## Research Article

# A New Class of Function with Finitely Many Fixed Points 

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The study introduced a generalized multiplier operator used as a tool to define and investigate a new class of function, TS $Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}(\omega, n)$ and its subclass $T S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}\left(\omega, n ; M_{i}\right)$. Various properties of the class of functions were investigated. The results extend some known results in literature.

## 1. Introduction and Preliminaries

Let

$$
\begin{equation*}
f(z)=z+\sum_{k=2}^{\infty} a_{k} z^{k} \tag{1}
\end{equation*}
$$

which are analytic in the open unit disk $\mathbb{U}=\{z \in \mathbb{C}:|z|<1\}$ and normalized by $f(0)=0$ and $f^{\prime}(0)=1$ be denoted by $S$.

We denote by $T$ the subclass of $S$ which are normalized univalent function of the form

$$
\begin{equation*}
f(z)=z-\sum_{k=2}^{\infty} a_{k} z^{k}, \quad a_{k}>0 \tag{2}
\end{equation*}
$$

in the open unit disk $\mathbb{U}$. The function of the form (2) was first introduced in [1].
1.1. Differential Operators (Multiplier Transformations). In 1975, Ruscheweyh with the use of convolution introduced a differential operator $D^{n}$ as a tool to investigate a certain class of univalent functions. The continuous extension and generalization of the tool have since then continue to gain attention among the function theorists. The differential operator and some of such extensions are discussed as follows.
1.1.1. Ruscheweyh Differential Operator (1975). Let $n \in \mathbb{N}_{0}$ $=\{0,1,2, \cdots\}$. The Ruscheweyh derivative of $n^{\text {th }}$ order denoted by $R^{n} f(z)$ is defined by

$$
\begin{equation*}
R^{n} f(z)=z+\sum_{k=2}^{\infty} B_{k}(n) a_{k} z^{k} \tag{3}
\end{equation*}
$$

where

$$
\begin{align*}
B_{k} & =B(n, k)=\binom{n+k-1}{n}=\frac{(n+1)(n+2) \cdots(n+k-1)}{(k-1)!} \\
& =\frac{(n+1)(n+2) \cdots(n+k-1)}{(k-1)!}=\frac{(n+1)_{k-1}}{(1)_{k-1}} . \tag{4}
\end{align*}
$$

Hence,

$$
\begin{equation*}
B(0, k)=\binom{k-1}{0}=\frac{(n+1)_{k-1}}{(1)_{k-1}}=1 \tag{5}
\end{equation*}
$$

So that $R^{0} f(z)$ is of the form (1).
1.1.2. Salagean Differential Operator (1983). Salagean defined an operator $D^{n}$ as follows.

Let $f(z) \in A$, the differential operator $D^{n} f(z)$ is defined as

$$
\begin{equation*}
D^{n} f(z)=z+\sum_{k=2}^{\infty} k^{n} a_{k} z^{k}, \quad n \in \mathbb{N}_{0} \tag{6}
\end{equation*}
$$

Since the introduction of differential operators in geometric function theory by Ruscheweyh in [2] and Salagean in [3], several attempts have been made to extend and generalize the operators by various authors and researchers in the field. According to Aldawish et al. in [4], the differential operator forms the link between function theory and mathematical physics. Perhaps, this is the reason why the study of differential operators is growing interest in Geometric Functions Theory and Application (GFTA). Some of the numerous differential operators (multiplier transformations) established by some authors are found in [5-11].

The Salagean differential operator (6) defined in [3] appears to be the most famous and most referred to in general.
1.2. A Generalized Multiplier Transformation. In this work, a generalized operator is defined and used to investigate the class $T S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}(\omega, n)$ and its subclass $T S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}\left(\omega, n ; M_{i}\right)$ with their geometric properties. The results improve some existing results in literature as pointed out in the work. The work was motivated by [12].

