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Research Article

Multiplicity of Solutions for a Class of Kirchhoff–Poisson Type Problem

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In this paper, we use the fountain theorems to investigate a class of nonlinear Kirchhoff–Poisson type problem. When the nonlinearity f satisfies the Ambrosetti–Rabinowitz's 4-superlinearity condition, or under some weaker superlinearity condition, we establish two theorems concerning with the existence of infinitely many solutions.

1. Introduction and Main Results

This paper is concerned with the existence and multiplicity of solutions for the nonlinear Kirchhoff–Poisson type problem:

$$\begin{cases} -\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\Delta u+\phi u=f(x,u), & x\in\Omega, \\ -\Delta\phi=u^2, & x\in\Omega, \\ u=\phi=0, & x\in\partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N=1, 2$ or 3), $a, b>0$ and $f:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function, satisfying some suitable conditions we will formulate later.

When $a=1$ and $b=0$, Equation (1) reduces to the boundary value problem:

$$\begin{cases} -\Delta u+\phi u=f(x,u), & x\in\Omega, \\ -\Delta\phi=u^2, & x\in\Omega, \\ u=\phi=0, & x\in\partial\Omega. \end{cases} \tag{2}$$

Knowledge of the solutions of System (2) is related to the study of stationary solutions $\psi(x,t)=e^{-it}u(x)$ for the nonlinear parabolic Schrödinger–Poisson system:

$$\begin{cases} -i\dfrac{\partial\psi}{\partial t}=-\Delta\psi+\phi(x)\psi-|\psi|^{p-2}\psi, & x\in\Omega, \\ -\Delta\phi=|\psi|^2, & x\in\Omega, \\ \psi=\phi=0, & x\in\partial\Omega. \end{cases} \tag{3}$$

The first equation in Problem (3) is called Schrödinger equation, and modeling quantum nonrelativistic particles interacting with the eletromagnetic field are caused by the motion. A typical and important class of Schrödinger equations is reflected in the potential $\phi(x)$, which is depended on the charge of wave function itself, that is to say, as the Poisson equation in Equation (3) holds. For more applications of the physical relevance of the Schrödinger–Poisson system, we refer to [1–4] and references therein.

System (2) has been extensively studied after the seminal work of Benci and Fortunato [4]. There are many results about the existence and nonexistence of solutions, multiplicity of solutions, least energy solutions, radial and nonradial solutions, semiclassical limit, and concentrations of solution which are covered in the literature (see for instance [1–3, 5–11], and the references therein).

When the potential ϕ in Equation (1) vanishes, we get the following problem:

$$\begin{cases} -\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\Delta u=f(x,u), & x\in\Omega, \\ u=0, & x\in\partial\Omega. \end{cases} \quad (4)$$

In the case $b\neq 0$, Problem (4) is nonlocal due to the emergence of $b\int_{\Omega}|\nabla u|^2dx\Delta u$ and is involved in the following stationary analog of equation:

$$u_{tt}-\left(a+b\int_{\Omega}|\nabla u|^2\right)\Delta u=g(x,u), \quad (5)$$

which was proposed by Kirchhoff as a generalization of the classical D'Alembert's wave equation for the free vibration of elastic strings:

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0, \quad (6)$$

where L is the length of the string, h denotes the area of crosssection, E represents the Young modulus of the material, and P_0 is recorded as the initial tension. The Kirchhoff's model deals with the change of string length caused by transverse vibrations, we refer to [12–14] for early work. It is well-known that only after Lions [15] put forward an abstract analysis framework, Equation (6) attracted great attention, see [16–18] for example. Recently, many nonlinear analytical methods and techniques are employed to investigate the existence of sign-changing solutions to Problem (4) or similar Kirchhoff-type equations, and consequently, some interesting results were obtained. Let me cite a few examples, Alves et al. [19] and Ma and Rivera [20] obtained the existence of positive solution to this kind of problem by means of variational method. Perera and Zhang [21] utilized the method of Yang index and critical group to obtain the nontrivial solution of Equation (4). Also in [22], they revisited Equation (4) by means of the invariant set of descending flow and the existence of signed solutions and sign-changing solutions is considered. Analogous results were established in [23] by Mao and Zhang [23]. The authors in [24–27] studied Problem (4) or more general Kirchhoff-type equations, respectively, by using constraint variational methods and quantitative deformation lemma. Later, under some more weaken assumptions on f, with the aid of some new analytical skills and non-Nehari manifold method, Tang and Cheng [28] extended some results obtained in [26]. It is well-known that similar nonlocal problems can be used to model some physical and biological systems, where u describes a process depending on its own average value, such as population density, see [29–31] for example.

