Research Article
On the Heat and Wave Equations with the Sturm-Liouville Operator in Quantum Calculus

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Received 2 October 2022; Revised 29 November 2022; Accepted 12 December 2022; Published 19 January 2023

Academic Editor: Douglas R. Anderson

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In this paper, we explore a generalised solution of the Cauchy problems for the \(q\)-heat and \(q\)-wave equations which are generated by Jackson’s and the \(q\)-Sturm-Liouville operators with respect to \(t\) and \(x\), respectively. For this, we use a new method, where a crucial tool is used to represent functions in the Fourier series expansions in a Hilbert space on quantum calculus. We show that these solutions can be represented by explicit formulas generated by the \(q\)-Mittag-Leffler function. Moreover, we prove the unique existence and stability of the weak solutions.

1. Introduction

In the last decade, the theory of quantum groups and \(q\)-deformed algebras have been the subject of intense investigation. Many physical applications have been investigated on the basis of the \(q\)-deformation of the Heisenberg algebra (see [1, 2]). For instance, the \(q\)-deformed Schrödinger equations have been proposed in [3, 4], and applications to the study of \(q\)-deformed version of the hydrogen atom and of the quantum harmonic oscillator have been presented (see [5]). Fractional calculus and the \(q\)-deformed Lie algebras are closely related. A new class of fractional \(q\)-deformed Lie algebras is proposed, which for the first time allows a smooth transition between the different Lie algebras (see [6]).

The origin of the \(q\)-difference calculus can be traced back to the works by Jackson (see [7, 8]) and Carmichael (see [9]) from the beginning of the twentieth century, while basic definitions and properties can be found, e.g., in the monographs [10, 11] and the PhD thesis [12]. Recently, the fractional \(q\)-difference calculus has been proposed by Al-salam (see [13]) and Agarwal (see [14]). We can also mention papers [15, 16], where the authors investigated the explicit solutions to linear fractional \(q\)-differential equations with the \(q\)-fractional derivative, and in [17], the \(q\)-analogue nonhomogeneous wave equations were studied.

A motivation behind this work is to state some new results about the \(q\)-heat and \(q\)-wave equations associated to the \(q\)-Sturm-Liouville operator (see (10)). We attempt to extend the heat representation theory studied in some cases (see [18–20], etc.). We define a generalised solution of the Cauchy problem for these equations generated by the \(q\)-Mittag-Leffler function and the \(q\)-associated functions of a biorthogonal system (see (13)). We investigate the well-posedness of the Cauchy problem for the \(q\)-heat and \(q\)-wave equations for
operators with a discrete nonnegative spectrum acting on \( L^2_q[0,1] \). In particular, we prove both unique existence and stability of the corresponding generalised solution.

The paper is organized as follows: the main results are presented and proved in Section 3 and Section 4. In order to not disturb these presentations, we include in Section 2 some necessary Preliminaries.

## 2. Preliminaries

In this section, we recall some notations and basic facts in \( q \)-calculus. We will always assume that \( 0 < q < 1 \). The \( q \)-real number \([a]_q\) is defined by

\[
[a]_q = \frac{1-q^a}{1-q}
\]

The \( q \)-shifted factorial is defined by

\[
(a)_n = \begin{cases} \, 1, & n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n \in \mathbb{N}. \end{cases}
\]

Moreover, their natural expansions to the reals are

\[
(a-b)_q^n = a^n \frac{(b/a)_q - (a)_q}{(q^n/b/a)_q - (a)_q} = \frac{[a]_q}{[a^n]_q} \prod_{i=0}^{n-1} (1-aq^i).
\]

The Jackson’s \( q \)-difference operator \( D_q f(x) \) is defined by \([8, 12\text{ Section 2.1}])

\[
D_q f(x) = \frac{f(x)-f(qx)}{x(1-q)}.
\]

The \( q \)-derivative \( D_q \) of a product of the functions \( f \) and \( g \) as defined by

\[
D_q (fg)(x) = f(qx) D_q (g)(x) + D_q (f)(x) g(x).
\]

As given in \([10]\), two \( q \)-analogues of the exponential functions are defined by

\[
e^q_x = \frac{1}{(1-q)x; q)_{\infty}}, E^q_x = \frac{1}{- (1-q)x; q)_{\infty}}.
\]

