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Research Article

Fixed Point Results for Generalized (α, ψ)-Contraction Mapping in Rectangular b-Metric Spaces

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In this paper, we introduce generalized (α, ψ)-contraction mappings in the setting of rectangular b-metric spaces and established existence and uniqueness of fixed points for the mappings introduced. Our results extend and generalize related fixed point results in the existing literature. We derive some consequences and corollaries from our obtained results. Also, we provide examples in support of our main findings. Furthermore, we determined a solution to an integral equation by applying our obtained results.

1. Introduction

Fixed point theory is one of the most important topics in mathematics, especially in analysis. Due to its application in various disciplines like engineering, computer science, biological sciences, and economics, many researchers took interest in fixed point theory and its application. The Banach contraction principle is a fundamental result in fixed point theory [1]. For one century, due to its importance and simplicity, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle in several direction. One possible direction is the notion of best proximity point results. In this line of research, [2–4] obtained interesting best proximity point results and derived fixed results as consequences of their works in which Banach fixed point is one of them. Also, taking the key role of the notion of the metric in mathematics and hence in quantitative sciences, it has been extended and generalized in several distinct directions by many authors. In particular, b-metric spaces introduced by Bakhtin [5] and Czerwik [6] are one generalization of metric spaces. In 2000, Branciari [7] introduced the concept of rectangular metric space by replacing the sum on the right hand side of the triangular inequality in the definition of a metric space by a three-term expression proved an analogue of the Banach contraction principle in such space. Also, various fixed point results were established on such spaces (see [8–13] and references therein). Recently, George et al. [14] announced the notion of rectangular b-metric space as a generalization of metric, b-metric space, and rectangular metric space; many authors initiated and studied many existing fixed point theorems in such spaces (see [15–20]). Recently, in 2019, Baiya and Kaewcharoen [21] established fixed point theorems for generalized contractions in complete rectangular metric spaces and proved the existence of fixed points, and in 2020, Kari et al. [22] introduced the notion of $\theta - \phi$-contraction in b-rectangular metric spaces and studied the existence and uniqueness of fixed point for the mappings introduced. Inspired and motivated by the works of Baiya and Kaewcharoen [21] and Kari et al. [22] in this paper, we introduce a new notion of generalized (α, ψ)-contraction mappings and establish some fixed point results for such mappings in complete rectangular b-metric spaces. Our results extend and generalize related fixed point results in the existing literature.

2. Preliminaries

In this section, we recall some basic definitions and results that will be used to prove our main results.

Notation: We need the following symbols and class of functions to prove certain results of this section:

(1) $\mathfrak{R}^+ = [0,\infty)$;

(2) $\mathbb{N}$ is the set of all natural numbers;

(3) $\Psi_1 = \{\psi = (0,\infty) \longrightarrow (1,\infty)$, such that, ψ is nondecreasing, for each sequence $\{t_n\} \subset (0,\infty)$, $\psi(t_n) \longrightarrow 1$ if and only if $t_n \longrightarrow 0$ and there exist $r \in (0, 1)$ and $l \in (0,\infty]$ such that $\lim_{t\longrightarrow 0} \psi(t) - 1/t^r = l\}$;

(4) $\Psi_2 = \{\psi = (0,\infty) \longrightarrow (1,\infty)$, such that, ψ is continuous and nondecreasing$\}$.

Definition 1 (see [23]). Let X be a nonempty set and $d : X \times X \longrightarrow \mathfrak{R}^+$ be a function satisfying the following conditions:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then, d is called a metric, and the pair (X, d) is called a metric space.

The following is the definition of the notion of b-metric space.

Definition 2 (see [6]). Let X be a nonempty set and $s \ge 1$ be a given real number. A function $d : X \times X \longrightarrow \mathfrak{R}^+$ is said to be a b-metric if and only if for all $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \le s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space.

Definition 3 (see [7]). Let X be a nonempty set, and let $d : X \times X \longrightarrow \mathfrak{R}^+$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \le [d(x, u) + d(u, v) + d(v, y)]$ (rectangular inequality).

Then, d is called a rectangular metric, and the pair (X, d) is called a rectangular metric space.

Definition 4 (see [14]). Let X be a nonempty set, $s \ge 1$ be a given real number, and $d : X \times X \longrightarrow \mathfrak{R}^+$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \le s[d(x, u) + d(u, v) + d(v, y)]$ (b-rectangular inequality).

Then, d is called a b-rectangular metric, and the pair (X, d) is called a b-rectangular metric space.

Definition 5 (see [14]). Let X be a b-rectangular metric space and $\{x_n\}$ be a sequence in X; we say that:

(i) $\{x_n\}$ b-rectangular converges to $x \in X$ if $d(x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$;

(ii) $\{x_n\}$ is a b-rectangular Cauchy sequence if $d(x_n, x_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$;

(iii) (X, d) is a b-rectangular complete if every b-rectangular Cauchy sequence in X is b-rectangular convergent.

Lemma 6 (see [22]). *Let X be a rectangular b-metric space, and let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ is not a Cauchy sequence, then, there exists $\varepsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$, such that n_k is the smallest index with $n_k > m_k > k$ for which $d(x_{m_k}, x_{n_k}) \ge \varepsilon$ and $d(x_{m_k}, x_{n_{k-1}}) < \varepsilon$. Then, the following hold:*

(1) $\varepsilon \le \liminf_{k\longrightarrow\infty} d(x_{m_k}, x_{n_k}) \le \limsup_{k\longrightarrow\infty} d(x_{m_k}, x_{n_k}) \le s\varepsilon$.

(2) $\varepsilon \le \liminf_{k\longrightarrow\infty} d(x_{m_{k+1}}, x_{n_k}) \le \limsup_{k\longrightarrow\infty} d(x_{m_{k+1}}, x_{n_k}) \le s\varepsilon$.

(3) $\varepsilon \le \liminf_{k\longrightarrow\infty} d(x_{m_k}, x_{n_{k+1}}) \le \limsup_{k\longrightarrow\infty} d(x_{m_k}, x_{n_{k+1}}) \le s\varepsilon$.

(4) $\varepsilon/s \le \liminf_{k\longrightarrow\infty} d(x_{m_{k+1}}, x_{n_{k+1}}) \le \limsup_{k\longrightarrow\infty} d(x_{m_{k+1}}, x_{n_{k+1}}) \le s^2\varepsilon$.

Definition 7 (see [24]). Let X be a nonempty set and $\alpha : X \times X \longrightarrow \mathfrak{R}^+$ be a function. A mapping $T : X \longrightarrow X$ is said to be α-admissible; if for all $x, y \in X$, $\alpha(x, y) \ge 1$ implies $\alpha(Tx, Ty)) \ge 1$.

Definition 8 (see [25]). Let X be a nonempty set and $\alpha : X \times X \longrightarrow \mathfrak{R}^+$ be a function. A mapping $T : X \longrightarrow X$ is said to be α-orbital admissible; if for all $x \in X$, $\alpha(x, Tx) \ge 1$ implies $\alpha(Tx, T^2x) \ge 1$.

Definition 9 (see [25]). Let X be a nonempty set and $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow \mathfrak{R}^+$ be a function. We say that T is triangular α-orbital admissible if

(i) T is α-orbital admissible;

(ii) For all $x, y \in X, \alpha(x, y) \ge 1$ and $\alpha(y, Ty) \ge 1$ imply that $\alpha(x, Ty) \ge 1$.

Lemma 10 (see [25]). *Let X be a nonempty set, $T : X \longrightarrow X$, and $\alpha : X \times X \longrightarrow \mathfrak{R}^+$. Suppose that T is a triangular α-orbital admissible mapping and assume that there exists x_0*

$\in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then, $\alpha(x_n, x_m) \geq 1$ for all m, $n \in \mathbb{N}$ with $n < m$.

Theorem 11 (see [26]). *Let (X, d) be a complete rectangular metric space and $T : X \longrightarrow X$. Suppose that there exist $\psi \in \Psi_1$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$*

$$d(Tx, Ty) \neq 0 \text{ implies } \psi(d(Tx, Ty)) \leq [\psi(R(x, y))]^{\lambda}, \quad (1)$$

where

$$R(x, y) = \max \{d(x, y), d(x, Tx), d(y, Ty)\}. \quad (2)$$

Then, T has a fixed point.

