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Research Article

Discretization Fractional-Order Biological Model with Optimal Harvesting

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In this paper, a discretization of a three-dimensional fractional-order prey-predator model has been investigated with Holling type III functional response. All its fixed points are determined; also, their local stability is investigated. We extend the discretized system to an optimal control problem to get the optimal harvesting amount. For this, the discrete-time Pontryagin's maximum principle is used. Finally, numerical simulation results are given to confirm the theoretical outputs as well as to solve the optimality problem.

1. Introduction

Recently, population models have been given attention by researchers due to their importance and their relation to real life. In fact, there are different processes to describe population models, namely, ordinary and partial differential equations, difference equations, and fractional-order differential equations. Fractional-order derivative systems play a significant role in real-life problems due to their ability to provide accurate descriptions of different linear and nonlinear phenomena. We refer to the most recent works that are done by many researchers [1–5]. The mathematical behavior of fractional-order prey-predator models is also discussed and investigated in [6–10] and references therein. Discrete-time mathematical models hit many branches of pure and applied mathematics; also, they depict many real situations in life due to exhibiting and producing more plentiful dynamical behavior than those that have been seen in continuous time models of the same types. These types of models can be used to study and investigate problems in biology, economics, engineering, etc. For more details, see [11–17].

In the last decades, there are many articles that deal with the subject of harvesting in different approaches for details (see [18–21] and references therein). We also refer to the excellent book that is written by Agrawal et al. [22]. In this book, recent important results of infection diseases and methods with new approximations are analyzed and developed. In [23], the authors investigated and discussed coronavirus disease in a mathematical model that deals with coronavirus, and an effective control policy for the COVID-19 outbreak is set.

The study of biological models with harvesting, in particular, predator-prey models, is an important subject in the commercial harvesting industry and for many scientific communities including biology, ecology, and economics. The fundamental problem in commercial exploitation of renewable resources is to get the optimal gain between present and future harvests, so that employing optimal control theory in this field is a very useful tool that can help to make decisions in various problems in real world including biological situations [24, 25].

This work is organized as follows. In Section 2, a fractional-order system is discretized and all its equilibrium points are found. In Section 3, the local stability of all its equilibria is discussed. Section 4 deals with the employment of the discrete-time Pontryagin's maximum principle to solve the optimal control solution and its corresponding optimal state solutions. In Section 5, numerical simulations are done to confirm the theoretical outcomes. Finally, conclusion is given in Section 6.

2. The Discrete-Time Model and Its Fixed Points

First, the authors considered and analyzed in [26] the following fractional-order three-dimensional prey-predator model with the functional response of Holling type III:

$$\begin{aligned} D^{\alpha}x(t) &= x(1-x) - \frac{ax^2y}{1+ax^2}, \\ D^{\alpha}y(t) &= \frac{bx^2y}{1+ax^2} - \frac{dy^2z}{1+dy^2} - m_1y - hy, \\ D^{\alpha}z(t) &= \frac{cy^2z}{1+dy^2} - m_2z, \end{aligned} \tag{1}$$

where X, Y, and Z are the density of the prey population, the middle predator population, and the top predator population. m_1, m_2, a, and b are the death rates for the middle predator population and top predator population, the capture rate of prey and middle predator, and the conservation rate of prey X, respectively. The parameters d and c are the conservation rates of middle predator Y to the top predator Z. The parameter h represents the harvesting rate. For more details about the interpretations of the parameters, see [26]. Here, D^{α} is called the α-order Caputo differential operator [27, 28].

In this work, we investigate the dynamic behavior of the discretization of the system (1); then, we extend the discretized system to the optimal control problem in the next section. The discretization process of the system (1) is given as follows:

Assume that $x(0)=x_0$, $y(0)=y_0$, and $z(0)=z_0$ are the initial conditions of system (1), so that the discretization of the system (1) with piecewise constant arguments is given as

$$\begin{aligned} D^{\alpha}x(t) &= x([t/s]s)(1-x([t/s]s)) - \frac{ax([t/s]s)^2y([t/s]s)}{1+ax([t/s]s)^2}, \\ D^{\alpha}y(t) &= \frac{bx([t/s]s)^2y([t/s]s)}{1+ax([t/s]s)^2} - \frac{dy([t/s]s)^2z([t/s]s)}{1+dy([t/s]s)^2} \\ &\quad - m_1y([t/s]s) - hy([t/s]s), \\ D^{\alpha}z(t) &= \frac{cy([t/s]s)^2z([t/s]s)}{1+dy([t/s]s)^2} - m_2z([t/s]s). \end{aligned} \tag{2}$$

First, let $t \in [0,s)$, so $t/s \in [0,1)$. Thus, we obtain

$$\begin{aligned} D^{\alpha}x(t) &= x_0(1-x_0) - \frac{ax_0{}^2y_0}{1+ax_0{}^2}, \\ D^{\alpha}y(t) &= \frac{bx_0{}^2y_0}{1+ax_0{}^2} - \frac{dy_0{}^2z_0}{1+dy_0{}^2} - m_1y_0 - hy_0, \\ D^{\alpha}z(t) &= \frac{cy_0{}^2z_0}{1+dy_0{}^2} - m_2z_0. \end{aligned} \tag{3}$$

