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Research Article

Boundary Value Problems for Liénard-Type Equations with Quadratic Dependence on the "Velocity"

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The estimates were obtained for the number of solutions for the Neumann and Dirichlet boundary value problems associated with the Liénard equation with a quadratic dependence on the "velocity." Sabatini's transformation is used to reduce this equation to a conservative one, which does not contain the derivative of an unknown function. Despite the one-to-one correspondence between the equilibria, the topological structure of the phase portraits of both equations can differ significantly.

1. Introduction

Nonlinear boundary value problems (BVP) for ordinary differential equations (ODE) are an integral part of many mathematical models of real phenomena. When calculating these models, it is important to know if a solution exists. In studying complex phenomena, multiple solutions can appear. To compute them, additional information is needed, such as the location, initial conditions, and oscillatory properties. The important problem of determining the number of solutions has attracted the attention of researchers for a long time. The nonlinear oscillation in physics and applied mathematics has been intensively studied in many articles. Such papers as [1, 2] presented analytical approximations to the periodic solutions and, in particular, to periodic solutions for oscillators described by ordinary differential equations with the odd-degree nonlinearity.

However, insufficient attention has been paid to a differential equation with even-degree nonlinearities. Although, for example, equations with quadratic nonlinearities have practical applications. As written in the work [3], this equation "has been used as a mathematical model of human eardrum oscillations." This fact motivated the search for an exact solution to a differential equation with quadratic nonlinearity. Equations with quadratic nonlinearities were studied in [4, 5] also.

In the articles [6, 7] the authors have studied the boundary value problems of the form

$$x'' + ax - b(t)x^n = 0, x(0) = x(T) = 0 \text{ or } x'(0) = x'(T) = 0, \tag{1}$$

where n is a positive integer (most results concern the cases $n = 2$, $n = 3$, and $n = 5$). The exact estimates of the number of solutions were obtained for autonomous equations of the form $x'' + ax - bx^n = 0$, and some results (mostly of computational nature) were stated for the case of $b(t)$ being piecewise constant function (in the articles, [7, 8]). The phase plane method was used extensively.

In this paper, our goal is different. We study the equation with quadratic term

$$x'' + f(x)x'^2 + g(x) = 0, \tag{2}$$

together with the two-point boundary conditions.

Equations of the form

$$x'' + f(x)x' + g(x) = 0 \tag{3}$$

are a classical object for investigation. The Liénard and Van-der-Pol equations fall into this class. Both arose from practice. Equations of the form (3) are rich in oscillatory behaviors. They are known to have (under suitable conditions) isolated periodic solutions. The problem of estimating the number of limit cycles for the case of polynomial functions $f(x)$ and $g(x)$ has attracted the attention of prominent researchers. In contrast, equation (2) can be reduced to a conservative equation. Our goal is to compare equation (2) with that for shorter equation

$$x'' + g(x) = 0. \tag{4}$$

We focus on the case $g(x) = ax - bx^3$, $a > 0$, $b > 0$, and consider two-point boundary conditions for both equations. In particular, we would like to compare the number of solutions to the respective BVPs.

For this, we make use of the special change of variables resulting in eliminating the middle term in (2). This technique was proposed by Sabatini ([9]) when studying isochronous problems. An equation in new variables has a simpler form and can be (formally) integrated. This transformation keeps the trivial solution. This is important because in various sources devoted to the study of multiple solutions of BVP, the following idea was exploited. Imagine that the oscillatory behavior of solutions can be measured around the trivial solution. If a comparison can be made with solutions far away from the trivial one, some conclusions can be made about the number of solutions for two-point boundary value problems. After reduction of the equation (2) to the form (4) using the above-mentioned variable change, another comparison can be made, namely, the equation in question versus the reduced equation. This approach will be considered in the next sections.

2. Reduction to Equation of the Form $u'' + h(u) = 0$

Consider equation (2). We suppose that f and g are continuous functions. Introduce the variable u by the transformation

$$u := \int_0^x e^{F(s)} ds, \tag{5}$$

where $F(x) = \int_0^x f(\xi) d\xi$. The primitive $F(x)$ can change sign arbitrarily, since there are no sign restrictions on $f(x)$, but $F(0) = 0$. Therefore $e^{F(x)}$ is a positive valued function with no restrictions on monotonicity. In particular, the integral $\int_0^{+\infty} e^{F(s)} ds$ or $\int_0^{-\infty} e^{F(s)} ds$ may be convergent, and the range of values of a new variable u can be bounded. Since $du/dx = e^{F(x)} > 0$, the function $u = u(x)$ is monotonically increasing, passing through zero, $u(0) = 0$. The inverse function $x(u)$ exists, but it can be defined on a bounded or a semi-bounded interval only.

The equation (2) takes the form

$$u'' + h(u) = 0, \tag{6}$$

where

$$h(u) = g(x(u)) e^{F(x(u))}. \tag{7}$$

The equation (6) can be considered instead of (2). There is a one-to-one correspondence between x and u.

3. Critical Points and Linearization

In the study of multiplicity of solutions to equations $x'' + ax - bx^3 = 0$ and/or $x'' + ax - bx^2 = 0$, a significant role was played by the linearization at the trivial solution $x(t) \equiv 0$. For instance, linearization of $x'' + ax - bx^3 = 0$ at the trivial solution yields the linear equation $y'' + ay = 0$, and estimations of the number of solutions of the Dirichlet and the Neumann problem were made in terms of this a.

Let us clarify this question in a general setting. Consider the conservative equation (4), where $g(x) = 0$ in several points x_i, including $x_0 = 0$. Then there is the trivial solution of the Dirichlet and the Neumann problems with zero boundary conditions. Linearization around the trivial solution yields

$$y'' + g_x(0) y = 0. \tag{8}$$

Notice that $g_x(0)$ need not be zero.

Consider now perturbed equation with the quadratic term (2): $x'' + f(x)x'^2 + g(x) = 0$, provided the same assumptions on $g(x)$. The critical points for an equivalent system $x' = y$, and $y' = -f(x)y^2 - g(x)$ are given by

$$y = 0, f(x)y^2 + g(x) = 0, \tag{9}$$

which reduces to $g(x) = 0$. So the critical points for equations (2) and (4) are the same.

Linearization of (2) at the trivial solution gives $y'' + [f_x x'^2 + g_x]|_{x\equiv 0} y + [2x'f]|_{x\equiv 0} y' = 0$ or $y'' + g_x(0) y = 0$, which is the same as (8).

The estimates for the equation $x'' + ax - bx^3 = 0$ (and/or $x'' + ax - bx^2 = 0$) were obtained by considering solutions of the Cauchy problems with small initial values and passing to a heteroclinic solution, which is slow.

