Hindawi
Abstract and Applied Analysis
Volume 2022, Article ID 9121239, 9 pages
https://doi.org/10.1155/2022/9121239

Research Article

On the Timelike Sweeping Surfaces and Singularities in Minkowski 3-Space $\mathbb{E}_1^3$

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Received 12 November 2021; Accepted 16 December 2021; Published 17 January 2022

Academic Editor: A. Ghareeb

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The Bishop frame or rotation minimizing frame (RMF) is an alternative approach to define a moving frame that is well defined even when the curve has vanished second derivative, and it has been widely used in the areas of computer graphics, engineering, and biology. The main aim of this paper is using the RMF for classification of singularity type of timelike sweeping surface and Bishop spherical Darboux image which is mightily concerning a unit speed spacelike curve with timelike binormal vector in $\mathbb{E}_1^3$.

1. Introduction

Kinematically, a sweeping surface is a surface traced by a one-parameter family of spheres with centers on a regular space curve, its directrix or spine. If the radii of the spheres are fixed, the sweeping surface is called tubular. There are several examples that we are familiar with, such as circular cylinder (spine is a line, the axis of the cylinder), right circular cone (spine is a line (the axis), radii of the spheres not constant), torus (directrix is a circle), and rotation surface (spine is a line). This visualization is a popularization of the classical notation of a partner of a planar curve [1–4]. One of the noteworthy facts linked with the sweeping surface is that the sweeping surface can be developable surface, that is, can be developed onto a plane without tearing and stretching. Therefore, sweeping surfaces have great usefulness in considerable product design which uses leather, paper, and sheet metal as materials (see, e.g., [5–8]). The developable surface can be represented using the Serret–Frenet frame of space curves from the viewpoint of singularity theory. In [9], Izumiya and Takeuchi defined the rectifying developable surfaces of space curves, where they proved that a regular curve is a geodesic of its rectifying developable surface and revealed the relationship between singularities of the rectifying developable surface and geometric invariants. Ishikawa investigated the relationship between the singularities of tangent developable surfaces and some types of space curves [10]. He also gave a classification of tangent developable surfaces by using the local topological property. There are several works about the singularity theory of developable ruled surfaces by using the Serret–Frenet frame of space curves, for example, [11–16]. However, the Serret–Frenet frame is undefined wherever the curvature vanishes, such as at points of inflection or along straight sections of the curve. A new frame is needed for the kind of mathematical analysis that is typically done with computer graphics. Therefore, Bishop [17] introduced the rotation minimizing frame (RMF) or Bishop frame, which could provide the desired means to ride along a space curve with vanished second 1derivative. After that, many research works linked to the RMF have been treated in the Euclidean space and Minkowski space [18–23].

In this paper, the classification of singularity type of timelike sweeping surfaces is studied with the RMF in $\mathbb{E}_1^3$. We present a new invariant related to the singularities of these sweeping surfaces. It is demonstrated that the generic singularities of this sweeping surface are cuspidal edge and swallowtail, and the types of these singularities can be characterized by this invariant, respectively. Afterwards, we have solved the problem of requiring the surface that is timelike sweeping surface and at the same time spacelike/timelike developable surface. Two examples are presented to explain the theoretical results.

2. Preliminaries

In this section, we give some definitions and basic concepts that we will use in this paper (see, for instance, [1, 8, 24]). Let $\mathbb{R}^3 = \{(p_1, p_2, p_3)|, p_i \in \mathbb{R}(i = 1, 2, 3)\}$ be a 3-dimensional Cartesian space. For any $\mathbf{q} = (q_1, q_2, q_3)$ and $\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{R}^3$, the pseudoscalar product of $\mathbf{q}$ and $\mathbf{p}$ is defined by

$$<\mathbf{q}, \mathbf{p}> = q_1 p_1 + q_2 p_2 - q_3 p_3. \tag{1}$$

We call $(\mathbb{R}^3, <,>)$ Minkowski 3-space. We write $\mathbb{E}_1^3$ instead of $(\mathbb{R}^3, <,>)$. We say that a nonzero vector $\mathbf{q} \in \mathbb{E}_1^3$ is spacelike, lightlike, or timelike if $<\mathbf{q}, \mathbf{q}> > 0$, $<\mathbf{q}, \mathbf{q}> = 0$, or $<\mathbf{q}, \mathbf{q}> < 0$, respectively. The norm of the vector $\mathbf{q} \in \mathbb{E}_1^3$ is defined to be $\|\mathbf{q}\| = \sqrt{|<\mathbf{q}, \mathbf{q}>|}$. For any two vectors $\mathbf{q}$, $\mathbf{p} \in \mathbb{E}_1^3$, we define the cross product by

$$\mathbf{q} \times \mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & -\mathbf{k} \\ q_1 & q_2 & q_3 \\ p_1 & p_2 & p_3 \end{vmatrix} = ((q_2 p_3 - q_3 p_2), (q_3 p_1 - q_1 p_3), -(q_1 p_2 - q_2 p_1)), \tag{2}$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the canonical basis of $\mathbb{E}_1^3$. The hyperbolic and Lorentzian unit spheres, respectively, are

$$\begin{aligned} \mathbb{H}_+^2 &= \{\mathbf{q} \in \mathbb{E}_1^3 \mid \|\mathbf{q}\|^2 = q_1^2 + q_2^2 - q_3^2 = -1\}, \\ \mathbb{S}_1^2 &= \{\mathbf{q} \in \mathbb{E}_1^3 \mid \|\mathbf{q}\|^2 = q_1^2 + q_2^2 - q_3^2 = 1\}. \end{aligned} \tag{3}$$

Let $\gamma = \gamma(s)$ be a unit speed spacelike curve with timelike binormal normal in $\mathbb{E}_1^3$; by $\kappa(s)$ and $\tau(s)$, we denote the natural curvature and torsion, respectively. Consider the Serret–Frenet frame $\{v_1(s), v_2(s), v_2(s)\}$ associated with curve $\gamma(s)$, then the Serret–Frenet formulae read

$$\begin{pmatrix} v_1'(s) \\ v_2'(s) \\ v_3'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} v_1(s) \\ v_2(s) \\ v_3(s) \end{pmatrix}, \tag{4}$$

where $v_1(s)(s) = \gamma'(s)$, $v_2(s) = \gamma''(s)/\|\gamma''(s)\|$, and $v_3(s) = -v_1(s) \times v_2(s)$ are called the unit tangent vector, the principal normal vector, and the binormal vector, respectively. Here, "prime" denotes the derivative with respect to the parameter s. The Serret–Frenet vector fields satisfy the relations

$$\begin{aligned} &<v_1, v_1> = <v_2, v_2> = 1, \\ &<v_3, v_3> = -1, \\ &v_1 \times v_2 = -v_3, \\ &v_1 \times v_3 = -v_2, \\ &v_2 \times v_3 = v_1. \end{aligned} \tag{5}$$

