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Research Article

Fixed-Point Theorems Involving Lipschitz in the Small

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Lipschitz in the small is a generalization of the Lipschitz condition. The Lipschitz condition guarantees the uniqueness of the solution of the initial value problems. A special Lipschitz condition in the small is a contraction in the small. Based on the Lipschitz in the small in this paper, fixed-point theorems involving contraction in the small will be presented. The results will be applied to develop Picard's theorem.

1. Introduction

A Lipschitz condition is an interesting condition, both in analysis and in its applications. In Picard's theorem, the Lipschitz condition guarantees the uniqueness of the solution of an initial value problem (IVP). Picard proved the guarantee of the uniqueness by using the equivalency of the solution of the IVP with the solution of its associated integral problem.

Picard's theorem can be proven by using the Banach contraction theorem or by constructing a sequence of functions that converges uniformly to the solution [1]. Contraction is a specific condition of the Lipschitz condition, where its Lipschitz constant is in the interval $[0, 1)$. Many researchers generalized the concept of the Lipschitz function to have fixed-point theorems, such as Hussain et al., Liu and Xu, Pata, Xu and Radenović [2–5].

The concept of the Lipschitz condition has been developed as the Lipschitz in the small [6]. We recall the definition of Lipschitz in the small in Definition 1.

Definition 1. Let (X, d) and (Y, ρ) be metric spaces. A function $G : X \longrightarrow Y$ is said to be Lipschitz in the small if there exist an $\eta > 0$ and a $K \geq 0$, such that for every $x, y \in X$, $d(x, y) < \eta$, we have

$$\rho(G(x), G(y)) \leq K\, d(x, y). \tag{1}$$

The constant K is called the constant of Lipschitz in the small of G on X.

From the definition, a Lipschitz function in the small is uniformly continuous, but the converse is not always true. Garrido and Jaramillo state that a uniformly continuous function can be approached by the Lipschitz function in the small or the local Lipschitz function.

A Lipschitz function is Lipschitz in the small but the converse is not always true [6]. Based on that fact, Garrido and Jaramillo define a small-determined metric space. A small-determined metric space is a metric space with the set of all the Lipschitz functions equal to the set of all Lipschitz functions in the small. One of the examples of a small-determined metric space is a quasiconvex metric space.

A metric space V is called quasiconvex if there exists a positive constant C, such that for every two points in $x, y \in V$ can be joined by a continuous path with its length not greater than C times the distance between those two points x and y.

Based on Garrido's results, Leung and Tang [7] give necessary and sufficient conditions on a subset A of X such that $f|A$ is the Lipschitz for every function f that is Lipschitz in the small on X [7].

In this paper, the influence of the Lipschitz in the small condition in Picard's theorem will be seen. After defining the contraction in the small, its characteristics due to fixed-point theorems will be presented. The results will be applied to develop Picard's theorem using contraction in the small. The proof of the uniqueness in Picard's theorem will be provided by using Gronwall's inequality.

We recall Gronwall's inequality as in Theorem 2.

Theorem 2. (Gronwall's inequality) [1]. *Let g and h be two continuous positive real-valued functions on an interval $[c, d]$ and $g(c) \geq 0$. If g satisfies*

$$g(x) \leq g(c) + \int_c^x h(t)\, g(t)\, dt, \tag{2}$$

for every $x \in [c, d]$, then

$$g(x) \leq g(c) e^{\int_c^x h(t) dt}, \tag{3}$$

for every $x \in [c, d]$.

2. Results

Based on the definition of Lipschitz in the small it will be defined as the contraction in the small. A fixed-point theorem using contraction will be developed by using contraction in the small. Furthermore, it will be applied to develop a new version of Picard's theorem.

Definition 3. A function $f : (X, d) \longrightarrow (Y, \rho)$ is said to be a contraction in the small if there exist $r > 0$ and $K \in [0, 1)$, such that for every $x, y \in X$, $d(x, y) < r$, we have

$$\rho(f(x), f(y)) \leq K\, d(x, y). \tag{4}$$

Contraction in the small is a special case of the Lipschitz condition with its constant less than 1. It is clear that every contraction function is a contraction in the small function. The converse is not always true. For example, let us consider the function $g : \mathbb{N} \longrightarrow \mathbb{R}$, where $g(n) = (1/4)n^2$, $n \in \mathbb{N}$. The function g is a contraction in the small on $\mathbb{N}$, but it is not a contraction on $\mathbb{N}$.