4. Case 1: Equation $x'' + \mu x'^2 + g(x) = 0$

Consider the equation

$$x'' + \mu x'^2 + g(x) = 0, \tag{10}$$

where $g(x)$ is given. We will assume that $g(x)$ is a particular cubic polynomial. To apply the Sabatini transformation, we

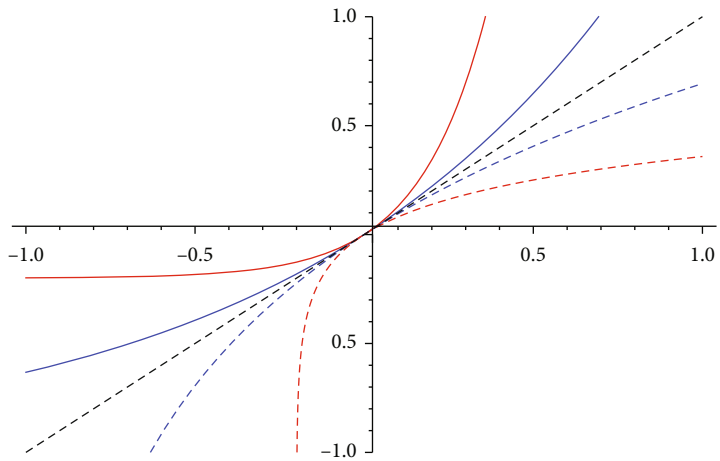

FIGURE 1: The graphs of $u(x) = 1/\mu(e^{\mu x} - 1)$ (solid), $x(u) = 1/\mu \ln(\mu u + 1)$ (dashed), blue for $\mu = 1$, and red for $\mu = 5$.

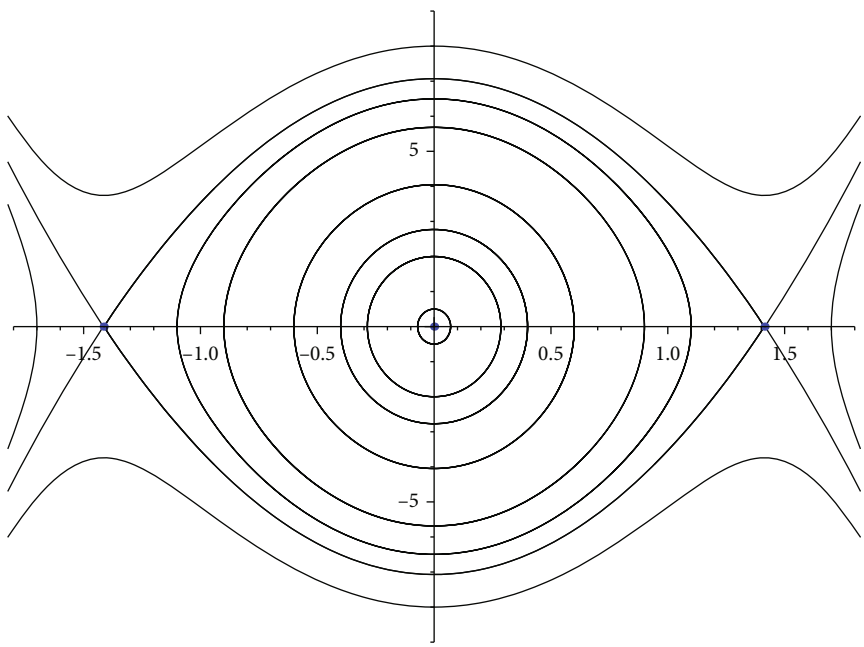

FIGURE 2: The phase portrait of system (23), $a = 50$, $b = 25$.

denote $F(x) = \int_0^x \mu d\xi = \mu x$. Introduce the new variable

$$u := \int_0^x e^{F(s)} ds = \int_0^x e^{\mu s} ds = 1/\mu \int_0^x e^{\mu s} d(\mu s) = 1/\mu(e^{\mu x} - 1). \tag{11}$$

Comparison will be made with the equation $x'' + g(x) = 0$. For μ positive (we assume this), $u = u(x)$ is the function, monotonically increasing from $-1/\mu$ to $+\infty$. The inverse function $x = x(u) = 1/\mu \ln(\mu u + 1)$ exists.

The equation (6) takes the form

$$\begin{aligned} u'' + g(x(u))e^{F(x(u))} &= u'' + g(x(u))e^{\mu(x(u))} = u'' + g(x(u))e^{\mu(1/\mu \ln(\mu u+1)))} \\ &== u'' + g(x(u))(\mu u + 1) = 0. \end{aligned} \tag{12}$$

In Figure 1, the functions $u(x)$ (solid) are depicted (blue for $\mu = 1$ and red for $\mu = 5$) together with the inverse functions $x(u)$ (dashed).

If $g(x) = ax - bx^3$, the equation (12) turns to

$$u'' + \frac{1}{\mu \ln(\mu u + 1)\left[a - \left(b/\mu^2 \ln^2(\mu u + 1)\right)\right](\mu u + 1)} = 0, \text{ where } -1/\mu < u < +\infty. \tag{13}$$

4.1. Comparison of $x'' + ax - bx^3 = 0$ and $x'' + \mu x'^2 + ax - bx^3 = 0$

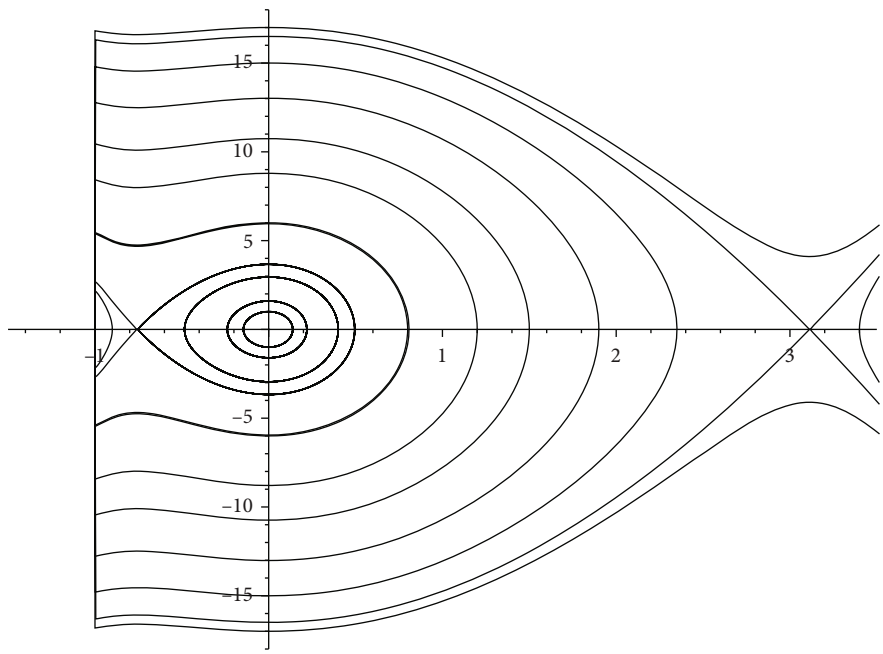

Figure 3: The phase portrait of system (24).