The Bishop frame or rotation minimizing frame (RMF) of $\gamma(s)$ is defined by the alternative frame equations

$$\begin{pmatrix} \xi' \\ \xi_1' \\ \xi_2' \end{pmatrix} = \begin{pmatrix} 0 & \mu_1 & -\mu_2 \\ -\mu_1 & 0 & 0 \\ -\mu_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \xi_1 \\ \xi_2 \end{pmatrix}, \tag{6}$$

where $\omega(s) = -\mu_2 \xi_1 + \mu_1 \xi_2$ is RMF Darboux vector. Here, the Bishop curvatures are defined by $\mu_1(s) = \kappa \cosh \varphi$, $\mu_2(s) = \kappa \sinh \varphi$. The relation matrix can be expressed as

$$\begin{pmatrix} \xi \\ \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \varphi & \sinh \varphi \\ 0 & \sinh \varphi & \cosh \varphi \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \tag{7}$$

where $\varphi(s) \geq 0$ is a hyperbolic angle. One can show that

$$\left. \begin{aligned} &\mu_1^2 - \mu_2^2 = \kappa^2, \varphi = \tanh^{-1}\left(\frac{\mu_2}{\mu_1}\right); \mu_1 \neq 0, \\ &\varphi(s) = -\int_{s_0}^{s} \tau ds + \varphi_0, \varphi_0 = \varphi(s_0). \end{aligned} \right\} \tag{8}$$

We define a Bishop spherical Darboux image $\mathbf{g}(s)$: $I \longrightarrow \mathbb{H}_+^2$ as

$$\mathbf{g}(s) = \frac{-\mu_2 \xi_1 + \mu_1 \xi_2}{\sqrt{\mu_1^2 - \mu_2^2}}. \tag{9}$$

Then, we define a new geometric invariant $\sigma(s) = \mu_2 \mu_1' - \mu_1 \mu_2'$.

A ruled surface in $\mathbb{E}_1^3$ is locally the map $\mathfrak{D}_{(\gamma,x)}$: $I \times \mathbb{R} \longrightarrow \mathbb{E}_1^3$ defined by

$$\mathfrak{D}_{(\gamma,x)}(s, t) = \alpha(s) + t\mathbf{x}(s), \mathrm{t} \in \mathbb{R}, \tag{10}$$

where $\alpha(s)$ is called the directrix curve and $\mathbf{x}(s)$ the director curve. The straight lines $t \longrightarrow \alpha(s) + t\mathbf{x}(s)$ are called rulings. It is well known that $\mathfrak{D}_{(\gamma,x)}$ is a developable surface iff $\det(\alpha'(s), \mathbf{x}(s), \mathbf{x}'(s)) = 0$.

Definition 1. A surface in the Minkowski 3-space $\mathbb{E}_1^3$ is called a timelike surface if the induced metric on the surface is a Lorentz metric and is called a spacelike surface if the induced metric on the surface is a positive definite Riemannian metric, i.e., the normal vector on spacelike (timelike) surface is a timelike (spacelike) vector.

3. Timelike Sweeping Surfaces and Singularities

In this section, the classification of singularity type of timelike sweeping surfaces is studied with the RMF in $\mathbb{E}_1^3$. Let $\gamma = \gamma(s)$ be a unit speed spacelike curve with timelike binormal as defined on the RMF frame. Then, we can give the parametric form of sweeping surface given by the spine curve $\gamma(s)$ as follows:

$$M : \mathbf{y}(s,t) = \gamma(s) + R(s)\mathbf{r}(t) = \gamma(s) + r_1(t)\xi_1(s) + r_2(t)\xi_2(s), \quad (11)$$

where $\mathbf{r}(t) = (0, r_1(t), r_2(t))^T$ is called planar profile (cross section); "T" represents transposition, with another parameter $t \in I \subseteq \mathbb{R}$. The semiorthogonal matrix $(s) = \{\xi(s), \xi_1(s), \xi_2(s)\}$ specifies the RMF along $\gamma(s)$. We will utilize "dot" to indicate the derivative with respect to the arc length parameter of the profile curve $\mathbf{r}(t)$.

The tangent vectors and the unit normal vector to the surface, respectively, are

$$\left.\begin{aligned} \mathbf{y}_t(s,t) &= \dot{r}_1\xi_1 + \dot{r}_2\xi_2, \\ \mathbf{y}_s(s,t) &= (1 - \mu_1 r_1 - \mu_2 r_2)\xi, \end{aligned}\right\} \quad (12)$$

$$\mathbf{n}(s,t) = \frac{\mathbf{y}_t \times \mathbf{y}_s}{\|\mathbf{y}_t \times \mathbf{y}_s\|} = \dot{r}_2\xi_1 + \dot{r}_1\xi_2.$$

From Equation (3.3), it follows that $\mathbf{n}(s,t)$ is contained in the normal plane of the spine curve $\gamma(s)$, since it is orthogonal to ξ. Thus, the normal of the profile curve $\mathbf{r}(t)$ and the surface normal are identical. Through this work, we will assume that the profile curve $\mathbf{r}(t)$ is a unit speed timelike curve, that is, $\dot{r}_1^2 - \dot{r}_2^2 = -1$. Thus, M is a timelike sweeping surface. From now on, we shall often not write the parameter s explicitly in our formulae.

Our aim of this work is the following theorem.

Theorem 2. *For the timelike sweeping surface Equation (3.1), with $\mu_1^2 - \mu_2^2 \neq 0$, one has the following*

(A)

(1) $g(s)$ is locally diffeomorphic to a timelike line $\{0\} \times \mathbb{R}$ at s_0 iff $\sigma(s_0) \neq 0$

(2) $g(s)$ is locally diffeomorphic to the cusp $C \times \mathbb{R}$ at s_0 iff $\sigma(s_0) \neq 0$ and $\sigma'(s_0) = 0$

(B)

(1) M is locally diffeomorphic to cuspidal edge CE at (s_0, t_0) iff $\mathbf{x} = \pm\mathbf{g}(s_0)$ and $\sigma(s_0) \neq 0$

(2) M is locally diffeomorphic to swallowtail SW at (s_0, t_0) iff $\mathbf{x} = \pm\mathbf{g}(s_0)$, $\sigma(s_0) \neq 0$, and $\sigma'(s_0) = 0$

Here, $C \times \mathbb{R} = \{(x_1, x_2) \mid x_1^2 = x_2^3\} \times \mathbb{R}$, $CE = \{(x_1, x_2, x_3) \mid x_1 = u, x_2 = v^2, x_3 = v^3\}$, and $SW = \{(x_1, x_2, x_3) \mid x_1 = u, x_2 = 3v^2 + uv^2,_3 = 4v^3 + 2uv\}$. The graphs of $C \times \mathbb{R}$, CE, and SW are seen in Figures 1–3.

