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Research Article

Uniform Convexity and Convergence of a Sequence of Sets in a Complete Geodesic Space

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In this paper, we first introduce two new notions of uniform convexity on a geodesic space, and we prove their properties. Moreover, we reintroduce a concept of the set-convergence in complete geodesic spaces, and we prove a relation between the metric projections and the convergence of a sequence of sets.

1. Introduction

There are a lot of works dealing with the relation between convergence of a sequence of sets and convergence of a sequence of projections corresponding to it. In particular, the following theorem on a reflexive and strictly convex real Banach space is one of the important results.

Theorem 1 (Tsukada [1]). *Let E be a strictly convex and reflexive real Banach space satisfying the Kadec-Klee property. Let $\{C_n\} \subset 2^E$ be a sequence of nonempty closed convex sets and $C_0 \subset E$ a nonempty closed convex subset such that $\{C_n\}$ converges to C_0 in the sense of Mosco. Then $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x \in E$ for any $x \in X$, where $P_K : E \longrightarrow K$ is the metric projection of E onto a nonempty closed convex subset K of X.*

Since a uniformly convex real Banach space is strictly convex, reflexive, and satisfying the Kadec-Klee property, this theorem is true in uniformly convex real Banach spaces. Moreover, since a real Hilbert space is a uniformly convex real Banach space, it is also true in real Hilbert spaces.

On the other hand, we know that a Hadamard space is another generalization of Hilbert spaces. It is defined as a complete metric space having a particular convexity structure and it also has various useful properties that Hilbert spaces have. Kimura [2] introduced Δ-Mosco convergence in complete geodesic spaces using a notion of asymptotic centre instead of weak convergence.

Theorem 2 (Kimura [2]). *Let X be a complete CAT(0) space. Let $\{C_n\} \subset 2^X$ be a sequence of nonempty closed convex sets and $C_0 \subset X$ a nonempty closed convex subset such that $\{C_n\}$ converges to C_0 in the sense of Δ -Mosco. Then $\{P_{C_n}x\}$ converges to $P_{C_0}x \in X$ for any $x \in X$, where $P_K : X \longrightarrow K$ is the metric projection of X onto a nonempty closed convex subset K of X.*

Moreover, Kimura and Satô [3] introduced Δ_κ-Mosco convergence in complete CAT(κ) spaces and obtain the following result:

Theorem 3 (Kimura and Satô [3]). *Let X be a complete admissible CAT(κ) space for $\kappa > 0$. Let $\{C_n\} \subset 2^X$ be a sequence of nonempty closed convex sets and $C_0 \subset X$ a nonempty closed convex subset such that $\{C_n\}$ converges to C_0 in the sense of Δ_κ -Mosco. Then $\{P_{C_n}x\}$ converges to $P_{C_0}x \in X$ for any $x \in X$, where $P_K : X \longrightarrow K$ is the metric* *projection of X onto a nonempty closed convex subset K of X.*

In this work, we introduce a new concept of the set-convergence and we obtain a similar result as above under the assumptions that both uniformly convex real Banach spaces and complete CAT(0) spaces have.

2. Preliminaries

A function $g : [0,\infty[\longrightarrow [0,\infty[$ is said to be a *gauge* if g is strictly increasing, continuous, and $g(0) = 0$. We know that if $\{\alpha_n\}$ is a real sequence of $[0,\infty[$ such that $g(\alpha_n) \longrightarrow 0$ for some gauge function g, then $\alpha_n \longrightarrow 0$.

Let E be a real Banach space. Then the following propositions are equivalent:

(i) E is uniformly convex

(ii) For $r > 0$, $z \in E$, and $\{x_n\}, \{y_n\} \subset E$, it holds that $\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0$ whenever

$$\lim_{n\longrightarrow\infty} \|x_n - z\| = \lim_{n\longrightarrow\infty} \|y_n - z\| = \lim_{n\longrightarrow\infty} \left\| \left(\frac{1}{2}x_n + \frac{1}{2}y_n\right) - z \right\| = r \tag{1}$$

(iii) For any $K > 0$, there exists a convex gauge function $\underline{g}_K : [0,\infty[\longrightarrow [0,\infty[$ such that

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\underline{g}_K(\|x-y\|) \tag{2}$$

for any $t \in [0,1]$, where $\|x\| \le K$ and $\|y\| \le K$ for $x, y \in E$

For more details about the properties of uniformly convex real Banach spaces, see [4].

Let (X,d) be a metric space and let $x, y \in X$. A *geodesic path* from x to y is a mapping $\gamma : [0, d(x,y)] \longrightarrow X$ such that $\gamma(0) = x$, $\gamma(d(x,y)) = y$ and $d(\gamma(s),\gamma(t)) = |s-t|$ for any s, $t \in [0, d(x,y)]$. Let $D \in]0,\infty]$. If for any $x, y \in X$ such that $d(x,y) < D$, a geodesic path from x to y exists, then we say that X is *D -geodesic*. Moreover, if such a geodesic path is unique for each pair of points, then X is said to be *D -uniquely geodesic*. In a D-uniquely geodesic space, the image of a geodesic path from x to y is called a *geodesic segment* joining x and y and it is denoted by $[x,y]$. For $x, y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that $d(x,z) = (1-t)d(x,y)$ and $d(y,z) = td(x,y)$. We denote it by $z = tx \oplus (1-t)y$. A geodesic triangle with vertices $x, y, z \in X$ is the union of geodesic segments $[x,y]$, $[y,z]$ and $[z,x]$. We denote it by $\Delta(x,y,z)$.

To define a CAT(κ) space, we use the following notation called a model space. For $\kappa = 0$, the two-dimensional model space $M_\kappa^2 = M_0^2$ is the two-dimensional Euclidean space $\mathbb{E}^2$ with the metric induced from the Euclidean norm. For $\kappa > 0$, M_κ^2 is the two-dimensional sphere $(1/\sqrt{\kappa})\mathbb{S}^2$ whose metric is a length of a minimal great arc joining each two points. For $\kappa < 0$, M_κ^2 is the two-dimensional hyperbolic space $(1/\sqrt{-\kappa})\mathbb{H}^2$ with the metric defined by a usual hyperbolic distance.

The diameter of M_κ^2 is denoted by D_κ, defined by

$$D_\kappa = \begin{cases} \infty & (\kappa \le 0), \\ \pi/\sqrt{\kappa} & (\kappa > 0). \end{cases} \tag{3}$$

We know that M_κ^2 is a D_κ-uniquely geodesic space for each $\kappa \in \mathbb{R}$.

Let $\kappa \in \mathbb{R}$. For $\Delta(x,y,z)$ in a geodesic space X satisfying that $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$, there exist points $\bar{x}, \bar{y}$, $\bar{z} \in M_\kappa^2$ such that $d(x,y) = d_{M_\kappa^2}(\bar{x},\bar{y})$, $d(y,z) = d_{M_\kappa^2}(\bar{y},\bar{z})$, $d(z,x) = d_{M_\kappa^2}(\bar{z},\bar{x})$. We call the triangle having vertices $\bar{x}$, $\bar{y}$ and $\bar{z}$ in M_κ^2 a comparison triangle of $\Delta(x,y,z)$. Notice that it is unique up to an isometry of M_κ^2. For a specific choice of comparison triangles, we denote it by $\bar{\Delta}(\bar{x},\bar{y},\bar{z})$. A point $\bar{p} \in [\bar{x},\bar{y}]$ is called a comparison point for $p \in [x,y]$ if $d(x,p) = d_{M_\kappa^2}(\bar{x},\bar{p})$.

Let $\kappa \in \mathbb{R}$ and X a D_κ-geodesic space. If for any $x, y, z \in X$ with $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$, for any $p, q \in \Delta(x,y,z)$, and for their comparison points $\bar{p}, \bar{q} \in \bar{\Delta}(\bar{x},\bar{y},\bar{z})$, the CAT($\kappa$) inequality

$$d(p,q) \le d_{M_\kappa^2}(\bar{p},\bar{q}) \tag{4}$$

holds, then we call X a CAT(κ) space. It is well known that any CAT(κ) space is also a CAT(κ') space whenever $\kappa < \kappa'$. Therefore, a CAT(κ) space is a CAT(0) space for any $\kappa < 0$.