Inspired ed by the papers mentioned above, the main goal of this paper is to show the existence and multiplicity of nontrivial solutions to Problem (1). The main tool is based on the classical fountain theorem in [32] and a variant version of the fountain theorem from Zou's [33] study. To the best of our knowledge, among the existing literatures, there is no such kind of result concerned with infinitely many solutions for Problem (1).

In this paper, we introduce the space $W=H_0^1(\Omega)$ endowed with the norm $\|u\|^2:=\int_{\Omega}|\nabla u|^2dx$. Throughout this paper, we denote by $|\cdot|_q$, the usual L^q-norm with $q\geq 1$. Since Ω is a bounded domain, it is well-known that $W\hookrightarrow L^q(\Omega)$ continuously for $q\in[1,2^*]$, and compactly for $q\in[1,2^*)$. Moreover, there exists $c_q>0$ such that:

$$|u|_q\leq c_q\|u\|, \ \forall u\in W. \quad (7)$$

We set by $0<\lambda_1<\lambda_2<\ldots$, the distinct eigenvalues of $-\Delta$ in $L^2(\Omega)$ with zero Dirichlet boundary conditions and denote by $e_1,e_2,e_3,\ldots$ the eigenfunctions corresponding to eigenvalues, respectively.

It is well-known that, by the Lax–Milgram theorem, for every $u\in H_0^1(\Omega)$, there exists a unique element $\phi_u\in H_0^1(\Omega)$ such that:

$$\phi_u(x)=\frac{1}{4\pi}\int_{\Omega}\frac{|u(y)|^2}{|x-y|}dy. \quad (8)$$

It is clear that the energy functional associated with Problem (1) can be expressed as follows:

$$\Gamma(u)=\frac{a}{2}\|u\|^2+\frac{b}{4}\|u\|^4+\frac{1}{4}\int_{\Omega}\phi_u u^2dx - \int_{\Omega}F(x,u)dx, \ u\in H_0^1(\Omega), \quad (9)$$

where $F(x,t)=\int_0^t f(x,s)ds$. It is not difficult to check that Γ is of class C^1:

$$\langle\Gamma'(u),v\rangle=(a+b\|u\|^2)\int_{\Omega}\nabla u\nabla vdx + \int_{\Omega}\phi_u uvdx-\int_{\Omega}f(x,u)vdx, \ \forall u,v\in H_0^1(\Omega), \quad (10)$$

and the critical points of Γ correspond to the weak solutions of Equation (1). Now, we state our main results as follows.

Theorem 1. *We assume the nonlinearity f satisfies the following conditions:*

(F_1) $f\in C(\overline{\Omega}\times\mathbb{R},\mathbb{R})$ *and for some* $4<q<2^*=\begin{cases}\frac{2n}{n-2}, & n\geq 3;\\ \infty, & n=1,2,\end{cases}$ *and some constant $C>0$ such that:*

$$|f(x,t)|\leq C(1+|t|^{q-1}). \quad (11)$$

(F_2) *There exist constants $\theta>4$ and $R>0$ such that:*

$$|t|\geq R\Rightarrow 0<\theta F(x,t)\leq tf(x,t), \ \forall x\in\Omega. \quad (12)$$

(F_3) *There holds* $f(x,-t)=-f(x,t)$, $\forall x\in\Omega$, $\forall t\in\mathbb{R}$.

Then Problem (1) admits a sequence of solutions $\{\omega_n\}$ such that $\Gamma(\omega_n)\longrightarrow\infty$ as $n\longrightarrow\infty$.