3.1. Lorentzian Bishop Height Functions. Now, we will define two different families of Lorentzian Bishop height functions that will be useful to study the singularities of M as follows: $\mathfrak{H} : I \times \mathbb{S}_1^2 \longrightarrow \mathbb{R}$, by $\mathfrak{H}(s, \mathbf{x}) = <\gamma(\mathbf{s}), \mathbf{x}>$. We call it the Lorentzian Bishop height function. We use the notation $\mathfrak{h}_{\mathbf{x}}(s) = \mathfrak{H}(s, \mathbf{x})$ for any fixed $\mathbf{x} \in \mathbb{S}_1^2$. We also define $\tilde{\mathfrak{H}} : I \times \mathbb{S}_1^2 \times \mathbb{R} \longrightarrow \mathbb{R}$, by $\tilde{\mathfrak{H}}(s, \mathbf{x}, w) = <\gamma, \mathbf{x}> -w$. We call it the extended Lorentzian Bishop height function of $\gamma(\mathbf{s})$. We denote that $\tilde{\mathfrak{h}}_{\mathbf{x}}(s) = \tilde{\mathfrak{H}}(s, \mathbf{x})$. From now on, we shall often not write the parameter s. Then, we have the following proposition.

Proposition 3. *Let $\gamma = \gamma(s)$ be a unit speed spacelike curve with timelike binormal normal and $\mu_1^2 - \mu_2^2 \neq 0$. Then, the following hold:*

(A)

(1) $\mathfrak{h}_{\mathbf{x}}'(s) = 0$ iff $\mathbf{x} = a_1\boldsymbol{\xi}_1 + a_2\boldsymbol{\xi}_2$ and $a_1^2 - a_2^2 = -1$

(2) $\mathfrak{h}_{\mathbf{x}}'(s) = \mathfrak{h}_x''(s) = 0$ iff $\mathbf{x} = \pm\mathbf{g}(s)$

(3) $\mathfrak{h}_{\mathbf{x}}'(s) = \mathfrak{h}_x''(s) = \mathfrak{h}_x'''(s) = 0$ iff $\mathbf{x} = \pm\mathbf{g}(s)$ and $\sigma(s) = 0$

(4) $\mathfrak{h}_{\mathbf{x}}'(s) = \mathfrak{h}_x''(s) = \mathfrak{h}_x'''(s) = \mathfrak{h}_{\mathbf{x}}^{(4)}(s) = 0$ iff $\mathbf{x} = \pm\mathbf{g}(s)$ and $\sigma(s) = \sigma'(s) = 0$

(5) $\mathfrak{h}_{\mathbf{x}}'(s) = \mathfrak{h}_x''(s) = \mathfrak{h}_x'''(s) = \mathfrak{h}_{\mathbf{x}}^{(4)}(s) = \mathfrak{h}_{\mathbf{x}}^{(5)}(s) = 0$ iff $\mathbf{x} = \pm\mathbf{g}(s)$ and $\sigma(s) = \sigma'(s) = \sigma''(s) = 0$

(B)

(1) $\tilde{\mathfrak{h}}_{\mathbf{x}}(s) = 0$ iff there exist $<\boldsymbol{\gamma}, \mathbf{x}> = w$

(2) $\tilde{\mathfrak{h}}_{\mathbf{x}}(s) = \tilde{\mathfrak{h}}_x'(s) = 0$ iff there exist $t \in \mathbb{R}$ such that $\mathbf{x} = \sinh t\boldsymbol{\xi}_1 + \cosh t\boldsymbol{\xi}_2$ and $<\boldsymbol{\gamma}, \mathbf{x}> = w$

(3) $\tilde{\mathfrak{h}}_{\mathbf{x}}(s) = \tilde{\mathfrak{h}}_{\mathbf{x}}'(s) = \tilde{\mathfrak{h}}_x''(s) = \tilde{\mathfrak{h}}_x''(s) = 0$ iff $\mathbf{x} = \pm\mathbf{g}(s)$, $<\boldsymbol{\gamma}, \mathbf{x}> = w$, and $\sigma(s) = 0$

(4) $\tilde{\mathfrak{h}}_{\mathbf{x}}(s) = \tilde{\mathfrak{h}}_{\mathbf{x}}'(s) = \tilde{\mathfrak{h}}_x''(s) = \tilde{\mathfrak{h}}_x''(s) = \tilde{\mathfrak{h}}_x'''(s) = 0$ iff $\mathbf{x} = \pm\mathbf{g}(s)$, $<\boldsymbol{\gamma}, \mathbf{x}> = w$, and $\sigma(s) = \sigma'(s) = 0$

(5) $\tilde{\mathfrak{h}}_{\mathbf{x}}(s) = \tilde{\mathfrak{h}}_{\mathbf{x}}'(s) = \tilde{\mathfrak{h}}_x''(s) = \tilde{\mathfrak{h}}_x''(s) = \tilde{\mathfrak{h}}_x'''(s) = \tilde{\mathfrak{h}}_x^{(4)}(s) = 0$ iff $\mathbf{x} = \pm\mathbf{g}(s)$, $<\boldsymbol{\gamma}, \mathbf{x}> = w$, and $\sigma(s) = \sigma'(s) = \sigma''(s) = 0$

(6) $\tilde{\mathfrak{h}}_{\mathbf{x}}(s) = \tilde{\mathfrak{h}}_{\mathbf{x}}'(s) = \tilde{\mathfrak{h}}_x''(s) = \tilde{\mathfrak{h}}_x''(s) = \tilde{\mathfrak{h}}_x'''(s) = \tilde{\mathfrak{h}}_x^{(4)}(s) = \tilde{\mathfrak{h}}_x^{(5)}(s) = 0$ iff $\mathbf{x} = \pm\mathbf{g}(s)$, $<\boldsymbol{\gamma}, \mathbf{x}> = w$, and $\sigma(s) = \sigma'(s) = \sigma''(s) = \sigma'''(s) = 0$

Proof. According to Equation (2.2), we have $\|\xi'\|^2 \neq 0$ iff $\mu_1^2 - \mu_2^2 \neq 0$.

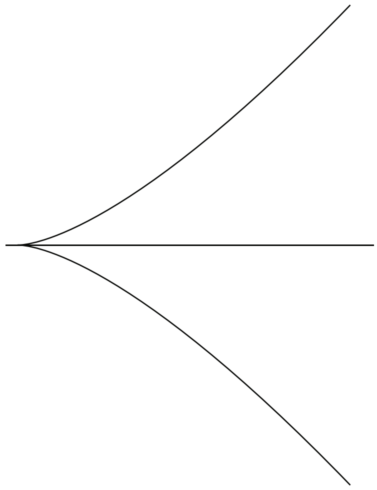

Figure 1: $C \times \mathbb{R}$.

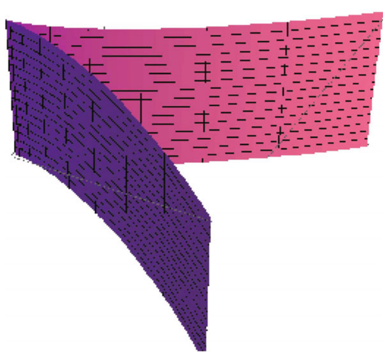

Figure 2: CE.

