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Research Article

A Positive Answer on Nirenberg's Problem on Expansive Mappings in Hilbert Spaces

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Nirenberg proposed a problem as to whether or not a continuous and expansive operator $T : X \longrightarrow X$ (where X is a Hilbert space) is surjective if $R(T)^\circ \neq \varnothing$. I shall give a positive answer for the problem provided that $R(T)^\circ$ is unbounded. For contents related to this paper, the reader is referred to the remarks and the study of Asfaw (2021). The present paper gives a complete answer for the problem that has been open for about 47 years.

1. Introduction and Preliminaries

In what follows, X denotes a real Hilbert space with an inner product $<\cdot>$, and the symbol $\|.\|$ denotes the norm in X. An operator $T : X \longrightarrow X$ is called "α-expansive" if there exists $\alpha > 0$ such that $\|Tx - Ty\| \geq \alpha \|x - y\|$ for all $x \in X$ and $y \in X$. The operator is called expansive if $\alpha = 1$. In 1974, Nirenberg [1] proposed a problem as to whether or not a continuous and expansive operator is surjective provided that $R(T)^\circ \neq \varnothing$.

The main goal of the present paper is to give a positive answer for the problem provided that $R(T)^\circ$ is unbounded. Clearly, T cannot be surjective if $R(T)^\circ$ is bounded.

Remark 1. Recently, I published a paper in [2] planning to give a proof of the problem. A minor error was found after publication. The editors decided a retraction based on their belief that the error is profound and not fixable. However, the error was minor, not profound and fixable. I submitted a corrigendum about the published paper but the editors decided to retract it based on the error in the original paper. Thus, the present paper reuses contents from [2]. The differences between the current and the retracted papers are given below.

(i) In Theorem 7, it is assumed that $R(T)$ has unbounded interior which was not assumed in [2]. It is obvious to see that T cannot be surjective if $R(T)^\circ$ is bounded

(ii) The sequences $p(n, m)$ and $q(n, m)$ used in Step 1 in [2] are not required (in the current paper) to show the (S_+) property of the operator $S + \lambda I$. Instead, the (S_+) property of $S + \lambda I$ is achieved based on Kirszbraun-Valentine [3, 4] extension theorem (c.f., $L^{10} - L^{27}$ on page 5 of the current paper). It is essential to mention that Valentine's [4] result is a generalization of Kirszbraun [3] extension theorem

Morel and Steinlein [5] constructed an example of such an operator T defined from $\ell^1(N)$ into $\ell^1(N)$ satisfying the stated conditions but fails to be surjective. In a Hilbert space setting (i.e., if $H = \ell^2(N)$), Szczepanski [6] constructed a family of nonsurjective continuous maps $\{F_\varepsilon : H \longrightarrow H\}_{\varepsilon>0}$ such that $R(F_\varepsilon)$ contains $B_1(0)$ and F_ε is expansive on each sphere and $\|F_\varepsilon x - F_\varepsilon y\| \geq \|x - y\|$ for all $x \in P_\varepsilon$ or $y \in P_\varepsilon$, where $P_\varepsilon = \{x \in H : 1 < \|x\| < 1 + \varepsilon\}$. In addition, Szczepanski [7] gave a negative answer for the problem in $X = \ell^2(N)$ by constructing a continuous and nonsurjective operator T satisfying weak expansive condition. Chang and Shujie [8], Theorem 7, gave a positive answer for the problem provided that $T : X \longrightarrow Y$ is Frèchet differentiable on X (where X is a real Banach space and Y is reflexive), and its derivative is locally bounded.

Recently, Kartsatos [9], Theorem 6 (with $\alpha = 1$ and $c \in (0, 1)$), and Xiang [10], Theorem 2.12 (with $c \in (-\infty,\alpha)$), gave surjectivity results if the condition $R(T)^\circ \neq \varnothing$ is replaced by strong monotonicity of $T + \alpha I$.

The following remarks on the recent result due to Ives and Preiss [11], are essential.

Remark 2. Ives and Preiss [11], Theorem 4, constructed an operator $\mathscr{P} : X \longrightarrow X$ (where $X = L^2(0,\infty)$) satisfying the following properties.

