\Omega^*)$ for $j \in (1,\infty]$. Hence,

$$\left\|e^{-\tau\mathscr{B}(\tilde{u}_T)}\right\|_{\mathfrak{L}(\mathscr{L}_p(\Omega^*))} \le 1, \tau \ge 0, j \in (1,\infty]. \tag{21}$$

Let us delineate,

$$\tilde{\Phi}(\tilde{u}_T) \coloneqq \int_0^T e^{-\tau\mathscr{B}(\tilde{u}_T)}\tilde{u}^0 d\tau, \tilde{u}_T \in \mathfrak{X}_T. \tag{22}$$

Then, (21) signifies that

$$\|\tilde{\Phi}(\tilde{u}_T)\|_\infty \le \int_0^T \left\|e^{-\tau\mathscr{B}(\tilde{u}_T)}\right\|_{\mathfrak{L}(\mathscr{L}_p(\Omega^*))} \|\tilde{u}^0\|_\infty d\tau \le T\|\tilde{u}^0\|_\infty \le \mathscr{R}_0, \tag{23}$$

with $\tilde{\Phi} : \mathfrak{X}_T \longrightarrow \mathfrak{X}_T$. Using the condition $2\gamma \in (n/p, 2)$ and (20) together with $1 \le \mathscr{K}(\mathscr{R}_0)$ and positive $\zeta(\mathscr{R}_0)$, such that

$$\left\|e^{-\tau\mathscr{B}(\tilde{u}_T)}\right\|_{\mathfrak{L}(\mathscr{L}_p(\Omega^*),\mathscr{W}_{p,D}^{2\gamma}(\Omega^*))} \le \mathscr{K}(\mathscr{R}_0)e^{\zeta(\mathscr{R}_0)\tau}\tau^{-\gamma}, \tau > 0. \tag{24}$$

Therefore,

$$\begin{aligned}\|\tilde{\Phi}(\tilde{u}_T)\|_{\mathscr{W}_{p,D}^{2\gamma}(\Omega^*)} &\le \int_0^T \left\|e^{-\tau\mathscr{B}(\tilde{u}_T)}\right\|_{\mathfrak{L}(\mathscr{L}_p(\Omega^*),\mathscr{W}_{p,D}^{2\gamma}(\Omega^*))} \|\tilde{u}^0\|_p d\tau,\\ &\le \left[\frac{\mathscr{K}(\mathscr{R}_0)}{1-\gamma} e^{\zeta(\mathscr{R}_0)T}T^{1-\gamma}\right] \left\||\tilde{u}^0\right|_p \le c(\mathscr{R}_0),\end{aligned} \tag{25}$$

$\tilde{\Phi}(\mathfrak{X}_T)$ is bounded in $\mathscr{W}_{p,D}^{2\gamma}(\Omega^*)$, the latter is being compact ingrained in $\mathfrak{C}(\bar{\Omega}^*)$ since $2\gamma \in n/p$. To validate the property of continuity of $\tilde{\Phi}$,

$$\begin{aligned}e^{-\tau\mathscr{B}(\tilde{u}_T)} - e^{-\tau\mathscr{B}(\tilde{v}_T)} &= -\int_0^\tau \frac{d}{\mathrm{dx}} e^{-(\tau-x)\mathscr{B}(\tilde{u}_T)}e^{-x\mathscr{B}(\tilde{v}_T)}\mathrm{dx},\\ &= -\int_0^\tau e^{-(\tau-x)\mathscr{B}(\tilde{u}_T)}e^{-x\mathscr{B}(\tilde{v}_T)}[-\mathscr{B}(\tilde{v}_T) + \mathscr{B}(\tilde{u}_T)]\mathrm{dx},\\ &= -\int_0^\tau e^{-(\tau-x)\mathscr{B}(\tilde{u}_T)}e^{-x\mathscr{B}(\tilde{v}_T)}(\tilde{\varphi}(\tilde{u}_T) - \tilde{\varphi}(\tilde{v}_T))\mathrm{dx},\end{aligned} \tag{26}$$