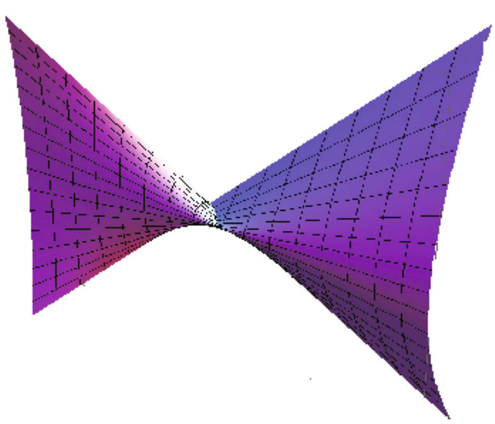

Figure 3: SW.

(A)

(1) Since $\mathfrak{h}_{\mathbf{x}}'(s) = <\xi, \mathbf{x}>$ and $\{\xi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2\}$ is RMF along $\boldsymbol{\gamma}(s)$, then there exist $a_1, a_2 \in \mathbb{R}$ such that $\mathbf{x} = a_1\boldsymbol{\xi}_1 + a_2\boldsymbol{\xi}_2$. By the condition that $\mathbf{x} \in \mathbb{H}_+^2$, we get $a_1^2 - a_2^2 = -1$. The converse direction also holds

(2) Since $\mathfrak{h}_{\mathbf{x}}''(s) = <\xi', \mathbf{x}> = <\mu_1\boldsymbol{\xi}_1 - \mu_2\boldsymbol{\xi}_2, \mathbf{x}> = 0$, we have $a_1\mu_1 + a_2\mu_2 = 0$. It follows from the fact $a_1^2 - a_2^2 = -1$ that $a_1 = \pm\mu_2/\sqrt{\mu_1^2 - \mu_2^2}$ and $a_2 = \mp\mu_1/\sqrt{\mu_1^2 - \mu_2^2}$. Therefore, we have

$$\mathbf{x} = \mp\left(\frac{1}{\sqrt{\mu_1^2 - \mu_2^2}}(-\mu_2\xi_1 + \mu_1\xi_2)\right)(s) = \mathbf{g}(s). \tag{13}$$

Therefore, $\mathfrak{h}_{\mathbf{x}}'(s) = \mathfrak{h}_x''(s) = 0$ iff $\mathbf{x} = \pm\mathbf{g}(s)$.

(3) Since $\mathfrak{h}_{\mathbf{x}}'''(s) = <\xi'', \mathbf{x}> = <(-\mu_1^2 + \mu_2^2)\xi + \mu_1'\boldsymbol{\xi}_1 - \mu_2'\boldsymbol{\xi}_2, \mathbf{x}> = 0$, by the conditions of (2), we have

$$\mp\left(\frac{1}{\sqrt{\mu_1^2 - \mu_2^2}}\left(-\mu_2\mu_1' + \mu_1\mu_2'\right)\right)(s) = \pm\left(\frac{\sigma}{\sqrt{\mu_1^2 - \mu_2^2}}\right)(s) = 0. \tag{14}$$

Thus, $\mathfrak{h}_{\mathbf{x}}'(s) = \mathfrak{h}_x''(s) = \mathfrak{h}_{\mathbf{x}}'''(s) = 0$ iff $\mathbf{x} = \pm\mathbf{g}(s)$ and $\sigma(s) = 0$.

(4) Since

$$\begin{aligned}\mathfrak{h}_{\mathbf{x}}^{(4)}(s) = <\xi''', \mathbf{x}> = <-3\left(\mu_1\mu_1' - \mu_2\mu_2'\right)\xi \\ + \left(\mu_1'' + \mu_1\left(-\mu_1^2 + \mu_2^2\right)\right)\xi_1 \\ - \left(\mu_2'' + \mu_2\left(-\mu_1^2 + \mu_2^2\right)\right)\xi_2, \mathbf{x}> = 0\},\end{aligned} \tag{15}$$

by the conditions of (7), we have

$$\mp\left(\frac{1}{\sqrt{\mu_1^2 - \mu_2^2}}\left(\mu_2\mu_1' - \mu_1\mu_2'\right)'\right)(s) = \pm\left(\frac{\sigma'}{\sqrt{\mu_2^2 - \mu_1^2}}\right)(s) = 0. \tag{16}$$

Thus, $\mathfrak{h}_{\mathbf{x}}'(s) = \mathfrak{h}_x''(s) = \mathfrak{h}_{\mathbf{x}}''(s) = \mathfrak{h}_{\mathbf{x}}^{(4)}(s) = 0$ iff $\mathbf{x} = \pm\mathbf{g}(s)$ and $\sigma(s) = \sigma'(s) = 0$.

(5) Since $\mathfrak{h}_{\mathbf{x}}^{(5)}(s) = <\xi^{(4)}, \mathbf{x} > 0$, we have

$$\begin{aligned}<\left(\left(\mu_1^2 - \mu_2^2\right)^2 - 4\left(\mu_1\mu_1'' - \mu_2\mu_2''\right) - 3\left(\mu_1'2 - \mu_2'2\right)\right)\xi \\ + \left(\mu_1''' - 5\mu_1\left(\mu_1'\mu_1 - \mu_2'\mu_2\right) + \mu_1'\left(-\mu_1^2 + \mu_2^2\right)\right)\xi_1 \\ - \left(\mu_2''' - 5\mu_2\left(\mu_1'\mu_1 - \mu_2'\mu_2\right) + \mu_2'\left(-\mu_1^2 + \mu_2^2\right)\right)\xi_2, \mathbf{x}> = 0\}.\end{aligned} \tag{17}$$

By using the conditions of (8), we have

$$\pm\frac{1}{\sqrt{\mu_1^2-\mu_2^2}}\left(\mu_1\mu_2'''-\mu_2\mu_1'''+\left(\mu_1\mu_2'-\mu_2\mu_1'\right)\left(-\mu_1^2+\mu_2^2\right)\right)(s)=0. \quad (18)$$

Therefore, $\mathfrak{h}_{\mathbf{x}}'(s)=\mathfrak{h}_x''(s)=\mathfrak{h}_{\mathbf{x}}'''(s)=\mathfrak{h}_{\mathbf{x}}^{(4)}(s)=\mathfrak{h}_{\mathbf{x}}^{(5)}=0$ iff $\mathbf{x}=\pm\mathbf{g}(s)$ and $\sigma(s)=\sigma'(s)=\sigma''(s)=0$.

(B)

Using the same computation as the proof of (A), we can get (B) (4) +.□ □

Proposition 4. *Let $\gamma=\gamma(s)$ be a unit speed spacelike curve with timelike binormal and $\mu_1^2-\mu_2^2\neq 0$. Then, we have $\sigma(s)=0$ iff $g(s)$ is a constant timelike vector.*

Proof. By simple calculations, we have

$$\mathbf{g}'(s)=\frac{\sigma(s)}{\left(\sqrt{\mu_1^2-\mu_2^2}\right)^3}(-\mu_1\xi_1+\mu_2\xi_2). \quad (19)$$

Thus, $\mathbf{g}'(s)=\mathbf{0}$ iff $\sigma(s)=\mu_2\mu_1'-\mu_1\mu_2'=0$ + .□ □

Proposition 5. *Let γ: $I\longrightarrow\mathbb{E}_1^3$ be a unit speed spacelike curve with timelike binormal and $\mu_1^2-\mu_2^2\neq 0$. Then, we have the following.*

(a) $\boldsymbol{\gamma}$ is a B-slant helix iff μ_2/μ_1 is constant

(b) $\boldsymbol{\xi}_2$ is a part of circle on $\mathbb{H}_+^2$ whose center is the timelike constant vector $\mathbf{g}_0$

Proof.