(i) $\mathscr{P}x = x$ for all $x \in \bar{B}_1(0)$

(ii) The $\mathscr{P}$ image of $\{x \in X : \|x\| \geq 1\}$ is nowhere dense in X

(iii) $\mathscr{P}$ is not surjective

We shall show that (i) and (ii) imply the boundedness of $\mathscr{R}(\mathscr{P})^\circ$. Indeed, let $F = \mathscr{P}(X \setminus B_1(0))$. The continuity and expansiveness of $\mathscr{P}$ and the closedness of $X \setminus B_1(0)$ show that F is closed. Since F is nowhere dense, it follows that $F^\circ = \varnothing$, i.e., $F = \partial F$ and $\overline{X \setminus F} = X$. Let $V = R(\mathscr{P})^\circ$. Then, for each $x \in V$, there exists a sequence $\{x_n\}$ in $X \setminus F$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$. Since V is nonempty and open containing x (V is a neighbourhood of x), there exists a positive integer N such that $x_n \in V$ for all $n \geq N$. In addition, we see that

$$R(\mathscr{P}) = \mathscr{P}(B_1(0)) \cup \mathscr{P}(X \setminus B_1(0)) = B_1(0) \cup F, \quad (1)$$

i.e., we have $V = V \cap (B_1(0) \cup F) = B_1(0) \cup (F \cap V)$. Since $x_n \notin F$ for all n (i.e., $x_n \notin F \cap V$ for all n) and $B_1(0)$ and $F \cap V$ are disjoint, we conclude that $x_n \in B_1(0)$ for all $n \geq N$. This shows that $x \in \bar{B}_1(0)$, i.e., $V \subseteq \bar{B}_1(0)$. Thus, the set V is bounded, i.e., $\mathscr{P}$ is not surjective.

Remark 3. Remark 2 shows that conditions (i) through (v) in Theorem 4 due to Ives and Preiss [11] imply that $R(T)$ has a bounded interior, i.e., T is nonsurjective. It is essential to mention that the result in [11] did not give a negative answer for Nirenberg's problem if $R(T)^\circ$ is unbounded. In addition, we point out that the case of boundedness of $R(T)^\circ$ can be ignored from the statement of Nirenberg's problem.

The following definition is needed in the squeal.

Definition 4. . Let Y be a real reflexive Banach space and Y^* be its dual space. Let G be a nonempty, bounded, and open subset of Y and $S : \bar{G} \longrightarrow Y^*$ be bounded on finite dimensional subsets of Y. Then, S is said to be "of type (S_+)" if a sequence $\{x_n\}$ in $\bar{G}$ satisfies $x_n \rightharpoonup x_0$ and $\limsup_{n\longrightarrow\infty} \langle Sx_n, x_n - x_0\rangle \leq 0$; then, $x_n \longrightarrow x_0$ and $Sx_n \rightharpoonup Sx_0$ as $n \longrightarrow \infty$.

Browder [12], Definition 5, introduced the following definition.

Definition 5. . Let G be a nonempty, bounded and open subset of a reflexive Banach space Y. A family of operators $\{S^t\}_{t\in[0,1]}$, defined from $\bar{G}$ into Y^*, is called a "homotopy of class (S_+)" if for sequences $\{x_n\}$ in $\bar{G}$ and $\{t_n\}$ in $[0, 1]$ such that $x_n \rightharpoonup x_0$, $t_n \longrightarrow t_0$ and $\limsup_{n\longrightarrow\infty} \langle S^{t_n} x_n, x_n - x_0\rangle \leq 0$, we have $x_n \longrightarrow x_0$ and $S^{t_n} x_n \rightharpoonup S^{t_0} x_0$ as $n \longrightarrow \infty$.

The operator $J : Y \longrightarrow 2^{Y^*}$ defined by

$$J(y) = \{y^* \in Y^* : \langle y^*, y\rangle = \|y\|^2, \|y\| = \|y^*\|\}, \quad (2)$$

is called the normalized duality mapping. It is well-known that J is single-valued, bounded, monotone, bicontinuous, surjective, and of type (S_+) if Y and Y^* are locally uniformly convex and real reflexive Banach spaces. The operator $J = I$ (where I is the identity operator on Y) if Y is a real Hilbert space. Browder [12], Theorems 3 and 4, constructed a degree mapping d for the class of operators of type (S_+) that satisfies the basic properties listed in Lemma 6.

