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Research Article

The Solvability and Explicit Solutions of Singular Integral–Differential Equations with Reflection

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This article deals with a classes of singular integral–differential equations with convolution kernel and reflection. By means of the theory of boundary value problems of analytic functions and the theory of Fourier analysis, such equations can be transformed into Riemann boundary value problems (i.e., Riemann–Hilbert problems) with nodes and reflection. For such problems, we propose a novel method different from classical one, by which the explicit solutions and the conditions of solvability are obtained.

1. Introduction

It is well-known that singular integral equations (SIEs) and boundary value problems for analytic functions are the main branches of complex analysis and have a lot of applications, e.g., in physics, engineering, elasticity theory, fluid dynamics, fracture mechanics, technology, and other fields. Muskhelishvilli, Chuan, and other authors [1–5] studied some classes of SIEs of convolution type with Cauchy kernel and Riemann–Hilbert problems (R–HPs), especially the solvable Noether theory. Litvinchuk, Li, and other authors [6–10], studied singular integral–differential equations (SIDEs) in which the class of differentiable functions was extended to the class of a Holder continuous functions and also studied the SIDEs in which the coefficients contain a first-kind discontinuity point. In [11–18], the authors proposed a general method for solving SIEs of Cauchy kernel and a convolution kernel with discontinuous property. This method involves converting these types of integral equations to R–HPs by using Fourier transform. In this paper, we use a novel method for solving several kinds of SIDEs of order m in class $\{0\}$. This work is organized as follows: In Section 2, we present some definitions, lemmas, and study the properties of the Fourier transforms and Cauchy transforms on a functions of class $\{0\}$. In Section 3, we adopt the Fourier transforms to convert SIDE with reflection into a R–HPs and obtain the solutions of the equation in class $\{0\}$. In Section 4, we solve singular integral–differential Wiener–Hopf equation with reflection in class $\{0\}$. This paper's results improve some of the results presented in [19–24], providing a theoretical framework for resolving physics-related problems.

2. Preliminaries

In this section, we present some definitions and lemmas.

Definition 1 [25]. We say that $F(x)$ is an element of a space of Holder continuous functions H on $[-N, N]$, if there exists some positive real number r such that for any $x_1, x_2 \in [-N, N]$, the condition $|F(x_2) - F(x_1)| \leq r|x_2 - x_1|^{\alpha}, (0<\alpha\leq 1)$ holds.

Definition 2 [2]. We say that the continuous function $F(x)$ belongs to $\tilde{H}$ if $F(x)$ satisfies (i) $F(x) \in H$ on $[-N, N]$ for any sufficient large positive number N and (ii) $|F(x_2) - F(x_1)| \leq k|\frac{1}{x2} - \frac{1}{x_1}|$ for any $|x_i|>N$ $(i = 1, 2)$, $k>0$.

Definition 3 [10]. If the function $F(x)$ satisfies the following conditions:

(i) $F(x) \in \tilde{H}$,

(ii) $F(x) \in L^1(R)$, where $L^1(R) = \{F(x)| \int_R |F(x)|dx<\infty\}$, $R = (-\infty, \infty)$, then we say that $F(x) \in \{\{0\}\}$. If $F(x)$ satisfies Holder condition on a neighborhood N_∞ of ∞, we say that $F(x) \in H(N_\infty)$.

Definition 4 [10]. The Fourier transform of a function $\varphi(x) \in L^1(R)$ is denoted by:

$$\mathbb{F}[\varphi(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) e^{isx} dx = \Phi(s), \tag{1}$$

and the inverse of Fourier transform of $\Phi(x)$ is defined by the following equation:

$$\mathbb{F}^{-1}[\Phi(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(x) e^{-isx} dx = \varphi(s). \tag{2}$$

It is easy to see that $\mathbb{F}^{-1}[\varphi(x)] = \mathbb{F}[\varphi(-x)] = \Phi(-x)$. If $\Phi(x) \in \{\{0\}\}$, then $\varphi(s) = \mathbb{F}^{-1}[\Phi(x)] \in \{0\}$.