(a) Suppose that $\sigma(s)=\mu_1\mu_2'-\mu_2\mu_1'=0$. Hence, we can write

$$\left(\frac{\mu_2}{\mu_1}\right)'=\frac{\mu_1\mu_2'-\mu_2\mu_1'}{\mu_1^2}=-\frac{\sigma(s)}{\mu_1^2}=0. \quad (20)$$

Thus, $\mu_2/\mu_1=$ constant, that is, $\boldsymbol{\gamma}$ is a B-slant helix.

(b) Since

$$<\mathbf{g},\xi_2>=\frac{\mu_1}{\sqrt{\mu_1^2-\mu_2^2}}<\left(-\frac{\mu_2}{\mu_1}\xi_1+\xi_2\right),\xi_2>=\frac{-1}{\sqrt{1-\mu_2^2/\mu_1^2}}=\text{const}. \quad (21)$$

This means that $\boldsymbol{\xi}_2$ is a part of circle on $\mathbb{H}_+^2$ whose center is the constant timelike vector $\mathbf{g}_0(s)$+.□ □

3.2. Unfolding of Functions by One Variable. In this subsection, we use some general results on the singularity theory for families of function germs. Let $\mathfrak{F}:(\mathbb{R}\times\mathbb{R}^r,(s_0,\mathbf{x}_0))\longrightarrow\mathbb{R}$ be a smooth function and $\mathfrak{f}(s)=\mathfrak{F}_{x_0}(s,\mathbf{x}_0)$. Then, $\mathfrak{F}$ is called an r-parameter unfolding of $\mathfrak{f}(s)$. We say that $\mathfrak{f}(s)$ has A_k-singularity at s_0 if $\mathfrak{f}^{(p)}(s_0)=0$ for all $1\le p\le k$ and $\mathfrak{f}^{(k+1)}(s_0)\neq 0$. We also say that $\mathfrak{f}$ has A_k-singularity ($k1$) at s_0. Let the $(k-1)$-jet of the partial derivative $\partial\mathfrak{F}/\partial x_i$ at s_0 be $j^{(k-1)}((\partial F/\partial x_i)(s,\mathbf{x}_0))(s_0)=\Sigma_{j=0}^{k-1}L_{ji}(s-s_0)^j$ (without the constant term), for $i=1,\cdots,r$. Then, $\mathfrak{F}(s)$ is called a p-versal unfolding if the $k\times r$ matrix of coefficients (L_{ji}) has rank $(k\le r)$. So, we write important set about the unfolding relative to the above notations. The discriminant set of $\mathfrak{F}$ is the set

$$\mathfrak{D}_{\mathfrak{F}}=\left\{\mathbf{x}\in\mathbb{R}^r \mid \text{there exists } s \text{ with } \mathfrak{F}(s,\mathbf{x})=\frac{\partial\mathfrak{F}}{\partial s}(s,\mathbf{x})=0 \text{ at } (s,\mathbf{x})\right\}. \quad (22)$$

The bifurcation set of $\mathfrak{F}$ is the set

$$\mathfrak{B}_{\mathfrak{F}}=\left\{\mathbf{x}\in\mathbb{R}^r \mid \text{there exists } s \text{ with } \frac{\partial\mathfrak{F}}{\partial s}(s,\mathbf{x})=\frac{\partial^2\mathfrak{F}}{\partial s^2}(s,\mathbf{x})=0 \text{ at } (s,\mathbf{x})\right\}. \quad (23)$$

We can also give the following theorem [12, 13].

Theorem 6. *Let $\mathfrak{F}:(\mathbb{R}\times\mathbb{R}^r,(s_0,x_0))\longrightarrow\mathbb{R}$ be an r-parameter unfolding of $\mathfrak{f}(s)$, which has the A_k singularity at s_0.*

Suppose that F is a p-versal unfolding.

(a) If $k=1$, then $\mathfrak{D}_{\mathfrak{F}}$ is locally diffeomorphic to $\{0\}\times\mathbb{R}^{r-1}$ and $\mathfrak{B}_{\mathfrak{F}}=\varnothing$

(b) If $k=2$, then $\mathfrak{D}_{\mathfrak{F}}$ is locally diffeomorphic to $\mathbf{C}\times\mathbb{R}^{r-2}$, and $\mathfrak{B}_{\mathfrak{F}}$ is locally diffeomorphic to $\{0\}\times\mathbb{R}^{r-1}$

(c) If $k=3$, then $\mathfrak{D}_{\mathfrak{F}}$ is locally diffeomorphic to $SW\times\mathbb{R}^{r-3}$, and $\mathfrak{B}_{\mathfrak{F}}$ is locally diffeomorphic to $\mathbf{C}\times\mathbb{R}^{r-2}$

Hence, we have the following fundamental proposition.

Proposition 7. *Let $\boldsymbol{\gamma}$: $I\longrightarrow\mathbb{E}_1^3$ be a unit speed spacelike curve with timelike binormal and $\mu_1^2-\mu_2^2\neq 0$. (1) If $\mathfrak{h}_{\mathbf{x}}(s)=\mathfrak{H}(s,\mathbf{x})$ has an A_k-singularity $(k=2,3)$ at $s_0\in\mathbb{R}$, then $\mathfrak{H}$ is a p-versal unfolding of $\mathfrak{h}_{\mathbf{x}_0}(s_0)$. (2) If $\tilde{\mathfrak{h}}_{\mathbf{x}}(s)=\tilde{\mathfrak{H}}(s,\mathbf{x},w)$ has an A_k-singularity $(k=2,3)$ at $s_0\in\mathbb{R}$, then $\tilde{\mathfrak{H}}$ is a p-versal unfolding of $\tilde{\mathfrak{h}}_{\mathbf{x}_0}(s_0)$.*

Proof. (see (4)).