thereby using (19), (21), and (24), we get

$$\begin{aligned}&\left\|e^{-\tau\mathscr{B}(\tilde{u}_T)} - e^{-\tau\mathscr{B}(\tilde{v}_T)}\right\|_{\mathfrak{L}(\mathscr{L}_p(\Omega^*),\mathscr{W}_{p,D}^{2\gamma}(\Omega^*))}\\ &\quad\le \int_0^\tau \left\|e^{-(\tau-x)\mathscr{B}(\tilde{u}_T)}\right\|_{\mathfrak{L}(\mathscr{L}_p(\Omega^*),\mathscr{W}_{p,D}^{2\gamma}(\Omega^*))} \|\tilde{\varphi}(\tilde{u}_T) - \tilde{\varphi}(\tilde{v}_T)\|_{\mathfrak{L}(\mathscr{L}_p(\Omega^*))} \left\|e^{-x\mathscr{B}(\tilde{v}_T)}\right\|_{\mathfrak{L}(\mathscr{L}_p(\Omega^*))} \mathrm{dx},\\ &\quad\le c(\mathscr{R}_0)e^{c(\mathscr{R}_0)T}\tau^{1-\gamma}\|\tilde{\varphi}(\tilde{u}_T) - \tilde{\varphi}(\tilde{v}_T)\|_\infty.\end{aligned} \tag{27}$$

The continuous embedding of $\mathscr{W}^{2\gamma}_{p,D}(\Omega^*)$ in $\mathfrak{C}(\bar{\Omega}^*)$

$$\begin{aligned}\|\tilde{\Phi}(\tilde{u}_T)-\tilde{\Phi}(\tilde{v}_T)\|_\infty &\le c\|\tilde{\Phi}(\tilde{u}_T)-\tilde{\Phi}(\tilde{v}_T)\|_{\mathscr{W}^{2\gamma}_{p,D}(\Omega^*)}\\ &\le c\int_0^T\left\|e^{-\tau\mathscr{B}(\tilde{u}_T)}-e^{-\tau\mathscr{B}(\tilde{v}_T)}\right\|_{\mathfrak{L}\left(\mathscr{L}_p(\Omega^*),\mathscr{W}^{2\gamma}_{p,D}(\Omega^*)\right)}\left\|\tilde{u}^0\right\|_p d\tau,\end{aligned} \tag{28}$$

using (25) and (27), we get

$$\begin{aligned}\|\tilde{\Phi}(\tilde{u}_T)-\tilde{\Phi}(\tilde{v}_T)\|_\infty &\le c\int_0^T c(\mathscr{R}_0)e^{c(\mathscr{R}_0)T}\tau^{1-\alpha}\,\|\tilde{\varphi}(\tilde{u}_T)-\tilde{\varphi}(\tilde{v}_T)\|_\infty\,\left\|\tilde{u}^0\right\|_p d\tau,\\ &\le c_1(\mathscr{R}_0)\,\|\tilde{\varphi}(\tilde{u}_T)-\tilde{\varphi}(\tilde{v}_T)\|_\infty,\end{aligned} \tag{29}$$

for $\tilde{u}_T, \tilde{v}_T \in \mathfrak{X}_T$. Hence, the required results on continuity $\tilde{\Phi}: \mathfrak{X}_T \longrightarrow \mathfrak{X}_T$ are established through (17). Using the precompact image of $\tilde{\Phi}\in\mathfrak{C}(\mathfrak{X}_T,\mathfrak{X}_T)$, the theorem due to Schauder shows the presence of point $\tilde{u}_T\in\mathfrak{X}_T$with the property $\tilde{u}_T=\tilde{\Phi}(\tilde{u}_T)$. Now, define $\tilde{u}(\tau)$ as

$$\tilde{u}(\tau) \coloneqq e^{-\tau\mathscr{B}(\tilde{u}_T)}\tilde{u}^0, \tau\in[0,T]. \tag{30}$$

in order to obtain the solution for (1). If $0\le\tilde{u}^0$, then $0\le\tilde{u}(\tau)$ for $\tau\in[0,T]$ because the semigroup is positive. This completes proof of part (i).

