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Research Article

Preopenness Degree in *RL*-Fuzzy Bitopological Spaces

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Based on the concept of pseudocomplement, we introduce a new representation of preopenness of L-fuzzy sets in RL-fuzzy bitopological spaces. The concepts of pairwise RL-fuzzy precontinuous and pairwise RL-fuzzy preirresolute functions are extended and discussed based on the (i, j)-RL-preopen gradation. Further, we follow up with a study of pairwise RL-fuzzy precompactness in RL-fuzzy bitopological spaces of an L-fuzzy set. We find that our paper offers more general results since RL-fuzzy bitopology is a generalization of L-bitopology, RL-bitopology, and L-fuzzy topology.

1. Introduction

The fuzzy set theory was introduced in the year 1965 by Zadeh [1]. The development of fuzzy set theory, since its introduction, has been dramatic and breathtaking! Thousands of research papers have appeared in various journals devoted entirely to theoretical and application aspects of fuzzy sets. Artificial intelligence, automata theory, computer science, control theory, decision-making, expert systems, medical diagnosis, neural networks, pattern recognition, robotics, and social sciences are a few fields where fuzzy sets find application. Within a short time since its introduction, the fuzzy sets have permeated almost every academic discipline and have made their way into consumer products! Apart from this, it is also used for the construction of machines by way of intelligent robotics (engines, cars, ships, turbines, etc.) and controls (Sendai subway train in Japan, etc.) as well as for military purposes.

In 1991, Bin Shahna [2] introduced the concept of α-open and preopen sets in the context of fuzzy sets, and he introduced a preliminary study of fuzzy strong semicontinuity and fuzzy precontinuity as well. Later, the concepts of fuzzy α-open sets, fuzzy preopen, fuzzy α-continuous mappings, and fuzzy precontinuous mappings have been generalized to the setting of fuzzy bitopological spaces in [3], where some of their fundamental properties have been studied.

Shi [4] presented the concept of an L-fuzzy preopen degree of L-fuzzy set in L-fuzzy topological spaces. In addition, he discussed the fundamental properties of L-fuzzy precontinuous and L-fuzzy preirresolute mappings. It has been found that Shi's operator is incredibly useful in introducing other gradations as well as in analyzing many topological characteristics [5].

The concept of RL-topology has recently been introduced and studied by H. Li and Q. Li [6] as an extension of L-topology. A detailed discussion is also presented concerning RL-continuous mapping and RL-compactness by means of an inequality. As a generalization of RL-topology and L-fuzzy topology, H. Li and Q. Li [7] defined RL-fuzzy topology. Further investigations are conducted regarding the RL-fuzzy compactness in RL-fuzzy topological spaces. As a consequence of their work, Zhang et al. [8] presented the Lindelöf property degree as well as the countable RL-fuzzy compactness degree of an L-subset. It is clear that the gradation of fuzzy compactness and Lindelöf property in the sense of Kubiak and Šostak are special cases of the corresponding degrees in RL-fuzzy topology.

In this paper, the pseudocomplement of L-fuzzy sets is put forward as a basis for the definition of (i, j)-RL-preopen degree in RL-fuzzy bitopology. Additionally, we introduce and discuss pairwise RL-fuzzy precontinuous, pairwise RL-fuzzy preirresolute mappings, and pairwise RL-fuzzy precompactness.

2. Preliminaries

Throughout this paper, $(L, \vee, \wedge, ')$ denotes a complete DeMorgan algebra [9, 10] and X is a nonempty crisp set. By L^X, we refer to the family of all L-fuzzy sets presented on X. The greatest and the smallest elements in L and L^X are $\mathbf{1}_L, \mathbf{0}_L$ and $\mathbf{1}_{L^X}, \mathbf{0}_{L^X}$, respectively. For any $\alpha, \beta \in L$, $\alpha \prec \beta$ means that the element α is wedge below β in L [11]. $J(L)$ (resp. $P(L)$) denotes the collection of nonzero coprime (resp. nonunit prime) members in L. The greatest minimal collection and the greatest maximal collection of α are denoted by $\alpha(\alpha)$ and $\beta(\alpha)$, respectively. Moreover, $\alpha^*(\alpha) = \alpha(\alpha) \cap J(L)$ and $\beta^*(\alpha) = \beta(\alpha) \cap P(L)$. The valuable L-fuzzy set A is an L-fuzzy set that achieves the condition $A \leq A'$. $\mathscr{V}_X^L$ refers to the family of valuable L-fuzzy sets on X, i.e, $\mathscr{V}_X^L = \{A \in L^X : A \leq A'\}$. Moreover, $\Psi_X^L(A) = \{B \in L^X : B \leq A\}$ for every $A \in \mathscr{V}_X^L$. Let $\psi : X \longrightarrow Y$ be a mapping, $A \in \mathscr{V}_X^L$, and $B \in \mathscr{V}_Y^L$, $\psi_{\vec{L}}|_A : \Psi_X^L(A) \longrightarrow L^Y$ with $D \in \Psi_X^L(A) \mapsto \psi_{\vec{L}}(D)$, is called an L-fuzzy function restriction (briefly, RL-fuzzy function), defined by $\psi_{\vec{L},A} : A \longrightarrow B$ with $\psi_{\vec{L}}(A) \leq B$. Moreover, $\psi_{\overleftarrow{L},A}(C) = \vee\{D \in \Psi_X^L(A): \psi_{\vec{L}}(D) \leq C\}$ for each $C \in \Psi_Y^L(B)$. Clearly, $\psi_{\overleftarrow{L},A}(C) = A \wedge \psi_{\overleftarrow{L}}(C)$. Let $A \in \mathscr{V}_X^L$ and $B \in \Psi_X^L(A)$, then $\langle_L^A B = A \wedge B'$ if $B \neq A$ and $\langle_L^A B = \mathbf{0}_{L^X}$ if $B = A$. The operation $\langle_L^A B$ is called the pseudocomplement of B with respect to A [6, 7]. The following proposition lists some of its properties:

Proposition 1 (see [6, 7]). *If $A \in \mathscr{V}_X^L$, $H, G \in \Psi_X^L(A)$, and $\{H_i\}_{i\in\Omega} \subseteq \Psi_X^L(A)$, then:*

(1) $\langle_L^A H = A \Leftrightarrow H \leq A'$

(2) $H \leq G \Rightarrow \langle_L^A G \leq \langle_L^A H$

(3) $\langle_L^A \wedge_{i\in\Omega} H_i = \vee_{i\in\Omega} \langle_L^A H_i$

(4) $\langle_L^A \wedge_{i\in\Omega} H_i \leq \wedge_{i\in\Omega} \langle_L^A H_i$, $\langle_L^A \wedge_{i\in\Omega} H_i = \wedge_{i\in\Omega} \langle_L^A H_i$ provided that $\vee_{i\in\Omega} H_i \neq A$

Lemma 2 (see [6]). *For any $A \in \mathscr{V}_X^L$, $B \in \mathscr{V}_Y^L$, $\psi_{\vec{L},A} : A \longrightarrow B$, $H \in \Psi_X^L(A)$, and $\mathscr{U} \subseteq \Psi_X^L(A)$, we have*

$$\bigvee_{y\in Y} \left(\psi_{\vec{L},A}(H)(y) \wedge \bigwedge_{G\in\mathscr{U}} G(y)\right) = \bigvee_{x\in X} \left(H(x) \wedge \bigwedge_{G\in\mathscr{U}} \psi_{\overleftarrow{L},A}(G)(x)\right). \tag{1}$$

Equivalently [8],

$$\bigwedge_{y\in Y} \left(\langle_L^A \psi_{\vec{L},A}(H)(y) \vee \bigvee_{G\in\mathscr{P}} G(y)\right) = \bigwedge_{x\in X} \left(\langle_L^A H(x) \vee \bigvee_{G\in\mathscr{P}} \psi_{\overleftarrow{L},A}(G)(x)\right). \tag{2}$$

A subcollection $\mathscr{T} \subseteq L^X$ is called an L-topology [9, 10, 12] (briefly, L-t) if $\mathscr{T}$ includes the smallest and the greatest L-fuzzy sets in L^X. Moreover, the subcollection $\mathscr{T}$ is closed for every suprema and finite infima. If $\mathscr{T}$ is an L-topology on X, then the paire $(X, \mathscr{T})$ is said to be an L-topological space on X. Furthermore, elements of $\mathscr{T}$ are said to be open, and their complements are said to be closed. For any mapping $\psi : (X, \mathscr{T}_1) \longrightarrow (Y, \mathscr{T}_2)$, ψ is said to be L-continuous iff $\psi_{\overleftarrow{L}}(C) \in \mathscr{T}_1$ for every $C \in \mathscr{T}_2$.