Moreover, we have that

\[
D_q e^q_x = e^q_x, D_q E^q_x = E^q_x q^q x, e^q_x E^q_x = 1.
\]

Due to the various types of \( q \)-differences introduced in quantum calculus, trigonometric functions have various \( q \)-analogues (see, \([21\text{ Section 2} [10], \text{Section 10 and} 12, \text{Section 2.12}]\). The following definition of cosine and sine will be useful in this investigation (see \([20]):

\[
\cos (z; q^2) = \sum_{k=0}^{\infty} (-1)^k q^{k^2} z^{2k}, \sin (z; q^2) = \sum_{k=0}^{\infty} (-1)^k q^{k+1} z^{2k+1}.
\]

where the \( q \)-analogue of the binomial coefficients \([n]_q!\) is defined by

\[
[n]_q! = \begin{cases} 1, & \text{if } n = 0, \\ [1]_q \times [2]_q \times \cdots \times [n]_q, & \text{if } n \in \mathbb{N}. \end{cases}
\]

The \( q \)-integral (or Jackson’s integral) is defined by (see \([8])

\[
\int_a^b f(x) d_q x = \int_0^b f(x) d_x q - \int_0^a f(x) d_x q,
\]

and a more general form is given by

\[
\int_a^b f(x) d_q x = \int_0^b f(x) d_x q - \int_0^a f(x) d_x q,
\]

for \(0 < a < b\).

The \( q \)-version of integration by parts reads

\[
\int_a^b f(x) D_q g(x) d_q x = [f g]_a^b - \int_a^b g(qx) D_q f(x) d_q x,
\]

and if \( f \equiv 1 \), then we get that

\[
\int_a^b D_q g(x) d_q x = g(b) - g(a).
\]

The \( q \)-Sturm-Liouville Problem. Let \( L^2_q[0,1] \) be the space of all real-valued functions defined on \([0,1]\) such that

\[
||f||_{L^2_q[0,1]} = \left( \int_0^1 |f(x)|^2 d_q x \right)^{1/2} < \infty.
\]

The space \( L^2_q[0,1] \) is a separable Hilbert space with the inner product:

\[
\langle f, g \rangle = \int_0^1 f(x) g(x) d_q x, f, g \in L^2_q[0,1].
\]

Now, we shortly describe the study introduced by Annaby and Mansour in of a basic \( q \)-Sturm-Liouville eigenvalue problem in a Hilbert space (see \([21\text{ Chapter 3}]\). In particular, they investigated the basic \( q \)-Sturm-Liouville equation:

\[
-\frac{1}{q} D_q^{-1} D_q y(x) + \nu(x)y(x) = \lambda y(x), 0 \leq x \leq 1 ; \lambda \in \mathbb{C},
\]

where \( \nu(\cdot) \) is defined on \([0,1]\) and continuous at zero. Let \( C^2_0[0,1] \) denotes the space of all functions \( y(\cdot) \) such that \( y
Thus, for $s \in \mathbb{R}$, we can also define the Sobolev spaces $W^s_{q,\mathcal{L}}$ associated to $\mathcal{L}$ in the following form:

$$W^s_{q,\mathcal{L}} := \left\{ f \in \mathcal{D}^{s,2}_{\mathcal{L}}[0,1] : \mathcal{L}^{s/2} f \in L^2_{q}[0,1] \right\},$$

with the norm $\| f \|_{W^s_{q,\mathcal{L}}} := \| \mathcal{L}^{s/2} f \|_{L^2_{q}[0,1]}$.

For $m \in \mathbb{N}_0$, we introduce the space $C^m_{q}([0,1] ; W^s_{q,\mathcal{L}}[0,1])$ defined by the norms

$$\| u \|_{C^m_{q}([0,T]; W^s_{q,\mathcal{L}}[0,1])} := \max_{0 \leq t \leq T} \| D^m_{q} u(t, \cdot) \|_{W^s_{q,\mathcal{L}}[0,1]},$$

where the $q$-partial differential operator $D_{q,t} u(t, x)$ with respect to $t$ has the following form:

$$D_{q,t} u(t, x) = \frac{u(t, x) - u(qt, x)}{(1-q)t}.$$

Notation: the symbol $M \leq K$ means that there exists $\gamma > 0$ such that $M \leq \gamma K$, where $\gamma$ is a constant.