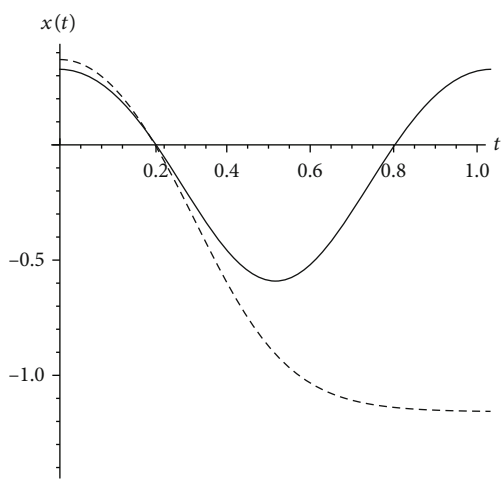

Figure 4: Graphs $x(t)$ for solutions of problems (15) and (30), $x_\alpha \approx 0.352$, $x_\alpha \approx 0.40228$ (dashed).

4.1.1. Number of Solution for the Neumann Problem. Consider the Neumann problem

$$x'' + \mu x'^2 + g(x) = 0, \tag{14}$$

$$x'(0) = x'(1) = 0. \tag{15}$$

The equation (14) turns to the equation

$$u'' + g(x(u))(\mu u + 1) = 0, \text{ where } g(x) = ax - bx^3, \tag{16}$$

or takes the form

$$u'' + 1/\mu \ln(\mu u + 1)\left[a - b/\mu^2 \ln^2(\mu u + 1)\right](\mu u + 1) = 0. \tag{17}$$

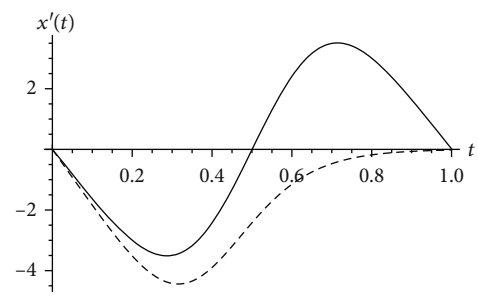

Figure 5: Graphs $x'(t)$ for solutions of problems (15) and (30), $x_\alpha \approx 0.352$, $x_\alpha \approx 0.40228$ (dashed).

Since $dx/dt = (dx/du)(du/dt) = (1/(du/dx))(du/dt) = e^{-F(x)}(du/dt)$, $e^{-F(x)} > 0$, the boundary conditions (15) for $u(t)$ are

$$u'(0) = u'(1) = 0. \tag{18}$$

Any solution $x(t)$ of problems (14) and (15) corresponds to a solution $u(t) = \int_0^{x(t)} e^{F(s)} ds$ of problems (17) and (18). Therefore, we can estimate the number of solutions of problems (17) and (18). Let us linearize equation (17) at the critical point $u = 0$. One has

$$\begin{aligned}
\frac{d(g(x(u)(\mu u + 1))}{du} &= \frac{dg(x(u))}{du}(\mu u + 1) + \frac{d(\mu u + 1)}{du} g(x(u)) \\
&= \frac{dg(x)}{dx}\frac{dx}{du}(\mu u + 1) + \mu g(x(u)) \\
&= \frac{d(ax - bx^3)}{dx}\frac{1}{du/dx}(\mu u + 1) + \mu g(x(u)) \\
&= \left(a - 3bx^2(u)\right)\frac{1}{e^{F(x)}}(\mu u + 1) + \mu g(x(u)) \\
&= \left(a - 3bx^2(u)\right) e^{-F(x)}(\mu u + 1) + \mu g(x(u)).
\end{aligned} \tag{19}$$

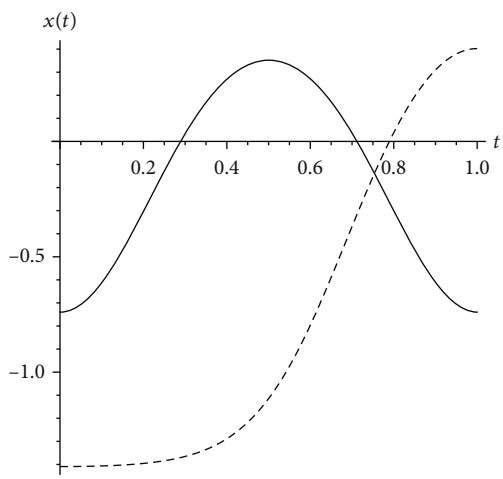

Figure 6: Graphs $x(t)$ for solutions of problems (15) and (30) $x_\alpha \approx -0.74$, $x_\alpha \approx -1.4092$ (dashed).

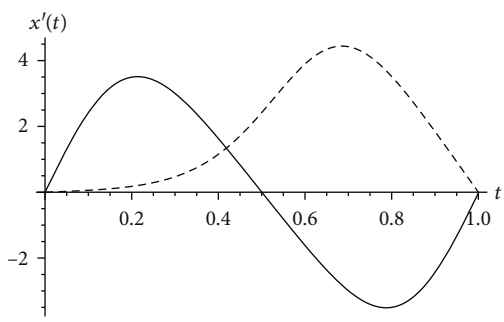

Figure 7: Graphs $x'(t)$ for solutions of problems (15) and (30) $x_\alpha \approx -0.74$, $x_\alpha \approx -1.4092$ (dashed).

Evaluating (19) at $u = 0$ and taking into account that $x(u)|_{u=0} = 0, F(0) = 0$, and $g(0) = 0$, one gets that the right side in (19) is a and the linearized equation for (17) at $u = 0$ is $y'' + ay = 0$. This corresponds to the linearization of equation

$$x'' = -ax + bx^3, a > 0, b > 0 \tag{20}$$

at $x = 0$.Hence the following statement is valid.

Theorem 1. *The number N of solutions of problems (14) and (15) is not less than the number n of solutions of problems (15) and (20).*

The number of nontrivial solutions of the BVP

$$x'' = -ax + bx^3, \quad x'(0) = 0, \quad x'(1) = 0 \tag{21}$$

is exactly 2n, where n is the number of extrema of a solution $x(t;\alpha)$ of the Cauchy problem $x(0) = 0, x'(0) = \alpha$ in the open interval $(0, 1)$ for $\alpha > 0$ and arbitrarily small. This follows from the monotonocity ([7]) of the period $T(\alpha)$, when α changes from zero to $\alpha_{heteroclinic}$, where the last value corresponds to the heteroclinic solution.

4.1.2. Phase Portraits. The comparison differential equation

$$x'' + g(x) = x'' + ax - bx^3 = 0, \tag{22}$$

and the equivalent system

$$\begin{cases} x' = y, \\ y' = -ax + bx^3 \end{cases} \tag{23}$$

have three critical points at $(-\sqrt{a/b}, 0)$,$(0, 0)$, and $(\sqrt{a/b}, 0)$.

