Discretization Fractional-Order Biological Model with Optimal Harvesting

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Received 16 March 2022; Revised 3 May 2022; Accepted 10 May 2022; Published 28 May 2022

Academic Editor: Simeon Reich

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In this paper, a discretization of a three-dimensional fractional-order prey-predator model has been investigated with Holling type III functional response. All its fixed points are determined; also, their local stability is investigated. We extend the discretized system to an optimal control problem to get the optimal harvesting amount. For this, the discrete-time Pontryagin’s maximum principle is used. Finally, numerical simulation results are given to confirm the theoretical outputs as well as to solve the optimality problem.

1. Introduction

Recently, population models have been given attention by researchers due to their importance and their relation to real life. In fact, there are different processes to describe population models, namely, ordinary and partial differential equations, difference equations, and fractional-order differential equations. Fractional-order derivative systems play a significant role in real-life problems due to their ability to provide accurate descriptions of different linear and nonlinear phenomena. We refer to the most recent works that are done by many researchers [1–5]. The mathematical behavior of fractional-order prey-predator models is also discussed and investigated in [6–10] and references therein. Discrete-time mathematical models hit many branches of pure and applied mathematics; also, they depict many real situations in life due to exhibiting and producing more plentiful dynamical behavior than those that have been seen in continuous time models of the same types. These types of models can be used to study and investigate problems in biology, economics, engineering, etc. For more details, see [11–17].

In the last decades, there are many articles that deal with the subject of harvesting in different approaches for details (see [18–21] and references therein). We also refer to the excellent book that is written by Agrawal et al. [22]. In this book, recent important results of infection diseases and methods with new approximations are analyzed and developed. In [23], the authors investigated and discussed coronavirus disease in a mathematical model that deals with coronavirus, and an effective control policy for the COVID-19 outbreak is set.

The study of biological models with harvesting, in particular, predator-prey models, is an important subject in the commercial harvesting industry and for many scientific communities including biology, ecology, and economics. The fundamental problem in commercial exploitation of renewable resources is to get the optimal gain between present and future harvests, so that employing optimal control theory in this field is a very useful tool that can help to make decisions in various problems in real world including biological situations [24, 25].

This work is organized as follows. In Section 2, a fractional-order system is discretized and all its equilibrium points are found. In Section 3, the local stability of all its equilibria is discussed. Section 4 deals with the employment of the discrete-time Pontryagin’s maximum principle to solve the optimal control solution and its corresponding optimal state solutions. In Section 5, numerical simulations are done to confirm the theoretical outcomes. Finally, conclusion is given in Section 6.
2. The Discrete-Time Model and Its Fixed Points

First, the authors considered and analyzed in [26] the following fractional-order three-dimensional prey-predator model with the functional response of Holling type III:

\[
\begin{align*}
D^\alpha x(t) &= x(1-x) - \frac{ax^2y}{1+ax^2}, \\
D^\alpha y(t) &= \frac{bx^2y}{1+ax^2} - \frac{dy^2z}{1+dy^2} - m_1y - hy, \\
D^\alpha z(t) &= \frac{cy^2z}{1+dy^2} - m_2z,
\end{align*}
\]

where \(X, Y,\) and \(Z\) are the density of the prey population, the middle predator population, and the top predator population. \(m_1, m_2, a,\) and \(b\) are the death rates for the middle predator population and top predator population, the capture rate of prey and middle predator, and the conservation rate of prey \(X,\) respectively. The parameters \(d\) and \(c\) are the conservation rates of middle predator \(Y\) to the top predator \(Z.\) The parameter \(h\) represents the harvesting rate. For more details about the interpretations of the parameters, see [26]. Here, \(D^\alpha\) is called the \(\alpha\)-order Caputo differential operator [27, 28].