The convolution of the functions $\varphi, \psi \in \{0\}$ which is defined by the following equation:

$$(\varphi \times \psi)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x-t)\psi(t) dt, \ t \in R, \tag{3}$$

belongs to $\{0\}$. From the convolution theorem [2], we have the following equation:

$$\mathbb{F}[(\varphi \times \psi)(x)] = \mathbb{F}\varphi(x).\mathbb{F}\psi(x) = \Phi(\boldsymbol{x})\Psi(\boldsymbol{x}), \tag{4}$$

where Φ, Ψ are the Fourier transforms of φ, ψ, respectively.

Definition 5 [25]. We define the operator T of the Cauchy principal value integral as follows:

$$T\varphi(x) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\varphi(s)}{s-x} ds. \tag{5}$$

Lemma 1. *If $\varphi(x) \in \{0\}$, then $\mathbb{F}[T\varphi(x)] = -\operatorname{sgn}(s)\Phi(s)$.*

Proof. Since

$$\begin{aligned} \mathbb{F}[T\varphi(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\varphi(y)}{y-x} dy\right] e^{isx} \\ dx &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{e^{isx}}{x-y} dx\right] \varphi(y) dy. \end{aligned} \tag{6}$$

From Lu [2], we have the following equation:

$$\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{e^{isx}}{x-y} dx = \begin{cases} e^{isy}, & s>0 \\ 0, & s=0 \\ -e^{isy}, & s<0 \end{cases}. \tag{7}$$

From Equations (6) and (7), we obtain the following equation:

$$\begin{aligned} \mathbb{F}[T\varphi(x)] &= -\operatorname{sgn}(s) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(y) e^{isy} dy, \\ \mathbb{F}[T\varphi(x)] &= -\operatorname{sgn}(s)\, \Phi(s). \end{aligned} \tag{8}$$

Similarly, we can also prove that:

$$\mathbb{F}[T\varphi(-x)] = -\operatorname{sgn}(s)\, \Phi(-s). \tag{9}$$

□

Lemma 2. *If $\varphi(x)$ and its derivative $\varphi^{(r)}(x)$ $(r = 1, \ldots, m)$ belongs to $\{0\}$, then:*

$$\mathbb{F}\left(\varphi^{(r)}(x)\right) = (-is)^r \Phi(s). \tag{10}$$

Proof. By using mathematical induction on r. When $r = 1$, we have the following equation:

$$\begin{aligned} \mathbb{F}(\varphi'(x)) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi'(x) e^{isx} \\ dx &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \varphi'(x) e^{isx} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \varphi'(x) e^{isx} dx. \end{aligned} \tag{11}$$

By using integration by parts, we obtain the following equation:

$$\begin{aligned} \mathbb{F}(\varphi'(x)) &= \frac{1}{\sqrt{2\pi}} \left[\varphi(0) + (-is) \int_{-\infty}^{0} \varphi(x) e^{isx} dx\right] \\ &\quad + \frac{1}{\sqrt{2\pi}} \left[-\varphi(0) + (-is) \int_{0}^{\infty} \varphi(x) e^{isx} dx\right] \\ &= \frac{1}{\sqrt{2\pi}} \left[(-is) \int_{-\infty}^{\infty} \varphi(x) e^{isx} dx\right] = (-is)\Phi(s). \end{aligned} \tag{12}$$

Let Equation (10) be true for $r = n$, i.e.:

$$\mathbb{F}\left(\varphi^{(n)}(x)\right) = (-is)^n \Phi(s). \tag{13}$$

For $r = n+1$, it is easy to obtain the following equation:

$$\mathbb{F}\left(\varphi^{(n+1)}(x)\right) = (-is)^{n+1} \Phi(s). \tag{14}$$

The Lemma is proved. □

Lemma 3. *If $\varphi(-x)$ and its derivative $\varphi^{(r)}(-x)$ $(r = 1, \ldots, m)$ belongs to $\{0\}$, then:*

$$\mathbb{F}\left(\varphi^{(r)}(-x)\right) = (is)^r \Phi(-s). \tag{15}$$

Proof. By using mathematical induction on r. When $r = 1$, we have the following equation:

$$\mathbb{F}(\varphi'(-x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi'(-x) e^{isx}$$
$$dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \varphi'(-x) e^{isx} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \varphi'(-x) e^{isx} dx. \tag{16}$$