The equation (16) and the equivalent system

$$\begin{cases} u' = v, \\ v' = -g(x(u))(\mu u + 1) \end{cases} \tag{24}$$

have also three critical points at $(u_1, 0)$,$(0, 0)$, and $(u_3, 0)$, where u_1 and u_3 are such that $x(u_1) = -\sqrt{a/b}$, and $x(u_3) = \sqrt{a/b}$. The behavior of solutions of (23) in the region between two heteroclinic trajectories and the behavior of solutions of (24) in the region inside the homoclinic solution are similar. All of them are periodic, and periods are defined for α small by the same linearization.

The phase portraits for systems (23) and (24), $a = 50$, $b = 25$, and $\mu = 1$, are depicted in Figures 2 and 3.

4.1.3. Solutions for the Neumann Problem. The number of solutions of the Neumann problem

$$x'' + \mu x'^2 + ax - bx^3 = 0, \tag{25}$$

(15) is not less than that for problems (15) and (20).

Consider the Cauchy problem (20),

$$x(0) = x_\alpha, x'(0) = 0, -\sqrt{\frac{a}{b}} < x_\alpha < \sqrt{\frac{a}{b}}, x_\alpha \neq 0. \tag{26}$$

Theorem 2. *Let i be a positive integer such that*

$$i\pi < \sqrt{a} < (i+1)\pi. \tag{27}$$

The Neumann problems (15) and (20) have exactly 2i nontrivial solutions such that $x(0) = x_\alpha$, $x'(0) = 0$, $-\sqrt{a/b} < x_\alpha < \sqrt{a/b}$, $x_\alpha \neq 0$.

The proof of Theorem 2 can be found in the articles [6].

Consider the Cauchy problem (25)

$$x(0) = x_\alpha, x'(0) = 0, -x_1 < x_\alpha < x_2, x_\alpha \neq 0, \tag{28}$$

where x_1 and x_2 are critical points of equation (25).

We know from Theorem 1 that the number of (nontrivial) solution of the BVP (15) and (25) is at least $2i$. We wish to illustrate this.

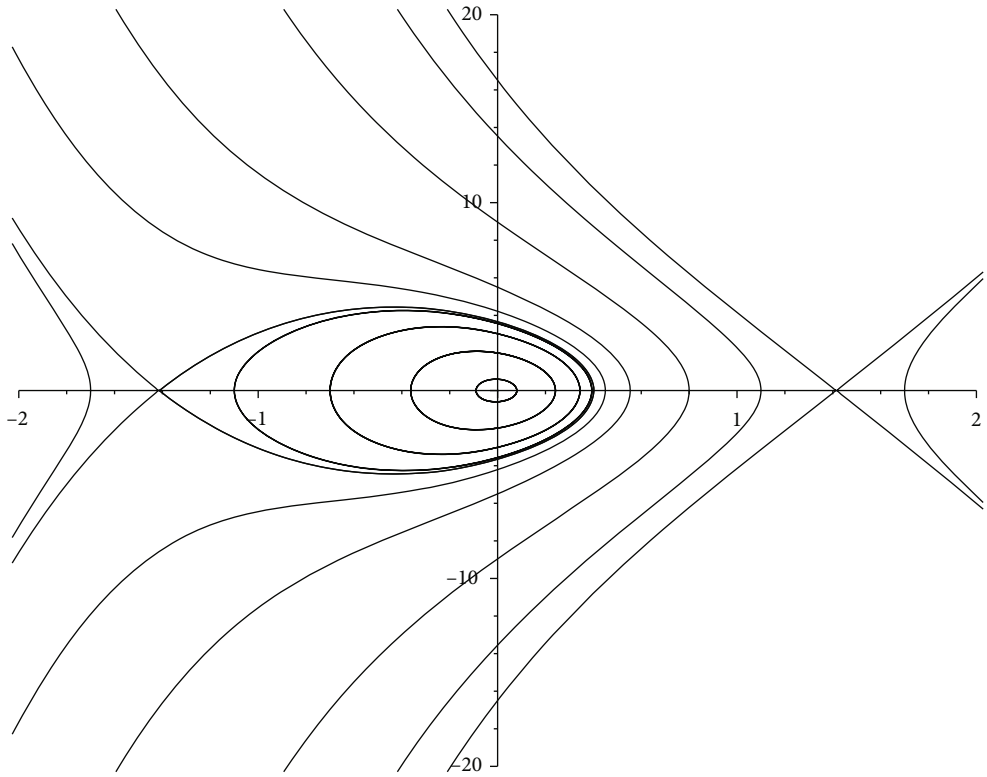

Figure 8: The phase portrait of $x'' + \mu x'^2 + ax - bx^3 = 0$, $a = 50$, $b = 25$, and $\mu = 1$.

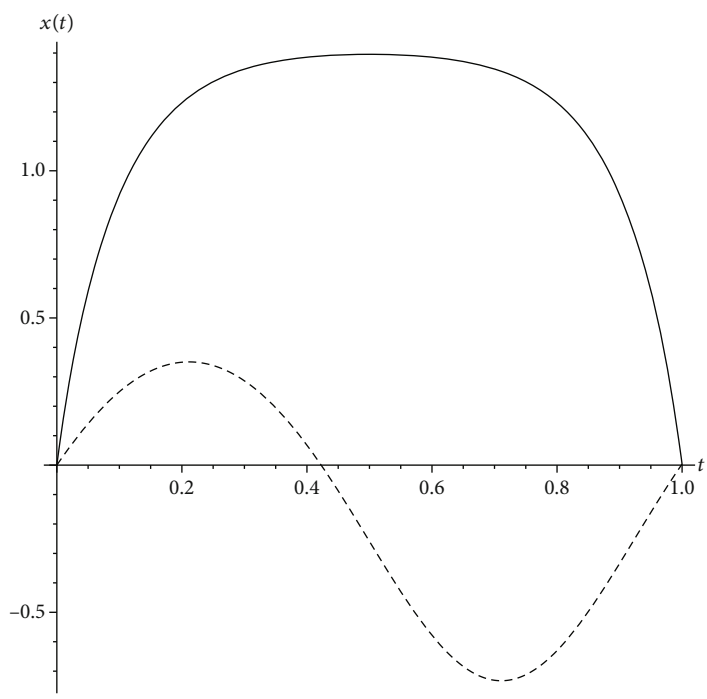

Figure 9: Graphs $x(t)$ for solutions of problems (30) and (31), $\alpha \approx 16.4802$, $\alpha \approx 3.1$ (dashed).