Part (ii). Now, assume a nonnegative $\tilde{\varphi}\in\mathfrak{C}(\mathfrak{R},\mathfrak{R})$ and that $a(1+|x|)\ge\tilde{\varphi}(x), x\in\mathfrak{R}$, for positive a. Let us presume $\tilde{u}^0\in\mathscr{L}_p(\Omega^*)$ with $n<2p$. We modify the set

$$\begin{aligned}\mathscr{R}_0 &\coloneqq T\left\|\tilde{u}^0\right\|_\infty,\\ \mathfrak{X}_T &\coloneqq \bar{\mathbb{B}}_{\mathscr{L}_p(\Omega^*)}(0,\mathscr{R}_0).\end{aligned} \tag{31}$$

□

One of the presumption on $\tilde{\varphi}$ includes

$$\tilde{\varphi}\in\mathfrak{BC}\left(\mathfrak{X}_T,\mathscr{L}_p(\Omega^*)\right). \tag{32}$$

Further, $\mathscr{W}^{2-2\varepsilon}_{p,D}(\Omega^*)$ maintains its continuity for small $n<2p$ in $\mathfrak{C}(\bar{\Omega}^*)$ over $\varepsilon>0$. Thus, for $\omega\in\mathscr{W}^2_{p,D}(\Omega^*)$, we have

$$\begin{aligned}\left\|\tilde{\Phi}(\tilde{u}_T)\omega\right\|_p &\le c\|\tilde{\varphi}(\tilde{u}_T)\|_p\|\omega\|_{\mathscr{W}^{2-2\varepsilon}_{p,D}(\Omega^*)},\\ &\le c\|\tilde{\varphi}(\tilde{u}_T)\|_p\|\omega\|_p^{\varepsilon}\|\omega\|^{1-\varepsilon}_{\mathscr{W}^2_{p,D}(\Omega^*)},\\ &\le c(\delta)\|\tilde{\varphi}(\tilde{u}_T)\|_p^{1/\varepsilon}\|\omega\|_p+\delta\|\omega\|_{\mathscr{W}^2_{p,D}(\Omega^*)},\end{aligned} \tag{33}$$

with meager positive δ. Hence, the relation (20) holds good if the equality (33) and the corresponding results are combined. Moreover, (21) holds for $j=p$. Defining $\tilde{\Phi}$ as in (22), by similar arguments as in previous part, a precompact image of $\tilde{\Phi}\in\mathfrak{C}(\mathfrak{X}_T,\mathfrak{X}_T)$ can be arrived (26) along with continuity that follows from the relations (21), (24), and (33) as given in the following:

$$\begin{aligned}&\left\|e^{-\tau\mathscr{B}(\tilde{u}_T)}-e^{-\tau\mathscr{B}(\tilde{v}_T)}\right\|_{\mathfrak{L}\left(\mathscr{L}_p(\Omega^*)\right)}\\ &\le\int_0^\tau\left\|e^{-(\tau-x)\mathscr{B}(\tilde{u}_T)}\right\|_{\mathfrak{L}\left(\mathscr{L}_p(\Omega^*)\right)}\|\tilde{\varphi}(\tilde{u}_T)-\tilde{\varphi}(\tilde{v}_T)\|_{\mathfrak{L}\left(\mathscr{L}_p(\Omega^*),\mathscr{W}^{2-2\varepsilon}_{p,D}(\Omega^*)\right)}\left\|e^{-x\mathscr{B}(\tilde{v}_T)}\right\|_{\mathfrak{L}\left(\mathscr{L}_p(\Omega^*),\mathscr{W}^{2-2\varepsilon}_{p,D}(\Omega^*)\right)}dx,\\ &\le c(\mathscr{R}_0)\,\|\tilde{\varphi}(\tilde{u}_T)-\tilde{\varphi}(\tilde{v}_T)\|_p.\end{aligned} \tag{34}$$

The required result is achieved by applying Schauder's fixed-point theorem.