The mapping $\mathscr{T} : L^X \longrightarrow L$ is called an L-fuzzy topology on the set X [13–15] if it achieves the following statements:

(1) $\mathscr{T}(\mathbf{0}_{L^X}) = \mathscr{T}(\mathbf{1}_{L^X}) = \mathbf{1}_L$

(2) $\mathscr{T}(H_1 \wedge H_2) \geq \mathscr{T}(H_1) \wedge \mathscr{T}(H_2)$, for every H_1, $H_2 \in L^X$

(3) $\mathscr{T}(\vee_{i\in\Omega} H_i) \geq \wedge_{i\in\Omega} \mathscr{T}(H_i)$, for every $\{H_i\}_{i\in\Omega} \subseteq L^X$

If $\mathscr{T}$ is an L-topology on X, then the pair $(X, \mathscr{T})$ is said to be an L-fuzzy topological space (briefly, L-fts). The degree of openness and the degree of closeness of $G \in L^X$ are represented by $\mathscr{T}(G)$ and $\mathscr{T}^*(G) = \mathscr{T}(G')$, respectively. The mapping $\psi : (X, \mathscr{T}_1) \longrightarrow (Y, \mathscr{T}_2)$ is said to be an L-fuzzy continuous if and only if $\mathscr{T}_1(\psi_{\overleftarrow{L}}(M)) \geq \mathscr{T}_2(M)$ for every $M \in L^Y$.

Definition 3 (see [6]). Let $A \in \mathscr{V}_X^L$ and $\mathscr{R} \subseteq \Psi_X^L(A)$; then $\mathscr{R}$ is called a relative L-topology (briefly, RL-t) A if it achieves the next conditions:

(1) $A \in \mathscr{R}$ and $H \in \mathscr{R}$, for each $H \leq A'$

(2) $H_1 \wedge H_2 \in \mathscr{R}$, for any $H_1, H_2 \in \mathscr{R}$

(3) $\vee_{i\in\Omega} H_i \in \mathscr{R}$, for any $\{H_i\}_{i\in\Omega} \subseteq \mathscr{R}$

If $\mathscr{R}$ is an RL-topology on A, then, the pair $(A, \mathscr{R})$ is called an RL-topological space on A (briefly, RL-ts). The relative open L-fuzzy sets (briefly, RL-open fuzzy set) are the members of $\mathscr{R}$ while their pseudocomplements are said to be RL-closed fuzzy sets, i.e., $\langle_L^A B \in \mathscr{R}$. The family of all R L-closed fuzzy sets with respect to $\mathscr{R}$ is denoted by $\langle_L^A \mathscr{R}$. Let $A \in \mathscr{V}_X^L$, $B \in \mathscr{V}_Y^L$ and $(A, \mathscr{R}_1)$, $(B, \mathscr{R}_2)$ be two RL-ts. The RL-fuzzy mapping $\psi_{\vec{L},A} : A \longrightarrow B$ is called an RL-continuous if and only if $\psi_{\overleftarrow{L},A}(C) \in \langle_L^A \mathscr{R}_1$ for each $C \in \langle_L^A \mathscr{R}_2$. Equivalently, $\psi_{\vec{L},A} : A \longrightarrow B$ is called an RL-continuous if and only if $\psi_{\overleftarrow{L},A}(H) \in \mathscr{R}_1$ for each $H \in \mathscr{R}_2$. If $\mathscr{R}_1$ and $\mathscr{R}_2$ are RL-topologies on $A \in \mathscr{V}_X^L$, then $(A, \mathscr{R}_1, \mathscr{R}_2)$ is called an RL-bitopological space (briefly, RL-bts). $\mathscr{R}_i$-RL-open (resp. closed) refers to the open (resp. closed) L-fuzzy set with respect to $(A, \mathscr{R}_i)$, where $i = 1, 2$. In case of $A = \mathbf{1}_X$, we will get back to L-topology and L-bitopology.

The concepts of β_α-cover, strong β_α-cover, Q_α-cover, α-shading, strong α-shading, α-remote collection, and strong α-remote family [16] are extended to RL-topological spaces in [17] as follows:

Definition 4 (see [17]). For each $A \in \mathscr{V}_X^L$, RL-topology $\mathscr{R}$ on A, $H \in \Psi_X^L(A)$, and $\alpha \in L_0$, a set $\mathscr{U} \subseteq \Psi_X^L(A)$ is said to be

(1) β_α-cover of H if $\alpha \in \beta(\langle_L^A H(x) \vee \vee_{G \in \mathscr{U}} G(x))$ for all $x \in X$ and $\mathscr{U}$ is said to be strong β_α-cover of B if $a \in \beta(\wedge_{x \in X}(\langle_L^A H(x) \vee \vee_{G \in \mathscr{U}} G(x)))$

(2) Q_α-cover of B if $\langle_L^A H(x) \vee \vee_{G \in \mathscr{U}} G(x) \geq \alpha$ for all $x \in X$

Definition 5 (see [17]). For each $A \in \mathscr{V}_X^L$, RL-topology $\mathscr{R}$ on A, $\alpha \in L_1$, and $H \in \Psi_X^L(A)$, a family $\mathscr{A} \subseteq \Psi_X^L(A)$ is said to be

(1) α-shading of H if $((\langle_L^A H(x) \vee \vee_{G \in \mathscr{A}} G(x)) \nleq \alpha$ for all $x \in X$

(2) strong α-shading of H if $\wedge_{x \in X}(\langle_L^A H(x) \vee \vee_{G \in \mathscr{A}} G(x)) \nleq \alpha$

(3) α-remote family of H if $(H(x) \wedge \wedge_{G \in \mathscr{A}} G(x)) \ngeq \alpha$ for all $x \in X$

(4) strong α-remote family of H if $\vee_{x \in X}(H(x) \wedge \wedge_{G \in \mathscr{A}} G(x)) \ngeq \alpha$

Theorem 6 (see [6]). *For each RL-ts $(A, \mathscr{R})$, the next statements are valid:*

(1) $A \in \langle_L^A \mathscr{R}$ and $H \in \langle_L^A \mathscr{R}$ for every $H \leq A'$

(2) $H_1 \vee H_2 \in \langle_L^A \mathscr{R}$ for every H_1, $H_2 \in \langle_L^A \mathscr{R}$

(3) $\wedge_{i \in I} H_i \in \langle_L^A \mathscr{R}$ for every $-H_i : i \in \Omega\} \subseteq \langle_L^A \mathscr{R}$

Definition 7 (see [7]). The mapping $\mathscr{R} : \Psi_X^L(A) \longrightarrow L$ such that $A \in \mathscr{V}_X^L$ is called an RL-fuzzy topology on A if $\mathscr{R}$ achieves the next statements:

(1) $\mathscr{R}(A) = \mathbf{1}_L$, for every $H \leq A'$, $\mathscr{R}(H) = \mathbf{1}_L$

(2) $\mathscr{R}(H_1 \wedge H_2) \geq \mathscr{R}(H_1) \wedge \mathscr{R}(H_2)$, for every H_1, $H_2 \in \Psi_X^L(A)$

(3) $\mathscr{R}(\vee_{i \in \Omega} H_i) \geq \wedge_{i \in \Omega} \mathscr{R}(H_i)$, for every $\{H_i\}_{i \in \Omega} \subseteq \Psi_X^L(A)$

If $\mathscr{R}$ is an RL-fuzzy topology on A, then the pair $(A, \mathscr{R})$ is called an RL-fuzzy topological space (briefly, RL-fts). For every $H \in \Psi_X^L(A)$, $\mathscr{R}(H)$ (resp. $\mathscr{R}(\langle_L^A H)$) refers to the degree of openness (resp. closeness) of H relative to $\mathscr{R}$, respectively. Moreover, if $\mathscr{R}(H) = \mathbf{1}_L$ (resp. $\mathscr{R}(\langle_L^A H) = \mathbf{1}_L$), then the RL-openness (resp. RL-closeness) of an L-fuzzy set H is confirmed. Clearly, if $A = \mathbf{1}_{L^X}$, then RL-fuzzy topology on A turn into Kubiak-Šostak's L-fuzzy topology. Further, if $\mathscr{R}$ is an RL-topology on A and $\chi_{\mathscr{R}} : \Psi_X^L(A) \longrightarrow L$ is a mapping defined by $\chi_{\mathscr{R}}(H) = \mathbf{1}_L$ if $H \in \mathscr{R}$, and $\chi_{\mathscr{R}}(H) = \mathbf{0}_L$ if $H \notin \mathscr{R}$, then $\chi_{\mathscr{R}}$ introduces a special RL-ft on A.