Example 1. Consider equation (20), where $a = 50$ and $b = 25$:

$$x'' = -50x + 25x^3, \tag{29}$$

with the initial conditions $x(0) = x_\alpha$, $x'(0) = 0$, and $0 < x_\alpha < \sqrt{2}$, and then the number of nontrivial solutions satisfying the boundary conditions (15) is two, and for initial conditions $x(0) = x_\alpha$, $x'(0) = 0$, and $-\sqrt{2} < x_\alpha < 0$, there are also

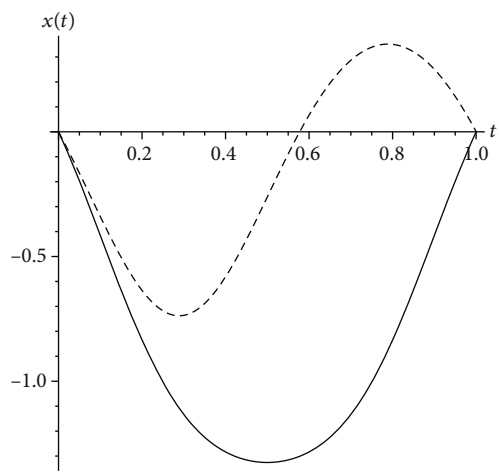

Figure 10: Graphs $x(t)$ for solutions of problems (30) and (31), $\alpha \approx -3.11$, $\alpha \approx -3.6605$ (dashed).

two solutions to the problem, totally four solutions. This is in accordance with Theorem 2. This is the case for $i = 2$ (namely, $2\pi < \sqrt{50} < 3\pi$) in the inequality (27).

Consider equation (25), where $a = 50$, $b = 25$, and $\mu = 1$:

$$x'' = -x'^2 - 50x + 25x^3 \tag{30}$$

with the initial conditions $x_\alpha > 0$, and then the number of nontrivial solutions satisfying the boundary conditions (15) is two. Graphs $x(t)$ for solutions of problems (15) and (30), where $x_\alpha > 0$, are depicted in Figure 4, but graphs $x'(t)$ are depicted in Figure 5. For initial conditions $x_\alpha < 0$, there are also two solutions to the problem. Graphs $x(t)$ for solutions of problems (12) and (30), where $x_\alpha < 0$, are depicted in Figure 6, but graphs $x'(t)$ are depicted in Figure 7. Totally the Neumann problems (15) and (30) have four solutions. This is in accordance with Theorem 1.

4.1.4. Number of Solution for the Dirichlet Problem. Question. As we see in Figure 8, two heteroclinic solutions have disappeared and were replaced by a homoclinic solution. The number of solutions for the Neumann problems for equations $x'' + ax - bx^3 = 0$ and $x'' + \mu x'^2 + ax - bx^3 = 0$ is, probably, the same. As to the Dirichlet problem with zero boundary conditions, extra solution(s) can appear outside of the region, bounded by homoclinic trajectory. Is this the case for the example in Figure 8 (and generally)?

Consider the Dirichlet problem (25),

$$x(0) = x(1) = 0. \tag{31}$$

Theorem 3. *Let i be a positive integer such that*

$$i\pi < \sqrt{a} < (i+1)\pi. \tag{32}$$

The Dirichlet problems (20) and (31) has exactly $2i$ nontrivial solutions such that $x(0) = 0$, $x'(0) = \alpha$, and $-\alpha_{\max} < \alpha < \alpha_{\max}$, where $\alpha_{\max} =: a/\sqrt{2b}$, $\alpha \neq 0$.

The proof of Theorem 3 can be found in the articles [6].

Example 2. Consider equation (29), with the initial conditions $x(0) = 0$, $x'(0) = \alpha$, and $0 < \alpha < 5\sqrt{2}$, and then the number of solutions satisfying the boundary conditions (31) is two, and for initial conditions $x(0) = 0$, $x'(0) = \alpha$, and $-5\sqrt{2} < \alpha < 0$, there are also two solutions to the problem, totally four solutions. This is in accordance with Theorem 3 for the case for $i = 2$ (namely, $2\pi < \sqrt{50} < 3\pi$) in the inequality (32).

Consider equation (30) with the initial conditions $0 < \alpha < \alpha_{\text{homoclinic}}$, then the number of solutions satisfying the boundary conditions (31) is one. The reason for this difference comparing with "short" equation (29) is that the phase portrait (depicted in Figure 8) is not symmetric with respect to the vertical axis, and the rotation is faster in the left half-plane. The graph $x(t)$ for a solution of problems (30) and (31), where $0 < \alpha < \alpha_{\text{homoclinic}} (\alpha \approx 3.1)$, is depicted in Figure 9. In the same picture, an additional solution of the problem with $\alpha_{\text{homoclinic}} < \alpha < \alpha_0 (\alpha \approx 16.4802)$ is shown. The value α_0 corresponds to the intersection point of a trajectory going to the critical point (right of the homoclinic region) with the x' axis. For the negative initial conditions $-\alpha_{\text{homoclinic}} < \alpha < 0$, there are two solutions to the problem. The graphs $x(t)$ for solutions of problems (30) and (31), where $\alpha < 0$, are depicted in Figure 10. Totally the Dirichlet problems (30) and (31) has four solutions. Of them three solutions are in the region bounded by the homoclinic trajectory, and one solution lies outside this region.

Theorem 4. *The Dirichlet problems (25) and (31) have at least $2i - 1$ nontrivial solutions for the initial conditions in the region bounded with homoclinic solutions. The extra solution(s) can appear outside of this region.*

The phase portrait is similar to that for a quadratic equation $x'' = -ax + bx^2$. Therefore, the proof about number of solutions is similar also.

Proof. Consider the phase portrait for equation (25) depicted in Figure 8. The trajectory of any nontrivial solution of problems (25) and (31) for $0 < \alpha < \alpha_{\text{homoclinic}}$ is located inside the region bounded by homoclinic orbits. Introduce the notation G for this region.

Consider solutions of the Cauchy problem (25),

$$x(0) = 0, x'(0) = \alpha, -\alpha_{\text{homoclinic}} < \alpha < \alpha_{\text{homoclinic}}, \alpha_{\text{homoclinic}} \neq 0. \tag{33}$$

Solutions for α small enough behave like solutions of the equation of variations $y'' = -ay$ around the trivial solution. Due to the assumption $i\pi \leq \sqrt{a} < (i+1)\pi$, solutions $y(t)$ along with solutions $x(t;\alpha)$ (for small enough α) have exactly i zeros in the interval $(0, 1)$, but for α large, the first zero $t_1(\alpha)$ tends to zero and stays in the interval. Therefore, for $0 < \alpha < \alpha_{\text{homoclinic}}$, there exist exactly $i - 1$ solutions.