Let $x = -y$ and similar to Lemma 1, we obtain the following equation:

$$\mathbb{F}\left(\varphi^{(r)}(-x)\right) = (is)^r \Phi(-s). \tag{17}$$

□

Lemma 4. *If $\varphi(\pm x)$ and its derivative $\varphi^{(r)}(\pm x)$ $(r = 1, \ldots, m)$ belongs to $\{0\}$, then:*

$$\mathbb{F}\left(T\varphi^{(r)}(x)\right) = -(-is)^r \operatorname{sgn} s\, \Phi(s) \tag{18}$$

$$\mathbb{F}\left(T\varphi^{(r)}(-x)\right) = -(is)^r \operatorname{sgn} s\, \Phi(-s). \tag{19}$$

Proof. From Lemmas 1–3, we obtain the following equation:

$$\begin{aligned}\mathbb{F}\left(T\varphi^{(r)}(x)\right) &= -\operatorname{sgn} s\, \mathbb{F}\left(\varphi^{(r)}(x)\right) = -(-is)^r \operatorname{sgn} s\, \Phi(s),\\ \mathbb{F}\left(T\varphi^{(r)}(-x)\right) &= -(is)^r \operatorname{sgn} s\, \Phi(-s).\end{aligned} \tag{20}$$

□

3. Singular Integral–Differential Equation of Convolution Type with Reflection

In this section, we solve the following SIDE with reflection:

$$\begin{aligned}\sum_{r=0}^{m} \Bigg\{ & a_r \varphi^{(r)}(x) + \widetilde{a}_r \varphi^{(r)}(-x) + \frac{b_r}{\pi i} \int_{-\infty}^{\infty} \frac{\varphi^{(r)}(y)}{y - x} dy - \frac{\widetilde{b}_r}{\pi i} \int_{-\infty}^{\infty} \frac{\varphi^{(r)}(-y)}{y - x} dy \\ & + \frac{c_r}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_r(x - y) \varphi^{(r)}(y) dy + \frac{\widetilde{c}_r}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h_r(x - y) \varphi^{(r)}(-y) dy \Bigg\} = g(x),\end{aligned} \tag{21}$$

where $x \in (-\infty, \infty)$ and the values $a_r, \tilde{a}_r, b_r, \tilde{b}_r, c_r, \tilde{c}_r$ $(r = 0, 1, \ldots, m)$ are real constants and the given functions $g(x), k_r(x), h_r(x)$ $(r = 0, 1, \ldots, m)$ belong to the class $\{0\}$. By taking Fourier transform to Equation (21) and using Lemmas 1–4, we obtain the following equation:

$$e_1(s)\, \Phi(s) + e_2(s)\, \Phi(-s) = G(s), \tag{22}$$

where

$$e_1(s) = \sum_{r=0}^{m} (a_r - \operatorname{sgn} s\, b_r + c_r K_r(s))(-is)^r, \tag{23}$$

$$e_2(s) = \sum_{r=0}^{m} \left(\widetilde{a}_r + \operatorname{sgn} s \widetilde{b}_r + \widetilde{c}_r H_r(s)\right)(is)^r. \tag{24}$$

$\Phi(s) = \mathbb{F}\varphi(x), G(s) = \mathbb{F}g(x), K_r(s) = \mathbb{F}k_r(x), H_r(s) = \mathbb{F}h_r(x)$ $(r = 0, 1, \ldots, m)$. The functions $\Phi(s), G(s), K_r(s), H_r(s) \in \{\{0\}\}$, these functions are continuous. By taking the limit of Equation (22) as $s \to 0$, where $\lim_{s \to \pm 0} s = \pm 1$, we have $G(0) = 0$. By putting $-s$ instead of s in Equation (22), we obtain the following equation:

$$e_2(-s)\Phi(s) + e_1(-s)\Phi(-s) = G(-s). \tag{25}$$

We solve the Equations (22)and (25) for the functions $\Phi(s), \Phi(-s)$.