Theorem 12 (see [21]). *Let (X, d) be a complete rectangular metric space, $T : X \longrightarrow X$, and $\alpha : X \times X \longrightarrow \mathfrak{R}^+$. Suppose that the following conditions hold:*

(i) There exist $\psi \in \Psi_2$ and $\lambda \in (0, 1)$ such that for all x, $y \in X$

$$d(Tx, Ty) \neq 0 \text{ implies } \alpha(x, y).\psi(d(Tx, Ty)) \leq [\psi(R(x, y))]^{\lambda}, \quad (3)$$

where

$$R(x, y) = \max \{d(x, y), d(x, Tx), d(y, Ty)\}; \quad (4)$$

(ii) There exists $x_0 \in X$ such that $\alpha(x_0Tx_0) \geq 1$;

(iii) T is a triangular α-orbital admissible mapping;

(iv) T is continuous.

Then, T has a fixed point.

Theorem 13 (see [21]). *Let (X, d) be a complete rectangular metric space and $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow \mathfrak{R}^+$ be a function. Suppose that the following conditions hold:*

(i) There exist $\psi \in \Psi_2$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$

$$d(Tx, Ty) \neq 0 \text{implies} \alpha(x, y).\psi(d(Tx, Ty)) \leq [\psi(R(x, y))]^{\lambda}, \quad (5)$$

where

$$R(x, y) = \max \{d(x, y), d(x, Tx), d(y, Ty)\}; \quad (6)$$

(ii) There exists $x_0 \in X$ such that $\alpha(x_0Tx_0) \geq 1$;

(iii) T is a triangular α-orbital admissible mapping;

(iv) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \longrightarrow x \in X$ as $n \longrightarrow \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N} \cup \{0\}$.

Then, T has a fixed point.

Theorem 14 (see [27]). *Let (X, d) be a complete rectangular metric space and $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow \mathfrak{R}^+$ be a function. Suppose that the following conditions hold:*

(i) There exist $\psi \in \Psi_1$ and $\lambda \in (0, 1)$ such that for all x, $y \in X$

$$d(Tx, Ty) \neq 0 \text{ implies } \alpha(x, y).\psi(d(Tx, Ty)) \leq [\psi(R(x, y))]^{\lambda}, \quad (7)$$

where

$$R(x, y) = \max \left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}\right\}; \quad (8)$$

(ii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2_{x_0}) \geq 1$;

(iii) T is a triangular α-orbital admissible mapping;

(iv) If $\{T^n_{x_0}\}$ is a sequence in X such that $\alpha(T^n_{x_0}, T^{n+1}_{x_0}) \geq 1$ for all n and $x_n \longrightarrow x \in X$ as $n \longrightarrow \infty$, then there exists a subsequence $\{T^{n(k)}_{x_0}\}$ of $\{T^n_{x_0}\}$ such that $\alpha(T^{n(k)}_{x_0}, x) \geq 1$ for all $k \in \mathbb{N} \cup \{0\}$. Then T has a fixed point z in X and $\{T^n_{x_0}\}$ converges to z.

3. Main Results

In this section, we introduce generalized (α, ψ)-contraction mapping in the setting of rectangular b-metric spaces and prove fixed point results for the mappings introduced.

We start this section by giving the definition of generalized (α, ψ)-contraction mapping in the setting of rectangular b-metric space as follows:

Definition 15. Let (X, d) be a rectangular b-metric space with parameter $s \geq 1$. Suppose that $\psi \in \Psi_2$, $\alpha : X \times X \longrightarrow \mathfrak{R}^+$ be a function and $\lambda \in (0, (1/s))$. A self-mapping $T : X \longrightarrow X$ is called generalized (α, ψ)-contraction mapping if it satisfies for all $x, y \in X$ in the following condition:

$$d(Tx, Ty) \neq 0 \text{ implies } \alpha(x, y)\psi\left(s^2 d(Tx, Ty)\right) \leq [\psi(M(x, y))]^{\lambda}, \quad (9)$$

where

$$M(x, y) = \max \left\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx), \frac{d(x, Tx).d(y, Ty)}{1 + d(x, y)}, \frac{d(x, Tx).d(y, Ty)}{1 + d(Tx, Ty)}, \frac{d(x, Tx)[1 + d(y, Tx)]}{1 + d(Tx, Ty)}\right\}. \quad (10)$$

Now, we state and prove the following fixed point theorem.

Theorem 16. *Let (X, d) be a complete rectangular b-metric space, $T : X \longrightarrow X$, and $\alpha : X \times X \longrightarrow \mathfrak{R}^+$.*

Suppose that the following conditions hold:

(i) T is a triangular α-orbital admissible mapping;

(ii) T is generalized (α, ψ)-contraction mapping;

(iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iv) T is continuous.

Then, T has a fixed point in X.

Proof. By (iii), there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. We define an iterative sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$, for $n \in \mathbb{N} \cup \{0\}$. Suppose that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in N$. Since $Tx_{n_0} = x_{n_0+1} = x_{n_0}$, the point x_{n_0} forms a fixed point of T that completes the proof.

From now on, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. By condition (iii), we have $\alpha(x_0, Tx_0) \geq 1$. Using Lemma 10, we obtain that

$$\alpha(x_n, x_{n+1}) \geq 1 \text{for alln} \in \mathbb{N} \cup \{0\}. \tag{11}$$

From (9) to (11), for all $n \in \mathbb{N} \cup \{0\}$, we have the following:

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &= \psi(d(Tx_{n-1}, Tx_n)) \leq \alpha(x_{n-1}, x_n) \\ &\cdot \psi\left(s^2 d(Tx_{n-1}, Tx_n)\right) \leq [\psi(M(x_{n-1}, x_n))]^\lambda \\ &< \psi(M(x_{n-1}, x_n)),\end{aligned} \tag{12}$$

where

$$\begin{aligned}M(x_{n-1}, x_n) &= \max \Bigg\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \\ &\quad d(x_n, Tx_{n-1}), \frac{d(x_{n-1}, Tx_{n-1}).d(x_n, Tx_n)}{1 + d(x_{n-1}, x_n)}, \\ &\quad \frac{d(x_{n-1}, Tx_{n-1}).d(x_n, Tx_n)}{1 + d(Tx_{n-1}, Tx_n)}, \\ &\quad \frac{d(x_{n-1}, Tx_{n-1})[1 + d(x_n, Tx_{n-1})]}{1 + d(Tx_{n-1}, Tx_n)}\Bigg\} \\ &= \max \Bigg\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ &\quad d(x_n, x_n) \frac{d(x_{n-1}, x_n).d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)}, \\ &\quad \frac{d(x_{n-1}, x_n).d(x_n, x_{n+1})}{1 + d(x_n, x_{n+1})}, \frac{d(x_{n-1}, x_n)[1 + d(x_n, x_n)]}{1 + d(x_n, x_{n+1})}\Bigg\} \\ &= \max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.\end{aligned} \tag{13}$$

If $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$, then by (12), we get $\psi(d(x_n, x_{n+1})) \leq [\psi(d(x_n, x_{n+1}))^\lambda < \psi(d(x_n, x_{n+1}))$, which is a contradiction. Hence, $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$. Using (12), we have $\psi(d(x_n, x_{n+1})) < \psi(d(x_{n-1}, x_n))$. Since ψ is nondecreasing, we have $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for all $n \geq 1$.

Hence, $\{d(x_n, x_{n+1})\}$ is decreasing sequence of nonnegative real numbers. Hence, $\{d(x_n, x_{n+1})\}$ converges to a nonnegative real number, say $r \geq 0$. That is

$$\lim_{n \longrightarrow \infty} d(x_n, x_{n+1}) = r \text{ and } d(x_n, x_{n+1}) \geq r. \tag{14}$$

We will prove that $r = 0$. Suppose that $r > 0$. Since ψ is nondecreasing, by using (12) and (14), we obtain that

$$\begin{aligned}1 &< \psi(r) \leq \psi(d(x_{n+1}, x_{n+2})) = \psi(d(Tx_n, Tx_{n+1})) \\ &\leq \alpha(x_n, x_{n+1})\psi\left(s^2 d(Tx_n, Tx_{n+1})\right) \leq [\psi(d(x_n, x_{n+1}))]^\lambda \\ &\leq [\psi(d(x_{n-1}, x_n))]^{\lambda^2} \vdots \leq [\psi(d(x_0, x_1))]^{\lambda^n}.\end{aligned} \tag{15}$$