Theorem 2. *Suppose that conditions (F_1) and (F_3) are satisfied. Furthermore, the following conditions hold true:*

- (H_1) $f(x,t)t\geq 0, \forall t\geq 0$; *and* $\liminf_{|t|\longrightarrow\infty}\frac{F(x,t)}{|t|^4}\longrightarrow\infty$ *uniformly in* $x\in\Omega$.
- (H_2) $f(x,t)=o(|t|)$ *as* $|t|\longrightarrow\infty$ *uniformly in* $x\in\Omega$; $\forall x\in\Omega$, *the function* $\frac{f(x,t)}{t^3}$ *is increasing in* $t\geq 0$.
- (H_3) $\tilde{F}(x,t):=\frac{1}{4}f(x,t)t-F(x,t)\longrightarrow\infty$ *as* $|t|\longrightarrow\infty$ *uniformly in* $x\in\Omega$.

Then, Problem (1) admits infinitely many solutions $\{\omega_n\}$ satisfying $\Gamma(\omega_n)\longrightarrow\infty$.

Remark 1. In order to prove Theorem 1, we shall use the Ambrosetti–Rabinowitz type 4-superlinear condition (F_2) to obtain the boundedness of (PS) sequences of the functional Γ. But, there are many functions which are 4-superlinear growth; however, it does not satisfy (F_2) for any $\theta>4$, hence, when (F_2) is not verified, it becomes more complicated to deal with. In Theorem 2, we employ Theorem 4 without (PS)-type assumption, to establish arbitrarily many solutions of Equation (1) under some weaker conditions than (F_2). We present a concrete example at the end of the proofs to explain the main results.

2. Proofs of Theorems 1 and 2

We first recall the following preliminary results, which are a collection of results from D'Aprile and Mugnai's [34] and Ruiz and Siciliano's [7] studies.

Lemma 1. *The function $\phi_u\in H_0^1(\Omega)$ given in Equation (8) has the following properties:*

- (i) $\phi_u\geq 0$, *and* $\phi_{tu}=t^2\phi_u, \forall t>0$,
- (ii) *there exists $C>0$ independent of u such that* $\|\phi_u\|\leq C\|u\|^2$:

$$\int_\Omega \phi_u u^2dx\leq C\|u\|^4, \tag{13}$$

- (iii) *if* $\omega_n\rightharpoonup u$ *in* $H_0^1(\Omega)$, *then* $\phi_{\omega_n}\rightharpoonup\phi_u$ *in* $H_0^1(\Omega)$:

$$\lim_{n\longrightarrow\infty}\int_\Omega \phi_{\omega_n}\omega_n^2dx=\int_\Omega\phi_u u^2dx. \tag{14}$$

Let W be a Banach space equipped with the norm $\|\cdot\|$ and $W=\overline{\oplus_{i\in\mathbb{N}}X_i}$ with $\dim X_i<\infty$ for any $i\in\mathbb{N}$. Denote by $Y_j=\oplus_{i=0}^{j}X_i$, $Z_j=\overline{\oplus_{i=j}^{\infty}X_i}$, and:

$$\begin{aligned}B_j&=\{u\in Y_j:\|u\|\leq\rho_j\},\\ N_j&=\{u\in Z_j:\|u\|=\gamma_j\}\text{ for }\rho_j>\gamma_j>0.\end{aligned} \tag{15}$$

Theorem 3 [32]. *Let $\varphi\in C^1(W,\mathbb{R})$ be a even functional. Assume that for each $j\in\mathbb{N}$, there exist $\rho_j>\gamma_j>0$ such that:*

- (D_1) $a_j=\max_{u\in Y_j,\|u\|=\gamma_j}\varphi(u)\leq 0$,
- (D_2) $b_j=inf_{u\in Z_j,\|u\|=\gamma_j}\varphi(u)\longrightarrow\infty, j\longrightarrow\infty$, *and*
- (D_3) φ *satisfies the* $(PS)_c$ *condition for every* $c>0$,

then φ has an unbounded sequence of critical values.

Now we define the C^1-functional $\Gamma_\lambda:W\longrightarrow\mathbb{R}$ defined by the following equation:

$$\Gamma_\lambda(u):=A(u)-\lambda B(u),\ \lambda\in[1,2]. \tag{16}$$

We suppose that:

- (E_1) Γ_λ maps bounded aggregate into bounded aggregate uniformly for $\lambda\in[1,2]$. Moreover, $\Gamma_\lambda(-u)=\Gamma_\lambda(u)$ for all $(\lambda,u)\in[1,2]\times W$.
- (E_2) $B(u)\geq 0$ for all $u\in W$; $A(u)\longrightarrow\infty$ or $B(u)\longrightarrow\infty$ as $\|u\|\longrightarrow\infty$, or
- (E_3) $B(u)\leq 0$ for all $u\in W$; $B(u)\longrightarrow-\infty$ as $\|u\|\longrightarrow\infty$.