The solution of (3) is reduced to

$$\begin{aligned} x_1(t) &= x_0 + I^{\alpha}\left(x_0(1-x_0) - \frac{ax_0{}^2y_0}{1+ax_0{}^2}\right) \\ &= x_0 + \frac{t^{\alpha}}{\alpha\Gamma(\alpha)}\left(x_0(1-x_0) - \frac{ax_0{}^2y_0}{1+ax_0{}^2}\right), \\ y_1(t) &= y_0 + I^{\alpha}\left(\frac{bx_0{}^2y_0}{1+ax_0{}^2} - \frac{dy_0{}^2z_0}{1+dy_0{}^2} - m_1y_0 - hy_0\right) \\ &= y_0 + \frac{t^{\alpha}}{\alpha\Gamma(\alpha)}\left(\frac{bx_0{}^2y_0}{1+ax_0{}^2} - \frac{dy_0{}^2z_0}{1+dy_0{}^2} - m_1y_0 - hy_0\right), \\ z_1(t) &= z_0 + I^{\alpha}\left(\frac{cy_0{}^2z_0}{1+dy_0{}^2} - m_2z_0\right) \\ &= z_0 + \frac{t^{\alpha}}{\alpha\Gamma(\alpha)}\left(\frac{cy_0{}^2z_0}{1+dy_0{}^2} - m_2z_0\right). \end{aligned} \tag{4}$$

Second, let $t \in [s, 2s)$, so $t/s \in [1,2)$. Hence, we get the following system:

$$\begin{aligned} D^{\alpha}x(t) &= x_1(1-x_1) - \frac{ax_1{}^2y_1}{1+ax_1{}^2}, \\ D^{\alpha}y(t) &= \frac{bx_1{}^2y_1}{1+ax_1{}^2} - \frac{dy_1{}^2z_1}{1+dy_1{}^2} - m_1y_1 - hy_1, \\ D^{\alpha}z(t) &= \frac{cy_1{}^2z_1}{1+dy_1{}^2} - m_2z_1, \end{aligned} \tag{5}$$

which have the following solution:

$$\begin{aligned} x_2(t) &= x_1(s) + I_s^{\alpha}\left(x_1(s)(1-x_1(s)) - \frac{ax_1^2(s)y_1(s)}{1+ax_1^2(s)}\right) \\ &= x_1(s) + \frac{(t-s)^{\alpha}}{\alpha\Gamma(\alpha)}\left(x_1(1-x_1) - \frac{ax_1^2(s)y_1(s)}{1+ax_1{}^2}\right), \\ y_2(t) &= y_1(s) + I_s^{\alpha}\left(\frac{bx_1^2(s)y_1(s)}{1+ax_1^2(s)} - \frac{dy_1^2(s)z_1(s)}{1+dy_1^2(s)} - m_1y_1(s) - hy_1(s)\right) \\ &= y_1(s) + \frac{(t-s)^{\alpha}}{\alpha\Gamma(\alpha)}\left(\frac{bx_1^2(s)y_1(s)}{1+ax_1^2(s)} - \frac{dy_1^2(s)z_1(s)}{1+dy_1^2(s)} - m_1y_1(s) - hy_1(s)\right), \\ z_2(t) &= z_1(s) + I_s^{\alpha}\left(\frac{cy_1^2(s)z_1(s)}{1+dy_1^2(s)} - m_2z_1(s)\right) \\ &= z_1(s) + \frac{(t-s)^{\alpha}}{\alpha\Gamma(\alpha)}\left(\frac{cy_1^2(s)z_1(s)}{1+dy_1^2(s)} - m_2z_1(s)\right), \end{aligned} \tag{6}$$

where $I_s^{\alpha} \equiv 1/\Gamma(\alpha)\int_s^t (t-\tau)^{\alpha-1}d\tau, \alpha > 0$. Thus, after repeating the discretization process n times, we obtain

$$\begin{aligned} x_{n+1}(t) &= x_n(ns) + \frac{(t-ns)^{\alpha}}{\alpha\Gamma(\alpha)} \\ &\quad \cdot \left(x_n(ns)(1-x_n(ns)) - \frac{ax_n^2(ns)y_n(ns)}{1+ax_n^2(ns)}\right), \end{aligned}$$

$$y_{n+1}(t) = y_n(ns) + \frac{(t-ns)^{\alpha}}{\alpha\Gamma(\alpha)} \cdot \left(\frac{bx_n^2(ns)y_n(ns)}{1+ax_n^2(ns)} - \frac{dy_n^2(ns)z_1(s)}{1+dy_n^2(ns)} - m_1 y_n(ns) - hy_n(ns)\right),$$

$$z_{n+1}(t) = z_1(ns) + \frac{(t-ns)^{\alpha}}{\alpha\Gamma(\alpha)}\left(\frac{cy_n^2(ns)z_n(ns)}{1+dy_n^2(ns)} - m_2 z_n(ns)\right), \tag{7}$$

where $t \in [ns, (n+1)s)$. For $t \longrightarrow (n+1)s$, then the previous system is reduced to

$$\begin{aligned} x_{n+1} &= x_n + \frac{s^{\alpha}}{\alpha\Gamma(\alpha)}\left(x_n(1-x_n) - \frac{a{x_n}^2 y_n}{1+a{x_n}^2}\right), \\ y_{n+1} &= y_n + \frac{s^{\alpha}}{\alpha\Gamma(\alpha)}\left(\frac{b{x_n}^2 y_n}{1+a{x_n}^2} - \frac{d{y_n}^2 z_n}{1+d{y_n}^2} - m_1 y_n - hy_n\right), \\ z_{n+1} &= z_n + \frac{s^{\alpha}}{\alpha\Gamma(\alpha)}\left(\frac{c{y_n}^2 z_n}{1+d{y_n}^2} - m_2 z_n\right). \end{aligned} \tag{8}$$

In order to get the fixed points of the considered system (8), we have to solve the following simultaneous equations:

$$\begin{aligned} x &= x + \frac{s^{\alpha}}{\alpha\Gamma(\alpha)}\left(x(1-x) - \frac{ax^2 y}{1+a{x_n}^2}\right), \\ y &= y + \frac{s^{\alpha}}{\alpha\Gamma(\alpha)}\left(\frac{bx^2 y}{1+ax^2} - \frac{dy^2 z}{1+dy^2} - m_1 y - hy\right), \\ z &= z + \frac{s^{\alpha}}{\alpha\Gamma(\alpha)}\left(\frac{cy^2 z}{1+dy^2} - m_2 z\right). \end{aligned} \tag{9}$$