Lemma 4. *If the function f is bounded and the Lipschitz in the small with K in Definition 3 is less than 1, then f is a contraction.*

Proof. Since f is bounded and Lipschitz in the small then f is Lipschitz. □

Lemma 5. *Let (X, d) and (Y, ρ) be metric spaces. If X is compact and if the function $f : X \longrightarrow Y$ is Lipschitz in the small then f is Lipschitz.*

Proof. The function f is Lipschitz in the small then f is uniformly continuous. Since X is compact, then f is bounded. As a corollary, f is Lipschitz. □

Corollary 6. *Let (X, d) and (Y, ρ) be metric spaces. If X is compact and if the function $f : X \longrightarrow Y$ is a contraction in the small, then f is a contraction.*

Proof. It is proven by Lemma 4. □

The relationships between a contraction in the small function and its fixed points will be discussed in Lemma 7 and Theorem 8.

Lemma 7. *Let (X, d) be a metric space and $x_0 \in X$. If $f : X \longrightarrow X$ is a contraction in the small and $f(x_0) = x_0$, then there exists a positive δ such that*

$$f(B(x_0, \delta)) \subseteq B(x_0, \delta). \tag{5}$$

Proof. Since f is a contraction in the small, there exist a positive K less than 1 and a positive r such that for every $p, q \in X$, $d(p, q) < r$,

$$d(f(p), f(q)) \leq K\, d(p, q). \tag{6}$$

Put $\delta = r$. Let $y \in f(B(x_0, \delta))$ be an arbitrary point. There is a point $x \in B(x_0, \delta)$ with $y = f(x)$. As a corollary, we have

$$d(y, x_0) = d(f(x), f(x_0)) \leq K\, d(x, x_0) < d(x, x_0). \tag{7}$$

This means that $y \in B(x_0, \delta)$. □

The converse of Lemma 7 is not always true. For example, let us consider the metric space $X = (0, 2)$ with its metric defined by $(u, v) = |u - v|$, for every $u, v \in X$. The function $f : X \longrightarrow \mathbb{R}$, where $f(x) = (x - 1)/3$, is a contraction in the small. For $x_0 = 1$ and $\delta = 2$, $f(B(1, 2)) = (0, 2/3) \subseteq (0, 2) = B(1, 2)$. However, $f(x_0) \neq x_0$.

Based on Lemma 7, it will be given a fixed-point theorem involving the contraction in the small in Theorem 8.

Theorem 8. *Let (X, d) be a complete metric space and $x_0 \in X$. If $f : X \longrightarrow X$ is a contraction in the small with constant r such that $f(\overline{B(x_0, r)}) \subseteq B(x_0, r)$, then there exists $u \in \overline{B(x_0, r)}$ as a fixed point of f. Moreover, if $u \in B(x_0, r)$, then u is a unique fixed point of f on $B(x_0, r)$.*

Proof. Since f is a contraction in the small, there exist $K \in [0, 1)$ and $r > 0$ such that for every $p, q \in X$, $d(p, q) < r$,

$$d(f(p), f(q)) \leq K\, d(p, q). \tag{8}$$

Let us define $x_1 = f(x_0)$ and $x_n = f(x_{n-1})$, for every $n \in \mathbb{N}$, $n > 1$. From the hypothesis, $x_n \in B(x_0, r)$ for every $n \in \mathbb{N}$. Let us consider that for every $n \in \mathbb{N}$,

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq K\, d(x_n, x_{n-1}) \leq K^n\, d(x_1, x_0) < K^n\, r. \tag{9}$$

As a corollary, for $N \geq n$, there is $m \in \mathbb{N}$, with $N = n + m$. Therefore,

$$d(x_N, x_n) = d(x_{n+m}, x_n) < r\left(K^{n+m-1} + K^{n+m-2} + \cdots + K^n\right) \le r K^n \sum_{i=0}^{\infty} K^i. \quad (10)$$