Let X be a CAT(κ) space for $\kappa \in \mathbb{R}$. If

$$d(u,v) < \frac{D_\kappa}{2} \tag{5}$$

for any $u, v \in X$, then we say that X is *admissible*. A CAT(κ) space is always admissible when $\kappa \le 0$.

A subset C of a D-uniquely geodesic space X is said to be *convex* if $tx \oplus (1-t)y \in C$ for every $x, y \in C$ and $t \in [0,1]$. For a subset S of D-uniquely geodesic space X, a *closed convex hull* of S is defined as the intersection of all closed convex sets including S, and we denote it by clco S.

Let X be a complete admissible CAT(κ) space for $\kappa \in \mathbb{R}$ and C a nonempty closed convex subset of X. Then for $x \in X$, there exists a unique point $y_x \in C$ such that

$$d(x,y_x) = d(x,C). \tag{6}$$

We call such a mapping defined by $P_C x = y_x$, the *metric projection* of X onto C.

Let X be a CAT(0) space. From the CAT(0) inequality, it is easy to see that

$$d(tx \oplus (1-t)y, z)^2 \le td(x,z)^2 + (1-t)d(y,z)^2 - t(1-t)d(x,y)^2 \tag{7}$$

for every $x, y, z \in X$ and $t \in [0,1]$.

The following lemma shows that a CAT(0) space has a similar property to the uniform convexity of Banach spaces.

Lemma 4. *Let X be a CAT(0) space. For $r > 0$, a point $z \in X$ and two sequences $\{x_n\}, \{y_n\} \subset X$, it holds that $\lim_{n \longrightarrow \infty} d(x_n, y_n) = 0$ whenever*

$$\lim_{n \longrightarrow \infty} d(x_n, z) = \lim_{n \longrightarrow \infty} d(y_n, z) = \lim_{n \longrightarrow \infty} d\left(\frac{1}{2}x_n \oplus \frac{1}{2}y_n, z\right) = r. \tag{8}$$

Proof. For $\{x_n\}, \{y_n\} \subset X$, $z \in X$ and $r \in]0,\infty[$, if

$$\lim_{n \longrightarrow \infty} d(x_n, z) = \lim_{n \longrightarrow \infty} d(y_n, z) = \lim_{n \longrightarrow \infty} d\left(\frac{1}{2}x_n \oplus \frac{1}{2}y_n, z\right) = r, \tag{9}$$

then

$$d\left(\frac{1}{2}x_n \oplus \frac{1}{2}y_n, z\right)^2 \le \frac{1}{2}d(x_n, z)^2 + \frac{1}{2}d(y_n, z)^2 - \frac{1}{4}d(x_n, y_n)^2, \tag{10}$$

which implies

$$d(x_n, y_n)^2 \le 2d(x_n, z)^2 + 2d(y_n, z)^2 - 4d\left(\frac{1}{2}x_n \oplus \frac{1}{2}y_n, z\right)^2. \tag{11}$$

Letting $n \longrightarrow \infty$, we have

$$\lim_{n \longrightarrow \infty} d(x_n, y_n) = 0. \tag{12}$$

This is the desired result. □

Let X be a metric space. For a point $x \in X$ and a nonempty subset $C \subset X$, the distance between them is defined by $d(x, C) = \inf_{y \in C} d(x, y)$.

Let X be a metric space. A function $f : X \longrightarrow]-\infty, \infty]$ is said to be *lower semicontinuous* if for $x_0 \in X$ and $\{x_n\} \subset X$,

$$f(x_0) \le \liminf_{n \longrightarrow \infty} f(x_n) \tag{13}$$

whenever $x_n \longrightarrow x_0$. Moreover, a function $f : X \longrightarrow]-\infty, \infty]$ is said to be *proper* if there exists a point $z_0 \in X$ such that $f(z_0) < \infty$ and the *domain of f* defined by

$$\operatorname{dom} f = \{x \in X \mid f(x) < \infty\}. \tag{14}$$

Let X be a uniquely geodesic space. A function $f : X \longrightarrow]-\infty, \infty]$ is said to be *convex* if for $x, y \in X$ and $t \in]0,1[$, f satisfies

$$f(tx \oplus (1-t)y) \le tf(x) + (1-t)f(y). \tag{15}$$

3. Uniform Convexity of a Complete Geodesic Space

In the following, we always suppose that for any x, y in geodesic space X, a geodesic joining x to y is unique.

A geodesic space X is said to satisfy the *condition* (D) if:

$$d(tx \oplus (1-t)y, z)^2 \le td(x,z)^2 + (1-t)d(y,z)^2 \tag{16}$$

for $x, y, z \in X$ and $t \in [0,1]$.

We introduce two new concepts of uniform convexity on a geodesic space.

Let X be a geodesic space. X is said to be *sequentially uniformly convex* if X satisfies the condition (D) and, for $r > 0$, a point $z \in X$ and two sequences $\{x_n\}, \{y_n\} \subset X$, it holds that

$$\lim_{n \longrightarrow \infty} d(x_n, y_n) = 0 \tag{17}$$

whenever

$$\lim_{n \longrightarrow \infty} d(x_n, z) = \lim_{n \longrightarrow \infty} d(y_n, z) = \lim_{n \longrightarrow \infty} d\left(\frac{1}{2}x_n \oplus \frac{1}{2}y_n, z\right) = r. \tag{18}$$

Let X be a geodesic space. X is said to be *uniformly convex* if for any $K > 0$ there exists a convex gauge function $\underline{g}_K : [0,\infty[\longrightarrow [0,\infty[$ such that for any $t \in [0,1]$,

$$d(tx \oplus (1-t)y, z)^2 \le td(x,z)^2 + (1-t)d(y,z)^2 - t(1-t)\underline{g}_K(d(x,y)), \tag{19}$$

where $x, y, z \in X$ with $d(x,z) \le K$ and $d(y,z) \le K$.

Uniformly convex real Banach spaces and CAT(0) spaces are uniformly convex in this sense.

Theorem 5. *Let X be a uniformly convex geodesic space. Then, X is sequentially uniformly convex.*

Proof. Let $x, y, z \in X$ and let $K = \max\{d(x,z), d(y,z)\} < \infty$. Then there exists a convex gauge function $\underline{g}_K : [0,\infty[\longrightarrow [0,\infty[$, satisfying that

$$d(tx \oplus (1-t)y, z)^2 \le td(x,z)^2 + (1-t)d(y,z)^2 - t(1-t)\underline{g}_K(d(x,y)) \tag{20}$$

for any $t \in [0,1]$ and hence

$$d(tx \oplus (1-t)y, z)^2 \leq td(x,z)^2 + (1-t)d(y,z)^2. \quad (21)$$

That is, X satisfies the condition (D).

Moreover, for $r > 0$, $z \in X$ and $\{x_n\}, \{y_n\} \subset X$, if

$$\lim_{n \longrightarrow \infty} d(x_n, z) = \lim_{n \longrightarrow \infty} d(y_n, z) = \lim_{n \longrightarrow \infty} d\left(\frac{1}{2}x_n \oplus \frac{1}{2}y_n, z\right) = r, \quad (22)$$

then there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, z) \leq r+1, d(y_n, z) \leq r+1 \quad (23)$$

for any $n \geq n_0$. From uniform convexity of X, there exists a convex gauge function $\underline{g}_{r+1}$, satisfying that

$$d\left(\frac{1}{2}x_n \oplus \frac{1}{2}y_n, z\right)^2 \leq \frac{1}{2}d(x_n, z)^2 + \frac{1}{2}d(y_n, z)^2 - \frac{1}{4}\underline{g}_{r+1}(d(x_n, y_n)). \quad (24)$$

It follows that

$$\begin{aligned} 0 &\leq \underline{g}_{r+1}(d(x_n, y_n)) \leq 2d(x_n, z)^2 + 2d(y_n, z)^2 \\ &\quad - 4d\left(\frac{1}{2}x_n \oplus \frac{1}{2}y_n, z\right)^2 \longrightarrow 2r^2 + 2r^2 - 4r^2 = 0. \end{aligned} \quad (25)$$