### 3. The $q$-Heat Equation

We start with a study of the following Cauchy problem:

$$D_{q,t} u(t, x) + \mathcal{L} u(t, x) = f(t, x), \quad x \in [0,1], t > 0,$$

with the initial condition

$$u(0, x) = \varphi(x), \quad x \in [0,1].$$

We say a generalised solution of the problem (14)-(15) is a function $u(t, x)$ such that they satisfy equation (14) and condition (15).

**Theorem 1.** We assume that $0 < T < \infty$. Let $\varphi \in W^2_{q,\mathcal{L}}[0,1]$ and $f \in C([0,T] ; W^2_{q,\mathcal{L}}[0,1])$. Then, there exists the generalised solution of $u$ to problem (14)-(15), and

$$u \in C_q^1 \left( [0,T] ; L^2_{q}[0,1] \right) \cap C \left( [0,T] ; W^2_{q,\mathcal{L}}[0,1] \right).$$

Moreover, this solution can be written in the following explicit form

$$u(t, x) = \sum_{k \in \mathbb{N}} \left( e^{-\lambda_k t} \phi_k + e^{-\lambda_k t} \int_0^t e^{\lambda_k q s} f(s, \cdot) \, ds \right) \phi_k(x).$$

**Proof.** Existence. Since the system of eigenfunctions $\{ \phi_k \}_{k=1}^{\infty}$ is a basis in $L^2_{q}[0,1]$ (see (11)), we seek for a function $u(t, x)$ in the form

$$u(t, x) = \sum_{k \in \mathbb{N}} u_k(t) \phi_k(x).$$

The space of $\mathcal{L}$-distributions $\mathcal{D}^s_{\mathcal{L}}[0,1] = L(C^\infty_{\mathcal{L}}[0,1], \mathbb{R})$ is the space of all linear continuous functionals on $C^\infty_{\mathcal{L}}[0,1]$. 

and $D_q y$ are continuous at zero. If $v = 0$, then we get the operator $\mathcal{L}$ in the following form:

$$\mathcal{L} := \begin{cases} \frac{1}{q} D_q y, & y \in C^\infty_{\mathcal{L}}[0,1], \\ y(0) = y(1) = 0, & \end{cases}$$

for $0 \leq x \leq 1$ and $\lambda \in \mathbb{R}$. The operator $\mathcal{L}$ is self-adjoint on $C^2_q[0,1] \cap L^2_q[0,1]$ (see [21], Theorem 3.4). A fundamental set of solutions of (10) are $\cos (\sqrt{\lambda} t^q)$ and $\sin (\sqrt{\lambda} t^q) / \sqrt{\lambda}$. Moreover, the eigenvalues $\{ \lambda_k \}_{k=1}^{\infty}$ are the zeros of $\sin (\sqrt{\lambda} t^q)$, where

$$\lambda_k = (1-q)^{-2} q^{-2k+2k^2}, \quad k = 0, 1, \cdots, (11)$$

and $\sum_{k=1}^{\infty} \mu_k < \infty, \quad 0 \leq \mu_k \leq 1$, and

$$\lambda_0 := (1-q)^{-2} q \leq \lambda_k, \quad k = 1, 2, 3, \cdots.$$ (12)

Additionally, the corresponding set of eigenfunctions $\{ \sin (\sqrt{\lambda_k} t^q) / \sqrt{\lambda_k} \}_{k=1}^{\infty}$ is an orthogonal basis in $L^2_q[0,1]$. Thus, we can identify $f \in L^2_q[0,1]$ with its Fourier series:

$$f(x) = \sum_{k=1}^{\infty} (f, \phi_k) \phi_k(x),$$

where

$$\phi_k(x) = \frac{\sin (\sqrt{\lambda_k} x^q)}{\sqrt{\lambda_k}}.$$ (13)