From Equations (22) and (25), we obtain the matrix equation:

$$A(s)X(s) = Y(s), \tag{26}$$

where

$$A(s) = \begin{pmatrix} e_1(s) & e_2(s) \\ e_2(-s) & e_1(-s) \end{pmatrix}, X(s) = \begin{pmatrix} \Phi(s) \\ \Phi(-s) \end{pmatrix},$$
$$Y(s) = \begin{pmatrix} G(s) \\ G(-s) \end{pmatrix}. \tag{27}$$

Denote:

$$M(s) = \begin{pmatrix} e_1(s) & e_2(s) & G(s) \\ e_2(-s) & e_1(-s) & G(-s) \end{pmatrix}, \Delta(s) = \det A(s), \tag{28}$$

where

$$\begin{aligned}\Delta(s) &= \begin{vmatrix} e_1(s) & e_2(s) \\ e_2(-s) & e_1(-s) \end{vmatrix} \\ &= \sum_{l=0}^{m} \Bigg\{ \sum_{r=0}^{m} (-1)^r (is)^{r+l} (a_r a_l - b_r b_l + c_r c_l K_r(s) K_l(-s) \\ &\quad + a_r c_l (K_l(-s) + K_l(s)) + b_r c_l \operatorname{sgn} s\, (K_r(s) - K_l(-s))) \\ &\quad - \Big(\widetilde{a}_r \widetilde{a}_l - \widetilde{b}_r \widetilde{b}_l + \widetilde{c}_r \widetilde{c}_l H_r(s) H_l(-s) + \widetilde{a}_r \widetilde{c}_l (H_r(s) \\ &\quad + H_l(-s)) + \widetilde{b}_r \widetilde{c}_l \operatorname{sgn} s\, (H_l(-s) - H_r(s))\Big) \Bigg\}.\end{aligned} \tag{29}$$

Suppose there exists $a > 0$ such that for $|s| > a$, we have $\Delta(s) \neq 0$, hence the solution of the matrix Equation (26) is unique and is given by the following form:

$$\Phi(s)=\frac{1}{\Delta(s)}\begin{vmatrix} G(s) & e_2(s) \\ G(-s) & e_1(-s)\end{vmatrix}$$
$$=\Delta^{-1}(s)\left[\sum_{r=0}^{m}(is)^r\left(G(s)(a_r+\operatorname{sgn} s\, b_r+c_rK_r(-s))-G(-s)\left(\widetilde{a}_r+\operatorname{sgn} s\,\widetilde{b}_r+\widetilde{c}_rH_r(s)\right)\right)\right]. \tag{30}$$

By using the inverse Fourier transform $\mathbb{F}^{-1}$ to Equation (30), the solution of Equation (21) is given by $\varphi=\mathbb{F}^{-1}(\Phi)$ in class $\{0\}$.

When $|s|\le a$, we discuss the next three cases:

(1) If $\Delta(s)\neq 0$, then Equation (26) has a unique solution:

$$X(s)=(A(s))^{-1}Y(s). \tag{31}$$

(2) If for some points $\{s_1,s_2,\ldots,s_n\}\in[-a,a]$ such that $\Delta(s_i)=0$, $\{i=1,\ldots,n\}$ and rank $A(s_i)=$ rank $M(s_i)$ then Equation (26) has infinite solution:

$$X(s)=\begin{pmatrix}\Phi(s)\\ \Phi(-s)\end{pmatrix}. \tag{32}$$

(3) If $\Delta(\tilde{s}_i)=0, i\in\{1,\ldots,m\}$, rank $A(\tilde{s}_i)\neq$ rank $M(\tilde{s}_i)$, $\tilde{s}_i\in[-a,a]$, then by condition (33):

$$G(\widetilde{s}_i)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(x)e^{i\tilde{s}_i}x\,dx=0, \tag{33}$$

for each $i\in\{1,\ldots,m\}$ must be satisfied for the solution of Equation (26) exists.

The homogeneous Equation (21) has a linear independent solutions $[e^{i\tilde{s}_1}x, e^{i\tilde{s}_2}x,\ldots,e^{i\tilde{s}_m}x]$.