Theorem 8 (see [7]). *For every $A \in \mathscr{V}_X^L$ and RL -ft $\mathscr{R}$ on A. The mapping ${}_L^A\mathscr{R}$: $\Psi_X^L(A) \longrightarrow L$ defined by $\langle_L^A \mathscr{R}(H) = \mathscr{R}(\langle_L^A H)$ for every $H \in \Psi_X^L(A)$ achieves the next statements:*

(1) $\langle_L^A \mathscr{R}(A) = 1_L$, for every $H \leq A'$, $\langle_L^A \mathscr{R}(H) = 1_L$

(2) $\langle_L^A \mathscr{R}(H_1 \vee H_2) \geq \langle_L^A \mathscr{R}(H_1) \wedge \langle_L^A \mathscr{R}(H_2)$, for every H_1, $H_2 \in \Psi_X^L(A)$

(3) $\langle_L^A \mathscr{R}(\wedge_{i \in \Omega} H_i) \geq \wedge_{i \in I} \langle_L^A \mathscr{R}(H_i)$, for every $\{H_i\}_{i \in \Omega} \subseteq \Psi_X^L(A)$

$\langle_L^A \mathscr{R}$ is called an RL-fuzzy cotopology (briefly, RL-fct) on A.

Definition 9 (see [7]). Let $A \in \mathscr{V}_X^L$, $B \in \mathscr{V}_Y^L$ and $(A, \mathscr{R}_1)$, $(B, \mathscr{R}_2)$ be two RL-fts on A and B, respectively. The RL-fuzzy mapping $\psi_{L,A} : A \longrightarrow B$ is called an RL-fuzzy continuous if and only if

$$\mathscr{R}_1\left(\psi_{\overleftarrow{L},A}(C)\right) \geq \mathscr{R}_1(C), \tag{3}$$

that is,

$$\mathscr{R}_1\left(\left\langle_L^A \psi_{\overleftarrow{L},A}(C)\right) \geq \mathscr{R}_1\left(\langle_L^B C\right), \tag{4}$$

for every $C \in \Psi_Y^L(B)$. Further, if $(A, \langle_L^A \mathscr{R}_1)$ and $(B, \langle_L^B \mathscr{R}_2)$ are RL-fcts with respect to $(A, \mathscr{R}_1)$ and $(B, \mathscr{R}_2)$ respectively, then $\psi_{\overrightarrow{L},A}$ is called an RL-fuzzy continuous if and only if

$$\langle_L^A \mathscr{R}_1\left(\psi_{\overleftarrow{L},A}(C)\right) \geq \langle_L^B \mathscr{R}_2(C), \tag{5}$$

for every $C \in \Psi_X^L(B)$.

Definition 10 (see [17, 18]). Let $A \in \mathscr{V}_X^L$, and $(A, \mathscr{R})$ be an RL-fts on A. The mapping $Cl^{\mathscr{R}} : \Psi_X^L(A) \longrightarrow L^{J(\Psi_X^L(A))}$ given by

$$Cl^{\mathscr{R}}(H)(x_\lambda) = \bigwedge_{x_\lambda \leq G \geq H} \left\langle_L^A \left(\mathscr{R}\left(\langle_L^A G\right)\right)\right. \tag{6}$$

for every $x_\lambda \in J(\Psi_X^L(A))$ and $H \in \Psi_X^L(A)$ is said to be the induced RL-fuzzy closure operator by $\mathscr{R}$.

Definition 11 (see [7]). For every $A \in \mathscr{V}_X^L$ and $\mathscr{R}RL$-ft on A, an L-fuzzy set $B \in \Psi_X^L(A)$ is called an RL-fuzzy compact (briefly, RL-fc) if for every $\mathscr{P} \subseteq \Psi_X^L(A)$; the next inequality is true:

$$\bigvee_{G \in \mathscr{P}} \mathscr{R}\left(\langle_L^A G\right) \vee \bigvee_{x \in X}\left(H(x) \wedge \bigwedge_{G \in \mathscr{P}} G(x)\right) \geq \bigwedge_{\mathscr{S} \in 2^{\mathscr{P}}} \bigvee_{x \in X}\left(H(x) \wedge \bigwedge_{G \in \mathscr{S}} G(x)\right). \tag{7}$$

Theorem 12 (see [7]). *For $A = \mathbf{1}_{L^X}$, the next statements are true:*

(1) $\langle_L^A H = H'$, $H \in \Psi_X^L(A)$ iff $H \in L^X$

(2) RL-fc is turned into L-fc

(3) H is RL-fc iff H is L-fc

Theorem 13 (see [7]). *For every $A \in \mathscr{V}_X^L$ and RL-fts $(A, \mathscr{R})$, the next conclusions are true:*

(1) *If $H_1, H_2 \in \Psi_X^L(A)$ and H_1, H_2 are RL-fc, then $H_1 \vee H_2$ is RL-fc*

(2) *If $H_1, H_2 \in \Psi_X^L(A)$ such that H_1 is an RL-fc and H_2 is an RL-closed, then $H_1 \wedge H_2$ is an RL-fc*

3. The (i, j)-RL-Preopenness Degree of L-Fuzzy Set

If $\mathscr{R}_1$ and $\mathscr{R}_2$ be two RL-fuzzy topologies on $A \in \mathscr{V}_X^L$, the the triple $(A, \mathscr{R}_1, \mathscr{R}_2)$ is called an RL-fuzzy bitopological space (briefly, RL-fbts). Moreover, if S any topological property, then we refer to S with respect the RL-ft $\mathscr{R}_i$ by $\mathscr{R}_i$-S. An L-fuzzy set $H \in \Psi_X^L(A)$ of an RL-bts $(A, \mathscr{R}_l, \mathscr{R}_2)$ is said to be an (i, j)-RL-preopen if and only if $H \le \mathrm{Int}^{\mathscr{R}_i}(Cl^{\mathscr{R}_j}(H))$. In the remainder of this paper i, $j = 1, 2$ such that $i \neq j$.

Definition 14. Let $A \in \mathscr{V}_X^L$ and $(A, \mathscr{R}_1, \mathscr{R}_2)$ be an RL-fbts on A. The (i, j)-RL-preopenness gradation of $H \in \Psi_X^L(A)$ with respect to $\mathscr{R}_i$ and $\mathscr{R}_j$ is the mapping (i, j)-$\mathscr{P} : \Psi_X^L(A) \longrightarrow L$ given by

$$(i, j)\text{-}\mathscr{P}(H) = \bigwedge_{x_\lambda \prec H} \bigvee_{x_\lambda \prec M} \left\{ \mathscr{R}_i(H) \wedge \bigwedge_{y_\mu \prec M} \bigwedge_{y_\mu \nprec G \geq H} \langle_L^A \left(\mathscr{R}_j \left(\langle_L^A G \right) \right) \right\}. \tag{8}$$

The value (i, j)-$\mathscr{P}(H)$ introduces the gradation of (i, j)-RL-preopenness of H is and (i, j)-$\mathscr{P}^*(H) = (i, j)$-$\mathscr{P}(\langle_L^A H)$ introduces the gradation of (i, j)-RL-preclose-ness of H.