In this work, we investigate the dynamic behavior of the discretization of the system (1); then, we extend the discretized system to the optimal control problem in the next section. The discretization process of the system (1) is given as follows:

Assume that \(x(0) = x_0,\) \(y(0) = y_0,\) and \(z(0) = z_0\) are the initial conditions of system (1), so that the discretization of the system (1) with piecewise constant arguments is given as

\[
\begin{align*}
D^\alpha x(t) &= x([t/s][s]) - \frac{ax([t/s][s])^2y([t/s][s])}{1+ax([t/s][s])^2}, \\
D^\alpha y(t) &= \frac{bx([t/s][s])^2y([t/s][s])}{1+ax([t/s][s])^2} - \frac{dy([t/s][s])^2z([t/s][s])}{1+dy([t/s][s])^2} - m_1y([t/s][s]) - hy([t/s][s]), \\
D^\alpha z(t) &= \frac{cy([t/s][s])^2z([t/s][s])}{1+dy([t/s][s])^2} - m_2z([t/s][s]).
\end{align*}
\]

First, let \(t \in [0,s),\) so \(t/s \in [0,1).\) Thus, we obtain

\[
\begin{align*}
D^\alpha x(t) &= x_0(1-x_0) - \frac{ax_0^2y_0}{1+ax_0^2}, \\
D^\alpha y(t) &= \frac{bx_0^2y_0}{1+ax_0^2} - \frac{dy_0^2z_0}{1+dy_0^2} - m_1y_0 - hy_0, \\
D^\alpha z(t) &= \frac{cy_0^2z_0}{1+dy_0^2} - m_2z_0.
\end{align*}
\]

The solution of (3) is reduced to

\[
x_{n+1}(t) = x_n(t) + \int_0^t x_n\left(\frac{ax_0^2y_0}{1+ax_0^2}\right) \cdot \left(x_n(t)(1-x_n(t)) - \frac{ax_0^2y_0}{1+ax_0^2}\right) d\tau,
\]

where \(x_n(t) = x_n(0) + \int_0^t x_n(\tau) \cdot \left(x_n(\tau)(1-x_n(\tau)) - \frac{ax_0^2y_0}{1+ax_0^2}\right) d\tau,\) \(t > 0,\) \(n = 0, 1, 2, \ldots,\) \(\int_0^t \cdot d\tau = \int_0^t (t-x)^{\alpha-1} dx,\) \(\alpha > 0.\) Thus, after repeating the discretization process \(n\) times, we obtain

\[
\begin{align*}
x_{n+1}(t) &= x_n(t) + \frac{(t-n\alpha)^\alpha}{a\Gamma(\alpha)} \cdot \left(x_n(t)(1-x_n(t)) - \frac{ax_0^2y_0}{1+ax_0^2}\right),
\end{align*}
\]

Second, let \(t \in [s, 2s),\) so \(t/s \in [1, 2).\) Hence, we get the following system:

\[
\begin{align*}
D^\alpha x(t) &= x_1(1-x_1) - \frac{ax_1^2y_1}{1+ax_1^2}, \\
D^\alpha y(t) &= \frac{bx_1^2y_1}{1+ax_1^2} - \frac{dy_1^2z_1}{1+dy_1^2} - m_1y_1 - hy_1, \\
D^\alpha z(t) &= \frac{cy_1^2z_1}{1+dy_1^2} - m_2z_1,
\end{align*}
\]

which have the following solution:

\[
\begin{align*}
x_1(t) &= x_1(s) + \int_s^t x_1(s)(1-x_1(s)) \cdot \frac{ax_1^2y_1(s)}{1+ax_1^2} ds, \\
y_1(t) &= y_1(s) + \int_s^t y_1(s)(1-y_1(s)) \cdot \frac{ax_1^2y_1(s)}{1+ax_1^2} ds, \\
z_1(t) &= z_1(s) + \int_s^t z_1(s) \cdot \frac{cy_1^2z_1(s)}{1+dy_1^2} ds.
\end{align*}
\]
where $t \in [ns, (n + 1)s]$. For $t \rightarrow (n + 1)s$, then the previous system is reduced to

\begin{align*}
x_{n+1} &= x_n + \frac{s^a}{a(t)} \left( x_n(1 - x_n) - \frac{ax_n^2 y_n}{1 + ax_n^2} \right), \\
y_{n+1} &= y_n + \frac{s^a}{a(t)} \left( b x_n y_n - \frac{dy_n^2 z_n}{1 + dy_n^2} - m_1 y_n - h y_n \right), \\
z_{n+1} &= z_n + \frac{s^a}{a(t)} \left( c y_n^2 z_n - m_2 z_n \right).
\end{align*}

(8)

In order to get the fixed points of the considered system (8), we have to solve the following simultaneous equations:

\begin{align*}
x = x + \frac{s}{a(t)} \left( x(1 - x) - \frac{ax^2 y}{1 + ax^2} \right), \\
y = y + \frac{s}{a(t)} \left( bx^2 y - \frac{dy^2 z - m_1 y - h y}{1 + dy^2} \right), \\
z = z + \frac{s}{a(t)} \left( cy^2 z - m_2 z \right).
\end{align*}

(9)

Thus, all possible fixed points of system (8) are as follows:

(1) The trivial fixed point $E_0 = (0, 0, 0)$ and the fixed point $E_1 = (1, 0, 0)$ exist without any restriction on the parameters.