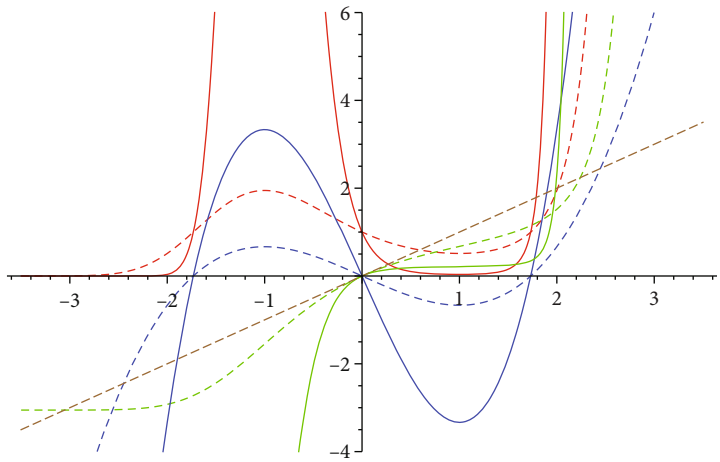

Figure 11: The graphs of $\mu((1/3)x^3 - x)$ (blue), $e^{\mu((1/3)x^3-x)}$ (red), and $u(x) = \int_0^x e^{\mu((1/3)s^3-s)}\, ds$ (green), $\mu = 1$ (dashed), and $\mu = 5$ (solid).

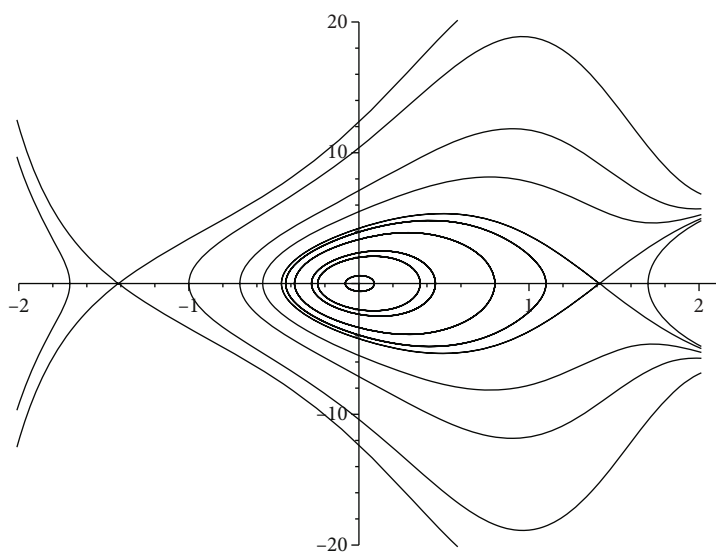

Figure 12: The phase portrait of $x'' + \mu(x^2 - 1)x'^2 + ax - bx^3 = 0$, $a = 50$, $b = 25$, $\mu = 1$.

Consider $-\alpha_{\text{homoclinic}} < \alpha < 0$. Solutions for α small enough have i zeros. These zeros move monotonically to the right as α increases in modulus and negative. Solutions $x(t;\alpha)$ with $-\alpha_{\text{homoclinic}} < \alpha$ and close enough to $-\alpha_{\text{homoclinic}}$ have no zeros in $(0, 1]$ since the respective trajectories are close to the homoclinic, therefore, exactly i solutions of problems (25) and (31).

The extra solution(s) can appear outside of this region if initial condition is $\alpha_{\text{homoclinic}} < \alpha < \alpha_0$. The value α_0 is the intersection point of the phase trajectory, which starts at the critical point, with the x' axis. □

5. Case 2: Equation $x'' + \mu(x^2 - 1)x'^2 + g(x) = 0$

The Liénard equation is

$$x'' - \mu(1 - x^2)x' + x = 0. \tag{34}$$

We take the coefficient of the quadratic term from (34) and consider the equation

$$x'' + \mu(x^2 - 1)x'^2 + g(x) = 0, \tag{35}$$

where $g(x)$ will be specified later. One has $f(x) = \mu(x^2 - 1)$, $F(x) = \mu((1/3)x^3 - x)$, and $u(x) = \int_0^x e^{\mu((1/3)s^3-s)} ds$.

In Figure 11, the functions $\mu((1/3)x^3 - x)$ (blue), $e^{\mu((1/3)x^3-x)}$ (red), and $u(x) = \int_0^x e^{\mu((1/3)s^3-s)} ds$ (green) are depicted for $\mu = 1$ (dashed) and $\mu = 5$ (solid).

The functions $u(x)$ have horizontal asymptote for $x \longrightarrow -\infty$, and the inverse functions $x(u)$ have vertical asymptote at $u_* = \int_0^{-\infty} e^{\mu((1/3)s^3-s)} ds$. This means that the phase plane for the respective equation $u'' + H(u) = 0$ is restricted to $u_* < u < +\infty$. This does not affect the number of solutions to the Neumann boundary value problem for the equation (35).

The phase portrait of $x'' + \mu(x^2 - 1)x'^2 + ax - bx^3 = 0$, $a = 50$, $b = 25$, $\mu = 1$, is depicted in Figure 12.

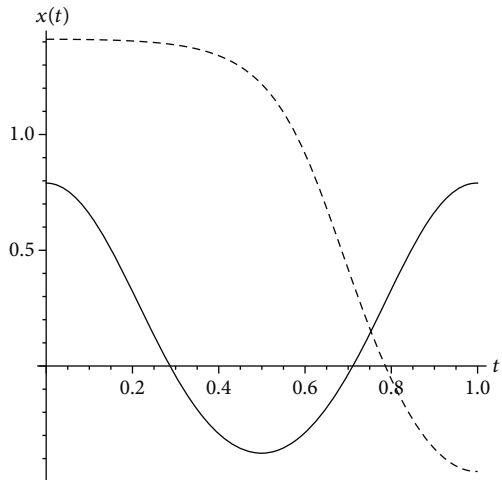

Figure 13: Graphs $x(t)$ for solutions of problems (15) and (37), $x_\alpha \approx 0.79$, $x_\alpha \approx 1.4115$ (dashed).

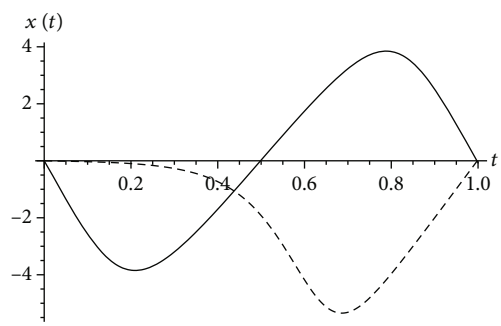

Figure 14: Graphs $x'(t)$ for solutions of problems (15) and (37), $x_\alpha \approx 0.79$, $x_\alpha \approx 1.4115$ (dashed).

5.1. Comparison of $x'' + ax - bx^3 = 0$ and $x'' + \mu(x^2 - 1)x'^2 + ax - bx^3 = 0$. Consider equation

$$x'' + \mu\left(x^2 - 1\right)x'^2 + g(x) = 0, \quad g(x) = ax - bx^3. \tag{36}$$

Generally, as in Case 1, we can say that the following is true.

Theorem 5. *The number N of solutions of problems (15) and(36) is not less than the number n of solutions of problems (15) and (20)*

The reason for this will be discussed in the Conclusions section.