For $j\geq 2$, we set the following equations:

$$T_j:=\left\{\gamma\in C(B_j,W):\gamma\text{ is odd},\gamma|_{\partial B_j}=id\right\}, \tag{17}$$

$$c_j(\lambda):=\inf_{\gamma\in T_j}\max_{u\in B_j}\Gamma_\lambda(\gamma(u)), \tag{18}$$

$$b_j(\lambda):=\inf_{u\in Z_j,\|u\|=\gamma_j}\Gamma_\lambda(u), \tag{19}$$

$$a_j(\lambda):=\max_{u\in Y_j,\|u\|=\rho_j}\Gamma_\lambda(u). \tag{20}$$

We also need the following variant version of the fountain theorem.

Theorem 4 [33]. *Suppose that (E_1) and (E_2) or (E_3) are satisfied. If $b_j(\lambda)>a_j(\lambda)$ for all $\lambda\in[1,2]$, then $c_j(\lambda)\geq b_j(\lambda)$ for all $\lambda\in[1,2]$. Moreover, for almost every (a.e.) $\lambda\in[1,2]$, there exists a sequence $\left\{\omega_n^j(\lambda)\right\}_{n=1}^{\infty}$ such that:*

$$\sup_n\left\|\omega_n^j(\lambda)\right\|<\infty,\ \Gamma'_\lambda\left(\omega_n^j(\lambda)\right)\longrightarrow 0\text{ and }\Gamma_\lambda\left(\omega_n^j(\lambda)\right)\longrightarrow c_j(\lambda),\text{ as }n\longrightarrow\infty. \tag{21}$$

Now, we are ready to prove our main results.

Proof of Theorem 1. It is obvious that $\Gamma(u)\in C^1(W,\mathbb{R})$. We have from (F_2) by integrating that:

$$c(|t|^\theta - 1) \le F(x,t),\ \forall t \in \mathbb{R}. \tag{22}$$

Take $\xi \in (\theta^{-1}, 4^{-1})$ and $\{\omega_n\} \subset W$, a $(PS)_c$-sequence of Γ. Thus, for n large enough, by Lemma 1—(i), we infer that:

$$\begin{aligned}
C + 1 + \|\omega_n\| \ge& \Gamma(\omega_n) - \xi\Gamma'(\omega_n)\omega_n \\
&= a\left(\frac{1}{2} - \xi\right)\|\omega_n\|^2 + b\left(\frac{1}{4} - \xi\right)\|\omega_n\|^4 + \left(\frac{1}{4} - \xi\right)\int_\Omega \phi_{\omega_n}|\omega_n|^2 dx \\
&\quad + \int_\Omega (\xi f(x,\omega_n)\omega_n - F(x,\omega_n))dx \\
&\ge a\left(\frac{1}{2} - \xi\right)\|\omega_n\|^2 + b\left(\frac{1}{4} - \xi\right)\|\omega_n\|^4 + (\beta\mu - 1)\int_\Omega F(x,\omega_n)dx - c \\
&\ge a\left(\frac{1}{2} - \xi\right)\|\omega_n\|^2 + b\left(\frac{1}{4} - \xi\right)\|\omega_n\|^4 + c(\xi\theta - 1)|\omega_n|_\theta^\theta - c \\
&\ge a\left(\frac{1}{2} - \xi\right)(\|z_n\|^2 + \lambda_1|y_n|_2^2) + b\left(\frac{1}{4} - \xi\right)(\|z_n\|^2 + \lambda_1|y_n|_2^2) + c(\xi\theta - 1)|\omega_n|_\theta^\theta - c,
\end{aligned} \tag{23}$$

where $C, c > 0$ are constants, and $\omega_n = y_n + z_n$, $y_n \in Y_n$, and $z_n \in Z_n$. From the fact that dim Y_n is finite, and all norms in Y are equivalent, we see that $\{\omega_n\}$ is bounded in W.