By taking limit as $n \longrightarrow \infty$ in the above inequality, we get $\psi(r) = 1$, which contradicts to the assumption that $\psi(t) > 1$ for all $t > 0$. Hence, we have $r = 0$, and therefore

$$\lim_{n \longrightarrow \infty} d(x_n, x_{n+1}) = r = 0. \tag{16}$$

Now, we shall prove that $x_n \neq x_m$ for all $n \neq m$ or $x_n \neq x_{n+p}$ for all $n, p \in \mathbb{N}$. Assume on the contrary that there exist $n, p \in \mathbb{N}$ such that $x_n = x_{n+p}$. Since $d(x_n, x_{n+1}) > 0$, for each $n \in \mathbb{N}$. Without loss of generality, we may assume that $p > 1$. Using (9) and (11), we obtain that

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &= \psi\left(d\left(x_{n+p}, x_{n+p+1}\right)\right) = \psi\left(d\left(Tx_{n+p-1}, Tx_{n+p}\right)\right) \\ &\leq \alpha\left(x_{n+p-1}, x_{n+p}\right)\psi\left(s^2 d\left(Tx_{n+p-1}, Tx_{n+p}\right)\right) \\ &\leq \left[\psi\left(M\left(x_{n+p-1}, x_{n+p}\right)\right)\right]^\lambda < \psi\left(M\left(x_{n+p-1}, x_{n+p}\right)\right),\end{aligned} \tag{17}$$

where

$$\begin{aligned}M\left(x_{n+p-1}, x_{n+p}\right) &= \max \Bigg\{d\left(x_{n+p-1}, x_{n+p}\right), d\left(x_{n+p-1}, Tx_{n+p-1}\right), \\ &\quad d\left(x_{n+p}, Tx_{n+p}\right), d\left(x_{n+p}, Tx_{n+p-1}\right), \\ &\quad \frac{d\left(x_{n+p-1}, Tx_{n+p-1}\right).d\left(x_{n+p}, Tx_{n+p}\right)}{1 + d\left(x_{n+p-1}, x_{n+p}\right)}, \\ &\quad \frac{d\left(x_{n+p-1}, Tx_{n+p-1}\right).d\left(x_{n+p}, Tx_{n+p}\right)}{1 + d\left(Tx_{n+p-1}, Tx_{n+p}\right)}, \\ &\quad \frac{d\left(x_{n+p-1}, Tx_{n+p-1}\right)\left[1 + d\left(x_{n+p}, Tx_{n+p-1}\right)\right]}{1 + d\left(Tx_{n+p-1}, Tx_{n+p}\right)}\Bigg\} \\ &= \max \Bigg\{d\left(x_{n+p-1}, x_{n+p}\right), d\left(x_{n+p-1}, x_{n+p}\right), \\ &\quad d\left(x_{n+p}, x_{n+p+1}\right), \frac{d\left(x_{n+p-1}, x_{n+p}\right).d\left(x_{n+p}, x_{n+p+1}\right)}{1 + d\left(x_{n+p-1}, x_{n+p}\right)}, \\ &\quad \frac{d\left(x_{n+p-1}, x_{n+p}\right).d\left(x_{n+p}, x_{n+p+1}\right)}{1 + d\left(x_{n+p}, x_{n+p+1}\right)}, \\ &\quad \frac{d\left(x_{n+p-1}, x_{n+p}\right)\left[1 + d\left(x_{n+p}, x_{n+p}\right)\right]}{1 + d\left(x_{n+p}, x_{n+p+1}\right)}\Bigg\} \\ &= \max \left\{d\left(x_{n+p-1}, x_{n+p}\right), d\left(x_{n+p}, x_{n+p+1}\right)\right\}.\end{aligned} \tag{18}$$

If $M(x_{n+p-1}, x_{n+p}) = d(x_{n+p}, x_{n+p+1})$, then from (17), we obtain that

$$\psi(d(x_n, x_{n+1})) = \psi(d(x_{n+p}, x_{n+p+1})) < \psi(d(x_{n+p}, x_{n+p+1})), \tag{19}$$

which is a contradiction. Hence, $M(x_{n+p-1}, x_{n+p}) = d(x_{n+p-1}, x_{n+p})$.

By (17), we obtain that

$$\psi(d(x_n, x_{n+1})) = \psi(d(x_{n+p}, x_{n+p+1})) < \psi(d(x_{n+p-1}, x_{n+p})). \tag{20}$$

Since ψ is nondecreasing from the above inequality, we get

$$d(x_n, x_{n+1}) < d(x_{n+p-1}, x_{n+p}). \tag{21}$$

By using (9), we have

$$\begin{aligned}\psi(d(x_{n+p-1}, x_{n+p})) &= \psi(d(Tx_{n+p-2}, Tx_{n+p-1})) \\ &\leq \alpha(x_{n+p-2}, x_{n+p-1})\psi(s^2 d(Tx_{n+p-2}, Tx_{n+p-1})) \\ &\leq [\psi(M(x_{n+p-2}, x_{n+p-1}))]^{\lambda} < \psi(M(x_{n+p-2}, x_{n+p-1})),\end{aligned} \tag{22}$$

where

$$\begin{aligned}M(x_{n+p-2}, x_{n+p-1}) &= \max \Big\{ d(x_{n+p-2}, x_{n+p-1}), d(x_{n+p-2}, Tx_{n+p-2}),. \\ &\quad d(x_{n+p-1}, Tx_{n+p-1}), d(x_{n+p-1}, Tx_{n+p-2}), \\ &\quad \frac{d(x_{n+p-2}, Tx_{n+p-2}).d(x_{n+p-1}, Tx_{n+p-1})}{1 + d(x_{n+p-2}, x_{n+p-1})}, \\ &\quad \frac{d(x_{n+p-2}, Tx_{n+p-2}).d(x_{n+p-1}, Tx_{n+p-1})}{1 + d(Tx_{n+p-2}, Tx_{n+p-1})}, \\ &\quad \frac{d(x_{n+p-2}, Tx_{n+p-2})[1 + d(x_{n+p-1}, Tx_{n+p-2})]}{1 + d(Tx_{n+p-2}, Tx_{n+p-1})} \Big\} \\ &= \max \Big\{ d(x_{n+p-2}, x_{n+p-1}), d(x_{n+p-2}, x_{n+p-1}), \\ &\quad d(x_{n+p-1}, x_{n+p}), \frac{d(x_{n+p-2}, x_{n+p-1}).d(x_{n+p-1}, x_{n+p})}{1 + d(x_{n+p-2}, x_{n+p-1})}, \\ &\quad \frac{d(x_{n+p-2}, x_{n+p-1}).d(x_{n+p-1}, x_{n+p})}{1 + d(x_{n+p-1}, x_{n+p})}, \\ &\quad \frac{d(x_{n+p-2}, x_{n+p-1})[1 + d(x_{n+p-1}, x_{n+p-1})]}{1 + d(x_{n+p-1}, x_{n+p})} \Big\} \\ &= \max \{ d(x_{n+p-2}, x_{n+p-1}), d(x_{n+p-1}, x_{n+p}) \}.\end{aligned} \tag{23}$$

If $M(x_{n+p-2}, x_{n+p-1}) = d(x_{n+p-1}, x_{n+p})$, then by (22), we obtain that

$$\psi(d(x_{n+p-1}, x_{n+p})) \leq \psi(d(x_{n+p-1}, x_{n+p}))^{\lambda} < \psi(d(x_{n+p-1}, x_{n+p})), \tag{24}$$

which is a contradiction. Hence,

$$M(x_{n+p-2}, x_{n+p-1}) = d(x_{n+p-2}, x_{n+p-1}). \tag{25}$$

By (22), we have

$$\psi(d(x_{n+p-1}, x_{n+p})) \leq [\psi(d(x_{n+p-2}, x_{n+p-1}))]^{\lambda} < \psi(d(x_{n+p-2}, x_{n+p-1})). \tag{26}$$

Since ψ is nondecreasing, we get

$$d(x_{n+p-1}, x_{n+p}) < d(x_{n+p-2}, x_{n+p-1}). \tag{27}$$

By continuing this process, we obtain the following inequality:

$$d(x_n, x_{n+1}) < d(x_{n+p-1}, x_{n+p}) < d(x_{n+p-2}, x_{n+p-1}) < \cdots < d(x_n, x_{n+1}), \tag{28}$$

which is a contradiction. Hence $x_n \neq x_m$ for all $n \neq m$.