Since $K \in [0.1)$, the sequence (x_n) is a Cauchy sequence in X. The completeness of X implies the existence of $u \in X$ such that (x_n) converges to u in X. Since $x_n \in B(x_0, r)$ for every n, then $u \in \overline{B(x_0, r)}$. From the hypothesis, $f(u) \in B(x_0, r)$. Therefore,

$$\begin{aligned} d(u, f(u)) &\le d\left(u, x_{n_0}\right) + d\left(x_{n_0}, x_{n_0+1}\right) + d\left(x_{n_0+1}, f(u)\right) \\ &\le (1+K) d\left(u, x_{n_0}\right) + d\left(x_{n_0}, x_{n_0+1}\right) \\ &< (1+K)\epsilon + 2\epsilon = (K+3)\epsilon. \end{aligned} \quad (11)$$

This means that u is a fixed point of the function f. Furthermore, if $v \in B(x_0, r)$ with $f(v) = v$, then

$$d(v, u) = d(f(v), f(u)) \le K\, d(u, v). \quad (12)$$

Since $K \in [0, 1)$, we have $d(v, u) = 0$. This means that $v = u$. □

Before we develop Picard's theorem using Lipschitz in the small let us consider that for every $(x, y), (x, y^*) \in \mathbb{R}^2$, the standard/usual metric in $\mathbb{R}^2$ will bring $d((x, y), (x, y^*)) < r$ that is equal with $|y - y^*| < r$.

Definition 9. Let $V \subseteq \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$. A function $f : V \longrightarrow \mathbb{R}$ is said to be Lipschitz in the small in the second component on V, if there exists a positive $r > 0$ and $K \ge 0$ such that for every $(x, y), (x, y^*) \in V$, $|y - y^*| < r$, we have

$$|f(x, y) - f(x, y^*)| \le K\,|y - y^*|. \quad (13)$$

Let $V \subseteq \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$. A function $f : V \longrightarrow \mathbb{R}$ is said to be a contraction in the small in the second component on V, if there exist a positive $r > 0$ and $K \in [0, 1)$ such that for every $(x, y), (x, y^*) \in V$, $|y - y^*| < r$, we have

$$|f(x, y) - f(x, y^*)| \le K\,|y - y^*|. \quad (14)$$

In the next discussion, the word domain refers to an open connected set in $\mathbb{R}^2$.

Let consider the initial value problem (IVP)

$$\begin{cases} \dfrac{dy}{dx} = f(x, y), \\ y(x_0) = y_0, \end{cases} \quad (15)$$

with f is a continuous real-valued function on a domain.

Picard's Theorem still holds if the condition of Lipschitz is replaced by a contraction in the small. The important thing in the proof is in choosing the value of $\alpha > 0$ such that the constructed sequence (ϕ_n) satisfies $|\phi_{n+1}(x) - \phi_n(x)| < r$, for every $|x - x_0| < \alpha$ and for every n. In the discussion, the function f is a nonzero function on the domain.

Theorem 10. *Let $D \subseteq \mathbb{R}^2$ be a domain, (x_0, y_0) be an interior point of D, and $f : D \longrightarrow \mathbb{R}$. If the function f is continuous on D and satisfies the contraction in the small in the second component on D, then there exist a positive α and a unique function*

$$\phi : [x_0 - \alpha, x_0 + \alpha] \longrightarrow \mathbb{R}, \quad (16)$$

such that ϕ is a solution of the initial value problem in (15) on $[x_0 - \alpha, x_0 + \alpha]$.

Proof. Since (x_0, y_0) is an interior point of D, there exist positive numbers $a, b \in \mathbb{R}$ such that

$$E = \left\{(x, y) \in \mathbb{R}^2 : |x - x_0| \le a, |y - y_0| \le b\right\} \subseteq D. \quad (17)$$

Since f is continuous on E and E is compact, then $|f|$ attains its maximum. Put $M = \max\{|f(x, y)| : (x, y) \in E\}$. Since f is a contraction in the small in the second component on D, there exist an $r > 0$ and $K \in [0, 1)$ such that for every $(x, y), (x, y^*) \in D$, $d((x, y), (x, y^*)) < r$, we have

$$|f(x, y) - f(x, y^*)| \le K|y - y^*|. \quad (18)$$

Put $\alpha = \min\{a, b/(b + M), r/M, r^2/(r+1)K\}$.