Therefore, there exists a subsequence of $\{\omega_n\}$, still denoted by itself, such that $\omega_n \rightharpoonup \omega$ in $H_0^1(\Omega)$. Thus, by using the Rellich theorem, we have $\omega_n \longrightarrow \omega$ in $L^p(\Omega)$; also we can infer that $f(x,\omega_n) \longrightarrow f(x,\omega)$ in $L^p(\Omega)$ with $p = \frac{q}{q-1}$ as a consequence of Theorem A.2 [32]. Next, we prove that $\{\omega_n\}$ admits a convergent subsequence. Notice that:

$$\begin{aligned}
&o_n(1) \\
&= \langle \Gamma'(\omega_n) - \Gamma'(\omega), \omega_n - \omega\rangle \\
&= a\int_\Omega |\nabla(\omega_n - \omega)|^2 dx + b\|\omega_n\|^2 \int_\Omega \nabla\omega_n \nabla(\omega_n - \omega)dx + b\|\omega\|^2 \int_\Omega \nabla\omega\nabla(\omega - \omega_n)dx \\
&\quad + \int_\Omega \phi_{\omega_n}\omega_n(\omega_n - \omega)dx - \int_\Omega \phi_\omega \omega(\omega_n - \omega)dx - \int_\Omega (f(x,\omega_n) - f(x,\omega))(\omega_n - \omega)dx \\
&= a\|\omega_n - \omega\|^2 + \int_\Omega (\phi_{\omega_n}\omega_n - \phi_\omega\omega)(\omega_n - \omega)dx + b\|\omega_n\|^2 \int_\Omega \nabla\omega_n\nabla(\omega_n - \omega)dx \\
&\quad + b\|\omega\|^2 \int_\Omega \nabla\omega\nabla(\omega - \omega_n)dx + b\|\omega_n\|^2 \int_\Omega \nabla\omega\nabla(\omega - \omega_n)dx \\
&\quad - b\|\omega_n\|^2 \int_\Omega \nabla\omega\nabla(\omega - \omega_n)dx - \int_\Omega (f(x,\omega_n) - f(x,\omega))(\omega_n - \omega)dx \\
&= (a + b\|\omega_n\|^2)\int_\Omega |\nabla(\omega_n - \omega)|^2 dx + b(\|\omega_n\|^2 - \|\omega\|^2)\int_\Omega \nabla\omega\nabla(\omega_n - \omega)dx \\
&\quad + \int_\Omega (\phi_{\omega_n}\omega_n - \phi_\omega\omega)(\omega_n - \omega)dx - \int_\Omega (f(x,\omega_n) - f(x,\omega))(\omega_n - \omega)dx,
\end{aligned} \tag{24}$$

from which, we get the following equation:

$$\begin{aligned}
&(a + b\|\omega_n\|^2)\int_\Omega |\nabla(\omega_n - \omega)|^2 dx \\
&= b(\|\omega\|^2 - \|\omega_n\|^2)\int_\Omega \nabla\omega\nabla(\omega_n - \omega)dx + \int_\Omega (\phi_{\omega_n}\omega_n - \phi_\omega\omega)(\omega - \omega_n)dx \\
&\quad + \int_\Omega (f(x,\omega_n) - f(x,\omega))(\omega_n - \omega)dx + o_n(1).
\end{aligned} \tag{25}$$

It is obvious that the first term of the right-hand side of Equation (25) converge to zero as $n\longrightarrow\infty$, by virtue of $\omega_n\rightharpoonup\omega$ in E and the boundedness of $\{\omega_n\}$ in W. For the second term of the right-hand side of Equation (25), we have by using Hölder inequality that:

$$\begin{aligned}\left|\int_\Omega(\phi_{\omega_n}\omega_n-\phi_\omega\omega)(\omega-\omega_n)dx\right| &= -\int_\Omega\phi_{\omega_n}(\omega_n-\omega)^2dx+\int_\Omega(\phi_{\omega_n}-\phi_\omega)\omega(\omega_n-\omega)dx\\ &\le\int_\Omega(\phi_{\omega_n}-\phi_\omega)\omega(\omega_n-\omega)dx\\ &\le|\phi_{\omega_n}-\phi_\omega|_3|\omega|_3|\omega_n-\omega|_3\\ &\le C|\phi_{\omega_n}-\phi_\omega|_3\|\omega\|\|\omega_n-\omega\|\longrightarrow0,\end{aligned}\tag{26}$$