(2) The top predator free fixed point $E_2 = (\bar{x}, y, 0)$ exists if $b > a(m_1 + h)$ and $0 < \bar{x} < 1$ where $\bar{x} = \sqrt{(m_1 + h)/(b - a(m_1 + h))}$ and $y = (1/ax)(1 - \bar{x})(1 + ax^2)$.

(3) The interior fixed point $E_3 = (x^*, y^*, z^*)$ exists if $c > m_2 d$ and $bx^2 y/(1 + ax^2) > (m_1 + h)$ where $y^* = \sqrt{m_2/(c - m_2 d)}$, $x^*$ is the positive root of the equation $ax^2 - ax^2 + (ay^* + 1)x - 1 = 0$, and $z^* = (1 + d)y^*/dy^* ((bx^2 y/(1 + ax^2)) - (m_1 + h))$.

Then, the Jacobian matrix $J$ of the system (8) evaluated at any point $(x, y, z)$ is given by

\begin{align*}
J &= \begin{bmatrix}
1 + m - 2mx - \frac{2amxy}{k^2} & - \frac{max^2}{k} & 0 \\
\frac{2mbxy}{k^2} & 1 + m \left( \frac{bx^2}{k} - \frac{2dyz}{k_1} - (m_1 + h) \right) & - \frac{mdy}{k_1} \\
0 & \frac{cmyz}{k_1} & 1 + \frac{cmy^2}{k_1} - mm_2
\end{bmatrix},
\end{align*}

(10)

where $k = (1 + ax^2)$, $k_1 = (1 + dy^2)$, and $m = s^a/a(t)$. Therefore, the characteristic equation of the Jacobian matrix is given by

\begin{align*}
p(\lambda) &= \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0, \\
a_2 &= -(A_1 + A_2 + A_3)a_1 \\
&= (A_1 A_3 + A_1 A_2 + A_1 A_2) + A_5 + A_4, \\
a_0 &= A_1 A_2 A_3 - A_1 A_5 - A_2 A_4 = -w - A_1 A_4, \text{ with } w = A_1 A_2 A_3 + A_1 A_5.
\end{align*}

\begin{align*}
A_1 &= 1 + m - 2mx - \frac{2amxy}{k^2}, \\
A_2 &= 1 + m \left( \frac{bx^2}{k} - \frac{2dyz}{k_1^2} - (m_1 + h) \right),
\end{align*}

(11)

In order to study stability analysis of the fixed points of the model (8), we give Definition 1 and Lemma 2.

**Definition 1** (see [29]). A fixed point $x^*$ of system $\tilde{x}_{t+1} = f(\tilde{x}_t)$ is said to be as follows:

...
Lemma 2 (see [29]). Let \( F(\lambda) = \lambda^2 + p\lambda + q \) be a polynomial of degree two. Suppose that \( F(1) > 0 \) and \( \lambda_1, \lambda_2 \) are the roots of \( F \); then

(1) \( |\lambda_1| < 1 \) and \( |\lambda_2| < 1 \), if and only if \( F(-1) > 0 \) and \( q < 1 \)

(2) \( |\lambda_1| > 1 \) and \( |\lambda_2| > 1 \), if and only if \( F(-1) > 0 \) and \( q > 1 \)

(3) \( |\lambda_1| > 1 \) and \( |\lambda_2| < 1 \) (or \( |\lambda_1| < 1 \) and \( |\lambda_2| > 1 \)), if and only if \( F(-1) < 0 \)

(4) \( \lambda_1 = -1 \) and \( |\lambda_2| \neq 1 \), if and only if \( F(-1) = 0 \) and \( p \neq 0, 2 \)

Theorem 3 discusses the dynamical behavior of the fixed points \( E_0, E_1, E_2, \) and \( E_3 \) of the system (8), respectively.