We now prove that $\{d(x_n, x_{n+2})\}$ is bounded. Since $\{d(x_n, x_{n+1})\}$ is bounded, there exists $B > 0$ such that

$$d(x_n, x_{n+1}) \leq B\, forall \mathrm{n} \in \mathbb{N} \cup \{0\}. \tag{29}$$

If $d(x_n, x_{n+2}) > B$ for all $n \in \mathbb{N} \cup \{0\}$, then from

$$\begin{aligned}M(x_{n-1}, x_{n+1}) &= \max \Big\{ d(x_{n-1}, x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1}), \\ &\quad d(x_{n+1}, Tx_{n-1}), \frac{d(x_{n-1}, Tx_{n-1})d(x_{n+1}, Tx_{n+1})}{1 + d(x_{n-1}, x_{n+1})}, \\ &\quad \frac{d(x_{n-1}, Tx_{n-1})d(x_{n+1}, Tx_{n+1})}{1 + d(Tx_{n-1}, Tx_{n+1})}, \\ &\quad \frac{d(x_{n-1}, Tx_{n-1})[1 + d(x_{n+1}, Tx_{n-1})]}{1 + d(Tx_{n-1}, Tx_{n+1})} \Big\} \\ &= \max \Big\{ d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2}), \\ &\quad d(x_{n+1}, x_n), \frac{d(x_{n-1}, x_n)d(x_{n+1}, x_{n+2})}{1 + d(x_{n-1}, x_{n+1})}, \\ &\quad \frac{d(x_{n-1}, x_n)d(x_{n+1}, x_{n+2})}{1 + d(x_n, x_{n+2})}, \frac{d(x_{n-1}, x_n)[1 + d(x_{n+1}, x_n)]}{1 + d(x_n, x_{n+2})} \Big\} \\ &= d(x_{n-1}, x_{n+1}).\end{aligned} \tag{30}$$

and Lemma 10, we obtain that

$$\begin{aligned}\psi(d(x_n, x_{n+2})) &= \psi(d(Tx_{n-1}, Tx_{n+1})) \\ &\leq \alpha(x_{n-1}, x_{n+1})\psi(s^2 d(Tx_{n-1}, Tx_{n+1})) \\ &\leq [\psi(M(x_{n-1}, x_{n+1}))]^{\lambda} = [\psi(d(x_{n-1}, x_{n+1}))]^{\lambda} \\ &< \psi(d(x_{n-1}, x_{n+1})).\end{aligned} \tag{31}$$

From above the inequality, we get for all $n \geq 1$

$$d(x_n, x_{n+2}) < d(x_{n-1}, x_{n+1}). \tag{32}$$

This implies that $\{d(x_n, x_{n+2})\}$ is decreasing. Therefore, $\{d(x_n, x_{n+2})\}$ is bounded.

If $d(x_n, x_{n+2}) \le B$ for some $n \in \mathbb{N} \cup \{0\}$, then from

$$\begin{aligned} M(x_n, x_{n+2}) = \max \Big\{ & d(x_n, x_{n+2}), d(x_n, Tx_n), d(x_{n+2}, Tx_{n+2}), d(x_{n+2}, Tx_n), \frac{d(x_n, Tx_n) d(x_{n+2}, Tx_{n+2})}{1 + d(x_n, x_{n+2})}, \\ & \frac{d(x_n, Tx_n) d(x_{n+2}, Tx_{n+2})}{1 + d(Tx_n, Tx_{n+2})}, \frac{d(x_n, Tx_n)[1 + d(x_{n+2}, Tx_n)]}{1 + d(Tx_n, Tx_{n+2})} \Big\} \\ = \max \Big\{ & d(x_n, x_{n+2}), d(x_n, x_{n+1}), d(x_{n+2}, x_{n+3}), d(x_{n+2}, x_{n+1}), \frac{d(x_n, x_{n+1}) d(x_{n+2}, x_{n+3})}{1 + d(x_n, x_{n+2})}, \\ & \frac{d(x_n, x_{n+1}) d(x_{n+2}, x_{n+3})}{1 + d(x_{n+1}, x_{n+3})}, \frac{d(x_n, x_{n+1})[1 + d(x_{n+2}, x_{n+1})]}{1 + d(x_{n+1}, x_{n+3})} \Big\} \le B \end{aligned} \tag{33}$$

and Lemma 10, we obtain that

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+3})) &= \psi(d(Tx_n, Tx_{n+2})) \le \alpha(x_n, x_{n+2}) \cdot \psi(s^2 d(Tx_n, Tx_{n+2})) \le [\psi(M(x_n, x_{n+2}))]^\lambda \\ &\le [\psi(B)]^\lambda < \psi(B). \end{aligned} \tag{34}$$

Therefore, $d(x_{n+1}, x_{n+3}) < B$. This implies that $\{d(x_n, x_{n+2})\}$ is bounded. We next prove that $\lim_{n \longrightarrow \infty} d(x_n, x_{n+2}) = 0$. Suppose that $\lim_{n \longrightarrow \infty} d(x_n, x_{n+2}) \ne 0$. So there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k \longrightarrow \infty} d(x_{n_k}, x_{n_{k+2}}) = a$ for some $a > 0$. Using (9) and Lemma 10, we have

$$\begin{aligned} \psi\left(d\left(x_{n_k}, x_{n_{k+2}}\right)\right) &= \psi\left(d\left(Tx_{n_{k-1}}, Tx_{n_{k+1}}\right)\right) \le \alpha\left(x_{n_{k-1}}, x_{n_{k+1}}\right) \cdot \psi\left(s^2 d\left(Tx_{n_{k-1}}, Tx_{n_{k+1}}\right)\right) \\ &\le \left[\psi\left(M\left(x_{n_{k-1}}, x_{n_{k+1}}\right)\right)\right]^\lambda < \psi\left(M\left(x_{n_{k-1}}, x_{n_{k+1}}\right)\right), \end{aligned} \tag{35}$$

where

$$\begin{aligned} M\left(x_{n_{k-1}}, x_{n_{k+1}}\right) = \max \Big\{ & d\left(x_{n_{k-1}}, x_{n_{k+1}}\right), d\left(x_{n_{k-1}}, Tx_{n_{k-1}}\right), d\left(x_{n_{k+1}}, Tx_{n_{k+1}}\right), d\left(x_{n_{k+1}}, Tx_{n_{k-1}}\right), \frac{d\left(x_{n_{k-1}}, Tx_{n_{k-1}}\right) d\left(x_{n_{k+1}}, Tx_{n_{k+1}}\right)}{1 + d\left(x_{n_{k-1}}, x_{n_{k+1}}\right)}, \\ & \frac{d\left(x_{n_{k-1}}, Tx_{n_{k-1}}\right) d\left(x_{n_{k+1}}, Tx_{n_{k+1}}\right)}{1 + d\left(Tx_{n_{k-1}}, Tx_{n_{k+1}}\right)}, \frac{d\left(x_{n_{k-1}}, Tx_{n_{k-1}}\right) \left[1 + d\left(x_{n_{k+1}}, Tx_{n_{k-1}}\right)\right]}{1 + d\left(Tx_{n_{k-1}}, Tx_{n_{k+1}}\right)} \Big\} \\ = \max \Big\{ & d\left(x_{n_{k-1}}, x_{n_{k+1}}\right), d\left(x_{n_{k-1}}, x_{n_k}\right), d\left(x_{n_{k+1}}, x_{n_{k+2}}\right), d\left(x_{n_{k+1}}, x_{n_k}\right), \frac{d\left(x_{n_{k-1}}, x_{n_k}\right) d\left(x_{n_{k+1}}, x_{n_{k+2}}\right)}{1 + d\left(x_{n_{k-1}}, x_{n_{k+1}}\right)}, \\ & \frac{d\left(x_{n_{k-1}}, x_{n_k}\right) d\left(x_{n_{k+1}}, x_{n_{k+2}}\right)}{1 + d\left(x_{n_k}, x_{n_{k+2}}\right)}, \frac{d\left(x_{n_{k-1}}, x_{n_k}\right) \left[1 + d\left(x_{n_{k+1}}, x_{n_k}\right)\right]}{1 + d\left(x_{n_k}, x_{n_{k+2}}\right)} \Big\}. \end{aligned} \tag{36}$$