where $|\phi_{\omega_n}-\phi_\omega|_3\longrightarrow0$ due to Lemma 1. Again by Hölder inequality, we get the following equation:

$$\int_\Omega|f(x,\omega_n)-f(x,\omega)(\omega_n-\omega)|dx\le|f(x,\omega_n)-f(x,\omega)|_q|\omega_n-\omega|_p\longrightarrow0,\tag{27}$$

when $n\longrightarrow\infty$. Therefore, the right-hand side of Equation (25) tends to zero, and so we infer to $\|\omega_n-\omega\|\longrightarrow0$, as $n\longrightarrow\infty$. This implies condition (D_3) is satisfied.

Using Equation (22), we obtain the following equation:

$$\begin{aligned}\Gamma(\omega_n)&\le\frac{a}{2}\|\omega_n\|^2+\frac{b}{4}\|\omega_n\|^4+\frac{1}{4}\int_\Omega\phi_{\omega_n}\omega_n^2dx-c|\omega_n|_\theta^\theta+c\\ &\le\frac{a}{2}\|\omega_n\|^2+\frac{b}{4}\|\omega_n\|^4+C\|\omega_n\|^4-c|\omega_n|_\theta^\theta+c.\end{aligned}\tag{28}$$

As all norms are equivalent on the finite dimensional space Y_k, (D_1) is satisfied for every sufficiently large $\rho_k>0$.

We next verify condition (D_2). By (F_1), we have the following equation:

$$|F(x,\omega)|\le c(1+|\omega|^q).\tag{29}$$

Define

$$\beta_j=\sup_{\omega\in Z_j,\|\omega\|=1}|\omega|_q,\tag{30}$$

such that on Z_j, we have the following equation:

$$\begin{aligned}I(\omega)&\ge\frac{a}{2}\|\omega\|^2+\frac{b}{4}\|\omega\|^4+\frac{1}{4}\int_\Omega\phi_\omega\omega^2dx-c|\omega|_q^q-c\\ &\ge\sqrt{\frac{ab}{2}}\|\omega\|^3-c\beta_j^q\|\omega\|^q-c.\end{aligned}\tag{31}$$

Set $\gamma_j:=\left(\frac{cq\beta_j^q}{\sqrt{(ab)/2}}\right)^{\frac{1}{3-q}}$. As in [32], we can infer to $\beta_j\longrightarrow0$, $j\longrightarrow\infty$, for any $\omega\in Z_j$ with $\|\omega\|=\gamma_j$, and so, we have the following equation:

$$\Gamma(\omega)\ge\sqrt{\frac{ab}{2}}\left(1-\frac{1}{q}\right)\left(\frac{cq\beta_j^q}{\sqrt{(ab)/2}}\right)^{\frac{3}{3-q}}-c\longrightarrow\infty,\tag{32}$$

as $j\longrightarrow\infty$. Thus, condition (D_2) is satisfied. Now, we have checked that all the conditions of Theorem 3 hold; hence, Problem (1) admits a sequence of solutions $\{\omega_j\}$ such that $\Gamma(\omega_j)\longrightarrow\infty$ and $j\longrightarrow\infty$.

Proof of Theorem 2. We introduce the auxiliary functional:

$$\begin{aligned}\Gamma_\lambda(\omega)&=\frac{a}{2}\|\omega\|^2+\frac{b}{4}\|\omega\|^4+\frac{1}{4}\int_\Omega\phi_\omega\omega^2dx\\ &\quad-\lambda\int_\Omega F(x,\omega)dx:=A(\omega)-\lambda B(\omega),\end{aligned}\tag{33}$$

for $\lambda\in[1,2]$, where $A(\omega)=\frac{a}{2}\|\omega\|^2+\frac{b}{4}\|\omega\|^4+\frac{1}{4}\int_\Omega\phi_\omega\omega^2dx$ and $B(\omega)=\int_\Omega F(x,\omega)dx$. Therefore, $B(\omega)\ge0$, $A(\omega)\longrightarrow\infty$ as $\|\omega\|\longrightarrow\infty$, and $\Gamma_\lambda(-\omega)=\Gamma_\lambda(\omega)$ for all $\lambda\in[1,2]$, $u\in W$.