The next corollary is a direct consequence of the above definition and Definition 10:

Corollary 15. *Let $(A, \mathscr{R}_1, \mathscr{R}_2)$ be an RL-fbts on $A \in \mathscr{V}_X^L$. Then, for every $H \in \Psi_X^L(A)$, we have*

$$(i, j)\text{-}\mathscr{P}(H) = \bigwedge_{x_\lambda \prec H} \bigvee_{x_\lambda \prec M} \left\{ \mathscr{R}_i(M) \wedge \bigwedge_{y_\mu \prec M} Cl^{\mathscr{R}_j}(H)\left(y_\mu\right) \right\}. \tag{9}$$

Theorem 16. *Let $\mathscr{R}_1, \mathscr{R}_2 : \Psi_X^L(A) \longrightarrow \{0_L, 1_L\}$ be RL-ts on $A \in \mathscr{V}_X^L$, and (i, j)-$\mathscr{P} : \Psi_X^L(A) \longrightarrow \{0_L, 1_L\}$ be the degree of (i, j)-RL-preopenness with respect to $\mathscr{R}_i$ and $\mathscr{R}_j$ with $i \neq j$. Then (i, j)-$\mathscr{P}(H) = 1_L$ if and only if H is an (i, j)-RL-preopen.*

Proof. The next inequality provides the proof:

$$\begin{aligned}
(i, j)\text{-}\mathscr{P}(H) = \mathbf{1}_L &\Leftrightarrow \bigwedge_{x_\lambda \prec H} \bigvee_{x_\lambda \prec C} \left\{ \mathscr{R}_i(C) \wedge \bigwedge_{y_\mu \prec C} Cl^{\mathscr{R}_j}(H)\left(y_\mu\right) \right\} \\
&= \mathbf{1}_L \Leftrightarrow \forall x_\lambda \prec H, \exists C \text{ such that } x_\lambda \prec C, \mathscr{R}_i(C) \\
&= \mathbf{1}_L \text{ and } \bigwedge_{y_\mu \prec C} Cl^{\mathscr{R}_j}(H)\left(y_\mu\right) \\
&= \mathbf{1}_L \Leftrightarrow \forall x_\lambda \prec H, \exists C \text{ such that } x_\lambda \prec C, \mathscr{R}_i(C) \\
&= \mathbf{1}_L \text{ and} \forall y_\mu \prec C, Cl^{\mathscr{R}_j}(H)\left(y_\mu\right) \\
&= \mathbf{1}_L \Leftrightarrow \forall x_\lambda \prec H, \exists C \text{ such that } x_\lambda \prec C, \mathscr{R}_i(C) \\
&= \mathbf{1}_L \text{ and } C \le Cl^{\mathscr{R}_j}(H) \Leftrightarrow H \\
&\le \mathrm{Int}^{\mathscr{R}_i}\left(Cl^{\mathscr{R}_j}(H)\right) \Leftrightarrow H \text{ is } (i, j)\text{-}RL\text{-preopen}.
\end{aligned} \tag{10}$$

□

Theorem 17. *Let $(A, \mathscr{R}_1, \mathscr{R}_2)$ be an RL-fbts on $A \in \mathscr{V}_X^L$ and (i, j)-$\mathscr{P}$ be the degree of (i, j)-RL-preopenness with respect to $\mathscr{R}_i$ and $\mathscr{R}_j$ with $i \neq j$. Then, for every $H \in \Psi_X^L(A)$, we have $\mathscr{R}_i(H) \le (i, j)$-$\mathscr{P}(H)$.*

Proof. The next inequality provides the proof:

$$\begin{aligned}
(i, j)\text{-}\mathscr{P}(H) &= \bigwedge_{x_\lambda \prec H} \bigvee_{x_\lambda \prec C} \left\{ \mathscr{R}_i(C) \wedge \bigwedge_{y_\mu \prec C} Cl^{\mathscr{R}_j}(H)\left(y_\mu\right) \right\} \\
&\geq \bigwedge_{x_\lambda \prec H} \left\{ \mathscr{R}_i(H) \wedge \bigwedge_{y_\mu \prec H} Cl^{\mathscr{R}_j}(H)\left(y_\mu\right) \right\} \\
&= \mathscr{R}_i(H) \wedge \mathbf{1}_L = \mathscr{R}_i(H).
\end{aligned} \tag{11}$$

□

Corollary 18. *Let $(A, \mathscr{R}_1, \mathscr{R}_2)$ be an RL-fbts on $A \in \mathscr{V}_X^L$ and (i, j)-$\mathscr{P}$ be the degree of (i, j)-RL-preopenness with respect to $\mathscr{R}_i$ and $\mathscr{R}_j$ with $i \neq j$. Then, for every $H \in \Psi_X^L(A)$, we have $\langle_L^A \mathscr{R}_i(H) \le (i, j)$-$\mathscr{P}^*(H)$.*

Theorem 19. *If $(A, \mathscr{R}_1, \mathscr{R}_2)$ be an RL-fbts on $A \in \mathscr{V}_X^L$, and (i, j)-$\mathscr{P}$ be the degree of (i, j)-RL-preopenness with respect to $\mathscr{R}_i$ and $\mathscr{R}_j$ with $i \neq j$, then (i, j)-$\mathscr{P}(\bigvee_{i \in \Omega} H_i) \geq \bigwedge_{i \in \Omega} (i, j)$-$\mathscr{P}(H_i)$ for every $\{H_i\}_{i \in \Omega} \subseteq \Psi_X^L(A)$.*

Proof. Let $\alpha \in L$ and $\alpha \prec \bigwedge_{i \in \Omega} (i, j)$-$\mathscr{P}(H_i)$. Then for any $i \in \Omega$ and for any $x_\lambda \prec H_i$, there exists $C_i \in \Psi_X^L(A)$ such that

$$x_\lambda \prec C_i, a \prec \mathscr{R}_i(C_i), \alpha \prec \bigwedge_{y_\mu \prec C_i} \bigwedge_{y_\mu \nprec D \geq H_i} \langle_L^A \left(\mathscr{R}_j \left(\langle_L^A D \right) \right). \tag{12}$$

Hence,

$$\alpha \leq \bigwedge_{i\in\Omega} \mathscr{R}(C_i) \leq \mathscr{R}_j\left(\bigvee_{i\in\Omega} C_i\right), \alpha \leq \bigwedge_{i\in\Omega y_\mu \prec C_i y_\mu \nleq D \geq H_i} \bigwedge \bigwedge \left\langle{}_L^A\left(\mathscr{R}_j\left(\left\langle{}_L^A D\right)\right)\right. . \tag{13}$$

By

$$\left\{x_\lambda : x_\lambda \prec \bigvee_{i\in I} C_i\right\} = \bigcup_{i\in I} - x_\lambda : x_\lambda \prec C_i\}, \tag{14}$$

we have

$$\begin{aligned}
&(i,j)\text{-}\mathscr{P}\left(\bigvee_{i\in I} H_i\right)\\
&= \bigwedge_{x_\lambda \prec \bigvee_{i\in\Omega} H_i} \bigvee_{x_\lambda \prec C} \left\{\mathscr{R}_i(C) \wedge \bigwedge_{y_\mu \prec C y_\mu \nleq D \geq} \bigwedge_{\bigvee_{i\in\Omega} A_i} \left\langle{}_L^A\left(\mathscr{R}_j\left(\left\langle{}_L^A D\right)\right)\right.\right\}\\
&\geq \bigwedge_{x_\lambda \prec \bigvee_{i\in\Omega} H_i} \left\{\mathscr{R}_i\left(\bigvee_{i\in\Omega} C_i\right) \wedge \bigwedge_{x_\lambda \prec \bigvee_{i\in\Omega} C_i x_\lambda \nleq D \geq} \bigwedge_{\bigvee_{i\in\Omega} A_i} \left\langle{}_L^A\left(\mathscr{R}_j\left(\left\langle{}_L^A D\right)\right)\right.\right\}\\
&= \bigwedge_{x_\lambda \prec \bigvee_{i\in\Omega} H_i} \left\{\mathscr{R}_i\left(\bigvee_{i\in\Omega} C_i\right) \wedge \bigwedge_{i\in\Omega x_\lambda \prec C_i x_\lambda \nleq D \geq} \bigwedge_{\bigvee_{i\in\Omega} H_i} \left\langle{}_L^A\left(\mathscr{R}_j\left(\left\langle{}_L^A D\right)\right)\right.\right\}\\
&\geq \bigwedge_{i\in\Omega x_\lambda \prec H_i} \left\{\mathscr{R}_i\left(\bigvee_{i\in\Omega} C_i\right) \wedge \bigwedge_{i\in\Omega x_\lambda \prec C_i x_\lambda \nleq D \geq H_i} \left\langle{}_L^A\left(\mathscr{R}_j\left(\left\langle{}_L^A D\right)\right)\right.\right\}\\
&\geq \alpha.
\end{aligned} \tag{15}$$

Then, $(i,j)\text{-}\mathscr{P}(\bigvee_{i\in\Omega} H_i) \geq \bigwedge_{i\in\Omega} (i,j)\text{-}\mathscr{P}(H_i)$. □