Since $\mathbf{x}=(x_0,x_1,x_2)\in\mathbb{H}_+^2$, $x_0^2+x_1^2-x_2^2=-1$, x_0, x_1, and x_2 cannot be all zero. Without loss of generality, we may assume that $x_2\neq 0$. Then, by $x_2=\sqrt{1+x_0^2+x_1^2}$, we have

$$\mathfrak{H}(s,\mathbf{x})=x_0\gamma_0(s)+x_1\gamma_1(s)-\sqrt{1+x_0^2+x_1^2}\gamma_2(s). \quad (24)$$

Thus, we have

$$\left.\begin{aligned}\frac{\partial \mathfrak{H}}{\partial x_0} &= \gamma_0(s) - \frac{x_0\gamma_2(s)}{\sqrt{1+x_0^2-x_1^2}}, \quad \frac{\partial \mathfrak{H}}{\partial x_1} = \gamma_1(s) - \frac{x_1\gamma_2(s)}{\sqrt{1+x_0^2+x_1^2}},\\ \frac{\partial^2 \mathfrak{H}}{\partial s\partial x_0} &= \gamma_0'(s) - \frac{x_0\gamma_2'(s)}{\sqrt{-1+x_0^2-x_1^2}}, \quad \frac{\partial^2 \mathfrak{H}}{\partial s\partial x_1} = \gamma_1'(s) - \frac{x_1\gamma_2'(s)}{\sqrt{1+x_0^2-x_1^2}}.\end{aligned}\right\} \tag{25}$$

Therefore, the 2-jets of $\partial\mathfrak{H}/\partial x_i$ at s_0 $(i=0,1)$ are as follows. Let $\mathbf{x}_0=(x_{00},x_{10},x_{20})\in\mathbb{H}_+^2$, and assume that $x_{20}\neq 0$, then

$$\left.\begin{aligned}j^1\left(\frac{\partial \mathfrak{H}}{\partial x_0}(s,\mathbf{x}_0)\right) &= \left(\gamma_0'(s) - \frac{x_0\gamma_2'(s)}{\sqrt{1+x_0^2-x_1^2}}\right)(s-s_0),\\ j^1\left(\frac{\partial \mathfrak{H}}{\partial x_1}(s,\mathbf{x}_0)\right) &= \left(\gamma_1'(s) - \frac{x_1\gamma_2'(s)}{\sqrt{1+x_0^2-x_1^2}}\right)(s-s_0),\end{aligned}\right\}$$

$$\left.\begin{aligned}&j^2\left(\frac{\partial \mathfrak{H}}{\partial x_0}(s,\mathbf{x}_0)\right) = \left(\gamma_0'(s) - \frac{x_{00}\gamma_2'(s)}{x_{20}}\right)(s-s_0) + \frac{1}{2}\left(\gamma_0'' - \frac{x_{00}\gamma_2''(s)}{x_{20}}\right)(s-s_0)^2,\\ &j^2\left(\frac{\partial \mathfrak{H}}{\partial x_1}(s,\mathbf{x}_0)\right) = \left(\gamma_1'(s) - \frac{x_{10}\gamma_2'(s)}{x_{20}}\right)(s-s_0),\\ &\frac{1}{2}\left(\gamma_1''(s) - \frac{x_{10}\gamma_2''(s)}{x_{20}}\right)(s-s_0)^2.\end{aligned}\right\} \tag{26}$$

(i) If $\mathfrak{h}_{\mathbf{x}_0}(s_0)$ has the A_2-singularity at s_0, then $\mathfrak{h}_{\mathbf{x}_0}'(s_0)=0$. So the $(2-1)\times 2$ matrix of coefficients (L_{ji}) is

$$A = \left(\left(\gamma_0'(s) - \frac{x_{00}\gamma_2'(s)}{x_{20}}\right)\gamma_1'(s) - \frac{x_{10}\gamma_2'(s)}{x_{20}}\right). \tag{27}$$

Suppose that the rank of the matrix A is zero, then we have

$$\gamma_0'(s) = \frac{x_{00}\gamma_2'(s)}{x_{20}}, \gamma_1'(s) = \frac{x_{10}\gamma_2'(s)}{x_{20}}. \tag{28}$$

Since $\|\gamma'(s_0)\| = \|\xi(s_0)\| = 1$, we have $\gamma_2'(s_0)\neq 0$ so that we have the contradiction as follows:

$$\begin{aligned}0 &= <\left(\gamma_0'(s_0),\gamma_1'(s_0),\gamma_2'(s_0)\right),(x_{00},x_{10},x_{20})>\\ &= \gamma_0'(s_0)x_{00} + \gamma_1'(s_0)x_{10} - \gamma_2'(s_0)x_{20}\\ &= \frac{x_{00}^2\gamma_2'(s_0)}{x_{20}} + \frac{x_{10}^2\gamma_2'(s_0)}{x_{20}} - \gamma_2'(s_0)x_{20}\\ &= \frac{\gamma_2'(s_0)}{x_{20}}\left(x_{00}^2+x_{10}^2-x_{20}^2\right) = \frac{\gamma_2'(s_0)}{x_{20}} \neq 0.\end{aligned} \tag{29}$$

Therefore, rank $(A)=1$, and $\mathfrak{H}$ is the (p) versal unfolding of $\mathfrak{h}_{\mathbf{x}_0}$ at s_0.

(ii) If $\mathfrak{h}_{\mathbf{x}_0}(s_0)$ has the A_3-singularity at $s_0\in\mathbb{R}$, then $\mathfrak{h}_{\mathbf{x}_0}'(s_0) = \mathfrak{h}_{\mathbf{x}_0}'{}'(s_0) = 0$, and by Proposition 3.

$$\mathbf{g}(s_0) = \frac{\mu_1}{\sqrt{\mu_1^2-\mu_2^2}}\left(-\frac{\mu_2}{\mu_1}\xi_1 + \xi_2\right), \tag{30}$$

where $\mu_1^2-\mu_2^2>0$, $\sigma'(s_0)=0$, and $\sigma''(s_0)\neq 0$. So, the $(3-1)\times 2$ matrix of the coefficients (L_{ji}) is

$$B = \begin{pmatrix} L_{11} & L_{12}\\ L_{21} & L_{22}\end{pmatrix} = \begin{pmatrix} \gamma_0'(s) + \dfrac{x_{00}\gamma_2'(s)}{x_{20}} & \gamma_1'(s) - \dfrac{x_{10}\gamma_2'(s)}{x_{20}}\\ \gamma_0'' + \dfrac{x_{00}\gamma_2''(s)}{x_{20}} & \gamma_1''(s) - \dfrac{x_{10}\gamma_2''(s)}{x_{20}}\end{pmatrix}. \tag{31}$$

For the purpose, we also require the 2×2 matrix B to be nonsingular, which always does. In fact, the determinate of this matrix at s_0 is

$$\begin{aligned}\det(B) &= \frac{1}{x_{20}}\begin{vmatrix} -\gamma_0' & \gamma_1' & \gamma_2'\\ -\gamma_0'' & \gamma_1'' & \gamma_2''\\ x_{00} & x_{10} & x_{20}\end{vmatrix} = \frac{1}{x_{20}} <\gamma'\times\gamma'',\mathbf{g_0}>\\ &= \mp\frac{1}{x_{20}\sqrt{\mu_1^2-\mu_2^2}} <\gamma'\times\gamma'',(-\mu_2\xi_1+\mu_1\xi_2)>.\end{aligned} \tag{32}$$

Since $\boldsymbol{\gamma}'=\xi$, we have $\boldsymbol{\gamma}''=\mu_1\boldsymbol{\xi}_1-\mu_2\boldsymbol{\xi}_2$. Substituting these relations to the above equality, we have

$$\det(B) = \pm\frac{\sqrt{\mu_1^2-\mu_2^2}}{x_{20}} \neq 0. \tag{33}$$

This means that rank $(B)=2$.