**The Sobolev Space Associated with $\mathcal{L}$**. The next step is to recall the essential elements of the Fourier analysis presented in [22–24], as well as its applications to the spectral properties of $\mathcal{L}$. The space $C^m_{\mathcal{L}}[0,1] := \bigcap_{j=0}^{m} \operatorname{Dom}(\mathcal{L}^m)$ is called the space of test functions for $\mathcal{L}$, where

$$\operatorname{Dom}(\mathcal{L}^m) := \left\{ f \in C^0_{\mathcal{L}}[0,1] : \mathcal{L}^j f \in \operatorname{Dom}(\mathcal{L}), \quad j = 0, 1, 2, \cdots, m-1 \right\}.$$
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for each fixed $0 < t < T < \infty$. The coefficients will then be given by the Fourier coefficients formula $u_k(t) = \langle u(t, \cdot) \phi_k \rangle$.

We can similarly expand the source function,

$$f(t, x) = \sum_{k \in \mathbb{N}} f_k(t) \phi_k(x), f_k(t) = (f(t, \cdot), \phi_k).$$

From (11) and (18), we have that

$$\mathcal{L} \phi_k(x) = \lambda_k \phi_k(x), k \in \mathbb{N}.$$

Hence,

$$\mathcal{L} u(t, x) = \sum_{k \in \mathbb{N}} u_k(t) \lambda_k \phi_k(x),$$

and

$$D_q u(t, x) = \sum_{k \in \mathbb{N}} D_q u_k(t) \phi_k(x).$$

Substituting (20) and (21) into the equation (14), we find that

$$\sum_{k \in \mathbb{N}} \left[ D_q u_k(t) + \lambda_k u_k(t) \right] \phi_k(x) = \sum_{k \in \mathbb{N}} f_k(t) \phi_k(x).$$

But then, due to the completeness,

$$D_q u_k(t) + \lambda_k u_k(t) = f_k(t), k \in \mathbb{N},$$

which are ODEs for the coefficients $u_k(t)$ of the series (18). Using the integrating factor $E_q^{\lambda t}$ and (2) and (3), we can rewrite the ODE as

$$E_q^{\lambda t} f_k(t) = E_q^{\lambda t} D_q u_k(t) + E_q^{\lambda t} \lambda_k u_k(t)$$

$$= E_q^{\lambda t} D_q u_k(t) + D_q \left[ E_q^{\lambda t} \right] u_k(t)$$

$$= D_q \left[ E_q^{\lambda t} u_k(t) \right].$$

Form (3), (5), and (24), we get that

$$\int_0^t D_q \left[ E_q^{\lambda t} u_k(t) \right] d_q s = \int_0^t E_q^{\lambda t} f_k(t) d_q s,$$

so that

$$E_q^{\lambda t} u_k(t) = u_k(0) + \int_0^t E_q^{\lambda t} f_k(s) d_q s,$$

which, in its turn, implies that

$$u_k(t) = \frac{u_k(0)}{E_q^{\lambda t}} + \frac{1}{E_q^{\lambda t}} \int_0^t E_q^{\lambda t} f_k(s) d_q s,$$

and we conclude that

$$u_k(t) = e^{-\lambda t} u_k(0) + e^{-\lambda t} \int_0^t E_q^{\lambda t} f_k(s) d_q s.$$

But the initial conditions (16) and (22) imply that $u_k(0) = \varphi_k$. Thus,

$$u_k(t) = e^{-\lambda t} \varphi_k + e^{-\lambda t} \int_0^t E_q^{\lambda t} f_k(s, \cdot) d_q s.$$

Therefore, the solution $u(t, x)$ can be written in the series form as

$$u(t, x) = \sum_{k \in \mathbb{N}} \left[ e^{-\lambda t} \varphi_k + e^{-\lambda t} \int_0^t E_q^{\lambda t} f_k(s, \cdot) d_q s \right] \phi_k(x),$$

so, also (17) is proved.