Corollary 20. *Let $(A, \mathscr{R}_1, \mathscr{R}_2)$ be an RL-fbts on $A \in \mathscr{V}_X^L$, and $(i,j)\text{-}\mathscr{P}$ be the degree of (i,j)-RL-preopenness with respect to $\mathscr{R}_i$ and $\mathscr{R}_j$ with $i \neq j$. Then, $(i,j)\text{-}\mathscr{P}^*(\bigwedge_{i\in\Omega} H_i) \geq \bigwedge_{i\in\Omega} (i,j)\text{-}\mathscr{P}^*(H_i)$ for any $\{H_i\}_{i\in\Omega} \subseteq \Psi_X^L(A)$.*

4. A New Representation of Pairwise Fuzzy Precontinuous (Preirresolute) Functions

Let $(A, \mathscr{R}_1, \mathscr{R}_2)$, $(B, \mathscr{R}_1^*, \mathscr{R}_2^*)$ be *RL*-fbts on $A \in \mathscr{V}_X^L$ and $B \in \mathscr{V}_Y^L$, respectively. The *RL*-fuzzy mapping $\psi_{L,A} : A \longrightarrow B$ is called pairwise *RL*-fuzzy continuous (briefly, *PRL*-fco) if and only if $\psi_{L,A} : (A, \mathscr{R}_1) \longrightarrow (B, \mathscr{R}_1^*)$ and $\psi_{L,A} : (A, \mathscr{R}_2) \longrightarrow (B, \mathscr{R}_2^*)$ are *RL*-fuzzy continuous. In a similar way, we define the concept pairwise *RL*-fuzzy open mapping (briefly, *PRL*-fo).

Definition 21. Let $(A, \mathscr{R}_1, \mathscr{R}_2)$ and $(B, \mathscr{R}_1^*, \mathscr{R}_2^*)$ be RL-fbts on $A \in \mathscr{V}_X^L$ and $B \in \mathscr{V}_Y^L$, respectively, and $(i,j)\text{-}\mathscr{P}_1$, $(i,j)\text{-}\mathscr{P}_2$ are the corresponding (i,j)-*RL*-preopenness degrees. An RL-fuzzy mapping $\psi_{L,A} : A \longrightarrow B$ is said to be

(1) pairwise *RL*-fuzzy precontinuous (briefly, *PRL*-fpco) if and only if $\mathscr{R}_i^*(M) \leq (i,j)\text{-}\mathscr{P}_1(\psi_{\overleftarrow{L,A}}(M))$ is true for every $M \in \Psi_X^L(B)$

(2) pairwise *RL*-fuzzy preirresolute (briefly, *PRL*-fpirr) if and only if $(i,j)\text{-}\mathscr{P}_2(M) \leq (i,j)\text{-}\mathscr{P}_1(\psi_{\overleftarrow{L,A}}(M))$ is true for every $M \in \Psi_X^L(B)$

Corollary 22. *Let $(A, \mathscr{R}_1, \mathscr{R}_2)$ and $(B, \mathscr{R}_1^*, \mathscr{R}_2^*)$ be RL-fbts on $A \in \mathscr{V}_X^L$ and $B \in \mathscr{V}_Y^L$, respectively, and $(i,j)\text{-}\mathscr{P}_1$, $(i,j)\text{-}\mathscr{P}_2$ be the corresponding (i,j)-RL-preopenness degrees. Then,*

(1) $\psi_{L,A}$ is PRL-fpco if and only if $\langle{}_L^B \mathscr{R}_i^*(M) \leq (i,j)\text{-}\mathscr{P}_1^*(\psi_{\overleftarrow{L,A}}(M))$ for every $M \in \Psi_X^L(B)$

(2) $\psi_{L,A}$ is PRL-fpirr if and only if $(i,j)\text{-}\mathscr{P}_2^*(M) \leq (i,j)\text{-}\mathscr{P}_1^*(\psi_{\overleftarrow{L,A}}(M))$ for every $M \in \Psi_X^L(B)$

Theorem 23. *Let $(A, \mathscr{R}_1, \mathscr{R}_2)$ and $(B, \mathscr{R}_1^*, \mathscr{R}_2^*)$ be RL-fbts on $A \in \mathscr{V}_X^L$ and $B \in \mathscr{V}_Y^L$, respectively, and $(i,j)\text{-}\mathscr{P}_1$, $(i,j)\text{-}\mathscr{P}_2$ be the corresponding (i,j)-RL-preopenness degrees. Then,*

(1) $\psi_{L,A} : (A, \mathscr{R}_1, \mathscr{R}_2) \longrightarrow (B, \mathscr{R}_1^*, \mathscr{R}_2^*)$ is PRL-fpco if and only if $\psi_{L,A} : (A, \mathscr{R}_{1[\alpha]}, \mathscr{R}_{2[\alpha]}) \longrightarrow (B, \mathscr{R}_{1[\alpha]}^*, \mathscr{R}_{2[\alpha]}^*)$ is PRL-pco for every $\alpha \in J(L)$

(2) $\psi_{L,A} : (A, \mathscr{R}_1, \mathscr{R}_2) \longrightarrow (B, \mathscr{R}_1^*, \mathscr{R}_2^*)$ is PRL-fpirr if and only if $\psi_{L,A} : (A, \mathscr{R}_{1[\alpha]}, \mathscr{R}_{2[\alpha]}) \longrightarrow (B, \mathscr{R}_{1[\alpha]}^*, \mathscr{R}_{2[\alpha]}^*)$ is PRL-pirr for every $\alpha \in J(L)$

Proof.

(1) If $M \in \mathscr{R}_{i\,[\alpha]}^*$ for every $M \in \Psi_X^L(B)$ and $\alpha \in J(L)$, then $\mathscr{R}_i^*(M) \geq \alpha$. Since $\psi_{L,A} : (A, \mathscr{R}_1, \mathscr{R}_2) \longrightarrow (B, \mathscr{R}_1^*, \mathscr{R}_2^*)$ is *PRL*-fpco, then $(i,j)\text{-}\mathscr{P}_1(\psi_{\overleftarrow{L,A}}(M)) \geq \mathscr{R}_i^*(M) \geq \alpha$, i.e., $(i,j)\text{-}\mathscr{P}_1(\psi_{\overleftarrow{L,A}}(M)) \geq \alpha$. Subsequently, $\psi_{\overleftarrow{L,A}}(M)$ is (i,j)-*RL*-preopen *L*-fuzzy set in $(A, \mathscr{R}_{1[\alpha]}, \mathscr{R}_{2[\alpha]})$. Thus, $\psi_{L,A} : (A, \mathscr{R}_{1[\alpha]}, \mathscr{R}_{2[\alpha]}) \longrightarrow (B, \mathscr{R}_{1[\alpha]}^*, \mathscr{R}_{2[\alpha]}^*)$ is *PRL*-pco mapping

Now, if $\mathscr{R}_i^*(M) \geq \alpha$ for every $M \in \Psi_X^L(B)$ and $\alpha \in J(L)$, then $M \in \mathscr{R}_{i\,[\alpha]}^*$. Since $\psi_{L,A} : (A, \mathscr{R}_{1[\alpha]}, \mathscr{R}_{2[\alpha]}) \longrightarrow (B, \mathscr{R}_{1[\alpha]}^*, \mathscr{R}_{2[\alpha]}^*)$ is *PRL*-pco, we have $\psi_{\overleftarrow{L,A}}(M)$ is (i,j)-*RL*-preopen in $(A, \mathscr{R}_{1[\alpha]}, \mathscr{R}_{2[\alpha]})$. Then, $(i,j)\text{-}\mathscr{P}_1(\psi_{\overleftarrow{L,A}}(M)) \geq \alpha$ for every $\alpha \in J(L) \cap J(\mathscr{R}_i^*(M))$, where $J(\mathscr{R}_i^*(M)) = -\alpha \in J(L) \mid \alpha \leq \mathscr{R}_i^*(M)\}$. Hence $(i,j)\text{-}\mathscr{P}_1(\psi_{\overleftarrow{L,A}}(M)) \geq \vee J(\mathscr{R}_i^*(M)) = \mathscr{R}_i^*(M)$.

