Hindawi
Abstract and Applied Analysis
Volume 2024, Article ID 2432993, 7 pages
https://doi.org/10.1155/2024/2432993

Research Article

Frequently Hypercyclic Semigroup Generated by Some Partial Differential Equations with Delay Operator

Cheng-Hung Hung

Cheng Shiu University, No.840, Cheng Cing Road, Kaohsiung 833, Taiwan

Correspondence should be addressed to Cheng-Hung Hung; hong838@yahoo.com.tw

Received 8 November 2023; Revised 16 February 2024; Accepted 17 April 2024; Published 17 May 2024

Academic Editor: Victor Kovtunenko

In this paper, under appropriate hypotheses, we have the existence of a solution semigroup of partial differential equations with delay operator. These equations are used to describe time–age-structured cell cycle model. We also prove that the solution semigroup is a frequently hypercyclic semigroup.

1. Introduction

Partial differential equations with delay have been studied for many years and arise in various applications, like biology, medicine, control theory, climate models, and many others. For example, the following mathematical problem involving a delayed nonlocal dynamical described by a particular partial differential equation as follows:

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = g(t, u, u_t), \tag{1}$$

where $u_t : \sigma \in [-\tau, 0] \to u(t+\sigma) \in X = L^p,\ \tau > 0$ and with an initial condition:

$$u(t, x) = \phi(t, x),\ \text{for } (t,\ x) \in [-\tau, 0] \times [0, 1]. \tag{2}$$

This equation is considered as a particular time–age-structured cell cycle model that was motivated by the biological process of hematological cells.

When $c(x) = x$ and $g(t, u, u_t) = \lambda u, \lambda$ is the constant, then equation can be written as follows:

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \lambda u,\ t \geq 0, 0 \leq x \leq 1, \tag{3}$$

with the initial condition:

$$u(0, x) = \nu(x), 0 \leq x \leq 1. \tag{4}$$

This equation is so called the Lasota equation (see, e.g., [1–5]) and the references cited therein.

This equation has been developed as a model of the dynamics of a self-reproducing cell population, such as the population of developing red blood cells (erythrocyte precursors). It also has been applicable to a conceptualization of abnormal blood cell production, such as leukemia. Although this equation is linear but the solution also has chaotic behavior and is studied by many authors

In this paper, we are interested in $g(t, u, u_t) = \Phi u_t, \Phi$ is a delay operator, and the first-order partial differential equation with delay is expressed as follows:

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \Phi u_t,\ t \geq 0, 0 \leq x \leq 1, \tag{5}$$

with an initial condition:

$$u(0) = w \text{ and } u_0 = f, \tag{6}$$

where Φ, w, and f see later hypotheses.

Here, we study the partial differential equations with delay in an abstract way.

Under the appropriate hypotheses that follow Bátkai and Piazzera's [6] study, we can switch the linear delay differential equations to be an abstract Cauchy problem in an appropriate Banach space. That means, we can use a semigroup approach to deal with those equations.

We introduce the abstract delay equation and hypotheses first. List the standing hypotheses as follows:

(H1) X is a Banach space.

(H2) Usually B just needs to be a closed, densely defined, linear operator, but here we choose $B = -x\frac{\partial u}{\partial x}$ to let the equations have the frequently hypercyclic property.

(H3) $f \in L^p([-1,0], X)$, $1 \le p < \infty$.

(H4) $\Phi \in W^{1,p}([-1,0], X) \to X$ is a bounded linear operator, called the delay operator.

(H5) $\Theta := X \times L^p([-1,0], X)$.

Under these hypotheses and for given elements $x \in X$ and $f \in L^p([-1,0], X)$.

The following initial value problem will be called an abstract delay equation:

$$(\mathrm{DE})_p \begin{cases} u'(t) = Bu(t) + \Phi u_t, t \ge 0 \\ u(0) = w \\ u_0 = f \end{cases} . \tag{7}$$

For a function $u : [-\tau, \infty) \to X$, $t \ge 0$ and $u_t : \sigma \in [-\tau, 0] \to u(t+\sigma) \in X$.

Chaotic phenomena are interesting and abundant topics in different areas and attract many mathematicians (see, e.g., [1, 2, 7–10]). Alberto Conejero et al. [7] introduced different kinds of chaotic operators, such as Devaney chaos, frequent hypercyclicity, and so on. In [7] the authors provide a lot of examples. Here, we are interesting in a tape of C_0-semigroup so called frequently hypercyclic semigroup. Motivated by Birkhoff's ergodic theorem, Bayart and Grivaux [11] introduced the notion of frequently hypercyclic operators trying to quantify the frequency with which an orbit meets the open set. This concept was extended to C_0-semigroup in [12].