Convergence. From (1), (4), and (5), we have that

$$e^{-\lambda x} = \frac{1}{1 - (1 - q) x} \leq \frac{1}{1 + (1 - q) x} \leq 1, \frac{E_q}{E_q^x} \leq E_q^x$$

for $x \in [0, 1]$. Hence, using for $0 < t < T < \infty$, (5), (23), and (25), we get that

$$|u_k(t)| \leq e^{-\lambda t} |\varphi_k| + \int_0^t \frac{E_q^{\lambda s}}{E_q^{\lambda t}} |f_k(s)| d_q s \leq |\langle \varphi, \phi_k \rangle|$$

$$+ T \max_{0 \leq s \leq T} |f(s, \cdot, \cdot)| \leq \max_{0 \leq s \leq T} \{1, T\}$$

$$\cdot \left( |\langle \varphi, \phi_k \rangle| + \max_{0 \leq s \leq T} |f(s, \cdot, \phi_k)| \right) \leq |\langle \varphi, \phi_k \rangle| + \max_{0 \leq s \leq T} |f(s, \cdot, \phi_k)|,$$

and

$$|D_q u_k(t)| \leq \lambda_k |u_k(t)| + |f_k(t)| \leq |\lambda_k \varphi, \phi_k)|$$

$$+ \lambda_k |f_k(t, \cdot, \phi_k)| + \lambda_k^{-1} \langle \lambda_k f_k(t, \cdot, \phi_k) \rangle \leq |\lambda_k \phi, \phi_k)|$$

$$+ (1 + \lambda_0)|\lambda_k f_k(t, \cdot, \phi_k)| \leq |\mathcal{L} \varphi, \phi_k)| + \max_{0 \leq s \leq T} |\langle \mathcal{L} f_k(s, \cdot), \phi_k \rangle|.$$

Hence,

$$|\mathcal{L} u(t, \cdot)| = |\langle \lambda_k u_k(t, \cdot), \phi_k \rangle| \leq |\lambda_k \varphi, \phi_k)$$

$$+ \max_{0 \leq s \leq T} |\langle \lambda_k f(s, \cdot), \phi_k \rangle| = |\langle \mathcal{L} \varphi, \phi_k \rangle| + \max_{0 \leq s \leq T} |\langle \mathcal{L} f(s, \cdot), \phi_k \rangle|.$$
that
\[
\|u(t, \cdot)\|_{L^2_q[t, \cdot][0, \cdot]}^2 = \sum_{k \in \mathbb{N}} |u_k(t)|^2 = \sum_{k \in \mathbb{N}} \sum_{\omega(k) \in \mathbb{N}} (\langle \phi, \phi_k \rangle)^2 + \max_{q \in \mathbb{N}} \sum_{k \in \mathbb{N}} \sum_{\omega(k) \in \mathbb{N}} (\langle f(s, \cdot), \phi_k \rangle)^2
\]
\[
= \|\phi\|_{W^2_q[0, \cdot]}^2 + \|f\|_{C([0, T] ; W^2_q[0, \cdot])}^2 < \infty,
\]
and
\[
\|D^q_{\alpha} u(t, \cdot)\|_{L^2_q[t, \cdot][0, \cdot]}^2 = \sum_{k \in \mathbb{N}} |D^q_{\alpha} u_k(t, \cdot)|^2 \leq \sum_{k \in \mathbb{N}} \sum_{\omega(k) \in \mathbb{N}} (\langle Lq \phi, \phi_k \rangle)^2
\]
\[
+ \max_{q \in \mathbb{N}} \sum_{k \in \mathbb{N}} \sum_{\omega(k) \in \mathbb{N}} (\langle Lq (s, \cdot), \phi_k \rangle)^2
\]
\[
= \|\phi\|_{W^2_q[0, \cdot]}^2 + \|f\|_{C([0, T] ; W^2_q[0, \cdot])}^2 < \infty,
\]
which mean that \( u \in C^2_q([0, T] ; L^2_q[0, 1]) \cap C([0, T] ; W^2_q[0, 1]) \).