(2) Let M be an (i,j)-*RL*-preopen *L*-fuzzy set with respect to $(B, \mathscr{R}_{1[\alpha]}^*, \mathscr{R}_{2[\alpha]}^*)$, then $(i,j)\text{-}\mathscr{P}_2(M) \geq \alpha$. Since $\psi_{L,A} : (A, \mathscr{R}_1, \mathscr{R}_2) \longrightarrow (B, \mathscr{R}_1^*, \mathscr{R}_2^*)$ is *PRL*-fpirr, then $(i,j)\text{-}\mathscr{P}_1(\psi_{\overleftarrow{L,A}}(M)) \geq (i,j)\text{-}\mathscr{P}_2(M) \geq \alpha$, so $(i,j)\text{-}\mathscr{P}_1(\psi_{\overleftarrow{L,A}}(M)) \geq \alpha$, then $\psi_{\overleftarrow{L,A}}(M)$ is $(i.j)$-*RL*-preopen *L*-fuzzy set with respect to $(A, \mathscr{R}_{1[\alpha]}, \mathscr{R}_{2[\alpha]})$. Hence, $\psi_{L,A} : (A, \mathscr{R}_{1[\alpha]}, \mathscr{R}_{2[\alpha]}) \longrightarrow (B, \mathscr{R}_{1[\alpha]}^*, \mathscr{R}_{2[\alpha]}^*)$ is *PRL*-pirr

Now, if $(i, j)\text{-}\mathscr{P}_2(M) \geq \alpha$ for every $\alpha \in J(L)$, then M is an α-$(i.j)$-RL-preopen with respect to $(B, \mathscr{R}^*_{1[\alpha]}, \mathscr{R}^*_{2[\alpha]})$. Since $\psi_{L,A} : (A, \mathscr{R}_{1[\alpha]}, \mathscr{R}_{2[\alpha]}) \longrightarrow (B, \mathscr{R}^*_{1[\alpha]}, \mathscr{R}^*_{2[\alpha]})$ is PRL-pirr, $f_{\overleftarrow{L,A}}(M)$ is α-(i, j)-RL-preopen with respect to $(A, \mathscr{R}_{1[\alpha]}, \mathscr{R}_{2[\alpha]})$. Therefore, $(i, j)\text{-}\mathscr{P}_1(\psi_{\overleftarrow{L,A}}(M)) \geq \alpha$ for all $\alpha \in J(L) \cap J((i, j)\text{-}\mathscr{P}_2(M))$, where $J((i, j)\text{-}\mathscr{P}_2(M)) = -\alpha \in J(L) \mid \alpha \leq (i, j)\text{-}\mathscr{P}_2(M)\}$. Hence, $(i, j)\text{-}\mathscr{P}_1(\psi_{\overleftarrow{L,A}}(M)) \geq \vee J((i, j)\text{-}\mathscr{P}_2(M)) = (i, j)\text{-}\mathscr{P}_2(M)$. □

Theorem 24. *Let $(A, \mathscr{R}_1, \mathscr{R}_2), (B, \mathscr{R}^*_1, \mathscr{R}^*_2)$ be RL-fbts on $A \in \mathscr{V}^L_X$ and $B \in \mathscr{V}^L_Y$, respectively. If $\psi_{L,A} : A \longrightarrow B$ is PRL-fco, then, $\psi_{L,A}$ is also PRL-fpco.*

Proof. Let $\psi_{L,A} : A \longrightarrow B$ be PRL-fco; then $\mathscr{R}^*_i(M) \leq \mathscr{R}_i(\psi_{\overleftarrow{L,A}}(M))$ for every $M \in \Psi^L_X(B)$ and $i = 1, 2$. Based on Theorem 17, we have

$$\mathscr{R}^*_i(M) \leq \mathscr{R}_i\left(\psi_{\overleftarrow{L,A}}(M)\right) \leq (i, j)\text{-}\mathscr{P}_1\left(\psi_{\overleftarrow{L,A}}(M)\right), \tag{16}$$

for every $M \in \Psi^L_X(B)$. Hence, $\psi_{L,A}$ is PRL-fpco. □

Theorem 25. *Let $(A, \mathscr{R}_1, \mathscr{R}_2)$, $(B, \mathscr{R}^*_1, \mathscr{R}^*_2)$ be two RL-fbts on $A \in \mathscr{V}^L_X$ and $B \in \mathscr{V}^L_Y$, respectively. If $\psi_{L,A} : (A, \mathscr{R}_1, \mathscr{R}_2) \longrightarrow (A, \mathscr{R}^*_1, \mathscr{R}^*_2)$ is PRL-fpirr, then $\psi_{L,A}$ is PRL-fprco.*

Proof. If $\psi_{L,A} : (A, \mathscr{R}_1, \mathscr{R}_2) \longrightarrow (B, \mathscr{R}^*_1, \mathscr{R}^*_2)$ be PRL-fpirr, then $(i, j)\text{-}\mathscr{P}_2(M) \leq (i, j)\text{-}\mathscr{P}_1(\psi_{\overleftarrow{L,A}}(M))$ for every $M \in \Psi^L_X(B)$. Based on Theorem 17, we have $\mathscr{R}_i(M) \leq (i, j)\text{-}\mathscr{P}_2(M) \leq (i, j)\text{-}\mathscr{P}_1(\psi_{\overleftarrow{L,A}}(M))$. Hence, $\psi_{L,A}$ is PRL-fpco. □

Theorem 26. *Let $(A, \mathscr{R}_1, \mathscr{R}_2)$, $(B, \mathscr{R}^*_1, \mathscr{R}^*_2)$, and $(C, \mathscr{R}^{**}_1, \mathscr{R}^{**}_2)$ be PRL-fbts on $A \in \mathscr{V}^L_X$, $B \in \mathscr{V}^L_Y$, and $C \in \mathscr{V}^L_Z$, respectively. If $\psi_{L,A} : (A, \mathscr{R}_1, \mathscr{R}_2) \longrightarrow (B, \mathscr{R}^*_1, \mathscr{R}^*_2)$ is PRL-fpco and $\phi_{L,B} : (B, \mathscr{R}^*_1, \mathscr{R}^*_2) \longrightarrow (C, \mathscr{R}^{**}_1, \mathscr{R}^{**}_2)$ is PRL-fco, then $(\phi \circ \psi)_{L,A} : (A, \mathscr{R}_1, \mathscr{R}_2) \longrightarrow (C, \mathscr{R}^{**}_1, \mathscr{R}^{**}_2)$ is PRL-fpco.*

Proof. Straightforward. □

5. A New Representation of Pairwise Fuzzy Precompactness

Definition 27. For any RL-fbt $(\mathscr{R}_1, \mathscr{R}_2)$ on $A \in \mathscr{V}^L_X$, an L-*fuzzy* set $H \in \Psi^L_X(A)$ is called a pairwise RL-fuzzy precompact (briefly, PRL -fpcom) with respect to $(\mathscr{R}_1, \mathscr{R}_2)$ if for every $\mathscr{R} \subseteq \Psi^L_X(A)$, the next inequality is true:

$$\begin{aligned}&\bigwedge_{D\in\mathscr{R}}(i, j)\text{-}\mathscr{P}(D) \wedge \bigwedge_{x\in X}\left(\langle^A_L H(x) \vee \bigvee_{D\in\mathscr{R}} D(x)\right)\\ &\quad\leq \bigvee_{\mathscr{Q}\in 2^{(\mathscr{R})}} \bigwedge_{x\in X}\left(\langle^A_L H(x) \vee \bigvee_{D\in\mathscr{Q}} D(x)\right),\end{aligned} \tag{17}$$

where $2^{(\mathscr{R})}$ denotes the family of all finite subfamilies of $\mathscr{R}$.