Theorem 3. For the system (8), the fixed point \( E_0 \) can be classified as follows:

(1) The point \( E_0 \) will never be a sink point

(2) The point \( E_0 \) is a source, if and only if \( m > \max \{2/(m_1 + h), 3/m_2\} \)

(3) The point \( E_0 \) is a saddle point, if and only if \( m < \min \{2/(m_1 + h), 3/m_2\} \)

(4) The point \( E_0 \) is a nonhyperbolic point, if and only if \( m = 2/(m_1 + h) \) or \( m = 3/m_2 \), where \( m = s^\epsilon/\alpha(x) \), \( m_1 = c_i k_{iR, i}^2 x_i \), and \( m_2 = R \theta c_2 \).

Proof. It is clear that the Jacobian matrix of system (8) at \( E_0 \) can be written as follows:

\[
J(E_0) = \begin{pmatrix}
(1 + m) & 0 & 0 \\
0 & 1 - m(m_1 + h) & 0 \\
0 & 0 & 1 - mm_2
\end{pmatrix}
\]

(12)

The characteristic equation of the Jacobian matrix \( J(E_0) \) is given by

\[
p(\lambda) = ((1 + m) - \lambda)((1 - m(m_1 + h)) - \lambda)((1 - mm_2) - \lambda).
\]

(13)

So that the eigenvalues of \( J(E_0) \) are \( \lambda_1 = (1 + m) \), \( \lambda_2 = 1 - m(m_1 + h) \), and \( \lambda_3 = 1 - mm_2 \). It is clear that \( |\lambda_1| \) is always greater than 1. Hence, the fixed point \( E_0 \) will never be a sink point. It is clear that if \( 0 < m < 2/(m_1 + h) \), then \( |1 - m(m_1 + h)| < 1 \). Therefore, \( |\lambda_2| < 1 \) if and only if \( m < 3/m_2 \), then \( |1 - mm_2| < 1 \). Hence, \( |\lambda_3| < 1 \). Therefore, all results can be obtained.

Theorem 4. For the system (8), the fixed point \( E_1 \) can be classified as follows:

(1) \( E_1 \) is a sink point if and only if \( m \in I_1 \cap I_2 \)

(2) \( E_1 \) is a source point if and only if \( m \in I_3 \)

(3) \( E_1 \) is a saddle point if and only if one of the following holds

\[
m \in I_4 \cap I_5, \quad m \in I_1 \cap I_5, \quad m \in I_1 \cap I_6.
\]

(14)

(4) \( E_1 \) is a nonhyperbolic point if and only if \( m = 2 \) or \( m = 3/m_2 \), where \( I_1 = (0, \min \{2, 3/m_2\}) \), \( I_2 = ((1 + a)((m_1 + h) - 2)/(m_1 + h)(1 + a)/(b), I_3 = (\max \{2, 3/m_2\}, \infty) \), \( I_5 = ((m_1 + h)(1 + a)/(b), \infty) \), and \( I_6 = (0, (1 + a)((m_1 + h) - 2)/(b)) \).

Proof. The Jacobian matrix of system (8) at \( E_1 \) is

\[
J(E_1) = \begin{pmatrix}
1 - m & -\frac{ma}{1 + a} & 0 \\
0 & 1 + \frac{mb}{1 + a} - (m_1 + h) & 0 \\
0 & 0 & 1 - mm_2
\end{pmatrix}
\]

(15)

Therefore, the characteristic equation of the Jacobian matrix \( J(E_1) \) is given by

\[
p(\lambda) = ((1 + m) - \lambda)((1 + \frac{mb}{1 + a} - (m_1 + h)) - \lambda)((1 - mm_2) - \lambda).
\]

(16)

So that the eigenvalues of \( J(E_1) \) are \( \lambda_1 = (1 - m) \), \( \lambda_2 = 1 + (mb/(1 + a)) - (m_1 + h) \), and \( \lambda_3 = 1 - mm_2 \). Now, if \( m < 2 \), then \( |1 - m| < 1 \). This implies that \( |\lambda_1| < 1 \) if \( (1 + a)((m_1 + h) - 2)/(m_1 + h)(1 + a)/(b) \). This gives \( |1 + (mb/(1 + a)) - (m_1 + h)| < 1 \). Therefore, \( |\lambda_2| < 1 \). Assume that \( 0 < m < 2/m_1 \). This implies that \( |1 - mm_2| < 1 \) and \( |\lambda_3| < 1 \). Therefore, all results are got.