Thus, all possible fixed points of system (8) are as follows:

(1) The trivial fixed point $E_0 = (0,0,0)$ and the fixed point $E_1 = (1,0,0)$ exist without any restriction on the parameters

(2) The top predator free fixed point $E_2 = (\bar{x}, \bar{y}, 0)$ exists if $b > a(m_1 + h)$ and $0 < \bar{x} < 1$ where $\bar{x} = \sqrt{(m_1+h)/(b - a(m_1+h))}$ and $\bar{y} = (1/ax)(1-\bar{x})(1 + a\bar{x}^2)$

(3) The interior fixed point $E_3 = (x^*, y^*, z^*)$ exists if $c > m_3 d$ and $bx^{*2}/(1+ax^{*2}) > (m_1+h)$ where $y^* = \sqrt{m_2/(c - m_2 d)}$, x^* is the positive root of the equation $ax^3 - ax^2 + (ay^* + 1)x - 1 = 0$, and $z^* = (1 + dy^{*2})/dy^*((bx^{*2}/(1+ax^{*2})) - (m_1+h))$

Then, the Jacobian matrix J of the system (8) evaluated at any point (x, y, z) is given by

$$(x,y,z) = \begin{bmatrix} 1 + m - 2mx - \dfrac{2amxy}{k^2} & -\dfrac{max^2}{k} & 0 \\ \dfrac{2mbxy}{k^2} & 1 + m\left(\dfrac{bx^2}{k} - \dfrac{2dyz}{k_1^2} - (m_1 + h)\right) & -\dfrac{mdy^2}{k_1} \\ 0 & \dfrac{cmyz}{k_1^2} & 1 + \dfrac{cmy^2}{k_1} - mm_2 \end{bmatrix}, \tag{10}$$

where $k = (1 + ax^2)$, $k_1 = (1 + dy^2)$, and $m = s^{\alpha}/\alpha\Gamma(\alpha)$. Therefore, the characteristic equation of the Jacobian matrix is given by

$$p(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0,$$

$$\begin{aligned} a_2 &= -(A_1 + A_2 + A_3)a_1 \\ &= (A_1A_3 + A_2A_3 + A_1A_2) + A_5 + A_4, \end{aligned}$$

$$a_0 = -A_1A_2A_3 - A_1A_5 - A_3A_4 = -w - A_3A_4 \text{ with } w = A_1A_2A_3 + A_1A_5,$$

$$A_1 = 1 + m - 2mx - \frac{2maxy}{k^2},$$

$$A_2 = 1 + m\left(\frac{bx^2}{k} - \frac{2dyz}{k_1^2} - (m_1 + h)\right),$$

$$A_3 = 1 + \frac{cmy^2}{k_1} - mm_2,$$

$$A_4 = \left(\frac{max^2}{k}\right)\left(\frac{2mbxy}{k^2}\right),$$

$$A_5 = \frac{2cdm^2y^3z}{k_1^3}. \tag{11}$$

In order to study stability analysis of the fixed points of the model (8), we give Definition 1 and Lemma 2.

Definition 1 (see [29]). A fixed point x^* of system $\vec{x}_{t+1} = f(\vec{x}_t)$ is said to be as follows:

(1) Saddle point, if at least one eigenvalue of the Jacobian matrix $J(x^*)$ is in modulus > 1

(2) Sink point, if all eigenvalues of the Jacobian matrix $J(x^*)$ are in modulus < 1

(3) Source point, if all eigenvalues of the Jacobian matrix $J(x^*)$ are in modulus > 1

(4) Nonhyperbolic point, if at least one eigenvalue of the Jacobian matrix $J(x^*)$ is in modulus $= 1$

Lemma 2 (see [29]). *Let $F(\lambda) = \lambda^2 + p\lambda + q$ be a polynomial of degree two. Suppose that $F(1) > 0$ and λ_1, λ_2 are the roots of F; then*

(1) $|\lambda_1| < 1$ and $|\lambda_2| < 1$, if and only if $F(-1) > 0$ and $q < 1$

(2) $|\lambda_1| > 1$ and $|\lambda_2| > 1$, if and only if $F(-1) > 0$ and $q > 1$

(3) $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$), if and only if $F(-1) < 0$

(4) $\lambda_1 = -1$ and $|\lambda_2| \neq 1$, if and only if $F(-1) = 0$ and $p \neq 0, 2$

Theorem 3 discusses the dynamical behavior of the fixed points E_0, E_1, E_2, and E_3 of the system (8), respectively.

Theorem 3. *For the system (8), the fixed point E_0 can be classified as follows:*

(1) The point E_0 will never be a sink point

(2) The point E_0 is a source, if and only if $m > \max\{2/(m_1 + h), 3/m_2\}$

(3) The point E_0 is a saddle point, if and only if $m < \min\{2/(m_1 + h), 3/m_2\}$

(4) The point E_0 is a nonhyperbolic point, if and only if $m = 2/(m_1 + h)$ or $m = 3/m_2$, where $m = s^\alpha/\alpha\Gamma(\alpha)$, $m_1 = c_1 k_0^2 R_0/a_1$, and $m_2 = R_0/c_2$.