5.1.1. Number of Solution for the Neumann Problem. The number of solution of problems (15) and (20) is defined in Theorem 2. Consider equation (36), where $a = 50$, $b = 25$, and $\mu = 1$:

$$x'' = -\left(x^2 - 1\right)x'^2 - 50x + 25x^3, \tag{37}$$

with the initial conditions $x_\alpha > 0$, and then the number of solutions satisfying the boundary conditions (15) is two. Graphs $x(t)$ for solutions of problems (15) and (37), where $0 < x_\alpha < x_{\text{homoclinic}}$ (in this example, $x_{\text{homoclinic}} \approx 1.42$), are depicted in Figure 13, but graphs $x'(t)$ are depicted in Figure 14. For initial conditions $-0.46 < x_\alpha < 0$, there are also two solutions to the problem. Graphs $x(t)$ for solutions of problems (15) and (37), where $x_\alpha < 0$, are depicted in Figure 15, but graphs $x'(t)$ are depicted in Figure 16. Totally the Neumann problems (15) and (37) have four solutions. This is the same number of solutions as in Theorem 2 for problems (15) and (20). Therefore the Theorem 5 is fulfilled.

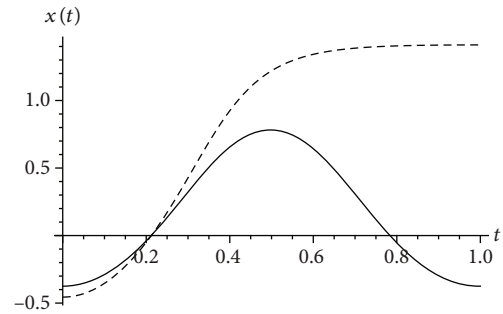

Figure 15: Graphs $x(t)$ for solutions of problems (15) and (30), $x_\alpha \approx -0.375$, $x_\alpha \approx -0.45568$ (dashed).

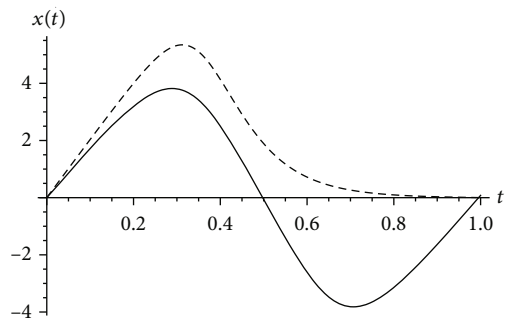

Figure 16: Graphs $x'(t)$ for solutions of problems (15) and (30), $x_\alpha \approx -0.375$, $x_\alpha \approx -0.45568$ (dashed).

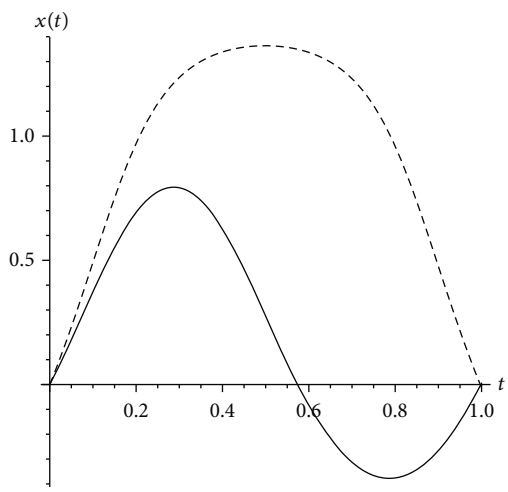

Figure 17: Graphs $x(t)$ for solutions of problem (31) and (37), $\alpha \approx 3.367$, $\alpha \approx 4.214$ (dashed).

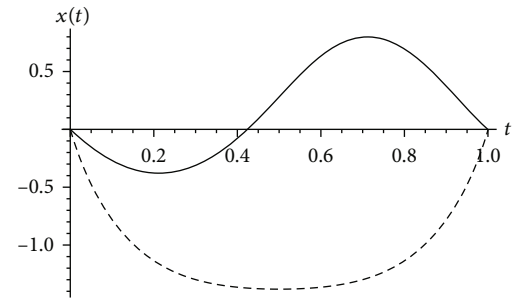

Figure 18: Graphs $x(t)$ for solutions of problem (31) and (37), $\alpha \approx -3.378$, $\alpha \approx -12.3317$ (dashed).

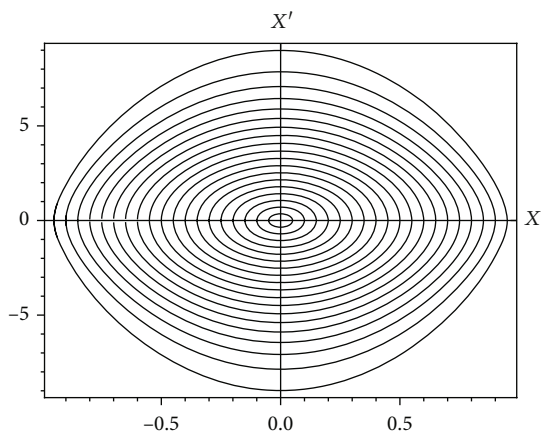

Figure 19: The central part of the phase portrait of the equation (38), where $g(x) = -50x + 25x^3$.

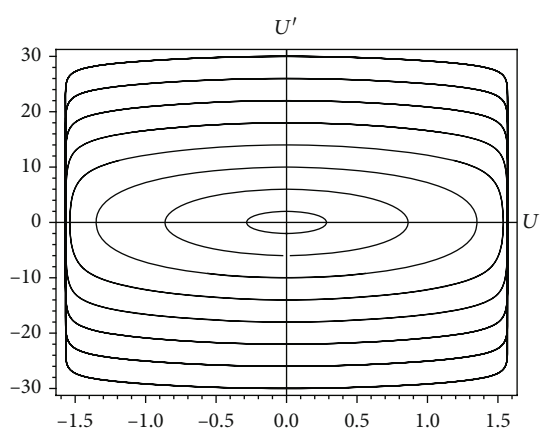

Figure 20: The central part of the phase portrait of the equation (41), where $a = 50$, and $b = 25$.

5.1.2. Number of Solution for the Dirichlet Problem. For Case 2, the number of solution for Dirichlet problem is the same as defined in Theorem 4 for Case 1: equation $x'' + \mu x'^2 + g(x) = 0$, $g(x) = ax - bx^3$. The Dirichlet problems (31) and (37) have three solutions in region bounded with homoclinic trajectory and one solution outside this region (see in Figures 17 and 18).