Theorem 5. For the system (8), the fixed point \( E_2 \) can be classified as follows:
(1) $E_3$ is a sink point, if and only if the following conditions hold:
\[
(2mau + m^2au - m^2a^2u^2)(1-x) > (m-2)b, \\
(2au + 2mabu - 2ma^2u^2)(1-x) < b.
\]
(17)

(2) $E_3$ is a source if and only if $(2au + 2mabu - 2ma^2u^2)(1-x) > b$

(3) $E_3$ is a saddle point if and only if $(2mau + m^2au - m^2a^2u^2)(1-x) < (m-2)b$

(4) $E_3$ is a nonhyperbolic point if and only if $(2au + m^2au - m^2a^2u^2)(1-x) = (m-2)b$ or $(2au + 2mabu - 2ma^2u^2)(1-x) = b$, where
\[
\sqrt{(m_1 + h)/(b - a(m_1 + h))}, \quad y = (1/a)(1-x)(1 + ax^2), \quad k = 1 + ax^2, \quad a = (m_1 + h).
\]

Proof. It is clear that the Jacobian matrix of system (8) at $E_2$ can be written as follows:
\[
J(E_2) = \begin{bmatrix}
1 - m + \frac{2mau(1-x)}{b} & -mau/b & 0 \\
2m(1-x)(b - au) & 1 - \frac{db(1-x)^2}{a^2u^2 + db^2(1-x)^2} & 0 \\
0 & 0 & 1 + \frac{mcy^2}{k_1} - mm_2
\end{bmatrix}
\]
(18)

Therefore, the characteristic polynomial of the Jacobian matrix at $E_2$ is given by
\[
f(\lambda) = \left(1 - m + \frac{2mau(1-x)}{b} - \lambda\right)F(\lambda) = 0,
\]
(19)

where $F(\lambda) = \lambda^2 + p\lambda + q$ such that $p = 2 + m - (2mau(1-x)/b)$ and $q = 1 - m + (2mau(1-x)/b) + 2m^2au(1-x) - (2ma^2u^2)/b(1-x)$.

Now, the first root of the characteristic polynomial is
\[
\lambda_1 = 1 + (mcy^2/(a^2u^2 + db^2(1-x)^2)) - mm_2, \quad \text{if} \quad 0 < m < 2((a^2u^2 + db^2(1-x)^2))/(c - d/m_1)b^2(1-x)^2 - m_1a^2u^2),
\]
then, we have $1 + (mcy^2/(a^2u^2 + db^2(1-x)^2)) - mm_2 < 1$ and $|\lambda_1| < 1$.

In order to investigate the other roots, we use Lemma 2 so that $F(1) > 0$ if and only if $1 + m - 2 - m + (2mau(1-x)/b) + 1 - m + (2mau(1-x)/b) + 2m^2au(1-x) - (2ma^2u^2/b)(1-x) > 0$. Thus, $F(1) > 0$ if and only if $2m^2au(1-x) - (2ma^2u^2/b)(1-x) > 0$. Since $0 < x < 1$, we get that $F(1)$ is always greater than zero.

Now, we have to show that $F(-1) < 0$ and $q < 1$. If $(2mau + m^2au - m^2a^2u^2)(1-x) > (m-2)b$, then $(2mau(b) + m^2au - (m^2a^2u^2/b)(1-x) > (m-2)$. Hence, $2 - m + (2mau(1-x)/b) + m^2au(1-x) - (m^2a^2u^2/b)(1-x) > 0$.

Therefore, $4 - 2m + (4mau(1-x)/b) + 2m^2au(1-x) - (2ma^2u^2/b)(1-x) > 0$.

This implies that $1 - m + 2 + (2mau(1-x)/b) + 1 - m + (2mau(1-x)/b) + 2m^2au(1-x) - (2ma^2u^2/b)(1-x) > 0$ and $F(-1) > 0$.

Now, if $(2au + 2mabu - 2ma^2u^2)(1-x) < b$, then $(2au/b + 2mabu - (2ma^2u^2/b)(1-x)) < 1$. Hence, $(2mau(1-x)/b) + 2m^2au(1-x) - (2ma^2u^2/b)(1-x) < 0$. Therefore, this implies that $1 - m + (2mau(1-x)/b) + 2m^2au(1-x) - (2ma^2u^2/b)(1-x) < 1$ and $q < 1$. Therefore, the results can be obtained.