Therefore, we obtain $\underline{g}_{r+1}(d(x_n, y_n)) \longrightarrow 0$ and hence $\lim_{n \longrightarrow \infty} d(x_n, y_n) = 0$, that is, X is sequentially uniformly convex. □

Theorem 6. *Let X be a sequentially uniformly convex geodesic space. For $r > 0$ and $x, y, z \in X$ with $x \neq y$, if $d(x,z) = d(y,z) = r$, then $d(1/2x \oplus 1/2y, z) < r$.*

Proof. For $r > 0$ and $x, y, z \in X$ with $x \neq y$, we suppose that $d(x,z) = d(y,z) = r$. If $r \leq d(1/2x \oplus 1/2y, z)$, then, since

$$r^2 \leq d\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right)^2 \leq \frac{1}{2}d(x,z)^2 + \frac{1}{2}d(y,z)^2 = r^2, \quad (26)$$

we have $d(1/2x \oplus 1/2y, z) = r$. From the sequential uniform convexity of X, we have $d(x,y) = 0$. This is a contradiction. Therefore, we have $d(1/2x \oplus 1/2y, z) < r$. This is the desired result. □

Theorem 7. *Let X be a sequentially uniformly convex complete geodesic space and let C be a nonempty closed convex subset of X. Then, for $x \in X$, there exists a unique point $y_0 \in C$ such that $d(x, y_0) = d(x, C)$.*

Proof. For $x \in X$, let $d = d(x, C)$. Then, for $n \in \mathbb{N}$, we can take a sequence $\{y_n\} \subset C$ such that

$$d \leq d(x, y_n) \leq d + \frac{1}{n}. \quad (27)$$

Then, we have $d(x, y_n) \longrightarrow d$. Suppose that $\{y_n\}$ is not a Cauchy sequence. That is, there exists $\varepsilon > 0$ such that for any $i \in \mathbb{N}$, there exist $m_i, n_i \geq i$ such that $d(y_{m_i}, y_{n_i}) \geq \varepsilon$. In this way, we take two subsequences $\{y_{m_i}\}, \{y_{n_i}\} \subset \{y_n\}$. Then,

$$\lim_{i \longrightarrow \infty} d\left(x, y_{m_i}\right) = \lim_{i \longrightarrow \infty} d\left(x, y_{n_i}\right) = d \quad (28)$$

and we have

$$d^2 \leq d\left(x, \frac{1}{2}y_{m_i} \oplus \frac{1}{2}y_{n_i}\right)^2 \leq \frac{1}{2}d\left(x, y_{m_i}\right)^2 + \frac{1}{2}d\left(x, y_{n_i}\right)^2 \longrightarrow d^2. \quad (29)$$

Hence, from the sequential uniform convexity of X, we have

$$\lim_{i \longrightarrow \infty} d\left(y_{m_i}, y_{n_i}\right) = 0. \quad (30)$$

This is a contradiction and thus $\{y_n\}$ is a Cauchy sequence. Since X is complete and C is closed, there exists $y_0 \in C$ such that $y_n \longrightarrow y_0$. Therefore, we have

$$d(x, y_0) = \lim_{n \longrightarrow \infty} d(x, y_n) = d = d(x, C). \quad (31)$$

Next, we show the uniqueness of y_0. Suppose that $y_0, z_0 \in C$ satisfy $y_0 \neq z_0$ and $d(x, y_0) = d(x, z_0) = d(x, C)$. Then, from Theorem 6, we have

$$d\left(x, \frac{1}{2}y_0 \oplus \frac{1}{2}z_0\right) < d(x, C). \quad (32)$$

This is a contradiction. Therefore, for $x \in X$, there exists a unique point $y_0 \in C$ such that $d(x, y_0) = d(x, C)$. □

Let X be a sequentially uniformly convex complete uniquely geodesic space and let C be a nonempty closed convex subset of X. Then for $x \in X$, there exists a unique point $y_x \in C$ such that

$$d(x, y_x) = d(x, C). \quad (33)$$

We call such a mapping $P_C : X \longrightarrow C$ defined by $P_C x = y_x$, the *metric projection* of X onto C.

Theorem 8. *Let X be a sequentially uniformly convex complete geodesic space and $\{C_n\} \subset 2^X$ a sequence of nonempty bounded closed convex subsets which is decreasing with respect to inclusion, that is, $C_{n+1} \subset C_n$ for any $n \in \mathbb{N}$. Then, $\bigcap_{n=1}^{\infty} C_n$ is nonempty.*

Proof. Since C_n is nonempty bounded closed convex subset for $n \in \mathbb{N}$, for $x \in X$, we can take a sequence $\{x_n\} \subset X$ by $x_n = P_{C_n} x$, where P_K is the metric projection of X onto a nonempty closed convex subset K of X. Then $\{d(x, x_n)\}$ is a bounded increasing real sequence and hence $\{d(x, x_n)\}$ has a limit $c \in [0, \infty[$. That is, we have

$$\lim_{n\longrightarrow\infty} d(x, x_n) = c. \tag{34}$$

First, we show that $\{x_n\}$ converges to some point $x_0 \in X$. If $c = 0$, then, since $d(x, P_{C_n} x) \longrightarrow 0$, we have $x_n \longrightarrow x_0$ as $x_0 = x$. Hence, we may suppose that $c > 0$. Suppose that $\{x_n\}$ is not a Cauchy sequence. That is, there exists $\varepsilon > 0$ such that for any $i \in \mathbb{N}$, there exist $m_i, n_i \geq i$ such that $d(x_{m_i}, x_{n_i}) \geq \varepsilon$. Without loss of generality, we can suppose that $m_i \geq n_i$. In this way, we take two subsequences $\{x_{m_i}\}, \{x_{n_i}\} \subset \{x_n\}$. Then,

$$\lim_{i\longrightarrow\infty} d(x, x_{m_i}) = \lim_{i\longrightarrow\infty} d(x, x_{n_i}) = c. \tag{35}$$

Since $x_{m_i}, x_{n_i} \in C_{n_i}$, we have

$$\begin{aligned} d(x, x_{n_i}) = d(x, P_{C_{n_i}} x) \leq d\left(x, \frac{1}{2} x_{m_i} \oplus \frac{1}{2} x_{n_i}\right)^2 \leq \frac{1}{2} d(x, x_{m_i})^2 \\ + \frac{1}{2} d(x, x_{n_i})^2 \end{aligned} \tag{36}$$

and thus

$$\lim_{i\longrightarrow\infty} d\left(x, \frac{1}{2} x_{m_i} \oplus \frac{1}{2} x_{n_i}\right) = c. \tag{37}$$

From the sequential uniform convexity of X, we have $\lim_{i\longrightarrow\infty} d(x_{m_i}, x_{n_i}) = 0$. This is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence and thus there exists $x_0 \in X$ such that $x_n \longrightarrow x_0$.

We show that $x_0 \in \bigcap_{n=1}^{\infty} C_n$. For $n_0 \in \mathbb{N}$, $x_n \in C_{n_0}$ for $n \geq n_0$ and thus $x_0 \in C_{n_0}$. Therefore, $x_0 \in \bigcap_{n=1}^{\infty} C_n$ and it completes the proof. □

4. Δ-Convergence

Let X be a metric space and $\{x_n\} \subset X$ a bounded sequence. An *asymptotic centre* $\mathrm{AC}(\{x_n\})$ of $\{x_n\}$ is defined by

$$\mathrm{AC}(\{x_n\}) = \left\{u \in X \mid \limsup_{n\longrightarrow\infty} d(u, x_n) = \inf_{x\in X} \limsup_{n\longrightarrow\infty} d(x, x_n)\right\}. \tag{38}$$

Lemma 9. *Let X be a sequentially uniformly convex complete geodesic space and let $\{x_n\} \subset X$ be a bounded sequence. Then, there exists a point $u \in X$ such that*

$$\limsup_{n\longrightarrow\infty} d(u, x_n) = \inf_{x\in X} \limsup_{n\longrightarrow\infty} d(x, x_n). \tag{39}$$

That is, $\mathrm{AC}(\{x_n\})$ is nonempty. Moreover, $\mathrm{AC}(\{x_n\})$ is bounded, closed and convex.