(2) Under the same notations as in (4), we have

$$\tilde{\mathfrak{H}}(s,\mathbf{x},x_2) = x_0\gamma_0(s) + x_1\gamma_1(s) - \sqrt{1+x_0^2+x_1^2}\gamma_2(s) - x_2. \tag{34}$$

We require the 2×3 matrix

$$G = \begin{pmatrix} \gamma_0'(s) - \dfrac{x_{00}\gamma_2(s)}{x_{20}} & \gamma_1'(s) - \dfrac{x_{10}\gamma_2'(s)}{x_{20}} & -1\\ \gamma_0'' - \dfrac{x_{00}\gamma_2''(s)}{x_{20}} & \gamma_1''(s) - \dfrac{x_{10}\gamma_2''(s)}{x_{20}} & 0\end{pmatrix}, \tag{35}$$

to have the maximal rank. By case (4) in Equation (3.14), the second row of G does not vanish, so rank $(G) = 2+$.□ □

Proof of Theorem 1 (see (4)). By Proposition 3, the bifurcation set of $\mathfrak{H}(s, \mathrm{x})$ is

$$\mathfrak{B}_{\mathfrak{H}} = \left\{ \frac{1}{\sqrt{\mu_1^2 - \mu_2^2}} (-\mu_2 \xi_1 + \mu_1 \xi_2) | s \in \mathbb{R} | s \in \mathbb{R} \right\}. \tag{36}$$

□

The assertion (4) of Theorem 2 follows from Propositions 3 and 7 and Theorem 6. The discriminant set of $\tilde{\mathfrak{H}}(s, \mathbf{x})$ is given as follows:

$$\mathfrak{D}_{\tilde{\mathfrak{H}}} = \{ \mathbf{x}_0 = \gamma + \sinh t \xi_1 + \cosh t \xi_2 \mid s \in \mathbb{R} \}. \tag{37}$$

The assertion (4) of Theorem 2 follows from Propositions 3 and 7 and Theorem 6 +.

Example 1. Given the spacelike helix

$$\gamma(s) = \left(\cosh s, \sqrt{2}s, \sinh s \right), -3 \le s \le 3. \tag{38}$$

It is easy to show that

$$\left. \begin{aligned} v_1(s) &= \left(\sinh s, \sqrt{2}, \cosh s \right), \\ v_2(s) &= (\cosh s, 0, \sinh s), \\ v_3(s) &= \left(\sqrt{2} \sinh s, 1, \sqrt{2} \cosh s \right), \\ \kappa(s) &= 1, \text{and } \tau(s) = \sqrt{2}. \end{aligned} \right\} \tag{39}$$

Then, $\varphi(s) = -\sqrt{2}s + \varphi_0$. If we choose $\varphi_0 = 0$, for example, we have

$$\begin{aligned} \mu_1(s) &= \cosh \sqrt{2}s, \\ \mu_2(s) &= -\sinh \sqrt{2}s. \end{aligned} \tag{40}$$

We can calculate the geometric invariant

$$\begin{aligned} \sigma(s) &= -\sqrt{2}, \\ \sigma'(s) &= 0. \end{aligned} \tag{41}$$

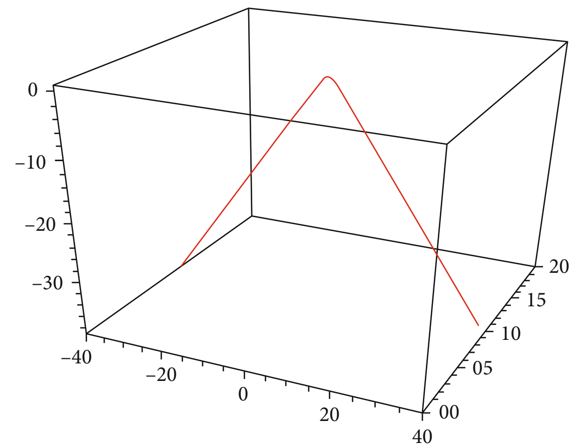

Figure 4: Bishop spherical Darboux image has a cusp as $s = 0$.

We also have

$$\begin{aligned} \xi_1 &= \begin{pmatrix} \xi_{11} \\ \xi_{12} \\ \xi_{13} \end{pmatrix} = \begin{pmatrix} \cosh\left(\sqrt{2}s\right) \cosh s - \sqrt{2} \sinh\left(\sqrt{2}s\right) \sinh s \\ -\sinh\left(\sqrt{2}s\right) \\ \cosh\left(\sqrt{2}s\right) \sinh s - \sqrt{2} \sinh\left(\sqrt{2}s\right) \cosh s \end{pmatrix}, \\ \xi_2 &= \begin{pmatrix} \xi_{21} \\ \xi_{22} \\ \xi_{23} \end{pmatrix} = \begin{pmatrix} -\sinh\left(\sqrt{2}s\right) \cosh s + \sqrt{2} \cosh\left(\sqrt{2}s\right) \sinh s \\ \cosh\left(\sqrt{2}s\right) \\ -\sinh\left(\sqrt{2}s\right) \sinh s + \sqrt{2} \cosh\left(\sqrt{2}s\right) \cosh s \end{pmatrix}. \end{aligned} \tag{42}$$

The timelike Bishop spherical Darboux image is shown in Figure 4)

$$\mathbf{g}(s) = \sinh \sqrt{2}s \begin{pmatrix} \xi_{11} \\ \xi_{12} \\ \xi_{13} \end{pmatrix} + \cosh \sqrt{2}s \begin{pmatrix} \xi_{21} \\ \xi_{22} \\ \xi_{23} \end{pmatrix}. \tag{43}$$

The timelike sweeping surface family is

$$\mathbf{y}(s, t) = \left(\cosh s, \sqrt{2}s, \sinh s \right) + r_1(t) \begin{pmatrix} \xi_{11} \\ \xi_{12} \\ \xi_{13} \end{pmatrix} + r_2(t) \begin{pmatrix} \xi_{21} \\ \xi_{22} \\ \xi_{23} \end{pmatrix}. \tag{44}$$

By choosing $r_1(t) = \cosh t$ and $r_2(t) = \sinh t$, then we immediately have a timelike sweeping surface (see Figure 5).

3.3. Developable Surfaces. Developable surfaces can be briefly introduced as special cases of ruled surfaces. Such surfaces are widely used, for example, in the manufacture of automobile body parts, airplane wings, and ship hulls. Therefore, we analyze the case that the profile curve $\mathbf{r}(t)$ degenerates into a

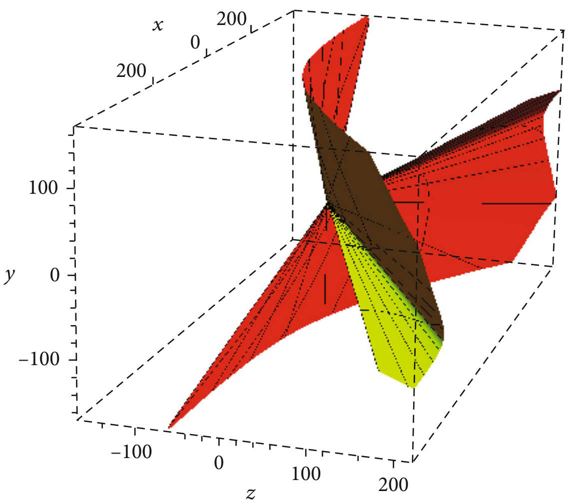

Figure 5: Timelike sweeping surface with spacelike helix singularity curve.