Theorem 1. *Equation (21) with condition $G(0)=0$ has the following solutions:*

(1) *If $\Delta(s)\neq 0$, $-\infty<s<\infty$, then from Equation (30), Equation (21) has a unique solution $\varphi(x)=\mathbb{F}^{-1}(\Phi(s))$ in class $\{0\}$.*

(2) *If $\Delta(s)=0$ and rank $A(s)=$ rank $M(s)$, $s=s_1,s_2,\ldots,s_n$ then Equation (21) has infinite solutions.*

(3) *If $\Delta(s)=0$ and rank $A(s)\neq$ rank $M(s)$, $s=\tilde{s}_1,\tilde{s}_2,\ldots,\tilde{s}_m$, then by Condition (33), Equation (21) has the following solution:*

$$\varphi(x)=\mathbb{F}^{-1}(\Phi(s))+\sum_{l=1}^{m}C_le^{i\tilde{s}_l}x, \tag{34}$$

in class $\{0\}$, where $C_l(1\le l\le m)$ are arbitrary constants.

3.1. Example. We give an example of the method used in this paper. In Equation (21), we assume that:

$$m=1, a_r=1, \widetilde{a}_r=1, b_r=1, \widetilde{b}_r=1, c_r=1, \widetilde{c}_r=0, r=0,1.$$
$$k_r(x)=g(x)=\frac{\sqrt{2\pi}}{2}e^{-x\operatorname{sgn}x}, r=0,1. \tag{35}$$

Obviously $k_r(x)$ continuous and even functions, by Fourier transformation, we have the following equation:

$$K_r(s)=G(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(2)^{-1}\sqrt{2\pi}\,e^{isx-x\operatorname{sgn}x}\,dx=\frac{1}{1+s^2},$$
$$r=0,1, \tag{36}$$

which belong to $\{\{0\}\}$, thus $k_r(x), g(x)\in\{0\}$. From Equations (29) and (30), we obtain the following equation:

$$\Delta(s)=\left(2+\frac{1}{1+s^2}\right), \Phi(s)=\frac{i}{2\left(s^2+\frac{3}{2}\right)(s+i)}. \tag{37}$$

By using the inverse Fourier transform $\mathbb{F}^{-1}$ to Φ, we get the following equation:

$$\varphi(x)=\mathbb{F}^{-1}(\Phi(s))=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(s)e^{-isx}\,ds=\sqrt{\frac{\pi}{2}}\left(3+\sqrt{6}\right)^{-1}$$
$$\exp\left(-\sqrt{\frac{3}{2}}|x|\right), \tag{38}$$

which is the solution of Equation (21).

4. Wiener–Hopf Equation with Reflection

In this section, we solve the following singular integral–differential Wiener–Hopf equation with reflection:

$$\sum_{r=0}^{m}\left\{A_r\varphi^{(r)}(x)+\frac{B_r}{\pi i}\int_0^{\infty}\frac{\varphi^{(r)}(y)}{y-x}dy-\frac{\widetilde{B}_r}{\pi i}\int_{-\infty}^{0}\frac{\varphi^{(r)}(-y)}{y-x}dy+\frac{C_r}{\sqrt{2\pi}}\int_0^{\infty}k_r(x-y)\varphi^{(r)}(y)dy+\frac{\widetilde{C}_r}{\sqrt{2\pi}}\int_{-\infty}^{0}h_r(x-y)\varphi^{(r)}(-y)dy\right\}=g(x), \tag{39}$$

where $A_r, B_r, C_r, \tilde{B}_r, \tilde{C}_r$ $(r = 0, 1, \ldots, m)$ are real constants and $k_r(x), h_r(x)$ $(r = 0, 1, \ldots, m), g(x) \in \{0\}$. By Fourier transforms, we define $K_r(s) = \mathbb{F}(k_r(x)), H_r(s) = \mathbb{F}(h_r(x))$, $G(s) = \mathbb{F}(g(x))$. The functions $K_r(s), H_r(s), G(s) \in \{\{0\}\}$. By extending $x \in R^+$ to $x \in R^-$ in Equation (39), we obtain the following form:

$$\sum_{r=0}^{m} \left\{ A_r \varphi_+^{(r)}(x) + \frac{B_r}{\pi i} \int_{-\infty}^{\infty} \frac{\varphi_+^{(r)}(y)}{y - x} dy - \frac{\widetilde{B}_r}{\pi i} \int_{-\infty}^{\infty} \frac{\varphi_+^{(r)}(-y)}{y - x} dy \right.$$
$$\left. + \frac{C_r}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_r(x - y) \varphi_+^{(r)}(y) dy + \frac{\widetilde{C}_r}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h_r(x - y) \varphi_+^{(r)}(-y) dy \right\}$$
$$= g(x) + \sum_{r=0}^{m} \varphi_-^{(r)}(x), t \in R, \tag{40}$$