Proof. It is clear that the Jacobian matrix of system (8) at E_0 can be written as follows:

$$J(E_0) = \begin{bmatrix} (1+m) & 0 & 0 \\ 0 & 1 - m(m_1 + h) & 0 \\ 0 & 0 & 1 - mm_2 \end{bmatrix}. \tag{12}$$

The characteristic equation of the Jacobian matrix $J(E_0)$ is given by

$$p(\lambda) = ((1+m) - \lambda)((1 - m(m_1 + h)) - \lambda)((1 - mm_2) - \lambda). \tag{13}$$

So that the eigenvalues of $J(E_0)$ are $\lambda_1 = (1+m)$, $\lambda_2 = 1 - m(m_1 + h)$, and $\lambda_3 = 1 - mm_2$. It is clear that $|\lambda_1|$ is always greater than 1. Hence, the fixed point E_0 will never be a sink point. It is clear that if $0 < m < 2/(m_1 + h)$, then $|1 - m(m_1 + h)| < 1$. Therefore, $|\lambda_2| < 1$. Now, if $m < 3/m_2$, then $|1 - mm_2| < 1$. Hence, $|\lambda_3| < 1$. Therefore, all results can be obtained. □

Theorem 4. *For the system (8), the fixed point E_1 can be classified as follows:*

(1) E_1 is a sink point if and only if $m \in I_1 \cap I_2$

(2) E_1 is a source point if and only if $m \in I_3$

(3) E_1 is a saddle point if and only if one of the following holds

$$\begin{aligned} &m \in I_4 \cap I_2, \\ &m \in I_1 \cap I_5, \\ &m \in I_1 \cap I_6. \end{aligned} \tag{14}$$

:

(4) E_1 is a nonhyperbolic point if and only if $m = 2$ or $m = 3/m_2$ or $m = 3/m_2$, where $I_1 = (0, \min\{2, 3/m_2\})$, $I_2 = ((1+a)((m_1 + h) - 2)/b, (m_1 + h)(1+a)/b)$, $I_3 = (\max\{2, 3/m_2, (m_1 + h)(1+a)/b\}, \infty)$, $I_4 = (\max\{2, 3/m_2\}, \infty)$, $I_5 = ((m_1 + h)(1+a)/b, \infty)$, and $I_6 = (0, (1+a)((m_1 + h) - 2)/b)$.

Proof. The Jacobian matrix of system (8) at E_1 is

$$J(E_1) = \begin{bmatrix} (1-m) & -\dfrac{ma}{1+a} & 0 \\ 0 & 1 + \dfrac{mb}{1+a} - (m_1 + h) & 0 \\ 0 & 0 & 1 - mm_2 \end{bmatrix}. \tag{15}$$

Therefore, the characteristic equation of the Jacobian matrix $J(E_1)$ is given by

$$p(\lambda) = ((1+m) - \lambda)\left(\left(1 + \frac{mb}{1+a} - (m_1 + h)\right) - \lambda\right)((1 - mm_2) - \lambda). \tag{16}$$

So that the eigenvalues of $J(E_1)$ are $\lambda_1 = (1-m)$, $\lambda_2 = 1 + (mb/(1+a)) - (m_1 + h)$, and $\lambda_3 = 1 - mm_2$. Now, if $<m < 2$, then $|1 - m| < 1$. This implies that $|\lambda_1| < 1$ if $(1+a)((m_1 + h) - 2)/b < m < (m_1 + h)(1+a)/b$. This gives $|1 + (mb/(1+a)) - (m_1 + h)| < 1$. Therefore, $|\lambda_2| < 1$. Assume that $0 < m < 2/m_3$. This implies that $|1 - mm_3| < 1$ and $|\lambda_3| < 1$. Therefore, all results are got. □

Theorem 5. *For the system (8), the fixed point E_2 can be classified as follows:*

(1) E_2 *is a sink point, if and only if the following conditions hold:*

$$\left(2mau + m^2au - m^2a^2u^2\right)(1-x) > (m-2)b,$$
$$\left(2au + 2mabu - 2ma^2u^2\right)(1-x) < b. \tag{17}$$

(2) E_2 *is a source if and only if* $(2au + 2mabu - 2ma^2u^2)(1-x) > b$

(3) E_2 *is a saddle point if and only if* $(2mau + m^2au - m^2a^2u^2)(1-x) < (m-2)b$

(4) E_2 *is a nonhyperbolic point if and only if* $(2mau + m^2au - m^2a^2u^2)(1-x) = (m-2)b$ *or* $(2au + 2mabu - 2ma^2u^2)(1-x) = b$, *where* $= \sqrt{(m_1+h)/(b-a(m_1+h))}$, $y = (1/ax)(1-x)(1+ax^2)$, $k = 1 + ax^2$, *and* $u = (m_1+h)$.

Proof. It is clear that the Jacobian matrix of system (8) at E_2 can be written as follows:

$$J(E_2) = \begin{bmatrix} 1 - m + \dfrac{2mau(1-x)}{b} & -\dfrac{mau}{b} & 0 \\ 2m(1-x)(b-au) & 1 & -\dfrac{db(1-x)^2}{a^2u^2 + db^2(1-x)^2} \\ 0 & 0 & 1 + \dfrac{mcy^2}{k_1} - mm_2 \end{bmatrix}. \tag{18}$$

Therefore, the characteristic polynomial of the Jacobian matrix at E_2 is given by

$$f(\lambda) = \left(1 - m + \frac{2mau(1-x)}{b} - \lambda\right)F(\lambda) = 0, \tag{19}$$

where $F(\lambda) = \lambda^2 + p\lambda + q$ such that $p = 2 + m - (2mau(1-x)/b)$ and $q = 1 - m + (2mau(1-x)/b) + 2m^2au(1-x) - (2m^2a^2u^2/b)(1-x)$.

Now, the first root of the characteristic polynomial is $\lambda_1 = 1 + (mcb^2(1-x)^2/(a^2u^2 + db^2(1-x)^2)) - mm_2$. If $0 < m < 2((a^2u^2 + db^2(1-x)^2)/((c-dm_2)b^2(1-x)^2 - m_2a^2u^2))$, then, we have $|1 + (mcb^2(1-x)^2/(a^2u^2 + db^2(1-x)^2)) - mm_2| < 1$ and $|\lambda_1| < 1$.