Let $\mu, \lambda, \eta, \sigma, \xi, \varepsilon \geq 0$ with $\omega>0$ such that $\xi<\omega$ and $n \in$ $\mathbb{N}_{0}=\mathbb{N} \cup\{0\}$. For $f \in S$, we define the following differential operator as follows:

$$
\begin{equation*}
D_{\mu, \lambda, \eta}^{0}(\sigma, \xi, \omega, \varepsilon) f(z)=f(z) \tag{7}
\end{equation*}
$$

$$
\begin{align*}
D_{\mu, \lambda, \eta}^{1}(\sigma, \xi, \omega, \varepsilon) f(z)= & \left(1-\frac{\lambda(\omega-\xi)^{\sigma}}{\mu+\omega}\right) f(z)+\frac{\lambda(\omega-\xi)^{\sigma}}{\mu+\omega} z f^{\prime}(z) \\
& +\frac{\eta+\varepsilon}{\mu+\omega} z^{2} f^{\prime \prime}(z) \\
\left(1-\frac{\lambda(\omega-\xi)^{\sigma}}{\mu+\omega}\right) f(z)= & \left(1-\frac{\lambda(\omega-\xi)^{\sigma}}{\mu+\eta}\right) z \\
& +\sum_{k=2}^{\infty}\left(1-\frac{\lambda(\omega-\xi)^{\sigma}}{\mu+\omega}\right) a_{k} z^{k},
\end{align*}
$$

$$
\begin{equation*}
\left(\frac{\eta+\varepsilon}{\mu+\omega}\right) z^{2} f^{\prime \prime}(z)=\sum_{k=2}^{\infty}\left(\frac{k(k-1)(\delta+\varepsilon)}{\mu+\omega}\right) a_{k} z^{k} \tag{11}
\end{equation*}
$$

Substituting (9), (10), and (11) in (8), we have

$$
\begin{align*}
& D_{\mu, \lambda, \eta}^{1}(\sigma, \xi, \omega, \varepsilon) f(z)=z+\sum_{k=2}^{\infty}\left\{1-\frac{\lambda(\omega-\xi)^{\sigma}}{\mu+\omega}+\frac{k \lambda(\omega-\xi)^{\sigma}}{\mu+\omega}+\frac{k(k-1)(\eta+\varepsilon)}{\mu+\omega}\right\} a_{k} z^{k} \\
& D_{\mu, \lambda, \eta}^{1}(\sigma, \xi, \omega, \varepsilon) f(z)=z+\sum_{k=2}^{\infty}\left(1+\frac{(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right]}{\mu+\omega}\right) a_{k} z^{k} \tag{12}
\end{align*}
$$

Similarly,

$$
\begin{equation*}
D_{\mu, \lambda, \eta}^{n}(\sigma, \xi, \omega, \varepsilon) f(z)=z\left(D_{\mu, \lambda, \eta}^{n-1}(\sigma, \xi, \omega, \varepsilon) f(z)\right) \tag{13}
\end{equation*}
$$

Thus,

$$
\begin{align*}
& D_{\mu, \lambda, \eta}^{n}(\sigma, \xi, \omega, \varepsilon) f(z)=z \\
& \quad+\sum_{k=2}^{\infty}\left(1+\frac{(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right]}{\mu+\omega}\right)^{n} a_{k} z^{k} \tag{14}
\end{align*}
$$

For convenience, for $f \in A$, we let

$$
\begin{equation*}
D_{\mu, \lambda, \eta}^{n}(\sigma, \xi, \omega, \varepsilon) f(z)=z+\sum_{k=2}^{\infty} \Phi(n, k) a_{k} z^{k} \tag{15}
\end{equation*}
$$

where $\Phi(n, k)=\left(1+(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right] / \mu+\omega\right)^{n}$.

Similarly, if $f \in T$ as defined in (2), then

$$
\begin{align*}
& D_{\mu, \lambda, \eta}^{n}(\sigma, \xi, \omega, \varepsilon) f(z) \\
& \quad=z-\sum_{k=2}^{\infty}\left(1+\frac{(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right]}{\mu+\omega}\right)^{n} a_{k} z^{k}, \tag{16}
\end{align*}
$$

$$
\begin{equation*}
D_{\mu, \lambda, \eta}^{n}(\sigma, \xi, \omega, \varepsilon) f(z)=z-\sum_{k=2}^{\infty} \Phi(n, k) a_{k} z^{k}, \tag{17}
\end{equation*}
$$