When a semigroup $\{S(t)\}_{t\ge0}$ is a frequently hypercyclic semigroup, then for every $t_0 > 0$ the operator $S(t_0)$ is frequently hypercyclic, but the chaotic semigroup does not necessarily satisfy this condition. By the results of Bayart and Bermúdez [13], there are chaotic C_0-semigroup $\{S(t)\}_{t\ge0}$ such that no single operator $S(t)$ is chaotic and a C_0-semigroup $\{S(t)\}_{t\ge0}$ containing a nonchaotic operator $S(t_0)$, $t_0 > 0$ and a chaotic operator $S(t_1)$ for some $t_1 > 0$. However, if a frequently hypercyclic semigroup $\{S(t)\}_{t\ge0}$ satisfies frequently hypercyclic criterion, then $S(t)$ is also chaotic for every $t > 0$ [12, Proposition 2.7]. This is one of the reason for us to study frequently hypercyclic semigroups.

The structure of this paper is following. In Section 2, we will introduce some useful terminologies and proposition. In Section 3, first, we prove that the semigroup $\{\mathfrak{J}_0(t)\}_{t\ge0}$ generated by A_0 (see later) is frequently hypercyclic in Theorem 1. Then, we describe the solution semigroup $\{U(t)\}_{t\ge0}$ using $\{\mathfrak{J}_0(t)\}_{t\ge0}$ and purtubation theorem. Then by constructing the new set having frequently hypercyclic characteristics with positive lower density, we prove the purtubation of a frequently hypercyclic semigroup is also a frequently hypercyclic semigroup in Theorem 2. In Section 4, we give some examples.

2. Terminologies

First, we introduce some useful terminologies and propositions. We recall that the lower density of a measurable set $M \subset R_+$ is defined by the following equation:

$$\underline{\mathrm{Dens}}(M) \equiv \lim_{N\to\infty} \inf \mu(M \cap [0, N])/N, \tag{8}$$

where μ is the Lebesgue measure on R_+. A C_0-semigroup $\{S(t)\}_{t\ge0}$ is called frequently hypercyclic on the sunspace $M \subseteq X$ if there exists $x \in M$ such that $\underline{\mathrm{Dens}}\,(\{t \in \mathbb{R}_+ : S(t)x \in U\}) > 0$ for any nonempty open set $U \subset M$.

The lower density of a set $A \subset \mathbb{N}$ is defined by the following equation:

$$\underline{\mathrm{Dens}}(A) \equiv \lim_{N\to\infty} \inf \#\{n \le N : n \in A\}/N. \tag{9}$$

An operator $S \in L(X)$ is said to be frequently hypercyclic on the sunspace $M \subseteq X$ if there exists $x \in M$ (called frequently hypercyclic vector) such that for any nonempty open set $U \subset M$, the set $\{n \in \mathbb{N} : S^n x \in U\}$ has positive lower density. If $x \in X$ is a frequently hypercyclic vector for $\{S(t)\}_{t\ge0}$, then, for every $t > 0$, the x is also a frequently hypercyclic vector for the operator $S(t)$, for detail see Mangino and Peris's [12] study.

Proposition 1. *See [14, Proposition 2.1].*

Let $\{S(t)\}_{t\ge0}$ be a C_0-semigroup on a separable Banach space X. Then, the following conditions are equivalent:

(1) *$\{S(t)\}_{t\ge0}$ is frequently hypercyclic,*

(2) *for every $t > 0$, the operator $S(t)$ is frequently hypercyclic, and*

(3) *there exist $t_0 > 0$ such that the operator $S(t_0)$ is frequently hypercyclic.*

By Proposition 1, to prove $S(t)$ is frequently hypercyclic just need to prove $S(t_0)$ is frequently hypercyclic operator for some fixed t_0.

3. The Frequently Hypercyclic Semigroup $\{S(t)\}_{t\ge0}$

In order to use the semigroup approach to deal with $(\mathrm{DE})_p$, we switch $(\mathrm{DE})_p$ to an abstract Chachy problem and hope the solution of $(\mathrm{DE})_p$ is equal to the solution of the abstract Chachy problem.

If $u : [-\tau, \infty) \to X$ is a classical solution of $(\mathrm{DE})_p$, then the function:

$$U : \begin{pmatrix} u(t) \\ u_t \end{pmatrix} \in \Theta, \tag{10}$$

is a classical solution of the abstract Chachy problem:

$$(\mathrm{ACP})_p \begin{cases} u'(t) = Au(t) \\ u(0) = \begin{pmatrix} x \\ f \end{pmatrix} \end{cases}, \tag{11}$$

with $A = \begin{pmatrix} B & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix}$, where $\frac{d}{d\sigma}$ denotes the distributional derivative domain:

$$D(A) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(B) \times W^{1,p}([-1,0,X]), f(0) = x \right\}. \tag{12}$$

Reversely, for every classical solution U of $(\mathrm{ACP})_p$, the function u is defined as follows:

$$u(t) = \begin{cases} (\Pi_1 \circ U)(t), \text{ if } t \geq 0 \\ f(t), \text{ if } t \in [-1, 0) \end{cases}. \tag{13}$$

Then, u is a classical solution of $(\mathrm{DE})_p$ and $(\Pi_2 \circ U)(t) = u_t$ for all $t \geq 0$, where Π_1 is the canonical project from Θ onto X and similarly Π_2 is the canonical project from Θ to $L^p([-1, 0], X)$.