Let us define $\phi(x_0) = y_0$ and

$$\phi_n(x) = y_0 + \int_{x_0}^{x} f(t, \phi_{n-1}(t))dt, x \in [x_0 - \alpha, x_0 + \alpha], \quad \text{for } n = 1, 2, 3, \cdots. \quad (19)$$

By using the induction method, it can be proven that for every $|x - x_0| < \alpha$, we have

$$\text{(i)} |\phi_n(x_0) - y_0| \le \int_{x_0}^{x} |f(t, \phi_{n-1}(t))|dt \le M\,|x - x_0| \le M\alpha \le b, \quad \text{for } n = 1, 2, \cdots, \quad (20)$$

$$\text{(ii)} |\phi_{n+1}(x) - \phi_n(x)| \le \frac{K^n M\,|x - x_0|^{n+1}}{(n+1)!} \le b\,\frac{(K\,|x - x_0|)^n}{n!}, \quad \text{for every } n. \quad (21)$$

From (20) and by considering that $K \in [0, 1)$ and $\alpha < 1$, by using the induction method, we can deduce that for every $|x - x_0| < \alpha$ and for every n,

$$|\phi_{n+1}(x) - \phi_n(x)| \le \frac{K^n M|x - x_0|^{n+1}}{(n+1)!} \le M\alpha\alpha^n \le M\alpha < r. \quad (22)$$

Let us consider that

$$
\begin{aligned}
|\phi_1(x) - \phi_0(x)| &\le \int_{x_0}^{x} |f(t, \phi_0(t))| dt \le M\,|x - x_0|, \\
|\phi_2(x) - \phi_1(x)| &\le \int_{x_0}^{x} |f(t, \phi_1(t)) - f(t, \phi_0(t))| dt \\
&\le \int_{x_0}^{x} K|\phi_1(t) - \phi_0(t)| dt \\
&\le \int_{x_0}^{x} KM\,|t - x_0| dt = KM\,\frac{1}{2}|x - x_0|^2.
\end{aligned} \tag{23}
$$

Let assume the formula is held for $n = m$, i.e.,

$$
|\phi_{m+1}(x) - \phi_m(x)| \le \frac{K^m M |x - x_0|^{m+1}}{(m+1)!}. \tag{24}
$$

Therefore,

$$
\begin{aligned}
&|\phi_{m+2}(x) - \phi_{m+1}(x)| \\
&\quad \le \int_{x_0}^{x} |f(t, \phi_{m+1}(t)) - f(t, \phi_m(t))| dt \\
&\quad \le \int_{x_0}^{x} K|\phi_{m+1}(t) - \phi_m(t)| dt \le \int_{x_0}^{x} K\,\frac{K^m\, M|t - x_0|^{m+1}}{(m+1)!} dt \\
&\quad = K\,\frac{K^m\, M|x - x_0|^{m+2}}{(m+2)!} = \frac{K^{m+1}\, M|x - x_0|^{m+2}}{(m+2)!}.
\end{aligned} \tag{25}
$$

The induction method gives the formula which holds for every n.

Moreover, for every $|x - x_0| < \alpha$ and for every n,

$$
|\phi_{n+1}(x) - \phi_n(x)| \le \frac{K^n\, M\,|x - x_0|^{n+1}}{(n+1)!} \le b\,\frac{(K|x - x_0|)^n}{n!} \le b\,\frac{(K\alpha)^n}{n!}. \tag{26}
$$

Therefore,

$$
|\phi_{n+1}(x) - \phi_0(x)| = \left|\sum_{i=0}^{n} (\phi_{i+1}(x) - \phi_i(x))\right| \le \sum_{i=0}^{n} b\,\frac{(K|x - x_0|)^i}{i!} \le \sum_{i=0}^{n} b\,\frac{(K\alpha)^i}{i!}. \tag{27}
$$

Since the series $\sum_{n=0}^{\infty} (K\alpha)^n/n!$ is convergent, via the Weierstrass M-test, then the sequence (ϕ_n) converges uniformly to a function, say ϕ, on $[x_0 - \alpha, x_0 + \alpha]$.