Lemma 6. *Let G be a nonempty, bounded, and open subset of a real reflexive Banach space Y. Let $f^* \in Y^*$. Then, the following basic properties hold.*

(i) *Normalization. $d(J, G, 0) = 1$ if $0 \in G$ and $d(J, G, 0) = 0$ if $0 \notin \bar{G}$.*

(ii) *Existence. Let $S : \bar{G} \longrightarrow Y^*$ be a bounded operator of type (S_+) such that $f^* \notin S(\partial G)$ and $d(S, G, f^*) \neq 0$. Then, $f^* \in S(G)$.*

(iii) *Homotopy Invariance. Let $\{S^t\}_{t\in[0,1]}$ be a homotopy of class (S_+) defined from $\bar{G}$ into Y^* such that $f^* \notin S^t(\partial G)$ for all $t \in [0, 1]$. Then, $d(S^t, G, f^*)$ is independent of $t \in [0, 1]$.*

(iv) *Invariance under Affine Homotopy. Let $S_i : \bar{G} \longrightarrow Y^*$ $(i = 1, 2)$ be a bounded operator of type (S_+) and $K^t = tS_1 + (1 - t)S_2$ for $t \in [0, 1]$. Then, $\{K^t\}_{t\in[0,1]}$ is a homotopy of class (S_+), and $d(K^t, G, f^*)$ is independent of $t \in [0, 1]$ provided that $f^* \notin K^t(\partial G)$ for all $t \in [0, 1]$*

2. Main Result

Theorem 7 gives a proof of the Nirenberg problem.

Theorem 7. *Let X be a real Hilbert space and $T : X \longrightarrow X$ be a continuous and α-expansive operator such that $R(T)^\circ$ is unbounded. Then, T is surjective.*

Proof. Choose $x_0 \in R(T)^\circ$. Then, there exists $r > 0$ such that $B_r(x_0) \subseteq R(T)$. Let $\tilde{T}x = Tx - x_0$. Then, the operator $\tilde{T} : X \longrightarrow X$ is α-expansive and continuous such that $R(T) - x_0 = R(\tilde{T})$. It follows that $R(T)^\circ$ is unbounded if and only if $R(\tilde{T})^\circ$ is unbounded. Clearly, the set $V = B_r(x_0) - x_0$ is open and $0 \in V \subseteq R(\tilde{T})$, i.e., $0 \in R(\tilde{T})^\circ$. In addition, we see that T is surjective if and only if $\tilde{T}$ is surjective. Thus, we assume without loss of generality that $0 \in R(T)^\circ$. Let $U = R(T)^\circ$ and $f \in X$. The continuity and α-expansive conditions imply that $R(T)$ is closed and

$$\|T^{-1}x - T^{-1}y\| \leq \alpha^{-1}\|x - y\|, \tag{3}$$

for all $x \in R(T)$ and $y \in R(T)$. Then, it follows that

$$\begin{aligned}\left\langle (T^{-1}x + \alpha^{-1}x) - (T^{-1}y + \alpha^{-1}y), x - y \right\rangle &= \left\langle T^{-1}x - T^{-1}y, x - y \right\rangle \\ &\quad + \alpha^{-1}\|x - y\|^2 \\ &\geq -\|T^{-1}x - T^{-1}y\|\|x - y\| \\ &\quad + \alpha^{-1}\|x - y\|^2 \\ &\geq -\alpha^{-1}\|x - y\|^2 + \alpha^{-1}\|x - y\|^2 \\ &= 0,\end{aligned} \tag{4}$$

for all $x \in R(T)$ and $y \in R(T)$, i.e., $T^{-1} + \alpha^{-1}I : R(T) \longrightarrow X$ is a continuous and monotone operator. It holds that $\bar{U} \subseteq R(T)$ because $R(T)$ is closed and $U \subseteq R(T)$. Let $S : \bar{U} \longrightarrow X$ be the restriction of $T^{-1} + \alpha^{-1}I$ on $\bar{U}$. Let $\lambda > 0$, and for each $f \in X$ and $\mu \in (0, 1)$, let $f_\mu = \mu^{-1}f$. Let

$$S_\lambda^f x = Sx + \lambda\left(x - f_\mu\right), x \in \bar{U}. \tag{5}$$

The monotonicity of S implies

$$\left\langle S_\lambda^f x - S_\lambda^f y, x - y \right\rangle \geq \lambda\|x - y\|^2, \tag{6}$$

for all $x \in \bar{U}$ and $y \in \bar{U}$. The monotonicity of S and the condition $0 \in \bar{U}$ yield

$$\begin{aligned}\left\langle S_\lambda^f x, x \right\rangle &= \left\langle Sx - S0 + S0 + \lambda\left(x - f_\mu\right), x \right\rangle \\ &\geq \lambda\|x\|^2 \\ &\quad - \left(\lambda\|f_\mu\| + \|S0\|\right)\|x\|,\end{aligned} \tag{7}$$