As we want, we can transform the problem of solving the delay equation $(\mathrm{DE})_p$ to solving $(\mathrm{ACP})_p$.

We write:

$$A = \begin{pmatrix} B & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix}, \tag{14}$$

as the sum $A = A_0 + C$, where $A_0 = \begin{pmatrix} B & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix}$, $C = \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix} \in L(D(A_0), \Theta)$, $1 \leq p < \infty$ and $D(A_0) = D(A) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(B) \times W^{1,p}([-1, 0], X), f(0) = x \right\}$.

From Bátkai and Piazzera [6], under appropriate assumption, we can see that A_0 is the generator of semigroup $\{\mathfrak{T}_0(t)\}_{t\geq 0}$ and the solution semigroup $\{\mathfrak{T}_0(t)\}_{t\geq 0}$ is given by the following equation:

$$\mathfrak{I}_0(t) = \begin{pmatrix} S(t) & 0 \\ S_t & T_0(t) \end{pmatrix}, \tag{15}$$

where the semigroup $\{S(t)\}_{t\geq 0}$ is generated by operator $B = -x\frac{d}{dx}$, $\{T_0(t)\}_{t\geq 0}$ is the nilpotent left semigroup on $L^p([-1, 0], X)$ and $S_t : X \to L^p([-1, 0], X)$ is defined by the following equation:

$$(S_t x)(-\tau) = \begin{cases} S(t-\tau)x, & \text{if } -t < -\tau \leq 0 \\ 0, & \text{if } -1 \leq -\tau \leq -t \end{cases}. \tag{16}$$

Remark 1. We consider the set $\Omega = \{v \in X : v(0) = 0\}$ is the subset X (as H1). The semigroup $\{S(t)\}_{t\geq 0}$ is generated by operator B is described by $S(t)x(\lambda) = x(\lambda e^{-t})$ and is frequently hypercyclic on Ω, for detail see Hung and Chang's [15] study.

Theorem 1. *If the solution semigroup $\{S(t)\}_{t\geq 0}$ is frequently hypercyclic, then the solution semigroup $\{\mathfrak{T}_0(t)\}_{t\geq 0}$ is also frequently hypercyclic on* $W = \left\{ \begin{pmatrix} y \\ g \end{pmatrix} \in \Omega \times W^{1,p}([-1, 0], X), g(0) = y, g(\omega)(s) = y(se^{-\omega}) \right\}$.

Proof. First, we note that W is nonempty. For example, $g(0)(s) = y(s) = s \in \Omega$, $g(\omega)(s) = se^{-\omega}$, and $0 \neq \begin{pmatrix} y \\ g \end{pmatrix} \in W$. □

According to Proposition 1, to prove $\mathfrak{T}_0(t)$ is frequently hypercyclic is equal to proving that $\mathfrak{T}_0(t_0)$ is frequently hypercyclic operator for some fixed $t_0 > 0$.

For every nonempty open subset $U = U_1 \times U_2 \subset W$, without losing the generality we can suppose $U_1 = \{z \in \Omega : \|z - y\|_X < \varepsilon_1\}$, $U_2 = \{h \in W^{1,p}([-1, 0], X) : \|h - g\|_p < \varepsilon_2\}$ for some $\varepsilon_1, \varepsilon_2 > 0$ and $\begin{pmatrix} y \\ g \end{pmatrix} \in W$. We need to check for U the existence of frequently hypercyclic vector $\begin{pmatrix} w \\ j \end{pmatrix} \in W$ such that the set $E = \left\{ n \in \mathbb{N}, \mathfrak{T}_0^n(t_0) \begin{pmatrix} w \\ j \end{pmatrix} \in U \right\}$ has positive lower density.

Since the semigroup $\{S(t)\}_{t\geq 0}$ generated by operator B is frequently hypercyclic, there exists a frequently hypercyclic vector $x \in \Omega$ such that for any neighborhood $\overline{U}$ with radius $r > 0$ and center $y = g(0)$, $(\overline{U} = \{z : \|z - y\|_X < r\})$ then the set $E_0 = \{n \in \mathbb{N}, S^n(t_0)x = S(nt_0)x \in \overline{U}\}$ has positive lower density.

Instead of proving $\mathfrak{T}_0^n(t_0) \begin{pmatrix} w \\ j \end{pmatrix}$ in U directly, we prove $\mathfrak{T}_0^n(t_0) \begin{pmatrix} w \\ j \end{pmatrix}$ belongs to a subset of U. Now, we construct such subset of U. Choose $r_1 > 0$ such that $\frac{1}{2}\min\{\varepsilon_1, \varepsilon_2\} > r_1$ and let $U' = \{z : \|z - y\|_X < r_1\}$, it is clear that $U' \subset U_1$. Thus, corresponding U', there exists a set $E_1 = \{n \in \mathbb{N}, S^n(t_0)x = S(nt_0)x \in U'\}$ and E_1 has positive lower density.