Before we study the behavior of the interior fixed point of the system (8), we need Lemma 6.

Lemma 6 (see [29]). Let $p(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$ be a cubic polynomial with real coefficients. Then, the necessary and sufficient conditions that all roots lie in the open disk $|\lambda| < 1$ are as follows:
\[
|a_2 + a_0| < 1 + a_1, \\
|a_2 - 3a_0| < 3 - a_1, \\
|a_2 + 6a_1a_0 - a_0^2| < a_1 < 1.
\]
(20)

Theorem 7. For the system (8), the fixed point $E_3$ is a stable point if the following conditions hold:
\[
c_1 < A_2A_4 < c_2, \\
d_1 < A_2A_4 < d_2, \\
A_2^2A_4^2 + (2\omega + A_3)A_4^2 + kk < 0,
\]
(21)

where $\omega = -(A_1 + A_2 + A_3)$, $kk = u^2 + a_1 + a_2w - 1$, $a_1 = A_1A_2A_3 + A_1A_3$, $a_2 = A_1A_2A_3 + A_1A_2A_3 + A_2A_3$, $a_0 = -(A_1A_2A_3 + A_1A_2A_3 + A_2A_3)$, $A_1 = 1 + m - 2m^2 - (2max/k^2)$, $A_2 = 1 + m/(b^2/k^2) - (2dyz/k^2) - (m_1 + h)$, $A_3 = 1 + (mcy/k^2) - mm_2$, $A_4 = (max/k^2)(2mbx/k^2)$, $A_5 = 2cd^2/yz/k_1^2$, $c_1 = 1 - (A_1A_2A_3 + A_1A_2A_3 + A_2A_3 - A_1A_2A_3 - A_1A_2A_3 + A_1A_2A_3)$, $c_2 = 1 - (A_1A_2A_3 + A_2A_3 + A_1A_2A_3 - A_1A_2A_3 - A_1A_2A_3 + A_1A_2A_3)$, $d_1 = (1/3)(-3 + (A_1A_3 + A_2A_3 + A_1A_2A_3 + A_2A_3 + A_1A_2A_3 - A_1A_2A_3 + A_2A_3)$, $d_2 = (1/3)(3 - (A_1A_3 + A_2A_3 + A_1A_2A_3 + A_2A_3 + A_1A_2A_3 - A_1A_2A_3 + A_2A_3)$.

Proof. The proof is based on Lemma 6.

The Jacobian matrix of system (8) at $E_3$ can be written as follows:

\[
J(E_3) = \begin{bmatrix}
1 - m + \frac{2mau(1-x)}{b} & -mau/b & 0 \\
2m(1-x)(b - au) & 1 - \frac{db(1-x)^2}{a^2u^2 + db^2(1-x)^2} & 0 \\
0 & 0 & 1 + \frac{mcy^2}{k_1} - mm_2
\end{bmatrix}
\]
(18)
Therefore, the characteristic polynomial of the Jacobian matrix $J(E_3)$ at $E_3$ is

$$p(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0.$$  \hspace{1cm} (23)

Now, let $c_1 < A_3A_4 < c_2$; this gives

$$-1 - (A_1A_3 + A_2A_4 + A_1A_4) - A_5 - A_4 - (A_1 + A_2 + A_3 + A_4) + A_5 + A_4 - (A_1 + A_2 + A_3) - A_1A_2 - A_3A_4 < 1 + (A_1A_3 + A_2A_4 + A_1A_4) + A_5 + A_4 - (A_1 + A_2 + A_3 + A_4) + A_5 + A_4 - (A_1 + A_2 + A_3 + A_4),$$

$$-1 - (A_1A_3 + A_2A_4 + A_1A_4) - A_5 - A_4 - (A_1 + A_2 + A_3 + A_4) + A_5 + A_4 - (A_1 + A_2 + A_3 + A_4) + A_5 + A_4 - (A_1 + A_2 + A_3 + A_4) + A_5 + A_4 - (A_1 + A_2 + A_3 + A_4).$$

Hence, $-1 - (A_1A_3 + A_2A_4 + A_1A_4) - A_5 - A_4 - (A_1 + A_2 + A_3 + A_4) + A_5 + A_4 - (A_1 + A_2 + A_3 + A_4) + A_5 + A_4 - (A_1 + A_2 + A_3 + A_4) + A_5 + A_4 - (A_1 + A_2 + A_3 + A_4) + A_5 + A_4 - (A_1 + A_2 + A_3 + A_4).$

Therefore, $-1 - a_1 < a_2 + a_0 < 1 + a_1$ and condition (1) in Lemma 6 hold.