Uniqueness. It only remains to prove the uniqueness of the solution. We assume the opposite; namely, that there exist functions \( u(t, x) \) and \( v(t, x) \), which are two different solutions of problem (14)-(15). Let \( 0 < t < T < \infty \). Then, we have that
\[
\begin{cases}
D^q_{\alpha} u(t, x) + L^q u(t, x) = f(t, x), & 0 < x < 1, \\
u(0, x) = \phi(x), & 0 \leq x \leq 1,
\end{cases}
\]
\[
\begin{cases}
D^q_{\alpha} v(t, x) + L^q v(t, x) = f(t, x), & 0 < x < 1, \\
v(0, x) = \phi(x), & 0 \leq x \leq 1.
\end{cases}
\]

We define \( W(t, x) = u(t, x) - v(t, x) \). Then, the function \( W(t, x) \) is a solution of the following problem
\[
\begin{cases}
D^q_{\alpha} w(t, x) + L^q w(t, x) = 0, & 0 < x < 1, \\
w(0, x) = 0, & 0 \leq x \leq 1.
\end{cases}
\]

From (18), it follows that \( W(t, x) \equiv 0 \), that is, \( u(x, t) \equiv v(x, t) \), and this contradiction to our assumption proves the uniqueness of the solution. The proof is complete.

4. The \( q \)-Wave Equation

In this section, we will seek for a generalised function \( u(t, x) \), which satisfies the following \( q \)-wave equation
\[
D^q_{\alpha} u(t, x) + L^q u(t, x) = f(t, x), 0 < x < 1,
\]
for \( 0 < t < T < \infty \) with the initial conditions
\[
u(0, x) = \psi(x), D^q_{\alpha} u(0, x) = \eta(x), 0 < x \leq 1.
\]

Theorem 2. We assume that \( 0 < T < \infty \). Let \( \psi, \eta \in W^2_q[0, 1] \) and \( f \in C^1_q([0, T] ; W^2_q[0, 1]) \). Then, there exists the generalised solution of problem (29)-(30):
\[
u \in C^2_q([0, T] ; L^2_q[0, T]) \cap C([0, T] ; W^2_q[0, 1]).
\]
Moreover, this solution can be written in the following explicit form:
\[
u(t, x) = \sum_{k \in \mathbb{N}} \left( \psi_k e_{2,1}(-\lambda_k t^2 ; q) + t \eta_k e_{2,1}(-\lambda_k t^2 ; q) \right.
\]
\[
+ \frac{1}{\lambda_k} \int_0^t \left( \lambda_k (t - s)^2 q \right) \frac{\Gamma_q(n+1)}{\Gamma_q(n+1)} f(s) d_q s \phi_k,
\]
where the \( q \)-Mittag-Leffler function \( e_{2,1}(\lambda_k (t - q^2 s^2 q^2) ; q) \) is given by (see [25] and [26], Section 7):
\[
e_{2,1}(\lambda_k (t - q^2 s^2 q^2) ; q) = \sum_{m=0}^{\infty} \lambda_k^m (t - q^2)_{q^2}^m \Gamma_q(m \alpha + \beta),
\]
for \( \alpha, \beta \in \mathbb{R} \) and \( 0 < s \leq t < \infty \), where the gamma function \( \Gamma_q(x) \) is defined by
\[
\Gamma_q(x) = \left( \frac{q^\alpha \Gamma_q(q^\alpha)}{q^\alpha} \right) (1 - q)^{-x}, \Gamma_q(n+1) = [n]_q!, n \in \mathbb{N}.
\]

Proof. Existence. By repeating the arguments in the proof of Theorem 1., we have the Cauchy type problem:
\[
D^q_{\alpha} u_k(t) + \lambda_k u_k(t) = f_k(t), k \in \mathbb{N}_0,
\]
with the initial conditions
\[
u_k(0) = \psi_k, D^q_{\alpha} u_k(0) = \eta_k, k \in \mathbb{N}_0,
\]
where \( f_k(t) = \langle f(t, \cdot), \phi_k \rangle \), \( \psi_k = \langle \psi(\cdot) \phi_k \rangle \), and \( \eta_k = \langle \eta(\cdot) \phi_k \rangle \).