Proof. Let $M = \inf_{x\in X} \limsup_{n\longrightarrow\infty} d(x, x_n)$ and define $\{C_k\} \subset 2^X$ by

$$C_k = \left\{u \in X \mid M \leq \limsup_{n\longrightarrow\infty} d(u, x_n) \leq M + \frac{1}{k}\right\} \tag{40}$$

for any $k \in \mathbb{N}$. Then, for any $k \in \mathbb{N}$, C_k is nonempty and bounded. If $\{u_m\} \subset C_k$ and $u_m \longrightarrow u_0$, then

$$\begin{aligned} M &\leq \limsup_{n\longrightarrow\infty} d(u_0, x_n) \leq d(u_0, u_m) + \limsup_{n\longrightarrow\infty} d(u_m, x_n) \\ &\leq d(u_0, u_m) + M + \frac{1}{k}. \end{aligned} \tag{41}$$

Letting $m \longrightarrow \infty$, we have

$$M \leq \limsup_{n\longrightarrow\infty} d(u_0, x_n) \leq M + \frac{1}{k} \tag{42}$$

and hence $u_0 \in C_k$. Moreover, for $u, v \in C_k$ and $t \in [0,1]$, we have

$$\begin{aligned} M^2 &\leq \limsup_{n\longrightarrow\infty} d(tu \oplus (1-t)v, x_n)^2 \leq t \limsup_{n\longrightarrow\infty} d(u, x_n)^2 \\ &\quad + (1-t) \limsup_{n\longrightarrow\infty} d(v, x_n)^2 \leq \left(M + \frac{1}{k}\right)^2. \end{aligned} \tag{43}$$

It implies $tu \oplus (1-t)v \in C_k$. Therefore, C_k is closed and convex for any $k \in \mathbb{N}$. Moreover, $\{C_k\}$ is decreasing with respect to inclusion. Hence, we have

$$\bigcap_{k=1}^{\infty} C_k = \bigcap_{k=1}^{\infty} \left\{u \in X \mid M \leq \limsup_{n\longrightarrow\infty} d(u, x_n) \leq M + \frac{1}{k}\right\} \neq \varnothing \tag{44}$$

and thus

$$\mathrm{AC}(\{x_n\}) = \bigcap_{k=1}^{\infty} \left\{u \in X \mid M \leq \limsup_{n\longrightarrow\infty} d(u, x_n) \leq M + \frac{1}{k}\right\} \neq \varnothing. \tag{45}$$

Also, we know that $\mathrm{AC}(\{x_n\})$ is closed and convex. □

If we suppose uniform convexity for X, we can prove that an asymptotic centre is a singleton.

Theorem 10. *Let X be a uniformly convex complete geodesic space and let $\{x_n\}$ be a bounded sequence of X. Then, $\mathrm{AC}(\{x_n\})$ is a singleton.*

Proof. Let $M = \inf_{x\in X} \limsup_{n\longrightarrow\infty} d(x, x_n)$. If $u, v \in \mathrm{AC}(\{x_n\})$ with $u \neq v$, then, since $\{x_n\} \subset X$ is bounded, there exists $K>0$ such that $d(u,x_n)\le K$ and $d(v,x_n)\le K$ for all $n\in\mathbb{N}$. Then, there exists a convex gauge function $\underline{g}_K$, and we have

$$d\left(\frac{1}{2}u\oplus\frac{1}{2}v,x_n\right)^2\le\frac{1}{2}d(u,x_n)^2+\frac{1}{2}d(v,x_n)^2-\frac{1}{4}\underline{g}_K(d(u,v)) \tag{46}$$

and hence

$$\begin{aligned}\limsup_{n\longrightarrow\infty} d\left(\frac{1}{2}u\oplus\frac{1}{2}v,x_n\right)^2&\le\frac{1}{2}\limsup_{n\longrightarrow\infty} d(u,x_n)^2\\&+\frac{1}{2}\limsup_{n\longrightarrow\infty} d(v,x_n)^2-\frac{1}{4}\underline{g}_K(d(u,v))\le M^2\\&-\frac{1}{4}\underline{g}_K(d(u,v)).\end{aligned} \tag{47}$$

Since $d(u,v)\neq 0$, we have $\underline{g}_K(d(u,v))>0$ and thus

$$\limsup_{n\longrightarrow\infty} d\left(\frac{1}{2}u\oplus\frac{1}{2}v,x_n\right)^2<M^2. \tag{48}$$

This is a contradiction. Therefore, $\mathrm{AC}(\{x_n\})$ is a singleton. □

Let $\{x_n\}\subset X$ be a bounded sequence and $x_0\in X$. We say $\{x_n\}$ *Δ -converges to a Δ-limit* x_0 if x_0 is the unique asymptotic centre of any subsequences of $\{x_n\}$, and we denote it by $x_n \rightharpoonup^{\Delta} x_0$.

Let X be a geodesic space. X is said to satisfy the condition (ΔC), if any nonempty closed convex subset $C\subset X$ is Δ-closed, that is, if $\{x_n\}\subset C$ and $x_n\rightharpoonup^{\Delta} x_0$, then $x_0\in C$.

CAT(0) spaces satisfy the condition (ΔC). Let E be a uniformly convex real Banach space. Then, the following propositions are equivalent:

(i) For any bounded sequence $\{x_n\}\subset E$, $\{x_n\}$ Δ-converges to $x_0\in E$ if and only if $\{x_n\}$ converges weakly to $x_0\in E$

(ii) E satisfies the condition (ΔC)

(iii) E satisfies Opial's condition

See [5] for details.

The following two theorems can be proved by the same method as the corresponding results in [6–8].

Theorem 11 (Bačák [6], Kirk and Panyanak [8]). *Let X be a uniformly convex complete geodesic space. Then, for any bounded sequence of X has a Δ -convergent subsequence.*

Corollary 12. *Let X be a uniformly convex complete geodesic space satisfying the condition (ΔC) and let C be a nonempty bounded closed convex subset of X. Then, for any sequence in C has a Δ -convergent subsequence and its Δ-limit belongs to C.*

Theorem 13 (Bačák [6], He, Fang, Lopez and Li [7]). *Let X be a uniformly convex complete geodesic space satisfying the condition (ΔC). Let $f: X\longrightarrow\,]-\infty,\infty]$ be a proper lower semicontinuous function and $\{x_n\}\subset X$ a sequence such that $x_n\rightharpoonup^{\Delta} x_0\in\operatorname{dom} f$. Then,*

$$f(x_0)\le\liminf_{n\longrightarrow\infty} f(x_n). \tag{49}$$

Corollary 14 (Δ-lower semicontinuity of the distance function). *Let X be a uniformly convex complete geodesic space satisfying the condition (ΔC) and let $z\in X$ be a point. Let $\{x_n\}\subset X$ be a sequence such that $x_n\rightharpoonup^{\Delta} x_0$. Then,*

$$d(x_0,z)\le\liminf_{n\longrightarrow\infty} d(x_n,z). \tag{50}$$

Lemma 15. *Let X be a uniformly convex complete geodesic space and let $\{x_n\}\subset X$ be a sequence such that Δ-converges to $x_0\in X$. Then,*

$$\frac{1}{2}x_n\oplus\frac{1}{2}x_0\overset{\Delta}{\rightharpoonup}x_0. \tag{51}$$

Proof. Since $\{x_n\}$ is bounded, so is $\{1/2x_n\oplus 1/2x_0\}$. For any subsequence $\{1/2x_{n_i}\oplus 1/2x_0\}$ of $\{1/2x_n\oplus 1/2x_0\}$, if $\{y_0\}=\mathrm{AC}(\{1/2x_{n_i}\oplus 1/2x_0\})$, then, we have

$$\begin{aligned}\limsup_{i\longrightarrow\infty} d(y_0,x_{n_i})&\le\limsup_{i\longrightarrow\infty} d\left(y_0,\frac{1}{2}x_{n_i}\oplus\frac{1}{2}x_0\right)\\&+\limsup_{i\longrightarrow\infty} d\left(\frac{1}{2}x_{n_i}\oplus\frac{1}{2}x_0,x_{n_i}\right)\le\limsup_{i\longrightarrow\infty} d\left(x_0,\frac{1}{2}x_{n_i}\oplus\frac{1}{2}x_0\right)\\&+\limsup_{i\longrightarrow\infty} d\left(\frac{1}{2}x_{n_i}\oplus\frac{1}{2}x_0,x_{n_i}\right)\le\frac{1}{2}\limsup_{i\longrightarrow\infty} d(x_0,x_{n_i})\\&+\frac{1}{2}\limsup_{i\longrightarrow\infty} d(x_0,x_{n_i})=\limsup_{i\longrightarrow\infty} d(x_0,x_{n_i}).\end{aligned} \tag{52}$$