Now, assume that $d_1 < A_3A_4 < d_2$. This gives

$$\frac{1}{3}(-3 + (A_1A_3 + A_2A_4 + A_1A_4) + A_5 + A_4$$

$$+ (A_1 + A_2 + A_3) - 3A_1A_2A_3 - 3A_1A_4) < A_3A_4$$

$$\frac{1}{3}(3 - (A_1A_3 + A_2A_4 + A_1A_4) - A_5 - A_4$$

$$+ (A_1 + A_2 + A_3) - 3A_1A_2A_3 - 3A_1A_4).$$

Hence, $-3 + (A_1A_3 + A_2A_4 + A_1A_4) + A_5 + A_4 + (A_1 + A_2 + A_3) - 3A_1A_2A_3 - 3A_1A_4 < 3 - (A_1A_3 + A_2A_4 + A_1A_4) - A_5 - A_4 + (A_1 + A_2 + A_3) - 3A_1A_2A_3 - 3A_1A_4,$

and $-3 + (A_1A_3 + A_2A_4 + A_1A_4) + A_5 + A_4 + (A_1 + A_2 + A_3) - 3A_1A_2A_3 - 3A_1A_4 < 3 - (A_1A_3 + A_2A_4 + A_1A_4)$

$- A_5 - A_4.$

Therefore, $-3 + a_1 < a_2 - 3a_0 < 3 - a_1$ and condition (2) in Lemma 6 hold.

Finally, if we have $A_3^2A_4^2 + 2wA_3A_4 + w^2 + a_1 + a_2w + a_3A_4 - 1 < 0,$

and $(-w - A_3A_4)^2 + a_1 - a_2(-w - A_3A_4) - 1 < 1$.

Hence, condition (3) in Lemma 6 holds. Therefore, all roots of the characterization polynomial are in the unit circle, and the point $E_3$ is locally stable. \hfill \square

3. Optimal Harvesting Strategy

In this section, we study the optimal control strategy to get the optimal harvesting amount; this will be done through the discrete version of Pontryagin’s maximum principle, so that the form of the objective functional that we have to minimize is

$$J(h_t) = \sum_{t=0}^{T-1} c_t^i h_t y_t.$$ \hspace{1cm} (27)

The parameter $h_t$ refers to the control variable, which represents the rate of harvesting at time $t$, with $0 \leq h_t \leq h_{\text{Max}},$ and $h_{\text{Max}}$ is the maximum harvesting. $c_1$ is a positive constant. In this optimal control, the state equations are

$$x_{i+1} = x_i + \frac{s^a}{aT(a)} \left( x_i(1 - x_i) - \frac{ax_i^2y_i}{1 + ax_i^2} \right),$$

$$y_{i+1} = y_i + \frac{s^a}{aT(a)} \left( \frac{bx_i^2y_i}{1 + ax_i^2} - \frac{dy_i^2z_i}{1 + dy_i^2} - m_1y_i - h_i y_i \right),$$

$$z_{i+1} = z_i + \frac{s^a}{aT(a)} \left( \frac{cy_i^2z_i}{1 + dy_i^2} - m_2z_i \right).$$ \hspace{1cm} (28)

The parameters $s, a, b, d, m_1,$ and $m_2$ are defined as before, and they have the same interpretations.

To solve the problem, we have to set the Hamiltonian function for $t = 0, 1, \cdots, T - 1$, which is given by

$$H_i = c_i h_i y_i + \lambda_{i+1} \left[ x_i + \frac{s^a}{aT(a)} \left( x_i(1 - x_i) - \frac{ax_i^2y_i}{1 + ax_i^2} \right) \right]$$

$$+ \mu_{i+1} \left[ y_i + \frac{s^a}{aT(a)} \left( \frac{bx_i^2y_i}{1 + ax_i^2} - \frac{dy_i^2z_i}{1 + dy_i^2} - m_1y_i - h_i y_i \right) \right]$$