where $\Phi(n, k)=\left(1+(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right] / \mu+\omega\right)^{n}$.
The operator defined in (14) reduces to (1) when $n=0$. Furthermore, it reduces to some existing operators in literature. Setting some parameters as follows:
(i) For $\lambda=\omega=1$ and $\sigma=\mu=\varepsilon=\eta=0$, we obtain the Salagean operator in [3]
(ii) When $\lambda=1, \mu=1, \sigma=0, \omega=1, \varepsilon=0$, and $\eta=0$, we get the Uralegaddi and Somanatha differential operator in [13]
(iii) The differential operator reduces to the Cho and Kim operator in [10] with $\lambda=1, \mu=1, \sigma=0, \varepsilon=$ 0 , and $\eta=0$
(iv) When $\mu=0, \omega=1, \sigma=0, \varepsilon=0$, and $\eta=0$, we get the Al-Oboudi differential operator in [14]
(v) When $\mu=1, \sigma=0, \varepsilon=0$, and $\eta=0$, we get multiplier transformation in [9]
(vi) Suppose $\mu=0, \omega=1, \sigma=0, \varepsilon=0, \eta=0$, and $\lambda=\alpha$ $+\delta / \gamma+\rho$ for $\alpha, \delta, \gamma, \rho \geq 0$ and provided that $\gamma+\rho$ $\neq 0$, then (14) reduces to the Alamri and Darus operator in [5]
(vii) When $\sigma=1$ and $\varepsilon=0$, we get the Amourah and Darus differential operator in [7]
(viii) Let $\lambda=\beta-\xi, \sigma=1, \varepsilon=0$, and $\eta=0$, we have the Amourah and Yousef differential operator in [8] provided $\beta>\xi$

In [15-17], the authors defined and studied different classes of univalent functions. Motivated by the work of Varma and Rosy in [12], we define the following class of univalent functions.

Definition 1 . Let $\omega \geq 0$, a function $f$ defined by (2) belongs to the class $S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}(\omega, n)$ if it satisfies the following condition:

$$
\begin{align*}
& \operatorname{Re}\left\{\frac{D_{\mu, \lambda, \eta}^{n}(\sigma, \xi, \omega, \varepsilon) f(z)}{z}\right\} \\
& \quad \geq \omega\left|\left[D_{\mu, \lambda, \eta}^{n}(\sigma, \xi, \omega, \varepsilon) f(z)\right]^{\prime}-\frac{D_{\mu, \lambda, \eta}^{n}(\sigma, \xi, \omega, \varepsilon) f(z)}{z}\right|, \quad \omega \geq 0 . \tag{18}
\end{align*}
$$

Investigation of univalent function is based on various classes and their geometric properties some of which are given in the following theorems and definitions.

Let $T \cap S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}(\omega, n)=T S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}(\omega, n)$. The necessary and sufficient condition for the functions in $S$ to be in TS $Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}(~(\omega, n)$ is investigated.

Remark 2. Let $T S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}\left(\omega, n ; M_{i}\right) \subset S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}(\omega, n)$. Then,
(i) $S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}(\omega, 0) \equiv S D(\alpha)$ and $T S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}\left(\omega, 0 ; M_{i}\right) \equiv T$ $S D(\alpha)$ studied by both Varma and Rosy and [12, 18]. The class of functions investigated in the work generalized some known classes of functions in literature
(ii) Similar classes of functions were also investigated in [19-21]
(iii) Different functions with finitely many fixed points were also considered in [22-24]

## 2. Main Results

Theorem 3. A function $f(z)$ defined by (16) is said to be in the class $T S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}(\omega, n)$ if and only if

$$
\begin{equation*}
\sum_{k=2}^{\infty}\left(1+\frac{(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right]}{\mu+\omega}\right)^{n}[1+\omega(k-1)]\left|a_{k}\right| \leq 1 . \tag{19}
\end{equation*}
$$