In order to investigate the other roots, we use Lemma 2 so that $F(1) > 0$ if and only if $1 + m - 2 - m + (2mau(1-x)/b) + 1 - m + (2mau(1-x)/b) + 2m^2au(1-x) - (2m^2a^2u^2/b)(1-x) > 0$. Thus, $F(1) > 0$ if and only if $2m^2au(1-x) - (2m^2a^2u^2/b)(1-x) > 0$. Since $0 < x < 1$, we get that $F(1)$ is always greater than zero.

Now, we have to show that $F(-1) > 0$ and $q < 1$. If $(2mau + m^2au - m^2a^2u^2)(1-x) > (m-2)b$, then $((2mau/b) + m^2au - (m^2a^2u^2/b))(1-x) > (m-2)$. Hence, $2 - m + (2mau(1-x)/b) + m^2au(1-x) - (m^2a^2u^2/b)(1-x) > 0$.

Therefore, $4 - 2m + (4mau(1-x)/b) + 2m^2au(1-x) - (2m^2a^2u^2/b)(1-x) > 0$.

This implies that $1 - m + 2 + (2mau(1-x)/b) + 1 - m + (2mau(1-x)/b) + 2m^2au(1-x) - (2m^2a^2u^2/b)(1-x) > 0$ and $F(-1) > 0$.

Now, if $(2au + 2mabu - 2ma^2u^2)(1-x) < b$, then $((2au/b) + 2mau - (2ma^2u^2/b))(1-x) < 1$. Hence, $((2mau/b) + 2m^2au - (2m^2a^2u^2/b))(1-x) < m$. Therefore, $-m + (2mau(1-x)/b) + 2m^2au(1-x) - (2m^2a^2u^2/b)(1-x) < 0$. This implies that $1 - m + (2mau(1-x)/b) + 2m^2au(1-x) - (2m^2a^2u^2/b)(1-x) < 1$ and $q < 1$. Therefore, the results can be obtained. ☐

Before we study the behavior of the interior fixed point of the system (8), we need Lemma 6.

Lemma 6 (see [29]). *Let* $p(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$ *be a cubic polynomial with real coefficients. Then, the necessary and sufficient conditions that all roots lie in the open disk* $|\lambda| < 1$ *are as follows:*

$$|a_2 + a_0| < 1 + a_1,$$
$$|a_2 - 3a_0| < 3 - a_1, \tag{20}$$
$$a_0^2 + a_1 - a_0a_2 < 1.$$

Theorem 7. *For the system (8), the fixed point* E_3 *is a stable point if the following conditions hold:*

$$c_1 < A_3A_4 < c_2,$$
$$d_1 < A_3A_4 < d_2, \tag{21}$$
$$A_3^2A_4^2 + (2w + a_2)A_3A_4 + kk < 0,$$

where $a_2 = -(A_1 + A_2 + A_3)$, $kk = w^2 + a_1 + a_2w - 1$, $a_1 = (A_1A_3 + A_2A_3 + A_1A_2) + A_5 + A_4$, $a_0 = -A_1A_2A_3 - A_1A_5 - A_3A_4 = -w - A_3A_4$ *such that* $w = A_1A_2A_3 + A_1A_5$, $A_1 = 1 + m - 2mx - (2maxy/k^2)$, $A_2 = 1 + m((bx^2/k) - (2dyz/k_1^2) - (m_1 + h))$, $A_3 = 1 + (cmy^2/k_1) - mm_2$, $A_4 = (max^2/k)(2mbxy/k^2)$, $A_5 = 2cdm^2y^3z/k_1^3$, $c_1 = -1 - (A_1A_3 + A_2A_3 + A_1A_2) - A_5 - A_4 - (A_1 + A_2 + A_3) - A_1A_2A_3 - A_1A_5$, $c_2 = 1 + (A_1A_3 + A_2A_3 + A_1A_2) + A_5 + A_4 - (A_1 + A_2 + A_3) - A_1A_2A_3 - A_1A_5$, $d_1 = (1/3)(-3 + (A_1A_3 + A_2A_3 + A_1A_2) + A_5 + A_4 + (A_1 + A_2 + A_3) - 3A_1A_2A_3 - 3A_1A_5)$, *and* $d_2 = (1/3)(3 - (A_1A_3 + A_2A_3 + A_1A_2) - A_5 - A_4 + (A_1 + A_2 + A_3) - 3A_1A_2A_3 - 3A_1A_5)$.

Proof. The proof is based on Lemma 6.

The Jacobian matrix of system (8) at E_3 can be written as follows:

$$J(E_3) = \begin{bmatrix} 1+m-2mx-\dfrac{2amxy}{k^2} & -\dfrac{max^2}{k} & 0 \\ \dfrac{2mbxy}{k^2} & 1+m\left(\dfrac{bx^2}{k}-\dfrac{2dyz}{k_1^2}-(m_1+h)\right) & -\dfrac{mdy^2}{k_1} \\ 0 & \dfrac{2cmyz}{k_1^2} & 1+\dfrac{cmy^2}{k_1}-mm_2 \end{bmatrix}. \quad (22)$$

Therefore, the characteristic polynomial of the Jacobian matrix $J(E_3)$ at E_3 is

$$p(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0. \quad (23)$$

Now, let $c_1 < A_3A_4 < c_2$; this gives

$$\begin{aligned} &-1-(A_1A_3+A_2A_3+A_1A_2)-A_5-A_4-(A_1+A_2+A_3) \\ &\quad -A_1A_2A_3-A_1A_5 < A_3A_4 < 1+(A_1A_3+A_2A_3+A_1A_2) \\ &\quad +A_5+A_4-(A_1+A_2+A_3)-A_1A_2A_3-A_1A_5. \end{aligned} \quad (24)$$

Hence, $-1-(A_1A_3+A_2A_3+A_1A_2)-A_5-A_4+(A_1+A_2+A_3)+A_1A_2A_3+A_1A_5 < -A_3A_4 < 1+(A_1A_3+A_2A_3+A_1A_2)+A_5+A_4+(A_1+A_2+A_3)+A_1A_2A_3+A_1A_5$, and $-1-(A_1A_3+A_2A_3+A_1A_2)-A_5-A_4 < -(A_1+A_2+A_3)-A_1A_2A_3-A_1A_5-A_3A_4 < 1+(A_1A_3+A_2A_3+A_1A_2)+A_5+A_4$.