Corollary 2. *Let $\tilde{u}$ be the solution to (1) provided by Theorem 1. If $p\ge2$, then*

$$\|\nabla\tilde{u}_T\|_2^2+\int_{\Omega^*}\tilde{\varphi}(\tilde{u}_T)|\tilde{u}_T|^2\,dx=\int_{\Omega^*}\left(\tilde{u}^0-\tilde{u}(T)\right)\tilde{u}_T\,dx\le2T\left\|\tilde{u}^0\right\|_2^2. \tag{35}$$

If $\tilde{u}^0\in\mathscr{W}^\gamma_p(\Omega^*)$ for some positive γ, then $\tilde{u}_T\in\mathscr{W}^2_p(\Omega^*)$

$$-\Delta\tilde{u}_T+\tilde{\varphi}(\tilde{u}_T)\tilde{u}_T=\tilde{u}^0-\tilde{u}(T)\ in\ \Omega^*, \tilde{u}_T=0\ on\ \partial\Omega^*. \tag{36}$$

Proof. The proof is the direct consequence of Theorem 1. □

3. Uniqueness Result

Following the methodology adopted in [24], we demonstrate the property of uniqueness for small data in (1) if $\mathscr{R}_0\coloneqq T\left\|\tilde{u}^0\right\|_\infty$ is small along with locally Lipschitz continuous

$$\tilde{\varphi}:\mathfrak{R}\longrightarrow\mathfrak{R}, \tag{37}$$

and

$$x\longmapsto\tilde{\varphi}(x)x, \tag{38}$$

is a function which is nondecreasing.

Assume that the existence of solutions is true and the relations (37) and (38) hold good, let the solutions of (1) be $\tilde{u}$ and $\tilde{v}$ with $\tilde{u}(0)=\tilde{u}^0=\tilde{v}(0)\in\mathscr{L}_\infty(\Omega^*)$. Fix $\omega_T\coloneqq$

$\tilde{u}_T - \tilde{v}_T$.

$$\Longrightarrow \|\tilde{\varphi}(\tilde{u}_T) - \tilde{\varphi}(\tilde{v}_T)\|_2 \leq L(\mathscr{R}_0)\|\tilde{u}_T - \tilde{v}_T\|_2, = L(\mathscr{R}_0)\|\omega_T\|_2, \quad (39)$$

for fixed value $L(\mathscr{R}_0)$, obtained by the definition on $\tilde{\varphi}$, which behaves as uniformly Lipschitz over its region of existence, (i.e., on the set $[-\mathscr{R}_0, \mathscr{R}_0]$). This leads to the fact (35) which entails that

$$\begin{aligned}\|\nabla\omega_T\|_2^2 &= -\int_{\Omega^*}[\tilde{u}_T - \tilde{v}_T](\tilde{\varphi}(\tilde{u}_T)\tilde{u}_T - \tilde{\varphi}(\tilde{v}_T)\tilde{v}_T)\mathrm{d}x \\ &\quad + \int_{\Omega^*}\omega_T[\tilde{v}(T) - \tilde{u}(T)]\mathrm{d}x, \\ &\leq \|\tilde{v}(T) - \tilde{u}(T)\|_2\|\omega_T\|_2.\end{aligned} \quad (40)$$