Proof. Suppose $f(z)$ defined by (16) satisfies (19). Then,

$$
\begin{align*}
\operatorname{Re} & \left\{\frac{D_{\mu, \lambda, \eta}^{n}(\sigma, \xi, \omega, \varepsilon) f(z)}{z}\right\} \\
& -\omega\left|\left[D_{\mu, \lambda, \eta}^{n}(\sigma, \xi, \omega, \varepsilon) f(z)\right]^{\prime}-\frac{D_{\mu, \lambda, \eta}^{n}(\sigma, \xi, \omega, \varepsilon) f(z)}{z}\right| \\
& =1-\left|\left(1-\sum_{k=2}^{\infty} \Phi(n, k) a_{k} z^{k-1}\right)-\right| \\
& -\omega\left|\left(1-\sum_{k=2}^{\infty} k \Phi(n, k) a_{k} z^{k-1}\right)-\left(1-\sum_{k=2}^{\infty} \Phi(n, k) a_{k} z^{k-1}\right)\right| \\
& =1-\left|\left(\sum_{k=2}^{\infty} \Phi(n, k) a_{k} z^{k-1}\right)\right| \\
& =1-\left|\sum_{k=2}^{\infty} \Phi \Phi(n, k) a_{k} z^{k-1}\right| \\
& -\omega\left|\sum_{k=2}^{\infty}(k-1) \Phi(n, k) a_{k} z^{k-1}\right|, \quad|z|=r<1 \Rightarrow \leq 1 \\
& -\sum_{k=2}^{\infty}|\Phi(n, k)|\left|a_{k}\right|-\omega \sum_{k=2}^{\infty-1}\left(k-1+\sum_{k=2}^{\infty} \Phi(n, k) a_{k} z^{k-1} \mid\right. \\
& -\sum_{k=2}^{\infty}|\Phi(n, k)|[1+\omega(k-1)]\left|a_{k}\right| \geq 0 .
\end{align*}
$$

Hence, $f \in \operatorname{TSQ}_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}\left(~\left(\omega, M_{j}\right)\right.$.
Conversely,

$$
\begin{align*}
& \operatorname{Re}\left\{\frac{D_{\mu, \lambda, \eta}^{n}(\sigma, \xi, \omega, \varepsilon) f(z)}{z}\right\} \\
& \quad-\omega\left|\left[D_{\mu, \lambda, \eta}^{n}(\sigma, \xi, \omega, \varepsilon) f(z)\right]^{\prime}-\frac{D_{\mu, \lambda, \eta}^{n}(\sigma, \xi, \omega, \varepsilon) f(z)}{z}\right|>0 \tag{21}
\end{align*}
$$

which implies that $\operatorname{Re}\left\{1-\sum_{k=2}^{\infty}|\Phi(n, k)|\left|a_{k}\right|\right\}-\omega \sum_{k=2}^{\infty}(k-1$ $)|\Phi(n, k)|\left|a_{k}\right|>0$.

Letting $z$ take real values and as $|z| \longrightarrow 1$, we get

$$
\begin{equation*}
1-\sum_{k=2}^{\infty}|\Phi(n, k)|\left|a_{k}\right|-\omega \sum_{k=2}^{\infty}|\Phi(n, k)|(k-1)\left|a_{k}\right| \geq 0 \tag{22}
\end{equation*}
$$

which implies $\sum_{k=2}^{\infty}|\Phi(n, k)|[1+\omega(k-1)]\left|a_{k}\right| \leq 1$ following (17).

Remark 4. For any function $f \in S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}(\omega, n)$,
$a_{k} \leq \frac{1}{\left(1+(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right] / \mu+\omega\right)^{n}[1+\omega(k-1)]}, \quad k \geq 2$.

The equality holds for
$f(z)=z-\sum_{k=2}^{\infty} \frac{1}{\left(1+(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right] / \mu+\omega\right)^{n}[1+\omega(k-1)]} z^{k}$.

The class is the generalization of the class $\operatorname{TSD}(\alpha)$ investigated in [12, 18, 25].

Corollary 5. A function $f(z)$ defined by (16) belongs to the class $f \in S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}(\omega, 0)$ if

$$
\begin{equation*}
\sum_{k=2}^{\infty}[1+\omega(k-1)]\left|a_{k}\right| \leq 1, \omega \geq 0 \tag{25}
\end{equation*}
$$

The class $S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}(\omega, 0) \equiv S D(\alpha)$ was investigated in [12, 18, 25].

Remark 6. Let $n=0$. Then, $f \in S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}(~(\omega, 0)$ with

$$
\begin{equation*}
a_{k} \leq \frac{1}{1+\omega(k-1)}, \quad k \geq 2 \tag{26}
\end{equation*}
$$

The equality holds for

$$
\begin{equation*}
f(z)=z-\sum_{k=2}^{\infty} \frac{1}{1+\omega(k-1)} z^{k} \tag{27}
\end{equation*}
$$

See $[12,18,25]$ and the articles cited therein for details.
2.1. Subclass of Analytic Functions with Finitely Many Fixed Coefficients. We now introduce the class of $T S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}(\omega, n$; $\left.M_{i}\right)$ as a subclass of $T S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}(\omega, n)$ with

$$
\begin{align*}
f(z)= & z-\sum_{i=2}^{t} \frac{M_{i}}{\left(1+(i-1)\left[\lambda(\omega-\xi)^{\sigma}+i(\eta+\varepsilon)\right] / \mu+\omega\right)^{n}[1+\omega(i-1)]} z^{i} \\
& -\sum_{k=t+1}^{\infty} a_{k} z^{k} . \tag{28}
\end{align*}
$$

In recent times, the authors in [12, 19-24, 26] have also investigated classes of univalent functions with finitely many fixed coefficients.