Therefore, $-1-a_1 < a_2+a_0 < 1+a_1$ and condition (1) in Lemma 6 hold.

Now, assume that $d_1 < A_3A_4 < d_2$. This gives

$$\begin{aligned} &\frac{1}{3}(-3+(A_1A_3+A_2A_3+A_1A_2)+A_5+A_4 \\ &\quad +(A_1+A_2+A_3)-3A_1A_2A_3-3A_1A_5) < A_3A_4 \\ &< \frac{1}{3}(3-(A_1A_3+A_2A_3+A_1A_2)-A_5-A_4 \\ &\quad +(A_1+A_2+A_3)-3A_1A_2A_3-3A_1A_5). \end{aligned} \quad (25)$$

Hence, $-3+(A_1A_3+A_2A_3+A_1A_2)+A_5+A_4+(A_1+A_2+A_3)-3A_1A_2A_3-3A_1A_5 < 3A_3A_4 < 3-(A_1A_3+A_2A_3+A_1A_2)-A_5-A_4+(A_1+A_2+A_3)-3A_1A_2A_3-3A_1A_5$, and $-3+(A_1A_3+A_2A_3+A_1A_2)+A_5+A_4 < -(A_1+A_2+A_3)+3A_1A_2A_3+3A_1A_5+3A_3A_4 < 3-(A_1A_3+A_2A_3+A_1A_2)-A_5-A_4$.

Therefore, $-3+a_1 < a_2-3a_0 < 3-a_1$ and condition (2) in Lemma 6 hold.

Finally, if we have $A_3^2A_4^2+(2w+a_2)A_3A_4+kk<0$, we get

$$A_3^2A_4^2+2wA_3A_4+w^2+a_1+a_2w+a_2A_3A_4-1<0, \quad (26)$$

and $(-w-A_3A_4)^2+a_1-a_2(-w-A_3A_4)<1$.

Hence, condition (3) in Lemma 6 holds. Therefore, all roots of the characterization polynomial are in the unit circle, and the point E_3 is locally stable. □

3. Optimal Harvesting Strategy

In this section, we study the optimal control strategy to get the optimal harvesting amount; this will be done through the discrete version of Pontryagin's maximum principle, so that the form of the objective functional that we have to minimize is

$$J(h_t) = \sum_{t=0}^{T-1} c_1h_ty_t. \quad (27)$$

The parameter h_t refers to the control variable, which represents the rate of harvesting at time t, with $0 \le h_t \le h_{\text{Max}}$, and h_{Max} is the maximum harvesting. c_1 is a positive constant. In this optimal control, the state equations are

$$\begin{aligned} x_{t+1} &= x_t + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(x_t(1-x_t)-\frac{\mathrm{a}x_t^{\,2}y_t}{1+ax_t^{\,2}}\right), \\ y_{t+1} &= y_t + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{bx_t^{\,2}y_t}{1+ax_t^{\,2}}-\frac{dy_t^{\,2}z_t}{1+dy_t^{\,2}}-m_1y_t-h_ty_t\right), \\ z_{t+1} &= z_t + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{cy_t^{\,2}z_t}{1+dy_t^{\,2}}-m_2z_t\right). \end{aligned} \quad (28)$$

The parameters s, a, b, d, m_1, and m_2 are defined as before, and they have the same interpretations.

To solve the problem, we have to set the Hamiltonian function for $t = 0, 1, \cdots, T-1$, which is given by

$$\begin{aligned} H_t = c_1h_ty_t &+ \lambda_{t+1}\left[x_t + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(x_t(1-x_t)-\frac{ax_t^{\,2}y_t}{1+ax_t^{\,2}}\right)\right] \\ &+ \mu_{t+1}\left[y_t + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{bx_t^{\,2}y_t}{1+ax_t^{\,2}}-\frac{dy_t^{\,2}z_t}{1+dy_t^{\,2}}-m_1y_t-h_ty_t\right)\right] \\ &+ \zeta_{t+1}\left[z_t + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{cy_t^{\,2}z_t}{1+dy_t^{\,2}}-m_2z_t\right)\right], \end{aligned} \quad (29)$$

where λ_t, μ_t, and ζ_t are the adjoint functions.

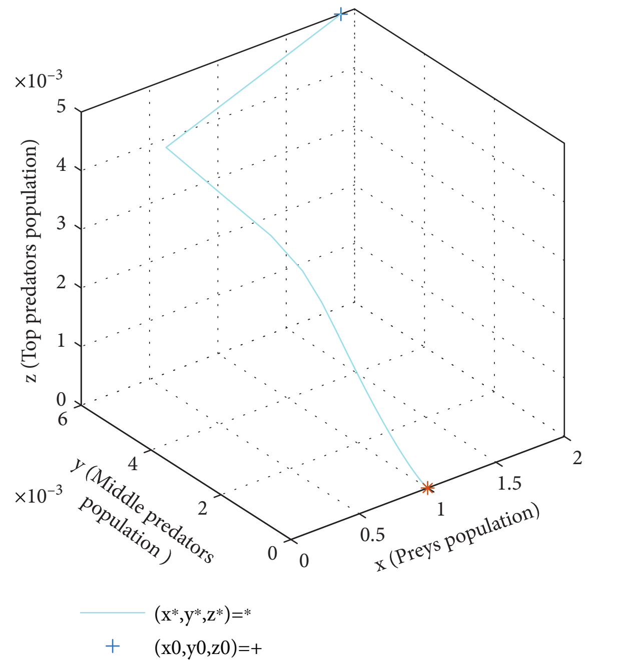

Figure 1: The plot shows the local stability of E_1 of the system (8).