Then, the solution to this Cauchy type in problem (29)-(30) is given (see [25], Example 6):
\[
u_k(t) = \psi_k e_{2,1}(-\lambda_k t^2 ; q) + t \eta_k e_{2,1}(-\lambda_k t^2 ; q)
\]
\[
+ \int_0^t (t - q^2)^2 q^2 \left( \lambda_k (t - q^2 s^2 q^2) ; q \right) f_k(s) d_q s.
\]
By using (2) and we find that

\[
D_{q,t} [e_{2,1} \left(-\lambda_k (t - q^2 s)^2 ; q \right)] \\
= -\lambda_k (t - q^2 s) \cdot \left(\sum_{m=0}^{\infty} (-\lambda_k)^m (2m+1) \cdot \frac{2m}{2m+1} \cdot q^2 s \right) \\
= \lambda_k (t - q^2 s) \cdot e_{2,2} \left(\lambda_k (t - q^2 s)^2 ; q \right).
\]

(37)

By applying (8) and using (36) and (37), we get that

\[
u_k(t) = \psi_k e_{2,1} \left(-\lambda_k t^2 ; q \right) + \frac{1}{\lambda_k} \int_0^t D_{q,s} [e_{2,1} \left(-\lambda_k (s - t)^2 ; q \right)] f_k(s) \, ds \\
= \psi_k e_{2,1} \left(-\lambda_k t^2 ; q \right) + \frac{1}{\lambda_k} \int_0^t f_k(s) e_{2,1} \left(-\lambda_k (s - t)^2 ; q \right) \, ds \\
- \lambda_k \frac{1}{\lambda_k} \int_0^t e_{2,1} \left(-\lambda_k (s - t)^2 ; q \right) D_{q,s} f_k(s) \, ds.
\]

(38)

Since \(e_{2,1} \left(-\lambda_k t^2 (q^2 ; q) \right) \equiv 0 \) (see [21], Theorem 7.12), by using (18) and (38), it follows that solution exists and can be written as

\[
u(t,x) = \sum_{k \in \mathbb{N}} \left(\psi_k e_{2,1} \left(-\lambda_k t^2 ; q \right) + \lambda_k e_{2,2} \left(\lambda_k t^2 ; q \right) \right) \\
- \lambda_k \frac{1}{\lambda_k} \int_0^t e_{2,1} \left(-\lambda_k (s - t)^2 ; q \right) D_{q,s} f_k(s) \, ds.
\]

i.e., on the explicit form (34).

Convergence. Firstly, using the results in [27], Lemma 6 and in [17], Lemma 1 for the q-trigonometric functions in (6), we see that \(e_{2,2} \left(-\lambda_k t^2 ; q \right)\) and \(e_{2,3} \left(-\lambda_k t^2 ; q \right)\) are also bounded with \(t > 0\). Then, forms (4), (12), and (32) follow that

\[
e_{2,2} \left(-\lambda_k t^2 ; q \right) \\
= \sum_{m=0}^{\infty} (-\lambda_k)^m \frac{1}{[2m+1] \cdot q} \\
\leq \sum_{m=0}^{\infty} \lambda_k^m \frac{1}{[2m+1] \cdot q} \\
\sin \left(\sqrt{\lambda_k T} ; t^2 \right) \\
= \frac{2T}{2T} \cdot \sqrt{\lambda_k} \\
\leq C_{2,q}.
\]

(39)

and

\[
|e_{2,1} \left(-\lambda_k t^2 ; q \right)| \\
\leq \frac{1}{\lambda_k^m} \left(\sum_{m=0}^{\infty} \frac{1}{[2m+1] \cdot q} \right) \\
\leq C_{2,q} /
\]

where \(C_{2,q} \) are any constant which only depends on \(q\). Next, by using (38), (39), and (40), we obtain that

\[
[u_k(t)] \leq \left|\psi, \phi_k\right| + \left|\eta, \phi_k\right| + \frac{1}{\lambda_k} \int_0^t \left|D_{q,s} f(s) \cdot \phi_k\right| \, ds \\
+ \frac{1}{\lambda_k} \int_0^t \left|\psi, \phi_k\right| \, ds \\
+ \frac{1}{\lambda_k} \int_0^t \left|\psi, \phi_k\right| \, ds \\
+ \frac{T}{\lambda_k} \left|\psi, \phi_k\right| + \frac{T}{\lambda_k} \left|\psi, \phi_k\right|
\]

(41)

where \(C_k := \text{max} \{1, (1/\lambda_k), (T/\lambda_k)\}\).