Theorem 7. A function $f(z)$ defined by (28) is said to be in the class $T S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}\left(\omega, n ; M_{i}\right)$ if

$$
\begin{align*}
& \sum_{k=t+1}^{\infty}\left(1+\frac{(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right]}{\mu+\omega}\right)^{n}[1+\omega(k-1)] a_{k} \\
& \quad \leq 1-\sum_{i=2}^{t} M_{i} \tag{29}
\end{align*}
$$

where $\propto \geq 0,0 \leq M_{i} \leq 1$, and $0 \leq \sum_{i=2}^{t} M_{i} \leq 1$.
Proof. Suppose $f \in T S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}\left(\omega, n ; M_{i}\right)$. Then, from (23),
$a_{i} \leq \frac{M_{i}}{\left(1+(i-1)\left[\lambda(\omega-\xi)^{\sigma}+j(\eta+\varepsilon)\right] / \mu+\omega\right)^{n}[1+\omega(i-1)]}$,

$$
\begin{equation*}
i=2,3, \cdots, t, 0 \leq M_{i} \leq 1,0 \leq \sum_{i=2}^{t} M_{i} \leq 1 \tag{30}
\end{equation*}
$$

which implies $\quad \sum_{i=2}^{t} M_{i}+\sum_{k=j+1}^{\infty}$ $\left(1+(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right] / \mu+\omega\right)^{n}[1+\omega(k-1)] a_{k} \leq 1$

Conversely,

$$
\begin{align*}
\operatorname{Re}\{ & \left\{\frac{D_{\mu, \lambda, \eta}^{n}(\sigma, \xi, \omega, \varepsilon) f(z)}{z}\right\} \\
& -\omega\left|\left[D_{\mu, \lambda, \eta}^{n}(\sigma, \xi, \omega, \varepsilon) f(z)\right]^{\prime}-\frac{D_{\mu, \lambda, \eta}^{n}(\sigma, \xi, \omega, \varepsilon) f(z)}{z}\right| \\
& \geq 1-\left|\frac{D_{\mu, \lambda, \eta}^{n}(\sigma, \xi, \omega, \varepsilon) f(z)}{z}-1\right| \\
& -\omega\left|\left[D_{\mu, \lambda, \eta}^{n}(\sigma, \xi, \omega, \varepsilon) f(z)\right]^{\prime}-\frac{D_{\mu, \lambda, \eta}^{n}(\sigma, \xi, \omega, \varepsilon) f(z)}{z}\right| \\
& =1-\sum_{k=2}^{\infty}\left(1+\frac{(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right]}{\mu+\omega}\right)^{n}\left|a_{k}\right| \\
& -\omega \sum_{k=2}^{\infty}\left(1+\frac{(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right]}{\mu+\omega}\right)^{n}(k-1)\left|a_{k}\right| \\
& =1-\sum_{i=2}^{t}\left(1+\frac{(i-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right]}{\mu+\omega}\right)^{n}[1+\omega(i-1)]\left|a_{j}\right| \\
& -\sum_{k=i+1}^{\infty}\left(1+\frac{(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right]}{\mu+\omega}\right)^{n}[1+\omega(k-1)]\left|a_{k}\right|=1 \\
& -\sum_{i=2}^{t} M_{i}-\sum_{k=i+1}^{\infty}\left(1+\frac{(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right]}{\mu+\omega}\right)^{n}[1+\omega(k-1)]\left|a_{k}\right| \geq 0, \tag{31}
\end{align*}
$$

by (29) and (30). Thus, $f \in T S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}\left(\omega, n ; M_{i}\right)$.