where (38) allows inequality to end. Now, owing to (21), (26), and (39), we get

$$\begin{aligned}\|\tilde{v}(T) - \tilde{u}(T)\|_2 &= \left\|\left(e^{-\tau\mathscr{B}(\tilde{u}_T)} - e^{-\tau\mathscr{B}(\tilde{v}_T)}\right)\tilde{u}^0\right\|_2, \\ &\leq \int_0^T \left\|e^{-(\tau-x)\mathscr{B}(\tilde{u}_T)}\right\|_{\mathfrak{L}(\mathscr{L}_2(\Omega^*))}\|\tilde{\varphi}(\tilde{u}_T) - \tilde{\varphi}(\tilde{v}_T)\|_2\left\|e^{-x\mathscr{B}(\tilde{v}_T)}\right\|_{\mathfrak{L}(\mathscr{L}_\infty(\Omega^*))}\|\tilde{u}^0\|_\infty dx, \\ &\leq \mathscr{R}_0 L(\mathscr{R}_0)\|\omega_T\|_2.\end{aligned} \quad (41)$$

So, (40) and (41) cover it

$$\|\Omega^*{}_T\|_2^2 \leq \mathscr{R}_0\, L(\mathscr{R}_0)\|\Omega^*{}_T\|_2^2 \leq c(\Omega^*)\mathscr{R}_0 L(\mathscr{R}_0)\|\nabla\omega_T\|_2^2, \quad (42)$$

where $c(\Omega^*)$ is borrowed from Poincare'sine quality. Now, if $\mathscr{R}_0 L(\mathscr{R}_0)c(\Omega^*) < 1$, it refers to $\omega_T \equiv 0$ in $\omega \Longrightarrow \tilde{u}_T = \tilde{v}_T$.

In the following proposition, we study the monotonicity condition and its effect on uniqueness theory.

Proposition 3. *Let $\tilde{\varphi}$ be unknown that satisfies (37), $2p > n$, and consider $\tilde{u}^0 \in \mathscr{L}_p(\Omega^*)$. There is $\mathscr{M} > 0$ such that (1) has a unique value*

$$\tilde{u} \in \mathfrak{C}([0,T], \mathscr{L}_p(\Omega^*)) \cap \mathfrak{C}^1((0,T], \mathscr{L}_p(\Omega^*)) \cap \mathfrak{C}\left((0,T], \mathscr{W}_p^2(\Omega^*)\right), \quad (43)$$

based on the inequality $T\|\tilde{u}^0\|_p \leq \mathscr{M}$.

Proof. The proof is obtained through Banach's fixed-point theorem. Fix $2\gamma \in (n/p, 2)$ and take $\mathscr{R}_0 > 0$ along with

$$\mathfrak{X}_T := \bar{\mathbb{B}}_{\mathscr{W}_{p,D}^{2\gamma}(\Omega^*)}(0, \mathscr{R}_0). \quad (44)$$

Notice that $\mathscr{W}_{p,D}^{2\gamma}(\Omega^*)$ embeds continuously into $\mathfrak{C}(\bar{\Omega}^*)$. Thus, as $\tilde{\varphi}$ is uniformly Lipschitz continuous on compact sets, there exists a constant $L(\mathscr{R}_0) > 0$ with $\tilde{\varphi}$

$$\|\tilde{\varphi}(\tilde{u}_T) - \tilde{\varphi}(\tilde{v}_T)\|_\infty \leq L(\mathscr{R}_0)\|\tilde{u}_T - \tilde{v}_T\|_{\mathscr{W}_{p,D}^{2\gamma}(\Omega^*)}, \tilde{u}_T, \tilde{v}_T \in \mathfrak{X}_T. \quad (45)$$

Especially from the proof of Theorem 1, we can write

$$\left\|\tilde{\Phi}(\tilde{u}_T)\right\|_{\mathscr{W}_{p,D}^{2\gamma}(\Omega^*)} \leq \frac{\mathscr{K}(\mathscr{R}_0)}{1-\gamma}e^{\zeta(\mathscr{R}_0)T}T^{1-\gamma}\left\|\tilde{u}^0\right\|_p, \quad (46)$$