Therefore, by using (7), (8), (30), (34), (33), and (41), we have that

\[
C_{2,q} \leq \left|\psi, \phi_k\right| + \left|\eta, \phi_k\right| + \frac{1}{\lambda_k} \int_0^t \left|D_{q,s} f(s) \cdot \phi_k\right| \, ds \\
+ \frac{1}{\lambda_k} \int_0^t \left|\psi, \phi_k\right| \, ds \\
+ \frac{T}{\lambda_k} \left|\psi, \phi_k\right| + \frac{T}{\lambda_k} \left|\psi, \phi_k\right|
\]

(42)

and

\[
C_{2,q} \leq \left|\psi, \phi_k\right| + \left|\eta, \phi_k\right| + \frac{1}{\lambda_k} \int_0^t \left|D_{q,s} f(s) \cdot \phi_k\right| \, ds \\
+ \frac{1}{\lambda_k} \int_0^t \left|\psi, \phi_k\right| \, ds \\
+ \frac{T}{\lambda_k} \left|\psi, \phi_k\right| + \frac{T}{\lambda_k} \left|\psi, \phi_k\right|
\]

(43)
Thus,

\[
\|u(t)\|^2_{L_q^2(0,1)} = \sum_{k \in \mathbb{N}} |u_k(t)|^2 \leq \sum_{k \in \mathbb{N}} \|\langle \psi, \phi_k \rangle\|^2 + \sum_{k \in \mathbb{N}} \|\langle \eta, \phi_k \rangle\|^2
\]

\[
+ \sum_{m=0}^{\infty} \sum_{0 \leq s \leq T} \left| \left. D_q^n f(t, \cdot), \phi_k \right| \right|^2
\]

\[
= \|\eta\|_{L_q^2(0,1)}^2 + \|\psi\|_{L_q^2(0,1)}^2
\]

\[
+ \|f\|_{C_q^1([0,T];W_q^2,0,1)}^2 < \infty,
\]

\[
\|D_qu(t)\|^2_{L_q^2(0,1)} = \sum_{k \in \mathbb{N}} |D_q^k u(t)|^2 \leq \sum_{k \in \mathbb{N}} \|\langle \psi, \phi_k \rangle\|^2
\]

\[
+ \sum_{k \in \mathbb{N}} \|\langle \eta, \phi_k \rangle\|^2
\]

\[
+ \sum_{m=0}^{\infty} \sum_{0 \leq s \leq T} \left| \left. D_q^n D_q u(t, \cdot), \phi_k \right| \right|^2
\]

\[
\leq \|\eta\|_{L_q^2(0,1)}^2 + \|\psi\|_{L_q^2(0,1)}^2
\]

\[
+ \|f\|_{C_q^2([0,T];W_q^2,0,1)}^2 < \infty,
\]

\[
\|D_q^2 u(t)\|^2_{L_q^2(0,1)} = \sum_{k \in \mathbb{N}} |D_q^k u(t)|^2 \leq \|\eta\|_{W_q^2,0,1}^2 + \|\psi\|_{W_q^2,0,1}^2
\]

\[
+ \|f\|_{C_q^3([0,T];W_q^2,0,1)}^2 < \infty,
\]

\[
\|\mathcal{L}u(t)\|_{H^1} = \sum_{k \in \mathbb{N}} |\langle \mathcal{L}u(t), \phi_k \rangle|\|^2
\]

\[
= \sum_{k \in \mathbb{N}} |\lambda_k u_k(t)|^2 \leq \|\eta\|_{W_q^2,0,1}^2
\]

\[
+ \|\psi\|_{W_q^2,0,1}^2
\]

\[
+ \|f\|_{C_q^2([0,T];W_q^2,0,1)}^2 < \infty,
\]

which means that \( u \in C_q^2([0,1];L_q^2(0,T)) \cap C_q^2([0, T] ; W_q^2,0,1) \).

Uniqueness. This part can be proved completely similar as the proof of Theorem 1. So we omit the details.

**Data Availability**

Data supporting this manuscript are available from Scopus, Web of Science, and Google Scholar.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Acknowledgments**

SS and NT were supported in parts by the MESRK (Ministry of Education and Science of the Republic of Kazakhstan) grant AP08052208.

**References**