for all $x \in \bar{U}$. Since $\bar{U}$ is unbounded, the right side of (7) approaches infinity as $\|x\| \longrightarrow \infty$. Then, there exists $R = R(f_\mu) > 0$ such that

$$\left\langle Sx + \lambda\left(x - f_\mu\right), x \right\rangle > 0, \tag{8}$$

for all $x \in \bar{U} \cap \partial B_R(0)$. Let $\Omega = U \cap B_R(0)$. We shall show that the operator $S + \lambda(I - f_\mu)$: $\bar{\Omega} \longrightarrow X$ is of type (S_+). We shall apply Kirszbraun-Valentine [3, 4] extension theorem. Indeed, the Lipschitz operator T^{-1} (with Lipschitz constant α^{-1}) can be extended to an operator $P : X \longrightarrow X$ with the same Lipschitz constant. Thus, the operator $\tilde{S} = P + \alpha^{-1}I + \lambda(I - f_\mu)$: $X \longrightarrow X$ is continuous and uniformly monotone, i.e., $\tilde{S}$ is of type (S_+). Let $\tilde{S}_{\bar{\Omega}}$ be the restriction of $\tilde{S}$ on $\bar{\Omega}$. We shall show that $\tilde{S}_{\bar{\Omega}}$ is of type (S_+). Indeed, let $x_n \in \bar{\Omega}$ for all n such that $x_n \rightharpoonup x_0 \in X$ as $n \longrightarrow \infty$ and

$$\limsup_{n \longrightarrow \infty}\left\langle \tilde{S}_{\bar{\Omega}}x_n, x_n - x_0 \right\rangle \leq 0. \tag{9}$$

It is enough to show that $x_n \longrightarrow x_0 \in \bar{\Omega}$. Since $\tilde{S} = \tilde{S}_{\bar{\Omega}}$ on $\bar{\Omega}$, it follows that

$$\limsup_{n \longrightarrow \infty}\left\langle \tilde{S}x_n, x_n - x_0 \right\rangle \leq 0. \tag{10}$$

The uniform monotonicity of $\tilde{S}$ with domain X (i.e., using (5), (6) and $\tilde{S} = S_\lambda^f$ on $\bar{\Omega}$) shows that

$$\begin{aligned}\left\langle \tilde{S}x_n - \tilde{S}x_0 + \tilde{S}x_0, x_n - x_0 \right\rangle &= \left\langle \tilde{S}x_n - \tilde{S}x_0, x_n - x_0 \right\rangle \\ &\quad + \left\langle \tilde{S}x_0, x_n - x_0 \right\rangle \\ &\geq \lambda\|x_n - x_0\|^2 \\ &\quad + \left\langle \tilde{S}x_0, x_n - x_0 \right\rangle,\end{aligned} \tag{11}$$

for all n. Thus, (9)–(11) yield

$$\limsup_{n \longrightarrow \infty}\left(\lambda\|x_n - x_0\|^2 + \left\langle \tilde{S}x_0, x_n - x_0 \right\rangle\right) \leq \limsup_{n \longrightarrow \infty}\left\langle \tilde{S}x_n, x_n - x_0 \right\rangle \leq 0. \tag{12}$$

Since $\langle \tilde{S}x_0, x_n - x_0 \rangle \longrightarrow 0$ as $n \longrightarrow \infty$ (because $x_n \rightharpoonup x_0$), (12) shows that

$$\limsup_{n \longrightarrow \infty} \lambda\|x_n - x_0\|^2 \leq 0, \tag{13}$$

i.e., $x_n \longrightarrow x_0$ as $n \longrightarrow \infty$. Clearly $x_0 \in \bar{\Omega}$ because $x_n \in \bar{\Omega}$ for all n. The continuity of $\tilde{S}_{\bar{\Omega}}$ implies that $\tilde{S}_\Omega x_n \longrightarrow \tilde{S}_\Omega x_0$ as $n \longrightarrow \infty$. Thus, the operator $\tilde{S}_{\bar{\Omega}} = S + \lambda(I - f_\mu)$: $\bar{\Omega} \longrightarrow X$ is bounded, continuous, and of type (S_+).