Since (ϕ_n) converges uniformly to ϕ on $[x_0 - \alpha, x_0 + \alpha]$, and for every n, ϕ_n is continuous on $[x_0 - \alpha, x_0 + \alpha]$, and the function ϕ is continuous on $[x_0 - \alpha, x_0 + \alpha]$. Furthermore,

$$
\lim_{n \longrightarrow \infty} \int_{x_0}^{x} \phi_n(t) dt = \int_{x_0}^{x} \lim_{n \longrightarrow \infty} \phi_n(t) dt. \tag{28}
$$

Therefore, for every $x \in [x_0 - \alpha, x_0 + \alpha]$,

$$
\begin{aligned}
\phi(x) = \lim_{n \longrightarrow \infty} \phi_{n+1}(x) &= y_0 + \lim_{n \longrightarrow \infty} \int_{x_0}^{x} f(t, \phi_n(t)) dt \\
&= y_0 + \int_{x_0}^{x} \lim_{n \longrightarrow \infty} f(t, \phi_n(t)) dt.
\end{aligned} \tag{29}
$$

This means that

$$
\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t)) dt, \tag{30}
$$

for every $x \in [x_0 - \alpha, x_0 + \alpha]$.

Since f is continuous, then ϕ is differentiable on $[x_0 - \alpha, x_0 + \alpha]$.

Moreover, $\phi(x_0) = y_0$. This means that ϕ is a solution of the IVP (15) on $[x_0 - \alpha, x_0 + \alpha]$.

Uniqueness. Let ϕ and ψ be two solutions of the IVP (15) on $[x_0 - \alpha, x_0 + \alpha]$,

$$
\begin{aligned}
\phi(x) &= y_0 + \int_{x_0}^{x} f(t, \phi(t)) dt, \\
\psi(x) = \phi(x) &= y_0 + \int_{x_0}^{x} f(t, \psi(t)) dt,
\end{aligned} \tag{31}
$$

for every $x \in [x_0 - \alpha, x_0 + \alpha]$.

$$
\begin{aligned}
|\phi(x) - \psi(x)| &= \left| \int_{x_0}^{x} f(t, \phi(t)) dt - \int_{x_0}^{x} f(t, \psi(t)) dt \right| \\
&\le \int_{x_0}^{x} K \lceil \phi(t) - \psi(t) \rceil dt.
\end{aligned} \tag{32}
$$

By using Gronwall's inequality with $g(x) = |\phi(x) - \psi(x)|$ and $h(x) = K$ for every $x \in [x_0, x_0 + \alpha]$, we get $0 \le |\phi(x) - \psi(x)| \le 0$. This means that $\phi(x) = \psi(x)$ for every $x \in [x_0, x_0 + \alpha]$. □

Similarly, for $x \in [x_0 - \alpha, x_0]$,we get $\phi(x) = \psi(x)$ for every $x \in [x_0 - \alpha, x_0]$. Therefore, we have $\psi = \phi$ on $[x_0 - \alpha, x_0 + \alpha]$.

Remark 11. Theorem 10 still holds if the condition contraction in the small in the second component on D is replaced by the contraction in the small.

The α in Theorem 10 may be extended as stated in Theorem 12.

Theorem 12. *Let $a, b \in \mathbb{R}$ and $D = \{(x, y) \in \mathbb{R}^2 : a < x < b, y \in \mathbb{R}\}$. If (x_0, y_0) is an interior point of D and $f : D \longrightarrow \mathbb{R}$ is a continuous function on D and satisfies the contraction in the small in the second component on D, then there exists a unique function ϕ as a solution of the initial value problem in (15) on (a, b). Moreover, if $a = -\infty$ and $b = \infty$, then the unique solution ϕ of the initial value problem in (15) exists on $(-\infty, \infty)$.*

Proof. From Theorem 10, there exist an $\alpha_0 > 0$ and a unique solution φ_0 of the initial value problem in (15) on $[x_0 - \alpha_0, x_0 + \alpha_0]$.