Letting $k \longrightarrow \infty$ in (35), we obtain that

$$\begin{aligned} \psi(a) &= \lim_{k \longrightarrow \infty} \psi\left(d\left(x_{n_k}, x_{n_{k+2}}\right)\right) \le \lim_{k \longrightarrow \infty} \left[\psi\left(M\left(x_{n_{k-1}}, x_{n_{k+1}}\right)\right)\right]^\lambda \\ &= [\psi(a)]^\lambda < \psi(a), \end{aligned} \tag{37}$$

which is a contradiction. Therefore,

$$\lim_{n \longrightarrow \infty} d(x_n, x_{n+2}) = 0. \tag{38}$$

We now prove that $\{x_n\}$ is a b-rectangular b-Cauchy sequence in X. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then, there exists $\varepsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$, such that n_k is the smallest index with $n_k > m_k > k$ for which

$$d\left(x_{m_k}, x_{n_k}\right) \ge \varepsilon. \tag{39}$$

This implies that

$$d\left(x_{m_k}, x_{n_{k-1}}\right) < \varepsilon. \tag{40}$$

By applying b-rectangular inequality, using (39) and (40), we get that

$$\begin{aligned} \varepsilon \le d\left(x_{m_k}, x_{n_k}\right) &\le sd\left(x_{m_k}, x_{n_{k-1}}\right) + sd\left(x_{n_{k-1}}, x_{n_{k-2}}\right) \\ &+ sd\left(x_{k-2}, x_{n_k}\right) < s\varepsilon + sd\left(x_{n_{k-1}}, x_{n_{k-2}}\right) + sd\left(x_{k-2}, x_{n_k}\right). \end{aligned} \tag{41}$$

Letting $k \longrightarrow \infty$ in above inequality, using (16) and (38), we get that

$$\varepsilon \le \liminf_{k \longrightarrow \infty} d\left(x_{m_k}, x_{n_k}\right) \le \limsup_{k \longrightarrow \infty} d\left(x_{m_k}, x_{n_k}\right) \le s\varepsilon. \tag{42}$$

Also,

$$\varepsilon \le d\left(x_{m_k}, x_{n_k}\right) \le sd\left(x_{m_k}, x_{m_{k+1}}\right) + sd\left(x_{m_{k+1}}, x_{n_{k+1}}\right) + sd\left(x_{n_{k+1}}, x_{n_k}\right). \tag{43}$$

Letting $k \longrightarrow \infty$ in above inequality, using (16) and Lemma 6, we obtain

$$\frac{\varepsilon}{s} \le \liminf_{k \longrightarrow \infty} d\left(x_{m_{k+1}}, x_{n_{k+1}}\right) \le \limsup_{k \longrightarrow \infty} d\left(x_{m_{k+1}}, x_{n_{k+1}}\right) \le s^2 \varepsilon. \tag{44}$$

For each $k \in \mathbb{N}$, we have

$$\begin{aligned} M\left(x_{m_k}, x_{n_k}\right) = \max \Bigg\{ & d\left(x_{m_k}, x_{n_k}\right), d\left(x_{m_k}, Tx_{m_k}\right), d\left(x_{n_k}, Tx_{n_k}\right), \\ & d\left(x_{n_k}, Tx_{m_k}\right), \frac{d\left(x_{m_k}, Tx_{m_k}\right) d\left(x_{n_k}, Tx_{n_k}\right)}{1 + d\left(x_{m_k}, x_{n_k}\right)}, \\ & \frac{d\left(x_{m_k}, Tx_{m_k}\right) d\left(x_{n_k}, Tx_{n_k}\right)}{1 + d\left(Tx_{m_k}, Tx_{n_k}\right)}, \\ & \frac{d\left(x_{m_k}, Tx_{m_k}\right)\left[1 + d\left(x_{n_k}, Tx_{m_k}\right)\right]}{1 + d\left(Tx_{m_k}, Tx_{n_k}\right)} \Bigg\} \\ = \max \Bigg\{ & d\left(x_{m_k}, x_{n_k}\right), d\left(x_{m_k}, x_{m_{k+1}}\right), d\left(x_{n_k}, x_{n_{k+1}}\right), \\ & d\left(x_{n_k}, x_{m_{k+1}}\right), \frac{d\left(x_{m_k}, x_{n_k}\right) d\left(x_{n_k}, x_{n_{k+1}}\right)}{1 + d\left(x_{m_k}, x_{n_k}\right)}, \\ & \frac{d\left(x_{m_k}, x_{m_{k+1}}\right) d\left(x_{n_k}, x_{n_{k+1}}\right)}{1 + d\left(x_{m_{k+1}}, x_{n_{k+1}}\right)}, \\ & \frac{d\left(x_{m_k}, x_{m_{k+1}}\right)\left[1 + d\left(x_{n_k}, x_{m_{k+1}}\right)\right]}{1 + d\left(x_{m_{k+1}}, x_{n_{k+1}}\right)} \Bigg\}. \end{aligned} \tag{45}$$

By using (16) and (42), we obtain that

$$\limsup_{k \longrightarrow \infty} M\left(x_{m_k}, x_{n_k}\right) \le s\varepsilon. \tag{46}$$

By (42) and (46), there exists a positive integer k_0 such that $d(x_{n_{k+1}}, x_{m_{k+1}}) > 0$ and $M(x_{n_k}, x_{m_k}) > 0$, for all $k \ge k_0$.

By Lemma 10 and using (9), for all $n_k > m_k > k > k_0$, we get that

$$\begin{aligned} \psi\left(s^2 d\left(x_{n_{k+1}}, x_{m_{k+1}}\right)\right) &= \psi\left(s^2 d\left(Tx_{n_k}, Tx_{m_k}\right)\right) \le \alpha\left(x_{n_k}, x_{m_k}\right) \\ &\quad \cdot \psi\left(s^2 d\left(Tx_{n_k}, Tx_{m_k}\right)\right) \\ &\le \left[\psi\left(M\left(x_{n_k}, x_{m_k}\right)\right)\right]^\lambda < \psi\left(M\left(x_{n_k}, x_{m_k}\right)\right). \end{aligned} \tag{47}$$

Letting $k \longrightarrow \infty$ in the above inequality, by (42), (44), and (46) and the continuity of ψ, we obtain that

$$\begin{aligned} \psi(s\varepsilon) &\le \limsup_{k \longrightarrow \infty} \psi\left(s^2 d\left(x_{m_{k+1}}, x_{n_{k+1}}\right)\right) \le \left[\limsup_{k \longrightarrow \infty} \psi\left(d\left(x_{n_k}, x_{m_k}\right)\right)\right]^\lambda \\ &\le [\psi(s\varepsilon)]^\lambda < \psi(s\varepsilon), \end{aligned} \tag{48}$$

which is a contradiction. Therefore, $\{x_n\}$ is a rectangular b-Cauchy sequence in X. Since X is complete rectangular b-metric space, it follows that $\{x_n\}$ converges to some $x \in X$. Since T is continuous, we have

$$x = \lim_{n \longrightarrow \infty} x_{n+1} = \lim_{n \longrightarrow \infty} Tx_n = Tx. \tag{49}$$

Therefore, x is a fixed point of T. □

By removing the continuity assumption in Theorem **16**, we get the following fixed point result.

Theorem 17. *Let (X, d) be a complete rectangular b-metric spaces, $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow \mathfrak{R}^+$.*

Suppose that the following conditions hold:

(i) T is generalized (α, ψ)-contraction mapping;

(ii) T is a triangular α-orbital admissible mapping;

(iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;

(iv) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \longrightarrow x \in X$ as $n \longrightarrow \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \ge 1$ for all $k \in \mathbb{N}$.

Then, T has a fixed point.