Let $U'' = \{h \in W^{1,p}([-1, 0], X) : h(0) = y' \in U'\}$, then:

$$\|h-g\|_p=\left(\int_{-1}^{0}(h(\lambda)(s)-g(\lambda)(s))^p d\lambda\right)^{\frac{1}{p}}=\left(\int_{-1}^{0}(y'(se^{-\lambda})-y(se^{-\lambda}))^p d\lambda\right)^{\frac{1}{p}}$$
$$\le\|y'-y\|_X\left(\int_{-1}^{0}1d\lambda\right)^{\frac{1}{p}}=\|y'-y\|_X<2r_1<\varepsilon_2. \quad (17)$$

From above inequalties, we have $U''\subset U_2$ and $U'\times U''\subset U_1\times U_2$.

Next, we need to find the set E, E_1 is natural suggestion and find a frequently hypercyclic vector, $\begin{pmatrix}x\\f\end{pmatrix}$ is natural suggestion for $\begin{pmatrix}w\\j\end{pmatrix}$.

From Equation (15), we have the following form:

$$\Im_0^n(t_0)\begin{pmatrix}x\\f\end{pmatrix}=\Im_0(nt_0)\begin{pmatrix}x\\f\end{pmatrix}=\begin{pmatrix}S(nt_0) & 0\\ S_{nt_0} & T_0(nt_0)\end{pmatrix}\begin{pmatrix}x\\f\end{pmatrix}=\begin{pmatrix}S(nt_0)x\\ S_{nt_0}x+T_0(nt_0)f\end{pmatrix}. \quad (18)$$

Then, we consider the second component of Equation (18), that is $S_{nt_0}x+T_0(nt_0)f$. From the perform of T_0, we have $T_0(nt_0)f(-\tau)=f(nt_0-\tau)=0$, where $n>n_0$ for some n_0 and we extent f to be zero out of $[-\tau,0]$.

Then, Equation (18) becomes:

$$\Im_0^n(t_0)\begin{pmatrix}x\\f\end{pmatrix}=\begin{pmatrix}S(nt_0)x\\ S_{nt_0}x\end{pmatrix}, \quad (19)$$

and we have the following equation:

$$\Im_0^n(t_0)\begin{pmatrix}x\\f\end{pmatrix}-\begin{pmatrix}y\\g\end{pmatrix}=\begin{pmatrix}S(nt_0)x-y\\ S_{nt_0}x-g\end{pmatrix},\ \text{for}\ \begin{pmatrix}y\\g\end{pmatrix}\in U'\times U''. \quad (20)$$

Since $S(nt_0)x\in U'$ in the first component of Equation (20) that is $\|S(nt_0)x-y\|<r_1$. Then, we move on the second component of Equation (20).

If g is the second component of $\begin{pmatrix}y\\g\end{pmatrix}\in W$ and $n\in E_1$, $n>n_0$, we get the following equation:

$$\begin{aligned}S_{nt_0}x(\lambda)-g(\lambda)&=S(nt_0+\lambda)x(s)-g(\lambda)(s)=x(se^{-nt_0-\lambda})-y(se^{-\lambda})\\&=\widetilde{S}(\lambda)S(nt_0)x-y(se^{-\lambda})\\&=\widetilde{S}(\lambda)y''-\widetilde{S}(\lambda)y,\end{aligned} \quad (21)$$

where we define an operator $\tilde{S}$ as $\tilde{S}(\lambda)\omega(s)=\omega(se^{-\lambda})$ and $y''=S(nt_0)x\in U'$. In particular, when $\omega=y$, we have $\tilde{S}(\lambda)y(s)=y(se^{-\lambda})=g(\lambda)(s)$.

From the above, we have the following equation:

$$\left\|S_{nt_0}x(\lambda)-g(\lambda)\right\|_p=\left\|\widetilde{S}(\lambda)y''(s)-\widetilde{S}(\lambda)y(s)\right\|_p=\|y''(se^{-\lambda})-y(se^{-\lambda})\|_p$$
$$=\left(\int_{-1}^{0}(y''(se^{-\lambda})-y(se^{-\lambda}))^p d\lambda\right)^{\frac{1}{p}}\le\|y''-y\|_X\left(\int_{-1}^{0}1d\lambda\right)^{\frac{1}{p}}=\|y''-y\|_X<r_1. \quad (22)$$

Since y'' in U'.