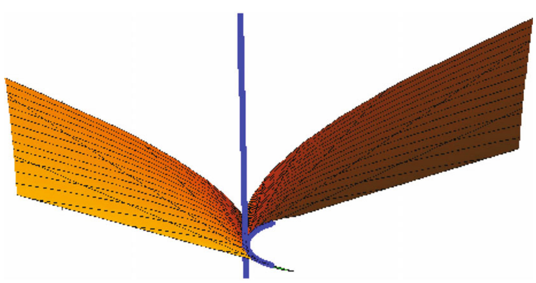

Figure 6: Timelike developable surface with its singular curve.

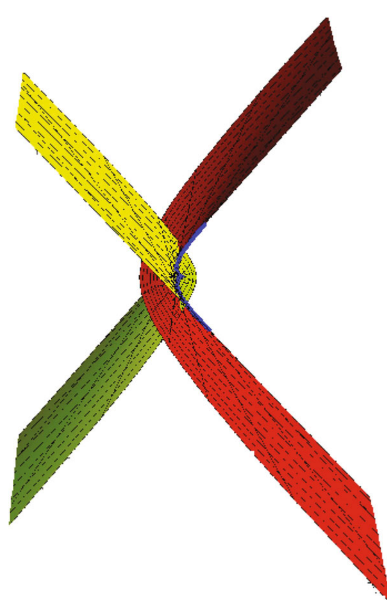

Figure 7: Spacelike developable surface with its singular curve.

timelike line. Then, we have the following timelike developable surface

$$\mathfrak{M} : \mathfrak{D}(s,t) = \gamma(s) + t\xi_2(s), t \in \mathbb{R}. \tag{45}$$

We also have the following spacelike developable surface

$$\mathfrak{M}^{\perp} : \mathfrak{D}^{\perp}(s,t) = \gamma(s) + t\xi_1(s), t \in \mathbb{R}. \tag{46}$$

It is clear that $\mathfrak{D}(s,0) = \gamma(s)$ (resp. $\mathfrak{D}^{\perp}(s,0)1 = \gamma(s)$), $0 \le s \le L$, that is, the surface $\mathfrak{M}$ (resp. $\mathfrak{M}^{\perp}$) interpolate the curve $\gamma(s)$. Also, we have

$$\begin{aligned} \frac{\partial \mathfrak{D}}{\partial s} \times \frac{\partial \mathfrak{D}}{\partial t} &= -(1 - t\mu_2)\xi_1(s). \\ \frac{\partial \mathfrak{D}^{\perp}}{\partial s} \times \frac{\partial \mathfrak{D}^{\perp}}{\partial t} &= -(1 - t\mu_1)\xi_2(s). \end{aligned} \tag{47}$$

Thus, we have that $\mathfrak{M}$ (resp. $\mathfrak{M}^{\perp}$) is nonsingular at (s_0, t_0) if and only if $1 - t_0\mu_2(s_0) \neq 0$ (resp. $1 - t_0\mu_1(s_0) \neq 0$). We designate $\mu(s)$ to represent $\mu_i(s)$ $(i = 1, 2)$, and based on Theorem 3.3 in [23], we can give the following corollary.

Corollary 8. *For the developable ruled surfaces $\mathfrak{D}(s,t)$ and $\mathfrak{D}^{\perp}(s,t)$, we have the following:*

(1) $\mathfrak{D}$ (resp. $\mathfrak{D}^{\perp}$) is locally diffeomorphic to the cuspidal edge CE $C(2,3) \times \mathbb{R}$ at (s_0, t_0) iff $\mu(s_0) = 0$ and $\mu'(s_0) \neq 0$

(2) $\mathfrak{D}$ (resp. $\mathfrak{D}^{\perp}$) is locally diffeomorphic to swallowtail SW at (s_0, t_0) iff $\mu(s_0) \neq 0$ and $\mu'(s_0) = 0$

Example 2. By making using of Example 1, we have the following:

(1) If $s_0 = 0$, then $\mu_2(s_0) = 0$ and $\mu_2'(s_0) \neq 0$. The timelike developable surface

$$\mathfrak{D}(s,t) = \left(\cosh s, \sqrt{2}s, \sinh s\right) + t \begin{pmatrix} -\sinh\left(\sqrt{2}s\right)\cosh s + \sqrt{2}\cosh\left(\sqrt{2}s\right)\sinh s \\ \cosh\left(\sqrt{2}s\right) \\ -\sinh\left(\sqrt{2}s\right)\sinh s + \sqrt{2}\cosh\left(\sqrt{2}s\right)\cosh s \end{pmatrix}, \quad t \in \mathbb{R} \tag{48}$$

is locally diffeomorphic to the cuspidal edge; see Figure 6. We can obtain the singular locus of $\mathfrak{M}$ as follows:

$$\mathfrak{S}(s) = \begin{pmatrix} 2\cosh s - \sqrt{2}\coth\left(\sqrt{2}s\right)\sinh s \\ \sqrt{2}s - \coth\left(\sqrt{2}s\right) \\ 2\sinh s - \sqrt{2}\coth\left(\sqrt{2}s\right)\cosh s \end{pmatrix}. \tag{49}$$

(2) If $s_0 = 0$, then $\mu_1(s_0) \neq 0$ and $\mu_1'(s_0) = 0$. The spacelike developable surface

$$\mathfrak{D}^{\perp}(s,t) = \left(\cosh s, \sqrt{2}s, \sinh s\right) + t\begin{pmatrix} \cosh\left(\sqrt{2}s\right)\cosh s - \sqrt{2}\sinh\left(\sqrt{2}s\right)\sinh s \\ -\sinh\left(\sqrt{2}s\right) \\ \cosh\left(\sqrt{2}s\right)\sinh s - \sqrt{2}\sinh\left(\sqrt{2}s\right)\cosh s \end{pmatrix}, \quad t \in \mathbb{R} \tag{50}$$

is locally diffeomorphic to swallowtail; see Figure 7. Also, the singular locus of $\mathfrak{M}^{\perp}$ is

$$\mathfrak{S}^{\perp}(s) = \begin{pmatrix} 2\cosh s - \sqrt{2}\tanh\left(\sqrt{2}s\right)\sinh s \\ \sqrt{2}s - \tanh\left(\sqrt{2}s\right) \\ 2\sinh s - \sqrt{2}\tanh\left(\sqrt{2}s\right)\cosh s \end{pmatrix}. \tag{51}$$

4. Conclusion

In this paper, we introduced the notion of timelike sweeping surfaces with rotation minimizing frames in Minkowski 3-space $\mathbb{E}_1^3$. By applying singularity, we classified the generic properties and present a new geometric invariant related to the singularities of this timelike sweeping surface. It leads to the fact that the generic singularities of this sweeping surface are cuspidal edge and swallowtail, and the types of these singularities can be characterized by this geometric invariant, respectively. Finally, some examples are presented to explain the theoretical results.

Data Availability

All of the data are available within the paper.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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