The result is sharp for

$$
\begin{align*}
f(z)= & z-\sum_{i=2}^{t} \frac{M_{i}}{\left(1+(i-1)\left[\lambda(\omega-\xi)^{\sigma}+j(\eta+\varepsilon)\right] / \mu+\omega\right)^{n}[1+\omega(i-1)]} z^{i} \\
& -\frac{\left(1-\sum_{i=2}^{t} M_{j}\right)}{\left(1+(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right] / \mu+\omega\right)^{n}[1+\omega(k-1)]} z^{k} . \tag{32}
\end{align*}
$$

Corollary 8. Let $T S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, 0, \eta}\left(\omega, n ; M_{i}\right)$. Then,

$$
\begin{equation*}
a_{k} \leq \frac{\left(1-\sum_{i=2}^{t} M_{i}\right)}{\left(1+(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right] / \mu+\omega\right)^{n}[1+\omega(k-1)]}, \quad k \geq t+1 . \tag{33}
\end{equation*}
$$

The result is sharp for $f(z)$ given by (32).
Corollary 9. For $T S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}\left(\omega, 0 ; M_{i}\right)$,

$$
\begin{equation*}
a_{k} \leq \frac{1-\sum_{i=2}^{t} M_{i}}{[1+\omega(k-1)]}, \quad k \geq t+1 \tag{34}
\end{equation*}
$$

The result is sharp for $f(z)$ given by

$$
\begin{equation*}
f(z)=z-\sum_{i=2}^{t} \frac{M_{i}}{1+\omega(i-1)]} z^{i}-\frac{\left(1-\sum_{i=2}^{t} M_{i}\right)}{1+\omega(k-1)} z^{k}, \quad k \geq 1 \tag{35}
\end{equation*}
$$

The class $\operatorname{TSQ}_{\sigma, \xi, \omega, \varepsilon}^{\mu, 0, \eta}\left(\varnothing, 0 ; M_{i}\right) \equiv \operatorname{TSD}(\alpha)$ was investigated in [12].

Theorem 10. The class $\operatorname{TSQ}_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}\left(\omega, n ; M_{i}\right)$ is convex.
Proof. Suppose there exist functions $f, g \in T S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}(\omega, n$; $M_{i}$ ) such that

$$
\begin{align*}
f(z)= & z-\sum_{i=2}^{t} \frac{M_{i}}{\left(1+(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right] / \mu+\omega\right)^{n}[1+\omega(k-1)]} z^{i} \\
& -\sum_{i=t+1}^{\infty} a_{k} z^{k} \tag{36}
\end{align*}
$$

$$
\begin{align*}
g(z)= & z-\sum_{i=2}^{t} \frac{M_{i}}{\left(1+(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right] / \mu+\omega\right)^{n}[1+\omega(k-1)]} z^{i} \\
& -\sum_{i=t+1}^{\infty} b_{k} z^{k} \tag{37}
\end{align*}
$$

where $0 \leq M_{i} \leq 1$ and $0 \leq \sum_{i=2}^{t} M_{i} \leq 1$. Then, let $0 \leq \zeta \leq 1$ and

$$
\begin{equation*}
h(z)=\zeta f(z)+(1-\zeta) g(z) \tag{38}
\end{equation*}
$$

So that by substituting (36) and (37) in (38), we have

$$
\begin{align*}
h(z)= & z-\sum_{i=2}^{\infty} \frac{M_{i}}{\left(1+(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right] / \mu+\omega\right)^{n}[1+\omega(k-1)]} z^{i} \\
& -\sum_{i=t+1}^{\infty}\left[\zeta a_{k}+(1-\zeta) b_{k}\right] z^{k} . \tag{39}
\end{align*}
$$

Such that if $\zeta=1$ and $\zeta=0$, equation (39) reduces to (36) and (37), respectively. Now,

$$
\begin{align*}
& \sum_{k=i+1}^{\infty}\left(1+\frac{(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right]}{\mu+\omega}\right)^{n}[1+\omega(k-1)]\left[\zeta a_{k}+(1-\zeta) b_{k}\right] \\
& =\zeta \sum_{k=i+1}^{\infty}\left(1+\frac{(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right]}{\mu+\omega}\right)^{n}[1+\omega(k-1)] a_{k} \\
& \quad+(1-\zeta) \sum_{k=i+1}^{\infty}\left(1+\frac{(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right]}{\mu+\omega}\right)^{n}[1+\omega(k-1)] b_{k} \\
& \quad \leq \zeta\left(1-\sum_{i=2}^{t} M_{i}\right)+(1-\zeta)\left(1-\sum_{i=2}^{t} M_{i}\right)=1-\sum_{i=2}^{t} M_{i} . \tag{40}
\end{align*}
$$

$$
\text { Thus, } h(z) \in T S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}\left(\omega, n ; M_{i}\right) .
$$