From the above conclusion, we get $S(nt_0)x-y$ in U', $S_{nt_0}x-g$ in U'', and:

$$\left\|\Im_0^n(t_0)\begin{pmatrix}x\\f\end{pmatrix}-\begin{pmatrix}y\\g\end{pmatrix}\right\|=\left\|\begin{pmatrix}S(nt_0)x-y\\ S_{nt_0}x-g\end{pmatrix}\right\|=\|S(nt_0)x-y\|_X+\left\|S_{nt_0}x-g\right\|_p<r_1+r_1<2r_1, \quad (23)$$

so $\mathfrak{J}_0^n(t_0)\begin{pmatrix}x\\f\end{pmatrix}\subset U'\times U''\subset U$. We get $\begin{pmatrix}x\\f\end{pmatrix}$ is a frequently hypercyclic vector and $E_1'=\{n\in\mathbb{N}, n\in E_1, n>n_0\}$ has postive lower density. Therefore, E_1' is the set that we wanted. This implies that $\mathfrak{J}_0(t_0)$ is a frequently hypercyclic operator on U and the theorem is proved.

Since we have the structure of $\{\mathfrak{J}_0(t)\}_{t\ge0}$, we can move on to study the existence of the solution semigroup of Equation (5), we add the condition (H6) and state as follows:

(H6) There exist some $t_0>0$ and $0\le q<1$ such that:

$$\int_0^{t_0}\|C\Im_0(r)x\|dr\le q\|x\|\ \text{for all}\ x\in D(A_0). \quad (24)$$

Under the condition (H6), we consider the perturbation case. The existence of solution semigroup $\{U(t)\}_{t\ge0}$ generated by A follows from Theorem 3.2 Miyadera–Voigt in [6].

The semigroup $\{U(t)\}_{t\ge0}$ is given by the Dyson–Phillips series and denoted as $U(t)x=\sum_{n=0}^{\infty}(V^n\mathfrak{J}_0)(t)x$ for all $x\in X$, $t\ge0$ and converging uniformly on compact subset of R^+. Here, $V\in L(C(R^+,L(X)))$ is the abstract Volterra operator defined by the following equation:

$$(VF)(t)z = \int_0^t F(t-s)C\Im_0(s)zds, \text{ for } t \geq 0, \tag{25}$$

and

$$(VF)(t)z = \lim_{n\to\infty} (VF)(t)z_n, \text{ for } z \in (X), t \geq 0. \tag{26}$$

Here, $F \in C(R^+, L(X)), z_n \in D(A_0)$ is a sequence such that $z = \lim_{n\to\infty} z_n$.

Let $F(t)$ to be $\mathfrak{J}_0(t)$, from Equation (25), we have $(V\mathfrak{J}_0)(t)z = \int_0^t \mathfrak{J}_0(t-s)C\mathfrak{J}_0(s)zds$ and we also have $V\mathfrak{J}_0$ is norm continuous for $t \geq 0$ for detail see Bátkai and Piazzera's [6] study. For any $0 < \beta < 1$, we can choose some $t_0 > 0$ such that $\|V\mathfrak{J}_0(t)\| \leq \beta$ for all $0 \leq t \leq t_0$.

We will use the frequently hypercyclic property of $\mathfrak{J}_0(t)$ to prove $U(t)$ is frequently hypercyclic and we write the result as Theorem 2.

Theorem 2. *When (H1)–(H6) are satisfied the solution semigroup $\{U(t)\}_{t\geq 0}$ generated by* A *is frequently hypercyclic.*

Proof. Without losing generality, we can suppose any nonempty open set as a neighborhood Z' with radius r and center at origin. To prove the frequently hypercyclic property of $U(t_0)$, we need to find a frequently hypercyclic vector x and a set $M = \{n \in \mathbb{N}, U^n(t_0)x \in Z'\}$ such that M has positive lower density for some particular operator $U(t_0)$. □

Since $\mathfrak{J}_0(t_0)$ is frequently hypercyclic operator, there exists a frequently hypercyclic vector $\overline{x} \in W$ such that for a neighborhood $\overline{Z}$ with radius $r' < \frac{1-q}{2(1-q)+\beta} r$ and center at origin, the set $\overline{M} = \{n \in \mathbb{N}, \mathfrak{J}_0^n(t_0)\overline{x} \in \overline{Z}\}$ has positive lower density.

$\overline{x}$ is a good candidate for the frequently hypercyclic vector of $U(t_0)$. Then, we figure out the set which has frequently hypercyclic property.

When $m = n+1$ for $n \in \overline{M}$ and let $\overline{y} = \mathfrak{J}_0^n(t_0)\overline{x} \in \overline{Z}$ we have $U^m(t_0)\overline{x} = U(mt_0)\overline{x} = \sum_{k=0}^{\infty} V^k\mathfrak{J}_0(mt_0)\overline{x} = \sum_{k=0}^{\infty} V^k\mathfrak{J}_0(t_0){\mathfrak{J}_0}^n(t_0)\overline{x} = \sum_{k=0}^{\infty} V^k\mathfrak{J}_0(t_0)\overline{y}$.