Theorem 28. *Let $(\mathscr{R}_1, \mathscr{R}_2)$ be RL-fbt on $A \in \mathscr{V}^L_X$. An L-fuzzy set $H \in \Psi^L_X(A)$ is called a PRL-fpcom with respect to $(\mathscr{R}_1, \mathscr{R}_2)$ if for every $\mathscr{W} \subseteq \Psi^L_X(A)$, we have*

$$\begin{aligned}&\bigvee_{D\in\mathscr{W}}(i, j)\text{-}\mathscr{P}\left(\langle^A_L D\right) \vee \bigvee_{x\in X}\left(H(x) \wedge \bigwedge_{D\in\mathscr{W}} D(x)\right)\\ &\quad\geq \bigwedge_{\mathscr{H}\in 2^{(\mathscr{W})}} \bigvee_{x\in X}\left(H(x) \wedge \bigwedge_{D\in\mathscr{H}} D(x)\right).\end{aligned} \tag{18}$$

Proof. Straightforward. □

Theorem 29. *If $(\mathscr{R}_1, \mathscr{R}_2)$ is an RL-fbt on $A \in \mathscr{V}^L_X$, and $B \in \Psi^L_X(A)$, then the following conditions are equivalent:*

(1) H is a PRL-fpcom

(2) For every $\alpha \in J(L)$, each strong α-remote family $\mathscr{R}$ of H with $\wedge_{D\in\mathscr{R}}(i, j)\text{-}\mathscr{P}^*(D) \nleq \alpha'$ has a finite subfamily $\mathscr{H}$ which is a (strong) α-remote family of H

(3) For every $\alpha \in J(L)$, in each strong α-remote family $\mathscr{R}$ of H with $\wedge_{D\in\mathscr{R}}(i, j)\text{-}\mathscr{P}^*(D) \nleq \alpha'$, there exists a finite subfamily $\mathscr{H}$ of $\mathscr{R}$ and $\beta \in \beta^*(\alpha)$ with $\mathscr{H}$ is a (strong) β-remote family of H

(4) For every $\alpha \in P(L)$, each strong α-shading $\mathscr{U}$ of H with $\wedge_{D\in\mathscr{U}}(i, j)\text{-}\mathscr{P}(D) \nleq \alpha$ has a finite subfamily $\mathscr{V}$ which is a (strong) α-shading of H

(5) For every $\alpha \in P(L)$, in every strong α-shading $\mathscr{U}$ of H with $\wedge_{D\in\mathscr{U}}(i, j)\text{-}\mathscr{P}(D) \nleq \alpha$, there exists a finite family $\mathscr{V}$ of $\mathscr{U}$ and $\beta \in \beta^*(\alpha)$ with $\mathscr{V}$ is a (strong) β-shading of H

(6) For every $\alpha \in J(L)$ and $\beta \in \beta^*(\alpha)$, every Q_α-cover $\mathscr{U}$ of H with $(i, j)\text{-}\mathscr{P}(D) \geq \alpha$ (for every $D \in \mathscr{U}$) has a finite subfamily $\mathscr{V}$ which is a Q_β-cover of H

(7) For every $\alpha \in J(L)$ and every $\beta \in \beta^*(\alpha)$, Q_α-cover $\mathscr{U}$ of H with $(i, j)\text{-}\mathscr{P}(D) \geq \alpha$ (for every $D \in \mathscr{U}$) has a finite subfamily $\mathscr{V}$ which is a (strong) β_α-cover of H

Proof. Straightforward. □

Theorem 30. *Let $(\mathscr{R}_1, \mathscr{R}_2)$ be an RL-fbt on $A \in \mathscr{V}^L_X$, $H \in \Psi^L_X(A)$ and $\beta(\alpha \wedge \beta) = \beta(\alpha) \wedge \beta(\beta)$ for all $\alpha, \beta \in L$; then, the following conditions are equivalent:*

(1) H is PRL-fpcom

(2) For every $\alpha \in J(L)$, every strong β_α-cover $\mathscr{U}$ of B with $\alpha \in \beta(\wedge_{D\in\mathscr{U}}(i, j)\text{-}\mathscr{P}(D))$ has a finite subfamily $\mathscr{V}$ which is a (strong) β_α-cover of H

(3) For every $\alpha \in J(L)$, in every strong β_α-cover $\mathscr{U}$ of H with $\alpha \in \beta(\wedge_{D\in\mathscr{U}}(i, j)\text{-}\mathscr{P}(D))$, there exists a finite subfamily $\mathscr{V}$ of $\mathscr{U}$ and $\beta \in J(L)$ such that $\alpha \in \beta^*(\beta)$ with $\mathscr{V}$ is a (strongly) β_β-cover of H

Proof. Straightforward. □

Definition 31. Let $A \in \mathscr{V}_X^L$, $B \in \Psi_X^L(A)$, $\alpha \in J(L)$, and $(A, \mathscr{R}_1, \mathscr{R}_2)$ be an RL-bts. An L-fuzzy set H is said to be an α-pairwise RL-fuzzy precompact (briefly, αPRL-fcom) if and only if for each $\beta \in \beta(\alpha)$, Q_α-(i,j)-RL-preopen cover $\mathscr{U}$ of H has a finite subfamily $\mathscr{V}$ which is a Q_β-(i,j)-RL-preopen cover of H.

Theorem 32. *Let $A \in \mathscr{V}_X^L$, and $(A, \mathscr{R}_1, \mathscr{R}_2)$ be an RL-bts. An L-fuzzy set $H \in \Psi_X^L(A)$ is PRL-fpcom if and only if H is αPRL-fpcom for every $\alpha \in J(L)$.*

Proof. If H is aPRL-fpcom, then for every $\alpha \in L_1$, $\beta \in \beta(\alpha)$, and $\mathscr{U}$ which can be any Q_α-(i,j)-RL-preopen cover of H, we obtain

$$\bigwedge_{x\in X}\left(\left\langle_L^A H(x) \vee \bigvee_{D\in\mathscr{U}} D(x)\right) \le \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \bigwedge_{x\in X}\left(\left\langle_L^A H(x) \vee \bigvee_{D\in\mathscr{V}} D(x)\right), \tag{19}$$

and $\alpha \le \wedge_{x\in X}(\langle_L^A H(x) \vee \vee_{D\in\mathscr{U}} D(x))$, so that

$$\alpha \le \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \bigwedge_{x\in X}\left(\left\langle_L^A H(x) \vee \bigvee_{D\in\mathscr{V}} D(x)\right). \tag{20}$$

By $\beta \in \beta(\alpha)$, we have

$$\beta \le \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \bigwedge_{x\in X}\left(\left\langle_L^A H(x) \vee \bigvee_{D\in\mathscr{V}} D(x)\right). \tag{21}$$

Then, there is $\mathscr{V} \in 2^{(\mathscr{U})}$ such that $\beta \le \wedge_{x\in X}(\langle_L^A H(x) \vee \vee_{D\in\mathscr{V}} D(x))$. It follows that $\mathscr{V}$ is Q_β-(i,j)-RL-preopen cover of H.

Now, suppose that every Q_α-(i,j)-RL-preopen cover $\mathscr{U}$ of H has a finite subfamily $\mathscr{V}$ which is a Q_α-(i,j)-RL-preopen cover of H for each $\beta \in \beta(\alpha)$. Then, $\alpha \le \wedge_{x\in X}(\langle_L^A H(x) \vee \vee_{D\in\mathscr{U}} D(x))$ implies that $\beta \le \wedge_{x\in X}(\langle_L^A H(x) \vee \vee_{D\in\mathscr{U}} D(x))$. Hence, $\alpha \le \wedge_{x\in X}(\langle_L^A H(x) \vee \vee_{D\in\mathscr{U}} D(x))$ implies that $\beta \le \vee_{\mathscr{V}\in 2^{(\mathscr{U})}} \wedge_{x\in X}(\langle_L^A H(x) \vee \vee_{D\in\mathscr{U}} D(x))$. So $\alpha \le \wedge_{x\in X}(\langle_L^A H(x) \vee \vee_{D\in\mathscr{U}} D(x))$ implies that

$$\bigvee_{\beta\in\beta(\alpha)} \beta \le \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \bigwedge_{x\in X}\left(\left\langle_L^A H(x) \vee \bigvee_{D\in\mathscr{U}} D(x)\right), \tag{22}$$

that is,

$$\alpha \le \bigwedge_{x\in X}\left(\left\langle_L^A H(x) \vee \bigvee_{D\in\mathscr{U}} D(x)\right) \tag{23}$$

yields

$$\alpha \le \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \bigwedge_{x\in X}\left(\left\langle_L^A H(x) \vee \bigvee_{D\in\mathscr{U}} D(x)\right). \tag{24}$$

Therefore,

$$\bigwedge_{x\in X}\left(\left\langle_L^A H(x) \vee \bigvee_{D\in\mathscr{U}} D(x)\right) \le \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \bigwedge_{x\in X}\left(\left\langle_L^A H(x) \vee \bigvee_{D\in\mathscr{V}} D(x)\right). \tag{25}$$

□

Theorem 33. *Let $A \in \mathscr{V}_X^L$ and $(A, \mathscr{R}_1, \mathscr{R}_2)$ be an RL-fbts. An L-fuzzy set $H \in \Psi_X^L(A)$ is a PRL-fpcom in $(A, \mathscr{R}_1, \mathscr{R}_2)$ iff H is an αPRL-fpcom in $(A, \mathscr{R}_{1[\alpha]}, \mathscr{R}_{2[\alpha]})$ for each $\alpha \in J(L)$.*