Clearly, the identity operator $I : \bar{\Omega} \longrightarrow X$ is bounded and of type (S_+). For each $\lambda > 0$, let $H_\lambda : [0, 1] \times \bar{\Omega} \longrightarrow X$ be given by

$$H_\lambda(t, x) = t\left(Sx + \lambda x - \lambda f_\mu\right) + (1 - t)x, (t, x) \in [0, 1] \times \bar{\Omega}. \tag{14}$$

Then, (iv) of Lemma 6 shows that the family $\{H_\lambda(t, .)\}_{t \in [0,1]}$ is an affine homotopy of class (S_+). Clearly, $0 \neq H_\lambda(0, x)$ for all $x \in \partial\Omega$. Thus, exactly one of the following holds.

(i) There exists $\lambda > 0$ such that for each $\lambda' \geq \lambda$, there exist $t_{\lambda'} \in (0, 1]$ and $x_{\lambda'} \in \partial\Omega$ such that $H_{\lambda''}(t_{\lambda'}, x_{\lambda'}) = 0$ for all $\lambda'' \geq \lambda'$

(ii) For each $\lambda > 0$ there exists $\lambda' \geq \lambda$ such that

$$H_{\lambda''}(t,x) \neq 0 \text{ for all } (t,x) \in [0,1] \times \partial\Omega \text{ and some } \lambda'' \geq \lambda'. \tag{15}$$

Suppose (i) holds, i.e., there exists $n > 0$ such that for each $n' \geq n$, this corresponds $t_{n'} \in (0,1]$, $x_{n'} \in \partial\Omega \subseteq \bar{\Omega} \subseteq \bar{U} \subseteq R(T)$, and $y_{n'} \in X$ such that $x_{n'} = Ty_{n'}$ and $H_{n''}(t_{n'}, x_{n'}) = 0$ for all $n'' \geq n'$, i.e., for each $n' \geq n$, we have

$$t_{n'}\left(y_{n'} + \alpha^{-1}Ty_{n'} + n''Ty_{n'} - n''f_\mu\right) + (1 - t_{n'})Ty_{n'} = 0, \tag{16}$$

for all $n'' \geq n'$. Replacing n by n_0, n' by n, and n'' by n' in (16), for each $n \geq n_0$, we get

$$t_n\left(y_n + \alpha^{-1}Ty_n + n'Ty_n - n'f_\mu\right) + (1 - t_n)Ty_n = 0, \tag{17}$$

for all $n' \geq n$. Next we choose a sequence of natural numbers $\{N_n\}$ such that $t_n \in (1/N_n, 1]$ for all n. Let $\lambda_n = N_n + (n-1)$ for all n. Clearly, $\lambda_n \geq n$ and $t_n \in (1/\lambda_n, 1]$ for all n (i.e., $t_n \lambda_n > 1$ for all n). Since $\lambda_n \geq n$ for all n, (17) implies

$$t_n\left(y_n + \alpha^{-1}Ty_n + \lambda_n Ty_n - \lambda_n f_\mu\right) + (1 - t_n)Ty_n = 0, \tag{18}$$

for all n. Clearly, $t_n \neq 0$ for all n. Dividing (18) by t_n for all n, we arrive at

$$\gamma_n^{-1} y_n + Ty_n = \lambda_n \gamma_n^{-1} f_\mu, \tag{19}$$

for all n, where $\gamma_n = \alpha^{-1} + \lambda_n + (1 - t_n)t_n^{-1}$. The boundedness of $\{x_n\}$ and T^{-1} shows that $\{y_n\}$ is bounded. Assume without loss of generality that $t_n \longrightarrow t_0 \in [0,1]$ as $n \longrightarrow \infty$. Suppose $t_0 = 0$. The sequence $\{t_n\lambda_n\}$ cannot converge to 0 because $t_n\lambda_n > 1$ for all n. Then, we consider either $\{t_n\lambda_n\}$ is unbounded or bounded. Suppose $\{t_n\lambda_n\}$ is unbounded, i.e., there exists a subsequence of $\{t_n\lambda_n\}$, denoted by $\{t_n\lambda_n\}$, such that $t_n\lambda_n \longrightarrow \infty$ as $n \longrightarrow \infty$. Since $t_0 = 0$, $\{y_n\}$ and $\{Ty_n\}$ are bounded, letting $n \longrightarrow \infty$ in (18) implies

$$t_n\lambda_n Ty_n - t_n\lambda_n f_\mu + Ty_n \longrightarrow 0, \tag{20}$$

as $n \longrightarrow \infty$, i.e., $c_n Ty_n \longrightarrow f_\mu$ as $n \longrightarrow \infty$, where $c_n = 1 + t_n\lambda_n / t_n\lambda_n$ for all n. Since $c_n \longrightarrow 1^+$ as $n \longrightarrow \infty$ and $\{Ty_n\}$ is bounded, we conclude that $x_n = Ty_n \longrightarrow f_\mu \in R(\bar{T}) = R(T)$ as $n \longrightarrow \infty$. Letting $\mu \longrightarrow 1^-$ gives $f \in R(T)$.