To estimate $U^m(t_0)\overline{x}$, we need to estimate $\mathfrak{J}_0(t_0)\overline{y}$ and $V^k\mathfrak{J}_0(t_0)\overline{y}$ for all $k \geq 1$.

From Equation (15), we have the following equation

$$\begin{aligned}\Im_0(t_0)\overline{y} &= \begin{pmatrix} S(t) & 0 \\ S_t & T_0(t) \end{pmatrix}\begin{pmatrix} \alpha \\ h \end{pmatrix} \\ &= \begin{pmatrix} S(t_0)\alpha \\ S_{t_0}\alpha + T_0(t)h \end{pmatrix}, \text{ for } \overline{y} = \begin{pmatrix} \alpha \\ h \end{pmatrix},\end{aligned} \tag{27}$$

We calculate the first component of $\mathfrak{J}_0(t_0)\overline{y}$, $S(t_0)\alpha$, as follows:

$$\|S(t_0)\alpha(\lambda)\|_X = \|\alpha(\lambda e^{-t_0})\|_X = \|\alpha\|_X. \tag{28}$$

The second component of $\mathfrak{J}_0(t_0)\overline{y}$ is $S_{t_0}\alpha + T_0(t_0)h$. We calculate their norms separately:

$$\begin{aligned}\|S_{t_0}\alpha\|_p = & \left(\int_{-1}^0 ((S_{t_0}\alpha)(\tau))^p d\tau\right)^{\frac{1}{p}} = \left(\int_{-1}^0 (S(t_0+\tau)\alpha(\lambda))^p d\tau\right)^{\frac{1}{p}} \\ &= \left(\int_{-t_0}^0 (S(t_0+\tau)\alpha(\lambda))^p d\tau\right)^{\frac{1}{p}} = \left(\int_{-t_0}^0 \left(\alpha\left(\lambda e^{-(t_0+\tau)}\right)\right)^p d\tau\right)^{\frac{1}{p}} \\ &\leq \left(\int_{-t_0}^0 \|\alpha\|_X^p d\tau\right)^{\frac{1}{p}} \leq \|\alpha\|_X \left(\int_{-t_0}^0 1 d\tau\right)^{\frac{1}{p}} \leq \|\alpha\|_X,\end{aligned} \tag{29}$$

and

$$\begin{aligned}\|T_0(t_0)h\|_p &= \left(\int_{-1}^0 (T_0(t_0)h(\tau))^p d\tau\right)^{\frac{1}{p}} = \left(\int_{-1}^0 (h(t_0+\tau))^p d\tau\right)^{\frac{1}{p}} \\ &= \left(\int_{-1}^{-t_0} (h(t_0+\tau))^p d\tau\right)^{\frac{1}{p}} = \left(\int_{-1-t_0}^0 (h(\lambda))^p d\lambda\right)^{\frac{1}{p}} \\ &\leq \left(\int_{-1}^0 (h(\lambda))^p d\lambda\right)^{\frac{1}{p}} = \|h\|_p \text{ since } h(\lambda) \in X \text{ for } \lambda \in [-1, 0].\end{aligned} \tag{30}$$

Combine the above estimates, we get the following equation:

$$\begin{aligned}\|\Im_0(t_0)\overline{y}\| &= \left\|\begin{pmatrix} S(t_0)\alpha \\ S_{t_0}\alpha + T_0(t)h \end{pmatrix}\right\| = \|S(t_0)\alpha\|_X + \left\|S_{t_0}\alpha + T_0(t)h\right\|_p \\ &\le \|S(t_0)\alpha\|_X + \left\|S_{t_0}\alpha\right\|_p + \|T_0(t)h\|_p \le \|\alpha\|_X + \|\alpha\|_X + \|h\|_p \le 2\left(\|\alpha\|_X + \|h\|_p\right) = 2\|\overline{x}\|.\end{aligned} \tag{31}$$

Then, we consider $V^n\mathfrak{J}_0(t_0)\overline{y}$ for all $n \ge 1$.

When $n = 1$, $\|V\mathfrak{J}_0(t_0)\overline{y}\| \le \beta\|\overline{y}\|$, we already know.

When $n = 2$, we get the following equation:

$$\begin{aligned}\|V^2\Im(t_0)\overline{y}\| &= \|V(V\Im(t_0))\overline{y}\| = \left\|\int_0^t V\Im_0(t-s)C\Im_0(s)\overline{y}ds\right\| \\ &\le \beta\int_0^t \|C\Im_0(s)\overline{y}\|ds \le \beta q\|\overline{y}\|.\end{aligned} \tag{32}$$

Moreover, $\|V^2\mathfrak{J}(t_1)\overline{y}\| \le \beta q\|\overline{y}\|$ for $0 \le t_1 \le t_0$.