$$+ \zeta_{i+1} \left[ z_i + \frac{s^a}{aT(a)} \left( \frac{cy_i^2z_i}{1 + dy_i^2} - m_2z_i \right) \right],$$ \hspace{1cm} (29)

where $\lambda_i, \mu_i,$ and $\zeta_i$ are the adjoint functions.
Now, according to the maximum principle of Pontryagin, the necessary conditions of the above problem are

\[ \lambda_t = \frac{\partial H_t}{\partial x_t} = \lambda_{t+1} \left[ 1 + \frac{s^a}{a \Gamma(a)} \left( 1 - 2x_t - \frac{2ax_t y_t}{(1 + ax_t^2)^2} \right) \right] \]

\[ \mu_t = \frac{\partial H_t}{\partial y_t} = \mu_{t+1} \left[ 1 + \frac{s^a}{a \Gamma(a)} \left( bx_t^2 \frac{2bx_t y_t}{(1 + ax_t^2)^2} \right) \right], \]

\[ \zeta_t = \frac{\partial H_t}{\partial z_t} = \zeta_{t+1} \left[ 1 + \frac{s^a}{a \Gamma(a)} \left( \frac{cy_t^2}{(1 + dy_t^2)^2} \right) \right]. \]

The characterization of the optimal solution \( h_t^* \) will be

\[ h_t^* = \begin{cases} 
0, & \text{if } S_t < 0, \\
\hat{h}_t, & \text{if } S_t > 0,
\end{cases} \]

where \( S_t \) is the switching function. The optimal control solutions and the corresponding state solutions at that \( t \) will be determined numerically.

4. Numerical Simulation

This section concerns with numerical simulations that confirm the theoretical results. For the fixed point \( E_1 \), we use the following set of value parameters: \( a = 0.7, c = 0.1, b = 0.1, d = 0.2, m_1 = 0.2, m_2 = 0.2, \hat{h} = 0.01, s = 0.8, \alpha = 0.98, x(1) = 1.9, y(1) = 0.006, z(1) = 0.005, \) and \( T = 200 \), so that according to Theorem 4, the fixed point \( E_1 \) is the local stability point. Figure 1 shows the local stability of \( E_1 \).
For the local stability of the fixed point $E_2$ of system (8), the following set of values of the parameter are used: $a = 0.5$, $c = 0.2$, $b = 0.3$, $d = 0.4$, $m_1 = 0.1$, $m_2 = 0.2$, $h = 0.01$, $s = 0.5$, $\alpha = 0.98$, $x(1) = 1.2$, $y(1) = 0.7$, $z(1) = 0.22$, and $T = 2000$, so that according to Theorem 5, the fixed point $E_2$ is the local stability point. Figure 2 shows that local stability.

For the local stability of the unique positive fixed point $E_3$ of system (8), we use the following values of parameters: $a = 0.7$, $c = 0.1$, $b = 0.2$, $d = 0.2$, $m_1 = 0.02$, $m_2 = 0.2$, $h = 0.01$, $s = 0.5$, $\alpha = 0.98$, $x(1) = 0.9$, $y(1) = 1.7$, $z(1) = 0.1$, and $T = 2000$, so that according to Theorem 5, the fixed point $E_3$ is the local stability point. Figure 3 shows the local stability of $E_3$.

For the control problems (27) and (28), we follow an iterative method which is found in [3, 12, 19, 30]. We choose the values of parameters as follows: $a = 0.7$, $b = 0.2$, $c = 0.1$, $c_1 = 0.01$, $d = 0.2$, $m_1 = 0.01$, $m_2 = 0.05$, $\alpha = 0.9$, and $T = 60$. The initial conditions are $(0.9, 1.7, 0.15)$; with these values
of parameters, we get the total optimal harvesting of $J_{opt} = 0.0531$. In Table 1, we compare the total optimal harvesting and other total harvesting strategies using the same values of the parameters. Figure 4 shows the optimal control variable as a function of time and the effect of optimal harvesting on the prey species, middle predator, and top predator species.

5. Conclusions

In this work, we have studied and investigated the first-order discretization of three-dimensional fractional-order differential biological systems. The existence and the stability of its equilibrium points are discussed. We have observed that some of them exist without any restriction on the parameters of the model. Optimal harvesting has been done to put the economic optimality at a certain level to get maximum profit gain. The analytical results that have been obtained in this paper are verified through numerical various numerical simulations.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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