Proof. As in the proof of Theorem 16, we can construct a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$, for all $n \in \mathbb{N} \cup \{0\}$ and $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$. The sequence $\{x_n\}$ is rectangular b-Cauchy, and since X is complete, there exists $x \in X$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$. By (iv), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \ge 1$ for all $k \in \mathbb{N} \cup \{0\}$. We can suppose that $x_{n_k} \ne Tx$ for all $k \in \mathbb{N} \cup \{0\}$. By (9), we obtain that

$$\begin{aligned} \psi\left(sd\left(x_{n_{k+1}}, Tx\right)\right) &= \psi\left(sd\left(Tx_{n_k}, Tx\right)\right) \le \alpha\left(x_{n_k}, x\right) \\ &\quad \cdot \psi\left(s^2 d\left(Tx_{n_k}, Tx\right)\right) \le \left[\psi\left(M\left(x_{n_k}, x\right)\right)\right]^\lambda \\ &< \psi\left(M\left(x_{n_k}, x\right)\right), \end{aligned} \tag{50}$$

where

$$\begin{aligned} M\left(x_{n_k}, x\right) = \max \Bigg\{ & d\left(x_{n_k}, x\right), d\left(x_{n_k}, Tx_{n_k}\right), d(x, Tx), d\left(x, Tx_{n_k}\right), \\ & \frac{d\left(x_{n_k}, Tx_{n_k}\right) d(x, Tx)}{1 + d\left(x_{n_k}, x\right)}, \frac{d\left(x_{n_k}, Tx_{n_k}\right) d(x, Tx)}{1 + d\left(Tx_{n_k}, Tx\right)}, \\ & \frac{d\left(x_{n_k}, Tx_{n_k}\right)\left[1 + d\left(x, Tx_{n_k}\right)\right]}{1 + d\left(Tx_{n_k}, Tx\right)} \Bigg\} \\ = \max \Bigg\{ & d\left(x_{n_k}, x\right), d\left(x_{n_k}, x_{n_{k+1}}\right), d(x, Tx), d\left(x, x_{n_{k+1}}\right), \\ & \frac{d\left(x_{n_k}, x_{n_{k+1}}\right) d(x, Tx)}{1 + d\left(x_{n_k}, x\right)}, \frac{d\left(x_{n_k}, x_{n_{k+1}}\right) d(x, Tx)}{1 + d\left(x_{n_{k+1}}, Tx\right)}, \\ & \frac{d\left(x_{n_k}, x_{n_{k+1}}\right)\left[1 + d\left(x, x_{n_{k+1}}\right)\right]}{1 + d\left(x_{n_{k+1}}, Tx\right)} \Bigg\}. \end{aligned} \tag{51}$$

Taking limit as $k \longrightarrow \infty$

$$\lim_{k \longrightarrow \infty} M\left(x_{n_k}, x\right) = d(x, Tx). \tag{52}$$

Now, we prove that $x = Tx$. Suppose that $x \ne Tx$. Therefore,

$$d(x, Tx) \le sd(x, x_{n_k}) + sd(x_{n_k}, x_{n_{k+1}}) + sd(x_{n_{k+1}}, Tx). \quad (53)$$

It follows that

$$d(x, Tx) \le \lim_{k \longrightarrow \infty} sd(x_{n_{k+1}}, Tx). \quad (54)$$

By using (50), (52), and (54) and ψ, we obtain that

$$\begin{aligned}\psi(d(x, Tx)) &\le \lim_{k \longrightarrow \infty} \psi(sd(x_{n_{k+1}}, Tx)) = \lim_{k \longrightarrow \infty} \psi(sd(Tx_{n_k}, Tx)) \\ &\le \lim_{k \longrightarrow \infty} \alpha(x_{n_k}, x)\psi(s^2 d(Tx_{n_k}, Tx)) \\ &\le \lim_{k \longrightarrow \infty} [\psi(M(x_{n_k}, x))]^\lambda = \left[\psi \lim_{k \longrightarrow \infty} (M(x_{n_k}, x))\right]^\lambda \\ &= [\psi(d(x, Tx))]^\lambda < \psi(d(x, Tx)),\end{aligned} \quad (55)$$

which is a contradiction. Thus, $x = Tx$, and hence, x is a fixed point of T. □

Now, we prove the uniqueness of fixed point. For this, we need the following additional condition.

(U): For all $x, y \in Fix(T)$, we have $\alpha(x, y) \ge 1$, where $Fix(T)$ denotes the set of all fixed points of T.

Theorem 18. *Adding condition (**U**) to the hypothesis of Theorem 16 (or Theorem 17), we obtain the uniqueness of fixed point of T.*

Proof. From the proof of Theorem 16 (or Theorem 17), $Fix(T) \ne \varnothing$. We argue by contradiction that there exist z, w be in $Fix(T)$ such that $z = Tz$ and $w = Tw$ with $z \ne w$. By condition (**U**), we have $\alpha(z, w) \ge 1$. So, by (9), we get

$$\begin{aligned}\psi(d(z, w)) = \psi(d(Tz, Tw)) &\le \alpha(z, w)\psi(s^2 d(Tz, Tw)) \\ &\le [\psi(M(z, w))]^\lambda < \psi(M(z, w)),\end{aligned} \quad (56)$$

where

$$\begin{aligned}M(z, w) = \max \Big\{& d(z, w), d(z, Tz), d(w, Tw), d(w, Tz), \\ & \frac{d(z, Tz)d(w, Tw)}{1 + d(z, w)}, \frac{d(z, Tz)d(w, Tw)}{1 + d(Tz, Tw)}, \\ & \frac{d(z, Tz)[1 + d(w, Tz)]}{1 + d(Tz, Tw)}\Big\} = \max \{d(z, w), d(z, z), \\ & d(w, w), d(w, z), \frac{d(z, z)d(w, w)}{1 + d(z, w)}, \frac{d(z, z)d(w, w)}{1 + d(z, w)}, \\ & \frac{d(z, z)[1 + d(w, z)]}{1 + d(z, w)}\Big\} = \max \{d(z, w), 0\} = d(z, w).\end{aligned} \quad (57)$$

From (56), we have

$$\psi(d(z, w)) \le [\psi(d(z, w))]^\lambda < \psi(d(z, w)), \quad (58)$$

which is a contradiction. Therefore, $z = w$. □

The following are corollaries to our main results.

Corollary 19. *Let (X, d) be a complete rectangular b-metric space, $T : X \longrightarrow X$, $\psi \in \Psi_2$, $\alpha : X \times X \longrightarrow \mathfrak{R}^+$, and $\lambda \in (0, 1)$. Suppose that the following conditions hold:*

(i) $d(Tx, Ty) \ne 0$ implies $\alpha(x, y)\psi(d(Tx, Ty)) \le [\psi(M(x, y))]^\lambda$,

where

$$\begin{aligned}M(x, y) = \max \Big\{& d(x, y), d(x, Tx), d(y, Ty), d(y, Tx), \\ & \frac{d(x, Tx).d(y, Ty)}{1 + d(x, y)}, \frac{d(x, Tx).d(y, Ty)}{1 + d(Tx, Ty)}, \\ & \frac{d(x, Tx)[1 + d(y, Tx)]}{1 + d(Tx, Ty)}\Big\};\end{aligned} \quad (59)$$

(ii) T is a triangular α-orbital admissible mapping;

(iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;

(iv) For every pair x and y of fixed points of T, $\alpha(x, y) \ge 1$;

(v) Either T is continuous or $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \longrightarrow x \in X$ as $n \longrightarrow \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \ge 1$ for all $k \in \mathbb{N}$

Then, T has a unique fixed point.

proof The result follows by taking $s = 1$ in Theorem 16 (or Theorem **17**).

Remark 20. By taking $M(x, y) = \max \{d(x, y), d(x, Tx), d(y, Ty)\}$ in Corollary 19, we get Thoerem 2.1 (or Theorem 2.2) of Baiya and Kaewcharoen [21]. Thus, this work generalizes the work of Baiya and Kaewcharoen [21].

Corollary 21. *Let (X, d) be a complete rectangular b metric space $T : X \longrightarrow X$, $\psi \in \Psi_2$, and $\lambda \in (0, 1)$. Suppose that the following condition hold:*

$$d(Tx, Ty) \ne 0 \text{ implies } \psi(d(Tx, Ty)) \le [\psi(M(x, y))]^\lambda, \quad (60)$$

where

$$M(x, y) = \max \{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\}. \quad (61)$$

Then, T has a unique fixed point.

proof. The result follows by taking $M(x,y)=\max\{d(x,y), d(x,Tx), d(y,Ty), d(y,Tx)\}$, $\alpha(x,y)=1$ for all $x,y\in X$ and $s=1$ in Theorem 16 (or Theorem **17**).

Corollary 22. *Let (X,d) be a complete rectangular b-metric space, $T:X\longrightarrow X$, $\psi\in\Psi_2$ and $\lambda\in(0,(1/s))$. Suppose that the following condition hold:*

$$d(Tx,Ty)\neq 0 \text{ implies } \psi\left(s^2 d(Tx,Ty)\right)\leq[\psi(M(x,y))]^{\lambda}, \quad (62)$$

where

$$M(x,y)=\max\left\{d(x,y), d(x,Tx), d(y,Ty), d(y,Tx), \frac{d(x,Tx).d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx).d(y,Ty)}{1+d(Tx,Ty)}, \frac{d(x,Tx)[1+d(y,Tx)]}{1+d(Tx,Ty)}\right\}. \quad (63)$$

Then, T has a unique fixed point.

proof The result follows by taking $\alpha(x,y)=1$ for all $x,y\in X$ in Theorem 16 (or Theorem 17).