From Equation (25) and by induction, we have the estimate:

$$\|V^n\Im(t_0)\overline{y}\| \le \beta q^{n-1}\|\overline{y}\|. \tag{33}$$

Consider $m \in M' = \{m \in \mathbb{N}, m = n+1, for\ n \in \overline{M}\}$, we have the following equation:

$$\begin{aligned}\|U^m(t_0)\overline{x}\| &= \|U(mt_0)\overline{x}\| = \left\|\sum_{k=0}^{\infty} V^k\Im_0(mt_0)\overline{x}\right\| \\ &= \left\|\sum_{k=0}^{\infty} V^k\Im_0(t_0)\Im^n(t_0)\overline{x}\right\| = \left\|\sum_{k=0}^{\infty} V^k\Im_0(t_0)\overline{y}\right\| \le \sum_{k=0}^{\infty}\left\|V^k\Im_0(t_0)\overline{y}\right\| \\ &\le \|\Im_0(t_0)\overline{y}\| + \sum_{k=1}^{\infty}\left\|V^k\Im_0(t_0)\overline{y}\right\| \le 2\|\overline{y}\| + \sum_{k=1}^{\infty}\beta q^{k-1}\|\overline{y}\| \\ &= \left(2 + \frac{\beta}{1-q}\right)\|\overline{y}\| \le \left(2 + \frac{\beta}{1-q}\right)r' < r.\end{aligned} \tag{34}$$

So $U^m(t_0)\overline{x} \subset Z'$ for all $m \in M'$. This implies $\overline{x}$ is a frequently hypercyclic vector and M' as we want. We finish the proof.

4. Examples

Example 1. We consider the following delay equation:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t) = -x\dfrac{\partial}{\partial x}u(t) + \displaystyle\int_{-1}^{0} u(t+\tau)d\tau\ , t \ge 0 \\ u(0) = w \\ u_0 = f, \end{cases} \tag{35}$$

where $w \in X, f \in L^p([-1,0],X), 1 \le p < \infty$.

Let $\Phi : C([-1,0],X) \to X$ be the bound linear operator given by Riemann–Stieltjes integral:

$$\Phi(g) = \int_{-1}^{0} g d\sigma,\ \text{for all } g \in C([-1,0],X) \to X. \tag{36}$$

Since $W^{1,p}([-1,0],X)$ is continuously embedded in $C([-1,0],X)$, Φ defines a bounded operator from $W^{1,p}([-1,0],X)$ to X. So, we can translate Equation (35) to $(\mathrm{DE})_p$ form.

To ensure the existence of the solution semigroup $U(t)$ of $(\mathrm{DE})_p$, we need to check (H6) is satisfied.

For $0 < t < 1$, we obtain the following equation:

$$\begin{aligned}&\int_0^t\left\|C\Im_0(r)\begin{pmatrix} x \\ f\end{pmatrix}\right\|dr = \int_0^t\left\|\begin{pmatrix} 0 & \Phi \\ 0 & 0\end{pmatrix}\begin{pmatrix} S(r) & 0 \\ S_r & T_0(r)\end{pmatrix}\begin{pmatrix} x \\ f\end{pmatrix}\right\|dr = \int_0^t\|\Phi(\mathrm{S}_r x + T_0(r)f)\|dr \\ &= \int_0^t\left\|\int_{-1}^{-r} f(r+\sigma)d\sigma + \int_{-r}^{0} S(r+\sigma)xd\sigma\right\|dr \\ &\le \int_0^t\int_{-1}^{-r}\|f(r+\sigma)\|d\sigma dr + \int_0^t\int_{-r}^{0}\|S(r+\sigma)x\|d\sigma dr \\ &\le \int_{-t}^{0}\int_{\sigma}^{0}\|f(s)\|dsd\sigma + \int_{-1}^{-t}\int_{\sigma}^{t+\sigma}\|f(s)\|dsd\sigma + \int_0^t\|x\|_X dr \\ &\le \int_{-t}^{0}(-\sigma)^{\frac{1}{p'}}\|f\|_p d\sigma + \int_{-1}^{-t}(t)^{\frac{1}{p'}}\|f\|_p d\sigma + t\|x\|_X \\ &\le \int_{-1}^{0}(t)^{\frac{1}{p'}}\|f\|_p d\sigma + t\|x\|_X = (t)^{\frac{1}{p'}}\|f\|_p + t\|x\|_X \le (t)^{\frac{1}{p'}}\left(\|f\|_p + \|x\|_X\right),\end{aligned} \tag{37}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. Finally, we conclude that $\int_0^t \left\| C\mathfrak{I}_0(r)\begin{pmatrix} x \\ f \end{pmatrix} \right\| dr \le (t)^{\frac{1}{p'}}\left(\|f\|_p + \|x\|_X \right)$.

Now choose t_0 small enough such that $t_0^{\frac{1}{p'}} < 1$, then condition (H6) is satisfied with $q \equiv t_0^{\frac{1}{p'}}$. Apply Theorem 2, we have $\{U(t)\}_{t\ge 0}$ is frequently hypercyclic.