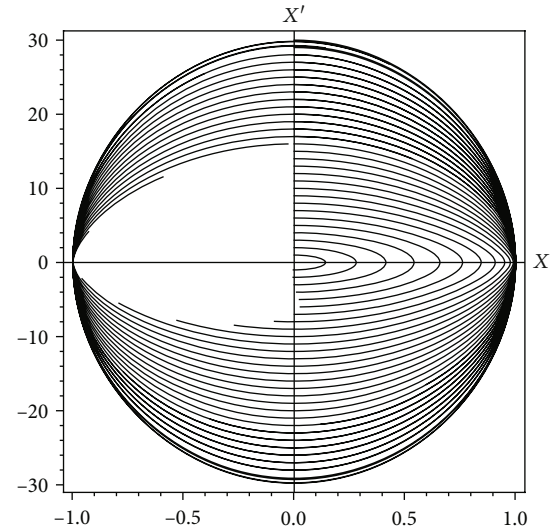

Figure 21: Solutions of the Cauchy problems (38), $x(0) = 0$, $x'(0) = i$, and $i = 1, 2, \cdots, 30$ on the interval $(0, \pi/7)$.

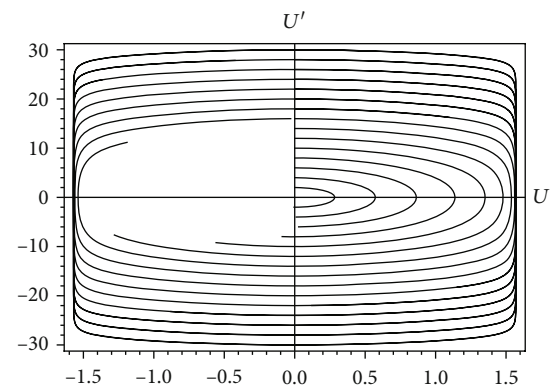

Figure 22: Solutions of the Cauchy problems (41), $u(0) = 0$, $u'(0) = i$, and $i = 2, 4, \cdots, 15$ on the interval $(0, \pi/7)$.

6. Case 3: Equation $x'' + (x/1 - x^2)x'^2 + g(x) = 0$

6.1. Transformation. The equation

$$x'' + \frac{x}{1 - x^2} x'^2 + g(x) = 0, \ |x| < 1 \tag{38}$$

was considered in the paper [10], where the oscillatory behaviors of solutions were compared for equation (38) and shortened equation $x'' + g(x) = 0$. The following result was provided.

Proposition 6. *The equation (38) by Sabatini's transformation turns to equation*

$$\begin{gathered} u'' + H(u) = 0, \\ H(u) = \frac{g(\sin(u))}{\cos u}, \\ u \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right). \end{gathered} \tag{39}$$

The following motivation was used.

$$\begin{aligned} f(x) &= \frac{x}{1-x^2}, \\ F(x) &= \int_0^x f(s)ds = \ln \frac{1}{\sqrt{1-x^2}}, \\ e^{F(x)} &= \frac{1}{\sqrt{1-x^2}}, \\ u = \Phi(x) &= \int_0^x e^{F(s)}ds = \int_0^x \frac{1}{\sqrt{1-s^2}}ds = \arcsin x, |x| < 1, \\ x &= x(u) = \sin u, |u| < \frac{\pi}{2}. \end{aligned} \tag{40}$$

Let $g(x) = -ax + bx^3$. Consider the respective equation

$$u'' + \frac{\text{asin}u - b\sin^3 u}{\cos u} = 0. \tag{41}$$

6.2. Number of Solutions. It is an easy matter to check that the linearized equation for the equation (39) around the trivial solution $u(t) \equiv 0$ is $v'' + av = 0$. Situation differs, however, from cases 1 and 2. The function $g(x)$ can have zeros outside the interval $(-1, 1)$, where equation (38) is defined. This is the case for $g(x) = -50x + 25x^3$. This function was used for comparison of results in the previous cases 1 and 2. Now the critical points $\pm\sqrt{2}$ are located outside the interval $(-1, 1)$.

In Figures 19 and 20 the central parts of the phase portraits of the equations (38) and (41) are depicted. The results of calculations, visualized in Figures 21 and 22, show that the angular speeds for the trajectories both for the equations (38) and (41) are not monotone with respect to $x'(0)$ and $u'(0)$. Therefore, periods of periodic solutions are not monotone functions of the initial data, and Theorem 3 and Theorem 4 are not applicable.

The following, however, is true.

Theorem 7. *Let $\sqrt{a/b} > 1$ and $\pi/\sqrt{a} < 1$.*

Then the Dirichlet problem

$$x'' + \frac{x}{1-x^2}x'^2 + ax - bx^3 = 0, |x| < 1, x(0) = 0, x(1) = 0, \tag{42}$$

and the Neumann problem

$$x'' + \frac{x}{1-x^2}x'^2 + ax - bx^3 = 0, |x| < 1, x'(0) = 0, x'(1) = 0 \tag{43}$$

have nontrivial solutions.

7. Conclusions

The Neumann (and generally the Dirichlet) boundary value problems for the equation

$$x'' + g(x) = 0, \quad g(x) = ax - bx^3, \quad a, b > 0 \tag{44}$$

are well studied. The exact number of solutions is known and it depends on the coefficient a only. The equation

$$x'' + f(x)x'^2 + g(x) = 0, \quad g(x) = ax - bx^3, a, b > 0 \tag{45}$$

can be significantly more complicated, due to the presence of $f(x)x'^2$. It appears nevertheless, that the number and the type of critical points for (44) and (45) coincide. The analysis of the equation (45) can be made easier by passing to the Newtonian equation

$$u'' + H(u) = 0, \quad H(u) = \left(ax(u) - bx^3(u)\right)e^{F(x(u))}, \tag{46}$$

where $x(u)$ is the inverse function to $u = \int_0^x e^{F(s)}ds$, $F(x)$ is the primitive of $f(x)$. The critical points of (46) and their types are the same as the ones for the equations (44) and (45). It can be shown, that the linearized equation for the trivial solution of (46) is also $v'' + av = 0$.

The exact number of solutions to the Neumann problem for (44) was established, using the monotonicity of the period function. This cannot be guaranteed for the equation (46).

If the variable $u = \int_0^x e^{F(x)}$ has asymptotes, the inverse variable $x(u)$ can be defined on a semibounded or bounded interval. The phase space for the modified equation $u'' + H(u) = 0$ can be a strip or a half-plane.

It may happen that the region between two heteroclinic trajectories (as in equation $x'' = -ax + bx^3$) becomes a region inside a homoclinic loop for the modified equation. Therefore, the phase potraits for the equations $x'' + g(x) = 0$ and $u'' + H(u) = 0$ generally are not topologically equivalent, despite of the fact that critical points are in one-to one correspondence and the characteristic types coincide.

For the Dirichlet problem, the number of solutions for the equations $x'' + g(x) = 0$ and $u'' + H(u) = 0$ may be the same, but their nature in phase plane can be quite different. The symmetry with respect to the vertical axis may be broken in the equation $u'' + H(u) = 0$.

Introducing the quadratic term into the equation $x'' + g(x) = 0$ may heavily influence the behavior of solutions and the structure of the phase plane.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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