Remark 23. By taking $M(x,y)=\max\{d(x,y), d(x,Tx), d(y,Ty), d(y,Tx)\}$ and $\lambda\in(0,1)$ in Corollary 22, we get Theorem 11 of Kari et al. [22].

Corollary 24. *Let (X,d) be a complete rectangular b-metric space, $T:X\longrightarrow X$, $\psi\in\Psi_2$, $\alpha:X\times X\longrightarrow\mathfrak{R}^+$ and $\lambda\in(0,1)$. Suppose that the following conditions hold:*

(i) $d(Tx,Ty)\neq 0$ implies $\alpha(x,y)\psi(d(Tx,Ty))\leq[\psi(d(x,y))]^{\lambda}$;

(ii) T is a triangular α-orbital admissible mapping;

(iii) There exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$;

(iv) For every pair x and y of fixed points of T, $\alpha(x,y)\geq 1$;

(v) Either T is continuous or $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$ and $x_n\longrightarrow x\in X$ as $n\longrightarrow\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x)\geq 1$ for all $k\in\mathbb{N}$.

Then, T has a unique fixed point.

proof. The result follows by taking $s=1$ and $M(x,y)=d(x,y)$ in Theorem 16 (or Theorem 17).

We now present examples for supporting our main results.

Example 1. Let X be a finite set defined as $X=\{1,2,3,4\}$.

Defined $d:X\times X\longrightarrow\mathfrak{R}^+$ as
$d(x,x)=0$ for all $x\in X$
$d(1,2)=d(2,1)=20$,
$d(2,3)=d(3,2)=d(1,3)=d(3,1)=2$,
$d(1,4)=d(4,1)=d(2,4)=d(4,2)=d(3,4)=d(4,3)=4$.

Clearly, (X,d) is complete rectangular b-metric spaces with $s=2$. But (X,d) is neither a metric nor a rectangular metric space because

$$\begin{aligned}20=d(1,2)&\geq d(1,3)+d(3,2)=2+2=4,\\ 20=d(1,2)&\geq d(1,3)+d(3,4)+d(4,2)=2+4+4=10.\end{aligned} \quad (64)$$

Let $T:X\longrightarrow X$ be the mapping defined by

$$T(x)=\begin{cases}2 & \text{if } x\neq 4\\ 3 & \text{if } x=4.\end{cases} \quad (65)$$

Define $\alpha:X\times X\longrightarrow[0,\infty)$, $\psi:(0,\infty)\longrightarrow(1,\infty)$ by

$$\alpha(x,y)=\begin{cases}1 & \text{if } x,y\in X\setminus\{4\}\\ \dfrac{1}{1200} & \text{otherwise.}\end{cases}; \quad (66)$$

and $\psi(t)=e^t$.

We next illustrate that all conditions in Theorem 16 and Theorem 17 hold. Taking $x_0=1$, we have $\alpha(x_0,Tx_0)=\alpha(1,T1)=\alpha(1,2)=1\geq 1$. We next prove that T is an α-orbital admissible.

Let $x\in X$ such that $\alpha(x,Tx)\geq 1$. Therefore, $x, Tx\in X\setminus\{4\}$ and then $x\in\{1,2,3\}$. By the definition of α, we obtain that

$$\begin{aligned}\alpha\left(T1,T^2 1\right)&=\alpha(2,2)=1\geq 1,\\ \alpha\left(T2,T^2 2\right)&=\alpha(2,2)=1\geq 1,\\ \alpha\left(T3,T^2 3\right)&=\alpha(2,2)=1\geq 1.\end{aligned} \quad (67)$$

It follows that T is an α-orbital admissible. Let $x,y\in X$ such that $\alpha(x,y)\geq 1$ and $\alpha(y,Ty)\geq 1$. By the definition of α, we have $x,y,Ty\in X\setminus\{4\}$. This yields
$\alpha(1,2)\geq 1$ and $\alpha(2,T2)\geq 1$ implies $\alpha(1,T2)\geq 1$,
$\alpha(1,3)\geq 1$ and $\alpha(3,T3)\geq 1$ implies $\alpha(1,T3)\geq 1$,
$\alpha(2,3)\geq 1$ and $\alpha(3,T3)\geq 1$ implies $\alpha(2,T3)\geq 1$,
$\alpha(2,1)\geq 1$ and $\alpha(1,T1)\geq 1$ implies $\alpha(2,T1)\geq 1$,
$\alpha(3,1)\geq 1$ and $\alpha(1,T1)\geq 1$ implies $\alpha(3,T1)\geq 1$,
$\alpha(3,2)\geq 1$ and $\alpha(2,T2)\geq 1$ implies $\alpha(3,T2)\geq 1$.

This implies that T is triangular α-orbital admissible. Let $\{x_n\}$ be a sequence such that $\alpha(x_n,x_{n+1})\geq 1$ for all $n\in N$ and $x_n\longrightarrow x$ as $n\longrightarrow\infty$. By the definition of α, for each $n\in N$, we get that $x_n\in X\setminus\{4\}=\{1,2,3\}$. We obtain that $x\in\{1,2,3\}$. Thus, we have $\alpha(x_n,x)\geq 1$ for each $n\in N$. We next prove that T is generalized (α,ψ)-contraction mapping. Let $x,y\in X$ be such that $d(Tx,Ty)\neq 0$. So we consider the following cases: $x\in\{1,2,3\}$ and $y=4$ or $y\in\{1,2,3\}$ and $x=4$. Since $d(x,y)=d(y,x)$, we divide the proof into three cases as follows:

Case 1 For $x = 1$ and $y = 4$, we have

$$\begin{aligned} M(1,4) &= \max \Big\{ d(1,4), d(1,T1), d(4,T4), d(4,T1), \frac{d(1,T1)d(4,T4)}{1+d(1,4)}, \frac{d(1,T1)d(4,T4)}{1+d(T1,T4)}, \\ &\quad \frac{d(1,T1)[1+d(4,T1)]}{1+d(T1,T4)} \Big\} \\ &= \max \left\{ 4, 20, 4, 16, \frac{80}{3}, \frac{100}{3} \right\} = \frac{100}{3}. \end{aligned} \tag{68}$$

This implies that

$$\begin{aligned} \alpha(1,4)\psi\left(s^2 d(T1,T4)\right) &= \frac{1}{1200}\psi(4d(2,3)) = \frac{1}{1200}e^8 \le e^{100/9} \\ &= \left[\psi\left(\frac{100}{3}\right)\right]^{1/3} = [\psi(M(1,4))]^{1/3}. \end{aligned} \tag{69}$$

Case 2 For $x = 2$ and $y = 4$, we have

$$\begin{aligned} M(2,4) &= \max \Big\{ d(2,4), d(2,T2), d(4,T4), d(4,T2), \frac{d(2,T2)d(4,T4)}{1+d(2,4)}, \frac{d(2,T2)d(4,T4)}{1+d(T2,T4)}, \\ &\quad \frac{d(2,T2)[1+d(4,T2)]}{1+d(T2,T4)} \Big\} = \max\{4, 0\} = 4. \end{aligned} \tag{70}$$

This implies that

$$\begin{aligned} \alpha(2,4)\psi\left(s^2 d(T2,T4)\right) &= \frac{1}{1200}\psi(4d(2,3)) = \frac{1}{1200}e^8 \\ &\le [\psi(4)]^{1/3} = [\psi(M(2,4))]^{1/3}. \end{aligned} \tag{71}$$

Case 3: For $x = 3$ and $y = 4$, we have

$$\begin{aligned} M(3,4) &= \max \Big\{ d(3,4), d(3,T3), d(4,T4), d(4,T3), \frac{d(3,T3)d(4,T4)}{1+d(3,4)}, \frac{d(3,T3)d(4,T4)}{1+d(T3,T4)}, \\ &\quad \frac{d(3,T3)[1+d(4,T3)]}{1+d(T3,T4)} \Big\} \\ &= \max \left\{ 4, 2, \frac{8}{5}, \frac{8}{3}, \frac{10}{3} \right\} = 4. \end{aligned} \tag{72}$$

This implies that

$$\begin{aligned} \alpha(3,4)\psi\left(s^2 d(T3,T4)\right) &= \frac{1}{1200}\psi(4d(2,3)) = \frac{1}{1200}e^8 \le e^{4/3} \\ &= [\psi(4)]^{1/3} = [\psi(M(3,4))]^{1/3}. \end{aligned} \tag{73}$$

Hence, all assumptions in Theorem 16 and Theorem 17 are satisfied, and thus, T has a fixed point which is $x = 2$. Finally let $x, y \in Fix(T)$. Clearly $x = y = 2$, therefore, by the definition of α, we have $\alpha(x,y) = \alpha(2,2) = 1 \ge 1$. So, all assumptions in Theorem 18 are satisfied, and thus, T has a unique fixed point which is $x = 2$

Example 2. Let $X = A \cup [1,2]$, where $A = \{0, 1/2, 1/3, 1/4, 1/5, 1/6\}$.