Example 2. The important special case of Example 1 is the following form:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t) = -x\dfrac{\partial}{\partial x}u(t) + \sum\limits_{k=0}^{n} B_k u(t+h_k)\ , t \ge 0 \\ u(0) = w \\ u_0 = f, \end{cases} \tag{38}$$

where $f \in W^{1,p}([-1,0],X), B_k \in L(X)$ and $h_k \in [-1,0]$ for $k = 0, 1, \ldots, n$.

The operator Φ is defined by the following equation:

$$\Phi(f) : \sum_{k=0}^{n} B_k f(h_k). \tag{39}$$

The result of Example 1 is coming from we can choose q small enough when Φ is defined as Equation (36). Similar results for Example 2 were proved by Kunisch and Schappacher [16].

Remark 2. $B = -x\frac{\partial}{\partial x}$ can extend to $B = -c(x)\frac{\partial}{\partial x}$, where $c(x)$ is a continuous function and satisfied the following equation:

$$c(0) = 0, c(x) > 0, \text{ for } x \in (0,1] \text{ and } \int_0^1 \frac{dx}{c(x)} = \infty. \tag{40}$$

The solution semigroup $\{ S(t) \}_{t\ge 0}$ generated by $B = -c(x)\frac{\partial}{\partial x}$ is also frequently hypercyclic, for details see Hung and Chang's [15] study.

5. Conclusion

By finding a frequently hypercyclic vector and the set has frequently hypercyclic characteristic with positive lower density, we have the results in Theorems 1 and 2. Finally, we give two examples satisfying Theorem 2, then they have the frequently hypercyclic solution semigroup.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

References

[1] Y.-H. Chang and C.-H. Hong, "The chaos of the solution semigroup for the quasi-linear Lasota equation," *Taiwanese Journal of Mathematics*, vol. 16, no. 5, pp. 1707–1717, 2012.

[2] C.-H. Hung, "Chaotic and hypercyclic properties of the quasi-linear Lasota equation," *Open Mathematics*, vol. 13, no. 1, pp. 372–379, 2015.

[3] A. L. Dawidowicz, N. Haribash, and A. Poskrobko, "On the invariant measure for the quasi-linear Lasota equation," *Mathematical Methods in the Applied Sciences*, vol. 30, no. 7, pp. 779–787, 2007.

[4] A. Lasota and T. Szarek, "Dimension of measures invariant with respect to the Ważewska partial differential equation," *Journal of Differential Equations*, vol. 196, no. 2, pp. 448–465, 2004.

[5] R. Rudnicki, "Chaos for some infinite-dimensional dynamical systems," *Mathematical Methods in the Applied Sciences*, vol. 27, no. 6, pp. 723–738, 2004.

[6] A. Bátkai and S. Piazzera, "Semigroups and linear partial differential equations with delay," *Journal of Mathematical Analysis and Applications*, vol. 264, no. 1, pp. 1–20, 2001.

[7] J. Alberto Conejero, C. Lizama, M. Murillo-Arcila, and A. Peris, "Linear dynamics of semigroups generated by differential operators," *Open Mathematics*, vol. 15, no. 1, pp. 745–767, 2017.

[8] G. Ruiz Goldstein, J. A. Goldstein, and M. Kaplin, "The chaotic Black–Scholes equation with time-dependent coefficients," *Archiv der Mathematik*, vol. 115, pp. 183–194, 2020.

[9] P. Zhu and Q. Yang, "Chaos of multi-dimensional linear hyperbolic PDEs," *Proceedings of the American Mathematical Society*, vol. 151, pp. 1593–1607, 2023.

[10] K.-G. Grosse-Erdmann and A. Peris, *Linear Chaos, Universitext*, Springer, London, 2011.

[11] F. Bayart and S. Grivaux, "Frequently hypercyclic operators," *Transactions of the American Mathematical Society*, vol. 358, no. 11, pp. 5083–5117, 2006.

[12] E. M. Mangino and A. Peris, "Frequently hypercyclic semigroups," *Studia Mathematica*, vol. 202, no. 3, pp. 227–242, 2011.

[13] F. Bayart and T. Bermúdez, "Semigroups of chaotic operators," *Bulletin of the London Mathematical Society*, vol. 41, no. 5, pp. 823–830, 2009.

[14] M. Kosti´c, *Chaos for Linear Operators and Abstract Differential Equations*, Nova Science Publishers, New York, 2020.

[15] C.-H. Hung and Y.-H. Chang, "Frequently hypercyclic properties of the age and maturity structured model of population," *Japan Journal of Industrial and Applied Mathematics*, vol. 36, pp. 461–472, 2019.

[16] K. Kunisch and W. Schappacher, "Necessary conditions for partial differential equations with delay to generate C_0-semigroups," *Journal of Differential Equations*, vol. 50, no. 1, pp. 49–79, 1983.