Next we assume that $\{t_n\lambda_n\}$ is bounded. Assume without loss of generality that there exists a subsequence, denoted again by $\{t_n\lambda_n\}$, such that $t_n\lambda_n \longrightarrow a \in [1,\infty)$ as $n \longrightarrow \infty$. Then, $c_n \longrightarrow (a+1)a^{-1}$ and

$$x_n = Ty_n \longrightarrow a(a+1)^{-1} f_\mu \in R(\bar{T}) = R(T), \tag{21}$$

as $n \longrightarrow \infty$. Since $d = a(a+1)^{-1} \in (0,1)$, letting $\mu \longrightarrow d$ gives $f \in R(T)$.

Next we consider the case $t_0 \in (0,1]$. Then, we have $\mu_n = \alpha^{-1} + (1 - t_n)t_n^{-1} \longrightarrow \mu_0 = \alpha^{-1} + (1 - t_0)t_0^{-1} \in \mathbb{R}$ and

$$\begin{aligned} \lambda_n \gamma_n^{-1} &= \frac{\lambda_n}{\alpha^{-1} + \lambda_n + (1 - t_n)t_n^{-1}} \\ &= \frac{N_n + (n-1)}{\mu_n + N_n + (n-1)} \\ &= \frac{1}{1 + \xi_n \mu_n}, \end{aligned} \tag{22}$$

for all n, where $\xi_n = (N_n + (n-1))^{-1}$. Since $\xi_n \longrightarrow 0$ as $n \longrightarrow \infty$ and $\mu_0 \in \mathbb{R}$, it follows that $\lambda_n\gamma_n^{-1} \longrightarrow 1$ as $n \longrightarrow \infty$. Letting $n \longrightarrow \infty$ in (19) yields $x_n = Ty_n \longrightarrow f_\mu \in R(\bar{T}) = R(T)$. Letting $\mu \longrightarrow 1^-$ shows that $f \in R(T)$. In all cases, each $f \in X$ satisfying (i) lies in $R(T)$.

Suppose (ii) holds. Then, there exists a subsequence of $\{n\}$, denoted again by $\{n\}$, such that $0 \in H_n(t, \partial\Omega)$ for all $t \in [0,1]$ and n. Then, (iv) of Lemma 6. shows that the family $\{H_n(t,.)\}_{t\in[0,1]}$ is a homotopy of class (S_+) defined from $\bar{\Omega}$ into X such that $0 \in H_n(t, \partial\Omega)$ for all $t \in [0,1]$ and n. Let d stand for Browder degree mapping. Applying (iii) of Lemma 6., we conclude that $d(H_n(t,.), \Omega, 0)$ is independent of $t \in [0,1]$, i.e., (i) and (iii) of Lemma 6. give

$$d\left(S + nI - nf_\mu, \Omega, 0\right) = d(I, \Omega, 0) = 1 \tag{23}$$

for all n. In addition, (ii) of Lemma 6. confirms that $n f_\mu \in (S + nI)(\Omega)$ for all n. Thus, for each positive integer n, there exist $x_n \in \Omega$ and $y_n \in X$ such that $x_n = Ty_n$ for all n and

$$\begin{aligned} nf_\mu &= Sx_n + nx_n = T^{-1}x_n + \alpha^{-1}x_n + nx_n \\ &= \left(n + \alpha^{-1}\right)Ty_n + y_n, \end{aligned} \tag{24}$$

for all n. Since $\{x_n\}$ and T^{-1} are bounded, it follows that $\{y_n\}$ is bounded. In addition, (24) gives

$$x_n = Ty_n = n\tau_n f_\mu - \tau_n y_n \longrightarrow f_\mu \in \bar{\Omega} \subseteq \bar{U} \subseteq R(T), \tag{25}$$

as $n \longrightarrow \infty$ because $\tau_n \longrightarrow 0$ and $n\tau_n \longrightarrow 1$, where $\tau_n = (n + \alpha^{-1})^{-1}$. Thus, letting $\mu \longrightarrow 1^-$ shows that $f \in R(T)$. Thus, each $f \in X$ satisfying (i) or (ii) lies in $R(T)$. This shows that T is surjective. The proof is completed. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of the paper.

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