Therefore, since $\mathrm{AC}(\{x_{n_i}\})=\{x_0\}$, we have $y_0=x_0$. Hence, x_0 is the unique asymptotic centre of any subsequence of $\{1/2x_n\oplus 1/2x_0\}$ and it completes the proof. □

Theorem 16 (Δ-Kadec-Klee property). *Let X be a uniformly convex complete geodesic space satisfying the condition (ΔC) and $\{x_n\}\subset X$ a sequence such that $x_n\rightharpoonup^{\Delta} x_0$ and $d(x_n,p)\longrightarrow d(x_0,p)$ for some $p\in X$. Then, $x_n\longrightarrow x_0$.*

Proof. Let $\{y_n\}\subset X$ be a sequence such that $y_n=x_0$ for any $n\in\mathbb{N}$. Since $x_n\rightharpoonup^{\Delta} x_0$, From Lemma 15, we have $1/2x_n\oplus 1/2x_0\rightharpoonup^{\Delta} x_0$. Then, since $d(\cdot,p)$ is Δ-lower semicontinuous and $d(x_n,p)\longrightarrow d(x_0,p)$, we have

$$\begin{aligned}d(x_0,p)^2&\le\liminf_{n\longrightarrow\infty} d\left(\frac{1}{2}x_n\oplus\frac{1}{2}x_0,p\right)^2\le\limsup_{n\longrightarrow\infty} d\left(\frac{1}{2}x_n\oplus\frac{1}{2}x_0,p\right)^2\\&\le\frac{1}{2}\limsup_{n\longrightarrow\infty} d(x_n,p)^2+\frac{1}{2}\limsup_{n\longrightarrow\infty} d(x_0,p)^2=d(x_0,p)^2\end{aligned} \tag{53}$$

and thus $\lim_{n\longrightarrow\infty} d(1/2x_n \oplus 1/2x_0, p) = d(x_0, p)$. Therefore, we obtain

$$\lim_{n\longrightarrow\infty} d(x_n, p) = \lim_{n\longrightarrow\infty} d(y_n, p) = \lim_{n\longrightarrow\infty} d\left(\frac{1}{2}x_n \oplus \frac{1}{2}y_n, p\right) = d(x_0, p). \tag{54}$$

If $d(x_0, p) = 0$, then $x_n \longrightarrow p = x_0$. If $d(x_0, p) > 0$, from sequential uniform convexity of X, we obtain $\lim_{n\longrightarrow\infty} d(x_n, y_n) = 0$. Since $y_n = x_0$ for any $n \in \mathbb{N}$, we have $x_n \longrightarrow x_0$. ☐

5. Convergence of a Sequence of Sets

Let $\{C_n\} \subset 2^X$ be a sequence of nonempty closed convex subsets of a uniformly convex complete geodesic space X. Δ-Mosco convergence is defined by using a notion of asymptotic centre by Kimura [2]. First, we define subsets $d\text{-}\mathrm{Li}_n C_n$ and $\Delta\text{-}\mathrm{Ls}_n C_n$ of X as follows: $x \in d\text{-}\mathrm{Li}_n C_n$ if and only if there exists $\{x_n\} \subset X$ such that $x_n \longrightarrow x$ and $x_n \in C_n$ for all $n \in \mathbb{N}$; $y \in \Delta\text{-}\mathrm{Ls}_n C_n$ if and only if there exist a bounded sequence $\{y_i\} \subset X$ and a subsequence $\{n_i\}$ of $\mathbb{N}$ such that $\{y\} = \mathrm{AC}(\{y_i\})$ and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If a subset C_0 of X satisfies that $C_0 = d\text{-}\mathrm{Li}_n C_n = \Delta\text{-}\mathrm{Ls}_n C_n$, we say that $\{C_n\}$ converges to C_0 in the sense of Δ-Mosco.

Here, we introduce a new concept of the set-convergence. We define subsets $\bar{\Delta}\text{-}\mathrm{Ls}_n C_n$ of X as follows: $y \in \bar{\Delta}\text{-}\mathrm{Ls}_n C_n$ if and only if there exists a bounded sequence $\{y_i\} \subset X$ and a subsequence $\{n_i\}$ of $\mathbb{N}$ such that $y_i \rightharpoonup^{\Delta} y$ and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$.

Since a convergent sequence is a Δ-convergent sequence, the inclusion

$$d\text{-}\mathrm{Li}_n C_n \subset \bar{\Delta}\text{-}\mathrm{Ls}_n C_n \tag{55}$$

is always true. If a subset C_0 of X satisfies that $C_0 = d\text{-}\mathrm{Li}_n C_n = \bar{\Delta}\text{-}\mathrm{Ls}_n C_n$, we say that $\{C_n\}$ converges to C_0 in the sense of $\bar{\Delta}$-Mosco. Furthermore, the following inclusion holds:

$$d\text{-}\mathrm{Li}_n C_n \subset \bar{\Delta}\text{-}\mathrm{Ls}_n C_n \subset \Delta\text{-}\mathrm{Ls}_n C_n. \tag{56}$$

Therefore, if $\{C_n\}$ converges to C_0 in the sense of Δ-Mosco, then $\{C_n\}$ converges to C_0 in the sense of $\bar{\Delta}$-Mosco.

Lemma 17. *Let X be a uniformly convex complete geodesic space satisfying the condition (ΔC) and $\{C_n\} \subset 2^X$ a sequence of nonempty closed convex sets which is decreasing with respect to inclusion, that is, $C_{n+1} \subset C_n$ for any $n \in \mathbb{N}$. if $C_0 = \bigcap_{n=1}^{\infty} C_n$ is nonempty, then $\{C_n\}$ converges to C_0 in the sense of $\bar{\Delta}$ -Mosco.*

Proof. We show that $\bar{\Delta}\text{-}\mathrm{Ls}_n C_n \subset C_0 \subset d\text{-}\mathrm{Li}_n C_n$. If $x \in C_0 = \bigcap_{n=1}^{\infty} C_n$, then $x \in C_n$ for any $n \in \mathbb{N}$. Let $\{x_n\}$ be a sequence of X such that $x_n = x$ for any $n \in \mathbb{N}$. Then, $x_n \longrightarrow x$ and $x_n \in C_n$ for any $n \in \mathbb{N}$. Therefore, we obtain $x \in d\text{-}\mathrm{Li}_n C_n$ and hence $C_0 \subset d\text{-}\mathrm{Li}_n C_n$.

Next, we show that $\bar{\Delta}\text{-}\mathrm{Ls}_n C_n \subset C_0$. If $y \in \bar{\Delta}\text{-}\mathrm{Ls}_n C_n$, then there exists a sequence $\{y_i\} \subset X$ and $\{n_i\} \subset \mathbb{N}$ such that $y_i \in C_{n_i}$ for any $i \in \mathbb{N}$ and $y_i \rightharpoonup^{\Delta} y$. For any $i_0 \in \mathbb{N}$, if $i \geq i_0$, then $\{y_i\} \subset C_{n_{i_0}}$. Therefore, from the condition (ΔC) of X, we have $y_i \rightharpoonup^{\Delta} y \in C_{n_{i_0}}$. Since $i_0 \in \mathbb{N}$ is arbitrarily, we obtain $y \in \bigcap_{n=1}^{\infty} C_n = C_0$. Hence, we have $\bar{\Delta}\text{-}\mathrm{Ls}_n C_n \subset C_0 \subset d\text{-}\mathrm{Li}_n C_n$.