where

$$\varphi_+^{(r)}(x) = \begin{cases} \varphi^{(r)}(x), & x \geq 0 \\ 0, & x < 0 \end{cases},$$
$$\varphi_-^{(r)}(x) = \begin{cases} 0, & x \geq 0 \\ \varphi^{(r)}(x), & x < 0 \end{cases}. \tag{41}$$

By taking Fourier transform to Equation (40), we obtain the following equation:

$$\widetilde{e_1}(s)\, \Phi^+(s) + \widetilde{e_2}(s)\, \Phi^-(s) + \widetilde{e_3}(s)\, \Phi^+(-s) = G(s), \tag{42}$$

where

$$\widetilde{e_1}(s) = \sum_{r=0}^{m} (A_r - B_r \operatorname{sgn}(s) + C_r K_r(s))(-is)^r, \tag{43}$$

$$\widetilde{e_2}(s) = \sum_{r=0}^{m} (-is)^r, \tag{44}$$

$$\widetilde{e_3}(s) = \sum_{r=0}^{m} \left(\widetilde{B}_r \operatorname{sgn}(s) + \widetilde{C}_r H_r(s) \right) (is)^r. \tag{45}$$

Substituting by $-s$ instead of s in Equation (42), we obtain the following equation:

$$\widetilde{e_3}(-s)\, \Phi^+(s) + \widetilde{e_2}(-s)\, \Phi^-(-s) + \widetilde{e_1}(-s)\, \Phi^+(-s) = G(-s). \tag{46}$$

Since $\Phi(\pm\infty) = 0, \Phi(s) \in \{\{0\}\}$ and

$$\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\Phi(y)}{y - s} dy = \Phi^+(s) + \Phi^-(-s). \tag{47}$$

Equations (42) and (46) can be reduced to the following system of SIEs of dimension 2 in class $\{\{0\}\}$:

$$R(s)E(s) + \frac{W(s)}{\pi i} \int_{-\infty}^{\infty} \frac{E(y)}{y - s} dy = V(s), \tag{48}$$

where

$$R(s) = \begin{pmatrix} \widetilde{e_1}(s) - \widetilde{e_3}(s) & \widetilde{e_3}(s) \\ \widetilde{e_3}(-s) & \widetilde{e_1}(-s) - \widetilde{e_2}(s) \end{pmatrix},$$
$$E(s) = \begin{pmatrix} \Phi(s) \\ \Phi(-s) \end{pmatrix}, \tag{49}$$

$$W(s) = \begin{pmatrix} \widetilde{e_1}(s) + \widetilde{e_2}(s) & -\widetilde{e_3}(s) \\ \widetilde{e_3}(-s) & -\widetilde{e_1}(-s) - \widetilde{e_2}(s) \end{pmatrix},$$
$$V(s) = \begin{pmatrix} 2G(s) \\ 2G(-s) \end{pmatrix}. \tag{50}$$

To solve Equation (48), we define a Hilbert transform $U(s)$ with the density function $E(y)$, that is:

$$U(s) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{E(y)}{y - s} dy. \tag{51}$$

From Gakhov [25], we have the following equation:

$$\begin{cases} U^+(s) + U^-(s) = \dfrac{1}{\pi i} \displaystyle\int_{-\infty}^{\infty} \frac{E(y)}{y - s} dy \\ U^+(s) - U^-(s) = E(s) \end{cases}. \tag{52}$$

Putting Equation (52) in Equation (48), we obtain the two-dimensional equation:

$$\vartheta(s) U^+(s) + \gamma(s) U^-(s) = V(s), \tag{53}$$

where

$$\vartheta(s) = R(s) + W(s), \gamma(s) = R(s) - W(s). \tag{54}$$

Suppose:

$$\det \gamma(s) \neq 0, \det \vartheta(s) \neq 0, s \in R. \quad (55)$$

By using the inverse of matrices $\gamma(s)$ and $\vartheta(s)$, we obtain the boundary value problem:

$$U^{+}(s) = Y(s)U^{-}(s) + X(s), \quad (56)$$

where

$$Y(s) = -(\vartheta(s))^{-1}\gamma(s), X(s) = \vartheta((s))^{-1}\ V(s) \quad (57)$$

The functions $Y(s), X(s) \in \tilde{H}$, and $\det Y(s) \neq 0$.