Theorem 11. Let

$$
\begin{aligned}
f_{t}(z)= & z-\sum_{i=2}^{t} \frac{M_{i}}{\left(1+(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right] / \mu+\omega\right)^{n}[1+\omega(k-1)]} z^{i}, \\
f_{k}(z)= & z-\sum_{i=2}^{t} \frac{M_{i}}{\left(1+(i-1)\left[\lambda(\omega-\xi)^{\sigma}+i(\eta+\varepsilon)\right] / \mu+\omega\right)^{n}[1+\omega(i-1)]} z^{i} \\
& -\sum_{k=t+1}^{\infty} \frac{\left(1-\sum_{k=2}^{t} M_{i}\right)}{\left(1+(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right] / \mu+\omega\right)^{n}[1+\omega(k-1)]} z^{k},
\end{aligned}
$$

$$
\begin{equation*}
k \geq i+1 \tag{41}
\end{equation*}
$$

Then, $f \in T S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}\left(\omega, n ; M_{i}\right)$ if and only if $f$ can be expressed in the form

$$
\begin{equation*}
f(z)=\sum_{k=t}^{\infty} \zeta_{k} f_{k}(z) \tag{42}
\end{equation*}
$$

where $\zeta_{k} \geq 0$ with $k \geq t$ and $\sum_{k=t}^{\infty} \zeta_{k} f_{k}(z)=1$.
Proof. If $f \in T$ can be expressed in form of (42). Then,

$$
\begin{align*}
f(z)= & z-\sum_{i=2}^{t} \frac{M_{i}}{\left(1+(i-1)\left[\lambda(\omega-\xi)^{\sigma}+i(\eta+\varepsilon)\right] / \mu+\omega\right)^{n}[1+\omega(i-1)]} z^{i} \\
& -\sum_{k=t+1}^{\infty} \frac{\zeta_{k}\left[1-\sum_{k=2}^{t} M_{i}\right]}{\left(1+(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right] / \mu+\omega\right)^{n}[1+\omega(k-1)]} z^{k} . \tag{43}
\end{align*}
$$

Now,

$$
\begin{align*}
& \sum_{k=t+1}^{\infty}\left(1+\frac{(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right]}{\mu+\omega}\right)^{n}[1+\omega(k-1)] \\
& \cdot\left\{\frac{\zeta_{k}\left[1-\sum_{k=2}^{t} M_{i}\right]}{\left(1+(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right] / \mu+\omega\right)^{n}[1+\omega(k-1)]}\right\} \\
& \quad=\sum_{k=t+1}^{\infty} \zeta_{k}\left[1-\sum_{k=2}^{t} M_{i}\right]=\left[1-\sum_{k=2}^{t} M_{i}\right] \sum_{k=t+1}^{\infty} \zeta_{k} \\
& \quad=\left[1-\sum_{k=2}^{t} M_{i}\right]\left(1-\zeta_{k}\right) \leq 1-\sum_{k=2}^{t} M_{i} \tag{44}
\end{align*}
$$

which implies $f \in T S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}\left(\omega, n ; M_{i}\right)$.
Conversely, for $k \geq t+1$, set

$$
\begin{equation*}
\zeta_{k}=\frac{\left(1+(k-1)\left[\lambda(\omega-\xi)^{\sigma}+k(\eta+\varepsilon)\right] / \mu+\omega\right)^{n}[1+\omega(k-1)] a_{k}}{1-\sum_{i=2}^{\infty} M_{i}}, \quad k \geq t+1, \tag{45}
\end{equation*}
$$

$$
\begin{equation*}
\zeta_{t}=1-\sum_{k=t+1}^{\infty} \zeta_{k} \tag{46}
\end{equation*}
$$

Then, $f$ can be represented as $f(z)=\sum_{k=t}^{\infty} \zeta_{k} f_{k}(z)$.

## 3. Conclusion

In this work, a generalized multiplier operator was defined and used to define a class of univalent function. Geometric properties of the class $T S Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}(\omega, n)$ and its subclass $T S$ $Q_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}\left(\omega, n ; M_{i}\right)$ were investigated. The results obtained are the generalization of many known results in literature. See [12, 18-21].

## Data Availability

No data was used.

## Conflicts of Interest

The authors declare that there are no competing interests.

## Authors' Contributions

Both authors contributed to the work and approved the final manuscript.

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