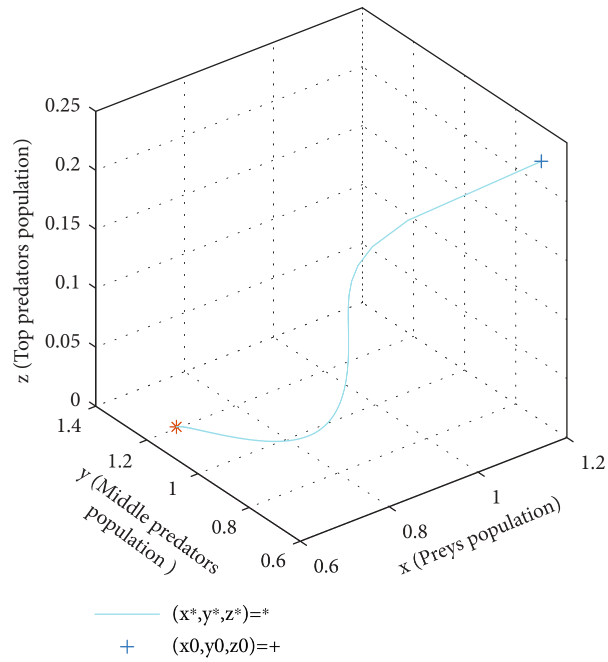

Figure 2: The plot shows the local stability of E_2 of the system (8).

Now, according to the maximum principle of Pontryagin, the necessary conditions of the above problem are

$$\lambda_t = \frac{\partial H_t}{\partial x_t} = \lambda_{t+1}\left[1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(1 - 2x_t - \frac{2ax_t y_t}{(1+ax_t{}^2)^2}\right)\right] + \mu_{t+1}\left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)\left(\frac{2bx_t y_t}{(1+ax_t{}^2)^2}\right),$$

$$\mu_t = \frac{\partial H_t}{\partial y_t} = c_1 h_t + \lambda_{t+1}\left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)\left(\frac{-ax_t{}^2}{1+ax_t{}^2}\right) + \mu_{t+1}\left[1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{bx_t{}^2}{1+ax_t{}^2} - \frac{2dy_t z_t}{(1+dy_t{}^2)^2} - m_1 - h_t\right)\right] + \zeta_{t+1}\left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)\left(\frac{2cy_t z_t}{(1+dy_t{}^2)^2}\right),$$

$$\zeta_t = \frac{\partial H_t}{\partial z_t} = \mu_{t+1}\left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)\left(\frac{-dy_t{}^2}{1+dy_t{}^2}\right) + \zeta_{t+1}\left[1 + \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)\left(\frac{cy_t{}^2}{(1+dy_t{}^2)} - m_2\right)\right]. \tag{30}$$

The characterization of the optimal solution $h_t{}^*$ will be

$$h_t{}^* = \begin{cases} 0, & \text{if } S_t < 0, \\ \mathrm{h}_{\mathrm{Max}}, & \text{if } S_t > 0, \end{cases} \tag{31}$$

where S_t is the switching function. The optimal control solutions and the corresponding state solutions at that t will be determined numerically.

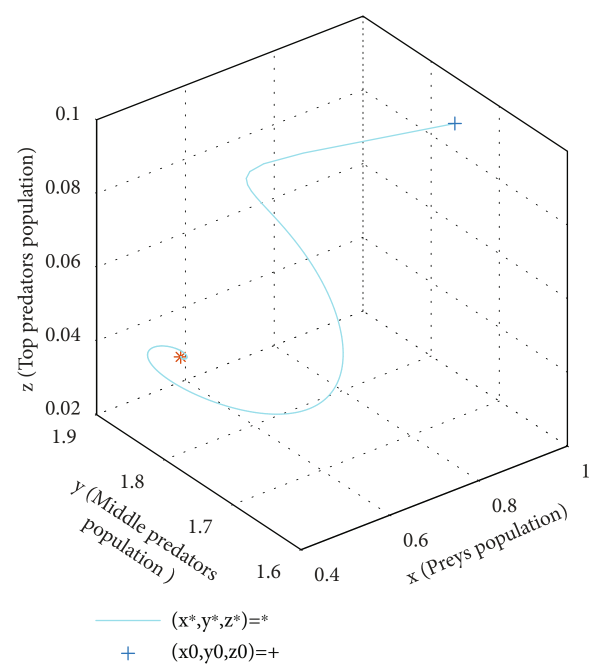

Figure 3: The plot shows the local stability of E_3 of the system (8).

4. Numerical Simulation

This section concerns with numerical simulations that confirm the theoretical results. For the fixed point E_1, we use the following set of value parameters: $a = 0.7$, $c = .01$, $b = 0.1$, $d = 0.2$, $m_1 = 0.2$, $m_2 = 0.2$, $h = 0.01$, $s = 0.8, \alpha = 0.98$, $x(1) = 1.9$, $y(1) = 0.006$, $z(1) = 0.005$, and $T = 200$, so that according to Theorem 4, the fixed point E_1 is the local stability point. Figure 1 shows the local stability of E_1.