Since $d\text{-}\mathrm{Li}_n C_n \subset \bar{\Delta}\text{-}\mathrm{Ls}_n C_n$ is always true, we have

$$\bar{\Delta}\text{-}\mathrm{Ls}_n C_n = C_0 = d\text{-}\mathrm{Li}_n C_n. \tag{57}$$

That is, $\{C_n\}$ converges to C_0 in the sense of $\bar{\Delta}$-Mosco. ☐

Theorem 18. *Let X be a uniformly convex complete geodesic space satisfying the condition (ΔC). Let $\{C_n\} \subset 2^X$ be a sequence of nonempty closed convex sets and C_0 a nonempty closed convex subset of X. If $\{C_n\}$ converges to C_0 in the sense of $\bar{\Delta}$-Mosco, then $\{P_{C_n} x\}$ converges to $P_{C_0} x \in X$ for any $x \in X$, where $P_K : X \longrightarrow K$ is the metric projection of X onto a nonempty closed convex subset K of X.*

Proof. Fix $x \in X$ arbitrarily. Since $P_{C_0} x \in C_0 = d\text{-}\mathrm{Li}_n C_n$, there exists a sequence $\{x_n\} \subset X$ such that $x_n \longrightarrow P_{C_0} x$ and $x_n \in C_n$ for every $n \in \mathbb{N}$. Since $d(x, P_{C_n} x) \leq d(x, x_n)$ for any $n \in \mathbb{N}$ and $\{x_n\}$ is bounded, $\{P_{C_n} x\}$ is also bounded. Moreover, letting $n \longrightarrow \infty$, we have

$$\limsup_{n\longrightarrow\infty} d\left(x, P_{C_n} x\right) \leq d\left(x, P_{C_0} x\right). \tag{58}$$

Here, we take a subsequence $\{P_{C_{n_i}} x\}$ of $\{P_{C_n} x\}$ arbitrarily. Since $\{P_{C_{n_i}} x\}$ is bounded, there exists a subsequence $\{P_{C_{n_{i_j}}} x\}$ of $\{P_{C_{n_i}} x\}$ and $p_0 \in X$ such that

$$P_{C_{n_{i_j}}} x \overset{\Delta}{\rightharpoonup} p_0. \tag{59}$$

Let $p_j = P_{C_{n_{i_j}}} x$ for any $j \in \mathbb{N}$. Then, since

$$\limsup_{j\longrightarrow\infty} d\left(x, p_j\right) \leq d\left(x, P_{C_0} x\right), \tag{60}$$

for any $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that for any $j \geq j_0$, it holds that

$$d\left(x, p_j\right) \leq d\left(x, P_{C_0} x\right) + \varepsilon. \tag{61}$$

Let $D = \{y \in X \mid d(x, y) \leq d(x, P_{C_0} x) + \varepsilon\}$. Then we have $p_j \in D$ for any $j \geq j_0$. Since D is closed and convex, from the condition (ΔC) of X, we obtain $p_j \rightharpoonup^{\Delta} p_0 \in D$. Therefore, we have

$$d(x, p_0) \leq d\left(x, P_{C_0} x\right) + \varepsilon. \tag{62}$$

Letting $\varepsilon \searrow 0$, we obtain

$$d(x, p_0) \le d(x, P_{C_0}x). \tag{63}$$

Since $p_0 \in \bar{\Delta}\text{-}\mathrm{Ls}_n C_n = C_0$, we have $d(x, P_{C_0}x) \le d(x, p_0)$ and thus

$$d(x, P_{C_0}x) = d(x, p_0). \tag{64}$$

Hence, we obtain $p_j \rightharpoonup^{\Delta} P_{C_0}x$. Since $d(x, \cdot)$ is Δ-lower semicontinuous and $p_j \rightharpoonup^{\Delta} P_{C_0}x$, we have

$$d(x, P_{C_0}x) \le \liminf_{j \longrightarrow \infty} d(x, p_j) \tag{65}$$

and hence

$$\begin{aligned} d(x, P_{C_0}x) &\le \liminf_{j \longrightarrow \infty} d(x, p_j) \le \limsup_{j \longrightarrow \infty} d(x, p_j) \\ &\le \limsup_{n \longrightarrow \infty} d(x, P_{C_n}x) \le d(x, P_{C_0}x). \end{aligned} \tag{66}$$

Therefore, $d(x, p_j) \longrightarrow d(x, P_{C_0}x)$. Since $p_j \rightharpoonup^{\Delta} P_{C_0}x$ and $d(x, p_j) \longrightarrow d(x, P_{C_0}x)$, from the Δ-Kadec-Klee property of X, we obtain $p_j \longrightarrow P_{C_0}x$. Since for any subsequence $\{P_{C_{n_i}}x\}$ of $\{P_{C_n}x\}$, there exists a subsequence $\{P_{C_{n_{i_j}}}x\}$ of $\{P_{C_{n_i}}x\}$ such that $P_{C_{n_{i_j}}}x \longrightarrow P_{C_0}x$, we have $P_{C_n}x \longrightarrow P_{C_0}x$. □

Let X be a uniformly convex complete geodesic space. If X has the condition (ΔC), we can show that convergence of a sequence of metric projections from $\bar{\Delta}$-Mosco convergence. If a sequence of sets which is decreasing with respect to inclusion, then we can show it on sequentially uniformly convex complete geodesic spaces without the condition (ΔC).

Theorem 19. *Let X be a sequentially uniformly convex complete geodesic space and $\{C_n\} \subset 2^X$ a sequence of nonempty closed convex sets which is decreasing with respect to inclusion, that is, $C_{n+1} \subset C_n$ for any $n \in \mathbb{N}$. Suppose that $C_0 = \bigcap_{n=1}^{\infty} C_n$ is nonempty. Then, $\{P_{C_n}x\}$ converges to $P_{C_0}x \in X$ for any $x \in X$, where $P_K : X \longrightarrow K$ is the metric projection of X onto a nonempty closed convex subset K of X.*

Proof. Fix $x \in X$ arbitrarily. Since $C_0 = \bigcap_{n=1}^{\infty} C_n \neq \emptyset$, if $p \in C_0$, then for any $n \in \mathbb{N}$, we have

$$d(x, P_{C_n}x) \le d(x, p) \tag{67}$$

and hence $\{d(x, P_{C_n}x)\}$ is bounded. Furthermore, since $C_{n+1} \subset C_n$ for any $n \in \mathbb{N}$, we have

$$d(x, P_{C_n}x) \le d(x, P_{C_{n+1}}x) \tag{68}$$

and thus $\{d(x, P_{C_n}x)\}$ is increasing. Therefore, $\{d(x, P_{C_n}x)\}$ has a limit

$$c = \lim_{n \longrightarrow \infty} d(x, P_{C_n}x). \tag{69}$$

First, we show that $\{P_{C_n}x\}$ converges to some point $x_0 \in X$. Since $P_{C_n}x \longrightarrow x$ if $c = 0$, we may assume that $c > 0$. Suppose that $\{P_{C_n}x\}$ is not a Cauchy sequence. That is, there exist $\varepsilon > 0$ such that for any $i \in \mathbb{N}$, there exist $m_i, n_i \ge i$ such that $d(P_{C_{m_i}}x, P_{C_{n_i}}x) \ge \varepsilon$. Without loss of generality, we can suppose that $m_i \ge n_i$. In this way, we take two sequences $\{P_{C_{m_i}}x\}, \{P_{C_{n_i}}x\} \subset \{P_{C_n}x\}$. Then, $\{P_{C_{m_i}}x\}$ and $\{P_{C_{n_i}}x\}$ satisfies

$$\lim_{i \longrightarrow \infty} d(x, P_{C_{m_i}}x) = \lim_{i \longrightarrow \infty} d(x, P_{C_{n_i}}x) = c. \tag{70}$$

Since $P_{C_{m_i}}x, P_{C_{n_i}}x \in C_{n_i}$, we obtain

$$\begin{aligned} d(x, P_{C_{n_i}}x)^2 &\le d\left(x, \frac{1}{2}P_{C_{m_i}}x \oplus \frac{1}{2}P_{C_{n_i}}x\right)^2 \\ &\le \frac{1}{2}d(x, P_{C_{m_i}}x)^2 + \frac{1}{2}d(x, P_{C_{n_i}}x)^2. \end{aligned} \tag{71}$$

Letting $i \longrightarrow \infty$, we obtain

$$\lim_{i \longrightarrow \infty} d\left(x, \frac{1}{2}P_{C_{m_i}}x \oplus \frac{1}{2}P_{C_{n_i}}x\right) = c. \tag{72}$$

From sequential uniform convexity of X, we have

$$\lim_{i \longrightarrow \infty} d(P_{C_{m_i}}x, P_{C_{n_i}}x) = 0. \tag{73}$$

This is contradictory to

$$d(P_{C_{m_i}}x, P_{C_{n_i}}x) \ge \varepsilon > 0 \tag{74}$$

and hence $\{P_{C_n}x\}$ is a Cauchy sequence. Therefore, since X is complete, there exists a point $x_0 \in X$ such that $P_{C_n}x \longrightarrow x_0$.