Let $\mu = indY\ (s)$, from Muskhelishvilli [5] and Gakhov [25], we have $p(z) = \begin{pmatrix} p_1^1 & p_1^2 \\ p_2^1 & p_2^2 \end{pmatrix}$ is the canonical solution matrix of the homogenous of Equation (56), and the general solution of Equation (56) is given by the following equation:

$$U(z) = p(z)\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{[p^{+}(y)]^{-1}X(y)}{(y-z)}dy + Q(z)\right), \quad (58)$$

where $Q(z)$ is a polynomial vector in two dimensions and:

$$p(z) = \exp\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln Y(y)}{(y-z)}dy\right), z \notin R. \quad (59)$$

Let $p^1 = (\, p_1^1\, p_1^2\,)$ and $p^2 = (\, p_2^1\, p_2^2\,)$ be the canonical solution system of the matrix $p(z)$. Let $ind\, p^j = \mu_j, \mu = \mu_1 + \mu_2$ and assume $\begin{pmatrix} \omega_1(y) \\ \omega_2(y) \end{pmatrix} = [p^{+}(y)]^{-1}X(y)$. Hence from Muskhelishvilli [5] and Gakhov [25], Equation (58) has the following form:

$$U(z) = \sum_{j=1}^{2} p^j(z)\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\omega_j(y)}{(y-z)}dy + \frac{Q_{\mu_{j-1}}(z)}{(z+i)^{\mu_j}}\right), z \notin R, \quad (60)$$

where $Q_{\mu_{j-1}}(z)$ is a polynomial of degree μ_{j-1} when $\mu_j \geq 0$, $Q_{\mu_{j-1}}(z) = 0$, if $\mu_j < 0$.

We assume that $\mu_1 \geq \mu_2$, hence we obtain the following results:

(i) If $\mu_1 \geq \mu_2 \geq 0, U(\infty) = 0$, then the solution $U(z)$ in Equation (60) has $\mu_1 + \mu_2$ arbitrary constants.

(ii) If $\mu_1 \geq 0 > \mu_2, U(\infty) = 0$, then the solution $U(z)$ in Equation (60) has μ_1 arbitrary constants, $Q_{\mu_{2-1}}(z) \equiv 0$ and to get a solution of Equation (56), the conditions,

$\int_{-\infty}^{\infty} (y+i)^r \omega_2\,(y)dy = 0,\ r = 0,\ 1,\ \ldots,\ -\mu_2 - 1,$ must be satisfied [5, 25].

(iii) If $0 > \mu_1 \geq \mu_2, U(\infty) = 0$, then the solution $U(z)$ in Equation (60) has the following solvability conditions
$\int_{-\infty}^{\infty} (y+i)^r \omega_j\,(y)dy = 0,\ r = 0,\ 1,\ \ldots,\ -\mu_j - 1,$
$j = 1,\ 2$ and $Q_{\mu_{j-1}}(z) \equiv 0\ (\,j = 1, 2)$.

From Equation (60), we obtain $U^{+}(t)$ and $U^{-}(t)$, and by substituting them in Equation (52), we get $E(s)$ the solution of Equation (48). Hence, we obtain the solution $\Phi(s)$ of Equations (42) and (46). Therefore, $\varphi(x) = \mathbb{F}^{-1}(\Phi(s)) \in \{0\}$ is a solution of Equation (39).

5. Conclusions

Two classes of SIDEs of the convolution type with reflection are studied in this research. We used the theory of Fourier analysis to find the solutions for Equations (21) and (39). The exact solution is obtained in class $\{0\}$. In this case, our method for solving these equations is novel as opposed to the classic Riemann–Hilbert methods.

Data Availability

Data are available upon request to A. S. Nagdy (email: abbas775@hotmail.com).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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