Proof. If $H \in \Psi_X^L(A)$ be a PRL-fpcom in $(A, \mathscr{R}_1, \mathscr{R}_2)$, then for every family $\mathscr{U} \subseteq \Psi_X^L(A)$, we have

$$\begin{aligned}\bigwedge_{D\in\mathscr{U}} (i,j)\text{-}\mathscr{P}(D) \wedge \bigwedge_{x\in X}\left(\left\langle_L^A H(x) \vee \bigvee_{D\in\mathscr{U}} D(x)\right) \\ \le \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \bigwedge_{x\in X}\left(\left\langle_L^A H(x) \vee \bigvee_{D\in\mathscr{V}} D(x)\right).\end{aligned} \tag{26}$$

Hence, for all $\alpha \in J(L)$ and $\mathscr{U} \subseteq ((i,j)\text{-}\mathscr{P})_{[\alpha]}$, we have that

$$\alpha \le \bigwedge_{x\in X}\left(\left\langle_L^A H(x) \vee \bigvee_{D\in\mathscr{U}} D(x)\right) \Rightarrow \alpha \le \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \bigwedge_{x\in X}\left(\left\langle_L^A H(x) \vee \bigvee_{D\in\mathscr{V}} D(x)\right). \tag{27}$$

Hence, for each $\beta \in \beta(\alpha)$, there is $\mathscr{V} \in 2^{(\mathscr{U})}$ such that $\beta \le \wedge_{x\in X}(\langle_L^A H(x) \vee \vee_{D\in\mathscr{V}} D(x))$, i.e., for each $\alpha \in J(L)$ and $\beta \in \beta(\alpha)$, each Q_α-(i,j)-RL-preopen cover $\mathscr{U}$ of H in $(A, \mathscr{R}_{1[\alpha]}, \mathscr{R}_{2[\alpha]})$ has a finite subfamily $\mathscr{V}$ which is a Q_α-$(i.j)$-RL-preopen cover. Then, for each $\alpha \in J(L)$, H is αPRL-fpcom in $(A, \mathscr{R}_{1[\alpha]}, \mathscr{R}_{2[\alpha]})$.

Now, suppose that for each $\alpha \in J(L)$, H is αPRL-fpcom in $(A, \mathscr{R}_{1[\alpha]}, \mathscr{R}_{2[\alpha]})$ and let $\alpha \le \wedge_{D\in\mathscr{U}}(i,j)\text{-}\mathscr{P}(D) \wedge \wedge_{x\in X}(\langle_L^A H(x) \vee \vee_{D\in\mathscr{U}} D(x))$ for each $\mathscr{U} \subseteq \Psi_X^L(A)$, then $\alpha \le \wedge_{D\in\mathscr{U}}\ (i,j)\text{-}\mathscr{P}(D)$ and $\alpha \le \wedge_{x\in X}(\langle_L^A H(x) \vee \vee_{D\in\mathscr{U}} D(x))$, i.e, $\mathscr{U} \subseteq ((i,j)\text{-}\mathscr{P})_{[\alpha]}$ and $\alpha \le \wedge_{x\in X}(\langle_L^A H(x) \vee \vee_{D\in\mathscr{U}} D(x))$. Then, for every $\beta \in \beta(\alpha)$, there is $\mathscr{V} \in 2^{(\mathscr{U})}$ suth that

$$\beta \le \bigwedge_{x\in X}\left(\left\langle_L^A H(x) \vee \bigvee_{D\in\mathscr{V}} D(x)\right). \tag{28}$$

Then,

$$\alpha \le \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \bigwedge_{x\in X}\left(\left\langle_L^A H(x) \vee \bigvee_{D\in\mathscr{V}} D(x)\right). \tag{29}$$

So that H is a PRL-fpcom in $(A, \mathscr{R}_1, \mathscr{R}_2)$. □

Lemma 34. *Let $A \in \mathscr{V}_X^L$ and $(A, \mathscr{R}_1, \mathscr{R}_2)$ be an RL-bitopological space, $\alpha \in J(L)$, and $H, G \in \Psi_X^L(A)$. If H is αPRL-fpcom and G is (i, j)-RL-preclosed, then $H \wedge G$ is αPRL-fpcom.*

As an immediate consequence from the above lemma, we have the following theorem:

Theorem 35. *Let $A \in \mathscr{V}_X^L$ and $(A, \mathscr{R}_1, \mathscr{R}_2)$ be an RL-fbts, and $H, G \in \Psi_X^L(A)$. If H is a PRL-fpcom and (i, j)-$\mathscr{P}^*(G) = 1_L$, then $H \wedge G$ is a PRL-fpcom.*

Lemma 36. *Let $A \in \mathscr{V}_X^L$ and $(A, \mathscr{R}_1, \mathscr{R}_2)$ be an RL-bts, $\alpha \in J(L)$, and $H, G \in \Psi_X^L(A)$. If H, G are αPRL-fpcom, then $H \vee G$ is αPRL-fpcom.*

Theorem 37. *Let $A \in \mathscr{V}_X^L$ and $(A, \mathscr{R}_1, \mathscr{R}_2)$ be an RL-fbts, and $H, G \in \Psi_X^L(A)$. If H, G are PRL-fpcom, then $H \vee G$ is PRL-fpcom.*

Proof. Straightforward. □

Lemma 38. *Let $(A, \mathscr{R}_1, \mathscr{R}_2), (B, \mathscr{R}_1^*, \mathscr{R}_2^*)$ be RL-bts's on $A \in \mathscr{V}_X^L$ and $B \in \mathscr{V}_Y^L$, respectively, $\alpha \in J(L)$, $H \in \Psi_X^L(A)$, and $\psi_{L,A} : A \longrightarrow B$ be a PRL-irresolute function. If H is αPRL-fpcom in $(A, \mathscr{R}_1, \mathscr{R}_2)$, then $\psi_{\vec{L},A}(H)$ is $\alpha PRL-fpcom$ in $(B, \mathscr{R}_1^*, \mathscr{R}_2^*)$.*

Theorem 39. *Let $(A, \mathscr{R}_1, \mathscr{R}_2)$ and $(B, \mathscr{R}_1^*, \mathscr{R}_2^*)$ be two RL-fbts's on $A \in \mathscr{V}_X^L$ and $B \in \mathscr{V}_Y^L$, respectively, $H \in \Psi_X^L(A)$, and $\psi_{L,A} : A \longrightarrow B$ be a PRL-fuzzy irresolute function. If H is a PRL-fpcom in $(A, \mathscr{R}_1, \mathscr{R}_2)$, then $\psi_{\vec{L},A}(H)$ is a PRL-fpcom in $(B, \mathscr{R}_1^*, \mathscr{R}_2^*)$.*

Proof. Let H be a PRL-fpcom in $(A, \mathscr{R}_1, \mathscr{R}_2)$. By Theorem 33, we have H which is αPRL-fpcom in $(A, \mathscr{R}_{1[\alpha]}, \mathscr{R}_{2[\alpha]})$ for every $\alpha \in J(L)$. By Theorem 33, $\psi_{L,A} : (A, \mathscr{R}_{1[\alpha]}, \mathscr{R}_{2[\alpha]}) \longrightarrow (B, \mathscr{R}^*_{1[\alpha]}, \mathscr{R}^*_{2[\alpha]})$ is PRL-irresolute. Therefore, based on Lemma 38, $\psi_{\vec{L},A}(H)$ is αPRL-fpcom in $(B, \mathscr{R}^*_{1[\alpha]}, \mathscr{R}^*_{2[\alpha]})$. Thus, $\psi_{\vec{L},A}(H)$ is PRL-fpcom in $(B, \mathscr{R}_1^*, \mathscr{R}_2^*)$. □

6. Conclusion

In this paper, we presented the gradation of preopenness of L-fuzzy sets in RL-fuzzy bitopological spaces relied on pseudo-complement. The new gradation is used to extend and characterize pairwise RL-fuzzy precontinuous and pairwise RL-fuzzy preirresolute functions. Moreover, we discussed pairwise RL-fuzzy precompactness of an L-fuzzy set in RL-fuzzy bitopological spaces. We think that our findings present more general results and it will open the way for many other studies.

Data Availability

No data were used to support this study.

Disclosure

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Conflicts of Interest

The authors declare that they have no conflict of interest.

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