(i) Put $x_1 = x_0 - \alpha_0$ and $y_1 = \varphi_0(x_0 - \alpha_0)$. It is clear that (x_1, y_1) is an interior point of D. By Theorem 10, there exist an $\alpha_1 > 0$ and a unique solution φ_1 on $[x_0 - \alpha_0 - \alpha_1, x_0 - \alpha_0]$ of the initial value problem

$$\begin{cases} \dfrac{dy}{dx} = f(x, y), \\ y(x_1) = y_1. \end{cases} \tag{33}$$

(ii) Put $x_2 = x_0 + \alpha_0$ and $y_2 = \varphi_0(x_0 + \alpha_0)$. It is clear that (x_2, y_2) is an interior point of D. Based on Theorem 10, there exist an $\alpha_2 > 0$ and a unique solution φ_2 on $[x_0 + \alpha_0, x_0 + \alpha_0 + \alpha_2]$ of the initial value problem

$$\begin{cases} \dfrac{dy}{dx} = f(x, y), \\ y(x_2) = y_2. \end{cases} \tag{34}$$

Continuing the process, we have $\alpha_k > 0$, $k \in \mathbb{N}$, $x_{2k+1} = x_{2k-1} + \alpha_{2k-1}$, $x_{2k} = x_{2(k-1)} + \alpha_{2(k-1)}$, $k = 1, 2, 3, \cdots$, and

(i) a unique solution φ_1 on $[x_0 - \alpha_0 - \alpha_1, x_0 - \alpha_0]$

(ii) unique solutions φ_{2k} on $[x_0 + \sum_{i=0}^{k-1} \alpha_{2i}, x_0 + \sum_{i=0}^{k} \alpha_{2i}]$, $k \in \mathbb{N}$

(iii) φ_{2k+1} on $[x_0 - \alpha_0 - \sum_{i=0}^{k} \alpha_{2i+1}, x_0 + \alpha_0 + \sum_{i=0}^{k-1} \alpha_{2i+1}]$, $k \in \mathbb{N}$

As a corollary, we have a unique solution φ, where

$$\varphi(x) = \begin{cases} \varphi_0(x), & x \in [x_0 - \alpha_0, x_0 + \alpha_0], \\ \varphi_1(x), & x \in [x_0 - \alpha_0 - \alpha_1, x_0 - \alpha_0], \\ \varphi_{2k}(x), & x \in \left[x_0 + \sum_{i=0}^{k-1} \alpha_{2i}, x_0 + \sum_{i=0}^{k} \alpha_{2i}\right], \quad \text{for some } k \in \mathbb{N}, \\ \varphi_{2k+1}(x), & x \in \left[x_0 - \alpha_0 - \sum_{i=0}^{k} \alpha_{2i+1}, x_0 + \alpha_0 + \sum_{i=0}^{k-1} \alpha_{2i+1}\right], \quad \text{for some } k \in \mathbb{N}, \\ & \varphi(x_0) = y_0. \end{cases} \tag{35}$$

For every $k = 1, 2, 3, \cdots$,

$$\begin{aligned} &\left[x_0 - \alpha_0 - \sum_{i=1}^{k} \alpha_{2i-1}, x_0 + \alpha_0 + \sum_{i=1}^{k} \alpha_{2i}\right] \\ &\quad \subseteq \left[x_0 - \alpha_0 - \sum_{i=1}^{k+1} \alpha_{2i-1}, x_0 + \alpha_0 + \sum_{i=1}^{k+1} \alpha_{2i}\right]. \end{aligned} \tag{36}$$

Let $a_k = x_{2k-1}$ and $b_k = x_{2k}$, $k = 1, 2, 3, \cdots$. We have

$$\lim_{k \longrightarrow \infty} a_k = a, \lim_{k \longrightarrow \infty} b_k = b. \tag{37}$$

□

3. Concluding Remarks

Theorem 8 has given a fixed-point theorem involving a contraction in the small. The proof of the developed Picard's Theorem has been provided in Theorem 10 by constructing a sequence of functions as in Picard approximation. The key is to have an α such that the contraction condition still holds for every n, $|\phi_{n+1}(x) - \phi_n(x)| < r$, for every $x \in [x_0 - \alpha, x_0 + \alpha]$.

Data Availability

No data were used in the research.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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