Define $d : X \times X \longrightarrow R^+$ by.
$d(x,y) = 0$ if and *only* if $x = y$
$d(x,y) = d(y,x)$ for *all* $x, y \in X$ *with*
$d(0,1/2) = d(1/3,1/4) = d(1/5,1/6) = 0.09$,
$d(0,1/3) = d(1/2,1/5) = d(1/4,1/5) = 0.04$,
$d(0,1/4) = d(1/2,1/3) = d(1/4,1/6) = 0.16$,
$d(0,1/5) = d(1/2,1/6) = d(1/3,1/6) = 0.25$,
$d(0,1/6) = d(1/2,1/4) = d(1/3,1/5) = 0.36$,
$d(x,y) = |x-y|^2$.

Then, (X,d) is a complete rectangular b-metric space with parameter $s = 3$. But, d is neither metric nor rectangular metric because

$$\begin{aligned} d\left(0, \frac{1}{6}\right) &= 0.36 > 0.29 = d\left(0, \frac{1}{3}\right) + d\left(\frac{1}{3}, \frac{1}{6}\right), \\ d\left(\frac{1}{3}, \frac{1}{5}\right) &= 0.36 > 0.34 = d\left(\frac{1}{3}, \frac{1}{4}\right) + d\left(\frac{1}{4}, \frac{1}{6}\right) + d\left(\frac{1}{6}, \frac{1}{5}\right). \end{aligned} \tag{74}$$

Define mapping $T : X \longrightarrow X$ by

$$T(x) = \begin{cases} \frac{1}{6} & if x \in [1,2] \\ \frac{1}{3} & if x \in A. \end{cases} \tag{75}$$

Define $\alpha : X \times X \longrightarrow [0,\infty)$, $\psi : (0,\infty) \longrightarrow (1,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & if x, y \in A \\ \frac{1}{2} & otherwise. \end{cases}; \tag{76}$$

and $\psi(t) = e^{\sqrt{t}}$ with $\lambda = 1/5 \in (0, 1/3)$.

We next illustrate that all conditions in Theorem 17 hold.

Clearly, T is a triangular α-orbital admissible mapping. Taking $x_0 = 1/2$, we have $\alpha(x_0, Tx_0) = \alpha(1/2, T1/2) = \alpha(1/2, 1/3) = 1 \ge 1$. Let $\{x_n\}$ be a sequence such that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in N$ and $x_n \longrightarrow x$ as $n \longrightarrow \infty$. By the definition of α, for each $n \in N$, we get that $x_n \in A = \{0, 1/2, 1/3, 1/4, 1/5, 1/6\}$. We obtain that $x \in \{0, 1/2, 1/3, 1/4, 1/5, 1/6\}$. Thus, we have $\alpha(x_n, x) \ge 1$ for each $n \in N$. We next prove T is generalized (α, ψ)-contraction mapping. Let $x, y \in X$ be such that $d(Tx, Ty) \ne 0$. So, we consider only the following case: $x \in A$ and $y \in [1,2]$ or $y \in A$ and $x \in [1,2]$. Since $d(x,y) = d(y,x)$, we have

$$\alpha(x,y)\psi(s^2d(Tx,Ty)) = \frac{1}{2}\psi(9d(Tx,Ty)) = \frac{1}{2}e^{1/2} \le e^{1/6} = [\psi(d(x,Tx))]^{1/5} = [\psi(M(x,y))]^{1/5}. \quad (77)$$

Hence, all assumptions in Theorem 17 are satisfied, and thus, T has a fixed point which is $x = 1/3$. Finally let $x, y \in F ix(T)$. Clearly $x = y = 1/3$; therefore, by the definition of α, we have $\alpha(x,y) = \alpha(1/3, 1/3) = 1 \ge 1$. So, all assumptions in Theorem 18 are satisfied, and thus, T has a unique fixed point which is $x = 1/3$.

3.1. Application to Integral Equation. In this section, we use Corollary 22 to show that there is a solution to the following integral equation:

$$x(t) = g(t) + \int_0^1 L(t,r,x(r))dr, \quad (78)$$

where $L : [0,1] \times [0,1] \times R \longrightarrow R^+$ and $g : [0,1] \longrightarrow R$ are continuous function. Let $X = C([0,1], R)$ be the set of all continuous functions defined on $[0,1]$. Define $d : X \times X \longrightarrow R^+$ by

$$d(x,y) = \max_{t\in[0,1]} (|x(t) - y(t)|)^2, \quad (79)$$

which for all $x, y \in X$, then (X, d) is a complete rectangular b-metric space with parameter $s = 2$. Now, we prove the following result.

Theorem 25. *Suppose the following hypothesis hold:*

(i) *there is continuous function* $\beta : [0,1] \times [0,1] \longrightarrow R_+$ *such that*

$$L(t,r,x(r)) - L(t,r,y(r))| \le \frac{1}{4}\beta(t,r)|x(r) - y(r)|; \quad (80)$$

(ii) *There is* $\lambda = 1/3 \in (0, 1/2)$ *such that*

$$\psi\left(\left(\int_0^1 \beta(t,r)dr\right)^2 d(x(t),y(t))\right) \le [\psi(d(x(t),y(t)))]^{1/3}. \quad (81)$$

Then, the integral Equation (78) has a unique solution in X.

Proof. For $x \in X$ and $t \in [0,1]$, define the mapping $T : X \longrightarrow X$ by $Tx(t) = g(t) + \int_0^1 L(t,r,x(r))dr.$.

For $x, y \in X$, from condition (i) and (ii), for all $t \in [0,1]$, we have

$$\begin{aligned}
\psi(s^2 d(Tx(t), Ty(t))) &= \psi(4(|Tx(t) - Ty(t)|)^2) \\
&= \psi\left(4\left(|g(t) + \int_0^1 L(t,r,x(r))dr - \left[g(t) + \int_0^1 L(t,r,y(r))dr|\right]\right)^2\right) \\
&= \psi\left(4\left(|\int_0^1 (L(t,r,x(r)) - L(t,r,y(r)))dr|\right)^2\right) \\
&\le \psi\left(4\left(\int_0^1 |L(t,r,x(r)) - L(t,r,y(r))| dr\right)^2\right) \\
&\le \psi\left(\left(\int_0^1 \beta(t,r)|x(r) - y(r)|dr\right)^2\right) \\
&= \psi\left(\left(\int_0^1 \beta(t,r)(|x(r) - y(r)|^2)^{1/2} dr\right)^2\right) \\
&\le \psi\left(\left(\int_0^1 \beta(t,r)(d(x(t),y(t)))^{1/2} dr\right)^2\right) \\
&= \psi d(x(t),y(t))\left(\int_0^1 \beta(t,r)dr\right)^2 \le [\psi(d(x(t),y(t)))]^{1/3} \\
&\le [\psi(M(x(t),y(t)))]^{1/3}.
\end{aligned} \quad (82)$$

Therefore, all conditions of Corollary 22 are satisfied and as a result the mapping T has a unique fixed point in X, which is a solution of the integral equation in (78). □

4. Conclusion

In 2020, Kari et al. [22] introduced the notion of $\theta - \phi$ -contraction in b-rectangular metric spaces and study the existence and uniqueness of fixed point for the mappings introduced. In 2019, Baiya and Kaewcharoen [21] established fixed point theorems for generalized contractions in complete rectangular metric spaces and proved the existence of fixed points. Inspired and motivated by the works of Kari et al. [22] and Baiya and Kaewcharoen [21] in this paper, we introduce generalized (α, ψ)-contraction mappings in rectangular b-metric spaces and prove the existence and uniqueness of fixed point for the mappings introduced. Our results extend and generalize related fixed point results in the existing literature. We have also supported the main result of this paper by applicable examples. Furthermore, we determined a solution to an integral equation by applying our obtained results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally. All authors read and approved the final manuscript.

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