Next, we show that $x_0 \in C_0$. For any $n_0 \in \mathbb{N}$, since $\{P_{C_n}x\} \subset C_{n_0}$ if $n \ge n_0$ and C_{n_0} is closed, we have $x_0 \in C_{n_0}$ and thus $x_0 \in \bigcap_{n=1}^{\infty} C_n$.

Finally, we show that $x_0 = P_{C_0}x$. For any $n \in \mathbb{N}$, form the property of the metric projection, we have

$$d(x, P_{C_n}x) \le d(x, P_{C_0}x) \le d(x, x_0). \tag{75}$$

Letting $n \longrightarrow \infty$, we obtain

$$d(x, P_{C_0}x) = d(x, x_0) \tag{76}$$

and hence $x_0 = P_{C_0}x$.

Consequently, we have $P_{C_n}x \longrightarrow P_{C_0}x$ for any $x \in X$. □

6. Convergence of a Sequence of Sets in CAT(κ) Spaces

In this section, we will consider CAT(κ) spaces for $\kappa \in \mathbb{R}$.

Let X be a metric space and let $\{x_n\}$ be a bounded sequence of X. The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ defined by

$$r(\{x_n\}) = \inf_{x\in X} \limsup_{n\longrightarrow\infty} d(x, x_n). \tag{77}$$

Let $\{C_n\} \subset 2^X$ be a sequence of nonempty closed convex subsets of a complete admissible CAT(κ) space for $\kappa \in \mathbb{R}$. Δ_κ-Mosco convergence is defined by Kimura and Satô [3]. First, we define subsets d-$\mathrm{Li}_n C_n$ and Δ_κ-$\mathrm{Ls}_n C_n$ of X as follows: d-$\mathrm{Li}_n C_n$ is the same as in the case of uniformly convex complete geodesic spaces, that is, $x \in d$-$\mathrm{Li}_n C_n$ if and only if there exists $\{x_n\} \subset X$ such that $x_n \longrightarrow x$ and $x_n \in C_n$ for all $n \in \mathbb{N}$; $y \in \Delta_\kappa$-$\mathrm{Ls}_n C_n$ if and only if there exist a sequence $\{y_i\} \subset X$ and a subsequence $\{n_i\}$ of $\mathbb{N}$ such that $r(\{y_i\}) < D_\kappa/2$, $\{y\} = \mathrm{AC}(\{y_i\})$ and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If a subset C_0 of X satisfies that $C_0 = d$-$\mathrm{Li}_n C_n = \Delta_\kappa$-$\mathrm{Ls}_n C_n$, it is said that $\{C_n\}$ converges to C_0 in the sense of Δ_κ-Mosco. The notion of Δ_κ-Mosco convergence coincides with Δ-Mosco convergence in complete CAT(κ) spaces for $\kappa \le 0$.

We define a subset $\bar{\Delta}_\kappa$-$\mathrm{Ls}_n C_n$ of X as follows: $y \in \bar{\Delta}_\kappa$-$\mathrm{Ls}_n C_n$ if and only if there exist a sequence $\{y_i\} \subset X$ and a subsequence $\{n_i\}$ of $\mathbb{N}$ such that $r(\{y_i\}) < D_\kappa/2$, $y_i \rightharpoonup^{\Delta} y$ and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If $\kappa \le 0$, then $\bar{\Delta}$-$\mathrm{Ls}_n C_n = \bar{\Delta}_\kappa$-$\mathrm{Ls}_n C_n$.

Since a convergent sequence is a Δ-convergent sequence, the inclusion

$$d\text{-}\mathrm{Li}_n C_n \subset \bar{\Delta}_\kappa\text{-}\mathrm{Ls}_n C_n \tag{78}$$

is always true. If a subset C_0 of X satisfies that $C_0 = d$-$\mathrm{Li}_n C_n = \bar{\Delta}_\kappa$-$\mathrm{Ls}_n C_n$, it is said that $\{C_n\}$ converges to C_0 in the sense of $\bar{\Delta}_\kappa$-Mosco. Furthermore, the following inclusion holds:

$$d\text{-}\mathrm{Li}_n C_n \subset \bar{\Delta}_\kappa\text{-}\mathrm{Ls}_n C_n \subset \Delta_\kappa\text{-}\mathrm{Ls}_n C_n. \tag{79}$$

Therefore, if $\{C_n\}$ converges to C_0 in the sense of Δ_κ-Mosco, then $\{C_n\}$ converges to C_0 in the sense of $\bar{\Delta}_\kappa$-Mosco.

$\bar{\Delta}_\kappa$-Mosco convergence coincides with $\bar{\Delta}$-Mosco convergence in complete CAT(κ) spaces for $\kappa \le 0$.

Since complete CAT(κ) space is uniformly convex and satisfies the condition (ΔC) for any $\kappa \le 0$, we have the following theorem:

Theorem 20. *Let X be a complete CAT(κ) space for $\kappa \le 0$. Let $\{C_n\} \subset 2^X$ be a sequence of nonempty closed convex sets and C_0 a nonempty closed convex subset of X. If $\{C_n\}$ is converges to C_0 in the sense of $\bar{\Delta}$-Mosco, then $\{P_{C_n}x\}$ converges to $P_{C_0}x \in X$ for any $x \in X$, where $P_K : X \longrightarrow K$ is the metric projection of X onto a nonempty closed convex subset K of X.*

We also have the following result [2].

Theorem 21 (Kimura [2]). *Let X be a complete CAT(κ) space for $\kappa \le 0$ and $P_K : X \longrightarrow K$ is the metric projection of X onto a nonempty closed convex subset K of X. Let $\{C_n\} \subset 2^X$ be a sequence of nonempty closed convex sets and C_0 a nonempty closed convex subset of X. If $\{P_{C_n}x\}$ converges to $P_{C_0}x \in X$ for any $x \in X$, then $\{C_n\}$ converges to C_0 in the sense of Δ-Mosco.*

Let X be a complete CAT(κ) space $\kappa \le 0$ and $P_K : X \longrightarrow K$ is the metric projection of X onto a nonempty closed convex subset K of X. Let $\{C_n\} \subset 2^X$ be a sequence of nonempty closed convex sets and C_0 a nonempty closed convex subset of X. Then, from Theorem 19 and Theorem 21, the following propositions are equivalent:

(i) $\{C_n\}$ converges to C_0 in the sense of Δ-Mosco

(ii) $\{C_n\}$ converges to C_0 in the sense of $\bar{\Delta}$-Mosco

(iii) $\{P_{C_n}x\}$ converges to $P_{C_0}x$ for any $x \in X$

That is, the notion of Δ_κ-Mosco convergence coincides with $\bar{\Delta}_\kappa$-Mosco convergence in complete CAT(κ) spaces for $\kappa \le 0$.

Since most of the results for CAT(κ) spaces for $\kappa > 0$ are easily deduced from that for CAT(1) spaces, in the following, we focus on CAT(1) spaces.

We first see the following known results for CAT(1) spaces.