Moreover, (27) and (45) combines to give

$$\begin{aligned}&\left\|e^{-\tau\mathscr{B}(\tilde{u}_T)} - e^{-\tau\mathscr{B}(\tilde{v}_T)}\right\|_{\mathfrak{L}\left(\mathscr{L}_p(\Omega^*), \mathscr{W}_{p,D}^{2\gamma}(\Omega^*)\right)} \\ &\quad \leq c(\mathscr{R}_0)e^{c(\mathscr{R}_0)T}T^{1-\gamma}\|\tilde{u}_T - \tilde{v}_T\|_{\mathscr{W}_{p,D}^{2\gamma}(\Omega^*)},\end{aligned} \quad (47)$$

for some $\tilde{u}_T, \tilde{v}_T \in \mathfrak{X}_T$ and positive $c(\mathscr{R}_0)$. Therefore,

$$\begin{aligned}&\left\|\tilde{\Phi}(\tilde{u}_T) - \tilde{\Phi}(\tilde{v}_T)\right\|_{\mathscr{W}_{p,D}^{2\gamma}(\Omega^*)} \\ &\quad \leq \int_0^T \left\|\tilde{u}^0\right\|_p \left\|e^{-\tau\mathscr{B}(\tilde{u}_T)} - e^{-\tau\mathscr{B}(\tilde{v}_T)}\right\|_{\mathfrak{L}\left(\mathscr{L}_p(\Omega^*), \mathscr{W}_{p,D}^{2\gamma}(\Omega^*)\right)} d\tau, \\ &\quad \leq c_1(\mathscr{R}_0)\left\|\tilde{u}^0\right\|_p e^{c(\mathscr{R}_0)T}T^{2-\gamma}\,\|\tilde{u}_T - \tilde{v}_T\|_{\mathscr{W}_{p,D}^{2\gamma}(\Omega^*)}.\end{aligned} \quad (48)$$

The inequality together with (46) shows that $\tilde{\Phi} : \mathfrak{X}_T \longrightarrow \mathfrak{X}_T$ behaves like a contraction mapping with $T\|\tilde{u}^0\|_p$ being very small. Consequently, if $T\|\tilde{u}^0\|_p$ is very small, then there exists a unique $\tilde{u}_T \in \mathfrak{X}_T$ with $\tilde{\Phi}(\tilde{u}_T) = \tilde{u}_T$. □

4. Conclusion

As discussed, the nonlinear force term on the right hand side plays an important role in establishing the existence of solution. It is clear that the cash flows to the market needs to be examined and must always be within limits. This fixed value must be compared with the actual price of the option and the investment decision that must be made.

Data Availability

No data were used to support the finding of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this work and read and approved the final version of the manuscript.

References

[1] G. Barles and H. E. Soner, "Option pricing with transaction costs and a nonlinear Black-Scholes equation," *Finance and Stochastics*, vol. 2, no. 4, pp. 369–397, 1998.

[2] M. Davis, V. G. Panas, and T. Zariphopoulou, "European option pricing with transaction costs," *SIAM Journal on Control and Optimization*, vol. 31, no. 2, pp. 470–493, 1993.

[3] B. Düring, *Black-Scholes Type Equations: Mathematical Analysis, Parameter Identification and Numerical Solution, [M.S. thesis]*, Universität Mainz, 2005.

[4] P. Wilmott, S. Howison, and J. Dewynne, *The Mathematics of Financial Derivatives: A Student Introduction*, Cambridge University Press, USA, 2012.

[5] G. O. Agyeman and F. T. Oduro, "Radial basis function in nonlinear Black-Scholes option pricing equation with transaction cost," *Asian Journal of Probability and Statistics*, vol. 5, no. 3, pp. 1–11, 2019.

[6] R. A. Adams and J. F. Fournier, *Sobolev Spaces*, Academic Press, New York, 2nd edition, 2003.

[7] M. Pilant and W. Rundell, "Fixed point methods for a nonlinear parabolic inverse coefficient problem," *Communications in Partial Differential Equations*, vol. 13, no. 4, pp. 469–493, 1988.