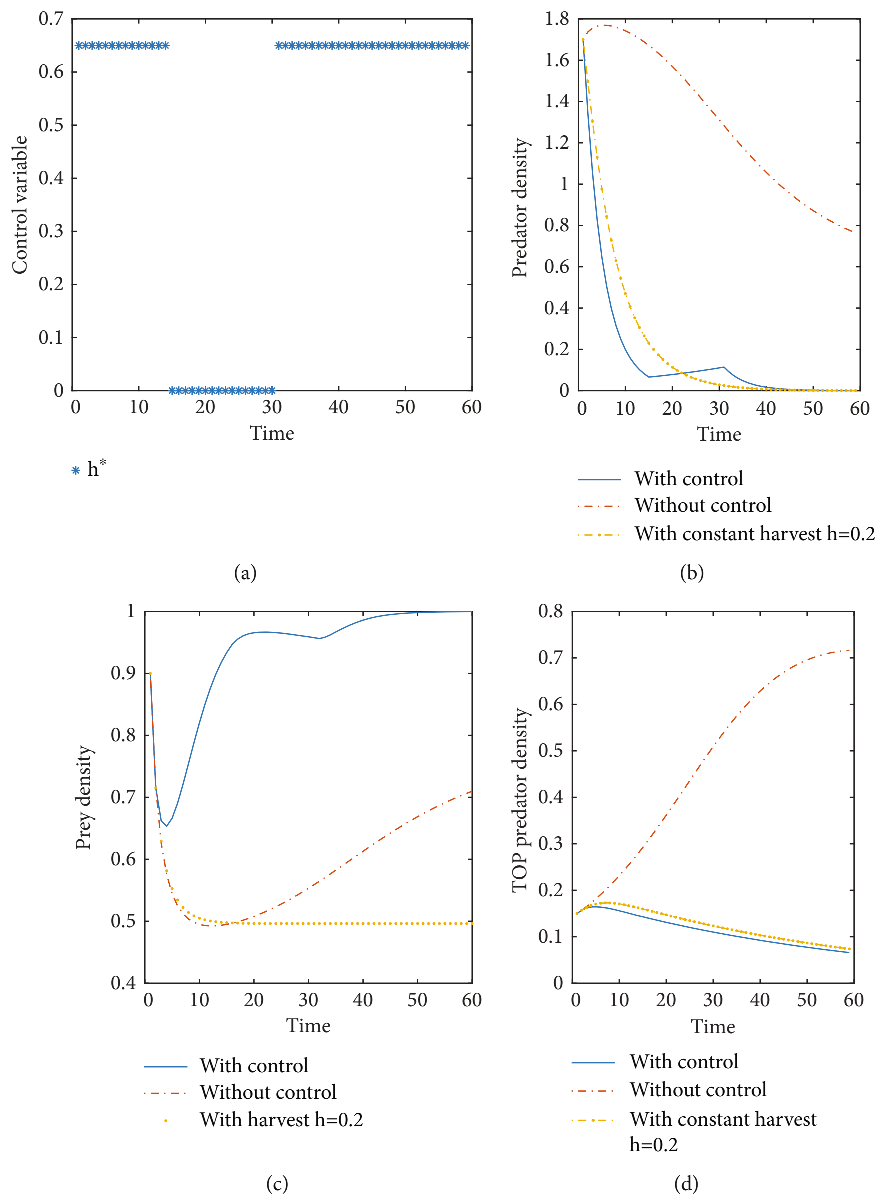

FIGURE 4: (a) The optimal control variable as a function of time. (b) The effect of optimal harvesting on the predator species. (c) The effect of optimal harvesting on the prey species. (d) The effect of optimal harvesting on the top predator species.

For the local stability of the fixed point E_2 of system (8), the following set of values of the parameter are used: $a = 0.5$, $c = 0.2$, $b = 0.3$, $d = 0.4$, $m1 = 0.1$, $m2 = 0.2$, $h = 0.01$, $s = 0.5$, $\alpha = 0.98$, $x(1) = 1.2$, $y(1) = 0.7$, $z(1) = 0.22$, and $T = 2000$, so that according to Theorem 5, the fixed point E_2 is the local stability point. Figure 2 shows that local stability.

For the local stability of the unique positive fixed point E_3 of system (8), we use the following values of parameters: $a = 0.7$, $c = 0.1$, $b = 0.2$, $d = 0.2$, $m1 = 0.02$, $m2 = 0.2$, $h = 0.01$, $s = 0.5$, $\alpha = 0.98$, $x(1) = 0.9$, $y(1) = 1.7$, $z(1) = 0.1$, and $T = 2000$, so that according to Theorem 5, the fixed point E_3 is the local stability point. Figure 3 shows the local stability of E_3.

For the control problems (27) and (28), we follow an iterative method which is found in [3, 12, 19, 30]. We choose

TABLE 1: This table compares the total optimal harvesting with other total harvesting strategies.

The control variable	Total net harvesting (J)
$h_t = h^*$	$J = 0.0531$
$h_t = 0.2$	$J = 0.0515$
$h_t = 0.3$	$J = 0.0520$
$h_t = 0.35$	$J = 0.0518$
$h_t = 0.4$	$J = 0.0516$

the values of parameters as follows: $a = 0.7$, $b = 0.2$, $c = 0.1$, $c_1 = 0.01$, $d = 0.2$, $m_1 = 0.01$, $m_2 = 0.05$, $\alpha = 0.9$, and $T = 60$. The initial conditions are (0.9, 1.7, 0.15); with these values of parameters, we get the total optimal harvesting of $J_{\text{opt}} = 0.0531$. In Table 1, we compare the total optimal harvesting and other total harvesting strategies using the same values of the parameters. Figure 4 shows the optimal control variable as a function of time and the effect of optimal harvesting on the prey species, middle predator, and top predator species.

5. Conclusions

In this work, we have studied and investigated the first-order discretization of three-dimensional fractional-order differential biological systems. The existence and the stability of its equilibrium points are discussed. We have observed that some of them exist without any restriction on the parameters of the model. Optimal harvesting has been done to put the economic optimality at a certain level to get maximum profit gain. The analytical results that have been obtained in this paper are verified through numerical various numerical simulations.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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