Theorem 22 (Espínola and Fernádez-León [9]). *Let X be a complete CAT(1) space. If a sequence $\{x_n\} \subset X$ satisfies $r(\{x_n\}) < \pi/2$, then there exists a Δ-convergent subsequence of $\{x_n\}$.*

Theorem 23 (Espínola and Fernádez-León [9]). *Let X be a complete admissible CAT(1) space and let $\{x_n\} \subset X$ be a sequence satisfies $r(\{x_n\}) < \pi/2$. Then,*

$$AC(\{x_n\}) \subset \bigcap_{k=0}^{\infty} clco\{x_k, x_{k+1}, x_{k+2}, \cdots\}. \tag{80}$$

Theorem 24 (He, Fang, Lopez, and Li [7]). *Let X be a complete CAT(1) space and let $\{x_n\} \subset X$ be a sequence such that $x_n \rightharpoonup^{\Delta} x_0 \in X$. Then for any $p \in X$ with $\limsup_{n\longrightarrow\infty} d(p, x_n) < \pi/2$, the following inequality holds:*

$$d(p, x_0) \le \liminf_{n\longrightarrow\infty} d(p, x_n). \tag{81}$$

Theorem 25 (Kimura and Satô [3]). *Let X be a complete admissible CAT(1) space and let $\{x_n\} \subset X$ be a sequence. Suppose that $x_n \rightharpoonup^{\Delta} x_0 \in X$ and $d(x_n, p) \longrightarrow d(x_0, p)$. Then $x_n \longrightarrow x_0$.*

Using these results, we obtain the relation between $\bar{\Delta}_1$-Mosco convergence of sets and corresponding sequence of metric projections.

Theorem 26. *Let X be a complete admissible CAT(1) space. Let $\{C_n\} \subset 2^X$ be a sequence of nonempty closed convex sets and C_0 a nonempty closed convex subset of X. If $\{C_n\}$ converges to C_0 in the sense of $\bar{\Delta}_1$-Mosco, then $\{P_{C_n}x\}$ converges to $P_{C_0}x \in X$ for any $x \in X$, where $P_K : X \longrightarrow K$ is the metric projection of X onto a nonempty closed convex subset K of X.*

Proof. Fix $x \in X$ arbitrarily. Since $P_{C_0}x \in C_0 = d\text{-Li}_n C_n$, there exists a sequence $\{x_n\} \subset X$ such that $x_n \longrightarrow P_{C_0}x$. Since $d(x, P_{C_n}x) \le d(x, x_n)$ for any $n \in \mathbb{N}$. Moreover, letting $n \longrightarrow \infty$, we obtain

$$\limsup_{n\longrightarrow\infty} d\left(x, P_{C_n}x\right) \le d\left(x, P_{C_0}x\right) < \frac{\pi}{2}. \tag{82}$$

Here, we take a subsequence $\{P_{C_{n_i}}x\}$ of $\{P_{C_n}x\}$ arbitrarily. From Theorem 22, there exists a subsequence $\{P_{C_{n_{i_j}}}x\}$ of $\{P_{C_{n_i}}x\}$ and $p_0 \in X$ such that

$$P_{C_{n_{i_j}}}x \overset{\Delta}{\rightharpoonup} p_0. \tag{83}$$

Let $p_j = P_{C_{n_{i_j}}}x$ for any $j \in \mathbb{N}$. Then, since

$$\limsup_{j\longrightarrow\infty} d\left(x, p_j\right) \le d\left(x, P_{C_0}x\right) < \frac{\pi}{2}, \tag{84}$$

for any $\varepsilon \in]0, (\pi/2)\text{-}d(x, P_{C_0}x)[$ there exists $j_0 \in \mathbb{N}$ such that for any $j \ge j_0$, it holds that

$$d\left(x, p_j\right) \le d\left(x, P_{C_0}x\right) + \varepsilon < \frac{\pi}{2}. \tag{85}$$

Let $D = \{y \in X \mid d(x, y) \le d(x, P_{C_0}x) + \varepsilon\}$. Then we have $p_j \in D$ for any $j \ge j_0$. Since D is closed and convex, from Theorem 23, we have $p_j \rightharpoonup^{\Delta} p_0 \in D$. Therefore, we have

$$d(x, p_0) \le d\left(x, P_{C_0}x\right) + \varepsilon. \tag{86}$$

Letting $\varepsilon \searrow 0$, we obtain

$$d(x, p_0) \le d\left(x, P_{C_0}x\right). \tag{87}$$

Since $p_0 \in \bar{\Delta}_1\text{-Ls}_n C_n = C_0$, we have $d(x, P_{C_0}x) \le d(x, p_0)$ and thus

$$d\left(x, P_{C_0}x\right) = d(x, p_0). \tag{88}$$

Hence, we obtain $p_j \rightharpoonup^{\Delta} P_{C_0}x$. From Theorem 24, we have

$$d\left(x, P_{C_0}x\right) \le \liminf_{j\longrightarrow\infty} d\left(x, p_j\right) \tag{89}$$

and hence

$$\begin{aligned} d\left(x, P_{C_0}x\right) &\le \liminf_{j\longrightarrow\infty} d\left(x, p_j\right) \le \limsup_{j\longrightarrow\infty} d\left(x, p_j\right) \\ &\le \limsup_{n\longrightarrow\infty} d\left(x, P_{C_n}x\right) \le d\left(x, P_{C_0}x\right). \end{aligned} \tag{90}$$

Therefore, $d(x, p_j) \longrightarrow d(x, P_{C_0}x)$. Since $p_j \rightharpoonup^{\Delta} P_{C_0}x$ and $d(x, p_j) \longrightarrow d(x, P_{C_0}x)$, from Theorem 25, we have $p_j \longrightarrow P_{C_0}x$. Since for any subsequence $\{P_{C_{n_i}}x\}$ of $\{P_{C_n}x\}$, there exists a subsequence $\{P_{C_{n_{i_j}}}x\}$ of $\{P_{C_{n_i}}x\}$ such that $P_{C_{n_{i_j}}}x \longrightarrow P_{C_0}x$, we have $P_{C_n}x \longrightarrow P_{C_0}x$. □

We also have the following result [3].

Theorem 27 (Kimura and Satô [3]). *Let X be a complete admissible CAT(1) space and $P_K : X \longrightarrow K$ is the metric projection of X onto a nonempty closed convex subset K of X. Let $\{C_n\} \subset 2^X$ be a sequence of nonempty closed convex sets and C_0 a nonempty closed convex subset of X. If $\{P_{C_n}x\}$ converges to $P_{C_0}x \in X$ for any $x \in X$, then $\{C_n\}$ converges to C_0 in the sense of Δ_κ -Mosco.*

Let X be a complete admissible CAT(1) space and $P_K : X \longrightarrow K$ is the metric projection of X onto a nonempty closed convex subset K of X. Let $\{C_n\} \subset 2^X$ be a sequence of nonempty closed convex sets and C_0 a nonempty closed convex subset of X. Then, from Theorem 26 and Theorem 27, the following propositions are equivalent:

(i) $\{C_n\}$ converges to C_0 in the sense of Δ_1-Mosco

(ii) $\{C_n\}$ converges to C_0 in the sense of $\bar{\Delta}_1$-Mosco

(iii) $\{P_{C_n}x\}$ converges to $P_{C_0}x$ for any $x \in X$

That is, Δ_1-Mosco convergence coincides with $\bar{\Delta}_1$-Mosco convergence in complete admissible CAT(1) spaces.

For the case where $\kappa > 0$, by using standard modification, we can obtain the same result. That is, Δ_κ-Mosco convergence coincides with $\bar{\Delta}_\kappa$-Mosco convergence in complete admissible CAT(κ) spaces for $\kappa > 0$.

Consequently, Δ_κ-Mosco convergence coincides with $\bar{\Delta}_\kappa$-Mosco convergence for every $\kappa \in \mathbb{R}$ and we can replace the definition. That is, if a subset C_0 of a complete admissible CAT(κ) space X for $\kappa \in \mathbb{R}$ satisfies that

$$C_0 = d\text{-Li}_n C_n = \bar{\Delta}_\kappa\text{-Ls}_n C_n, \tag{91}$$

then we can say that $\{C_n\}$ converges to C_0 in the sense of Δ_κ-Mosco.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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