[8] X. B. Shu and Y. Shi, "A study on the mild solution of impulsive fractional evolution equations," *Applied Mathematics and Computation*, vol. 273, pp. 465–476, 2016.

[9] A. Nanda and P. C. Das, "Determination of the non-linear reaction term in a coupled reaction-diffusion system," *Numerical Functional Analysis and Optimization*, vol. 21, no. 3-4, pp. 489–508, 2000.

[10] R. Sowrirajan and K. Balachandran, "Existence and uniqueness results for a semilinear Black-Scholes type equation," *Nonlinear Analysis:Real world Applications*, vol. 11, pp. 2796–2809, 2010.

[11] Y. Guo, M. Chen, X. B. Shu, and F. Xu, "The existence and Hyers-Ulam stability of solution for almost periodical fractional stochastic differential equation with fBm," *Stochastic Analysis and Applications*, vol. 39, no. 4, pp. 643–666, 2021.

[12] X. Ma, X. B. Shu, and J. Mao, "Existence of almost periodic solutions for fractional impulsive neutral stochastic differential equations with infinite delay," *Stochastics and Dynamics*, vol. 20, no. 1, p. 2050003, 2020.

[13] R. Abe and N. Ishimura, "Existence of solutions for the nonlinear partial diferential equation arising in the optimal investment problem," *Proceedings of the Japan Academy Series A Mathematical Sciences*, vol. 84, pp. 11–14, 2008.

[14] K. Ezzinbi, S. Ghnimi, and M. A. Taoudi, "Existence results for some partial integrodifferential equations with nonlocal conditions," *Glasnik Matematiĉki*, vol. 51, no. 71, pp. 413–430, 2016.

[15] C. Walker, "On positive solutions of some system of reaction-diffusion equations with nonlocal initial conditions," *Journal fur die Reine und Angewandte Mathematik*, vol. 660, pp. 149–179, 2011.

[16] P. Amster, C. G. Averbuj, M. C. Mariani, and D. Rial, "A Black-Scholes option pricing model with transaction costs," *Journal of Mathematical Analysis and Applications*, vol. 303, no. 2, pp. 688–695, 2005.

[17] F. E. Benth, K. H. Karlsen, and K. Reikvam, "A semilinear Black and Scholes partial differential equation for valuing American options," *Finance and Stochastics*, vol. 7, no. 3, pp. 277–298, 2003.

[18] Q. L. Luo and W. C. Sheng, "A unique solution to a semilinear Black-Scholes partial differential equation for valuing multi-assets of American options," *Journal of Shanghai University (English Edition)*, vol. 11, no. 4, pp. 344–350, 2007.

[19] M. C. Mariani, E. K. Ncheuguim, and I. Sengupta, "Solution to a nonlocal Black-Scholes equation," *Electronic Journal of Differential Equations*, vol. 2011, no. 158, pp. 1–10, 2011.

[20] X. J. Shi, L. Yang, and Z. H. Huang, "A fixed point method for the linear complementarity problem arising from American option pricing," *Acta Mathematicae Applicatiae Sinica English Series*, vol. 32, no. 4, pp. 921–932, 2016.

[21] O. G. Gaxiola and J. A. Santiago, "The Black-Scholes operator as the generator of a C_0-semigroup and applications," *International Journal of Pure and Applied Mathematics*, vol. 76, no. 2, 2012.

[22] C. Walker, "Strong solutions to a nonlocal-in-time semilinear heat equation," 2020, https://arxiv.org/abs/2007.05029.

[23] V. N. Starovoitov, "Initial boundary value problem for a nonlocal in time parabolic equation," *Siberian Electonic Mathematical Reports*, vol. 15, pp. 1311–1319, 2018.

[24] V. N. Starovoitov, "Boundary value problem for a global in time parabolic equation," 2020, http://arxiv.org/abs/2001.04058.