Claim 1. *There exists $\lambda_n\longrightarrow1$ as $n\longrightarrow\infty$, $c_j^*>b_j^*>0$, and $\{z_n\}_{n=1}^\infty\subset W$ such that:*

$$\Gamma'_{\lambda_n}(z_n)=0,\ \Gamma_{\lambda_n}(z_n)\in\left[b_j^*,c_j^*\right].\tag{34}$$

In fact, by conditions (H_1), for any $\ell>0$, we can choose a constant C_ℓ such that $F(x,\omega)\ge\ell|\omega|^4-C_\ell$ for all $\omega\in\mathbb{R}$.

Take $c_{j,r}$ such that $|\omega|_r\ge c_{j,r}\|\omega\|$, $\forall\omega\in Y_j$. Then, for $\omega\in Y_j$, we have the following equation:

$$\begin{aligned}\Gamma_\lambda(\omega) &\le \frac{a}{2}\|\omega\|^2 + \frac{b}{4}\|\omega\|^4 + \frac{1}{4}\int_\Omega \phi_\omega \omega^2 dx - \lambda\ell\int_\Omega |\omega|^4 dx + \lambda\int_\Omega C_\ell dx \\ &\le \frac{a}{2}\|\omega\|^2 + \frac{b}{4}\|\omega\|^4 + \frac{c}{4}\|\omega\|^4 - \ell\left(c_{j,4}\right)^4\|\omega\|^4 + c \longrightarrow -\infty,\end{aligned} \tag{35}$$

for $\rho_j > 0$ large enough, provided $\ell(c_{j,4})^4 > \frac{b+c}{4}$. Hence, we obtain the following equation:

$$a_j(\lambda) = \max_{\omega\in Y_j, \|\omega\|=\rho_\omega} \Gamma_\lambda(\omega) \le 0, \tag{36}$$

uniformly for $\lambda \in [1, 2]$ if $\rho_j > 0$ large enough.

On the other hand, by conditions (F_1), (H_2), we have, for any $\varepsilon > 0$, there exists $D_\varepsilon > 0$, such that:

$$|f(x,t)| \le D_\varepsilon |t|^{q-1} + \varepsilon|t|, \ \forall x \in \Omega, \ \forall t \in \mathbb{R}. \tag{37}$$

Let β_j be defined as Equation (30). Then, for each $\omega \in Z_j$ and $\varepsilon > 0$ small enough, we get the following equation:

$$\begin{aligned}\Gamma_\lambda(\omega) &= \frac{a}{2}\|\omega\|^2 + \frac{b}{4}\|\omega\|^4 + \frac{1}{4}\int_\Omega \phi_\omega\omega^2 dx - \lambda\int_\Omega F(x,\omega)dx \\ &\ge \frac{a}{2}\|\omega\|^2 + \frac{b}{4}\|\omega\|^4 + \frac{1}{4}\int_\Omega \phi_\omega\omega^2 dx - \frac{\lambda\varepsilon}{2}\int_\Omega |\omega|^2 dx - \frac{\lambda D_\varepsilon}{q}\int_\Omega |\omega|^q dx \\ &= \frac{a}{2}\|\omega\|^2 + \frac{b}{4}\|\omega\|^4 + \frac{1}{4}\int_\Omega \phi_\omega\omega^2 dx - \frac{\lambda\varepsilon}{2}|\omega|_2^2 - \frac{\lambda D_\varepsilon}{q}|\omega|_q^q \\ &\ge \frac{a}{4}\|\omega\|^2 + \frac{b}{4}\|\omega\|^4 - c|\omega|_q^q \\ &\ge \frac{a}{4}\|\omega\|^2 - c\beta_j^q\|\omega\|^q.\end{aligned} \tag{38}$$

Denote by $\gamma_j = \left(\frac{4cq\beta_j^q}{a}\right)^{\frac{1}{2-q}}$, then for $\omega \in Z_j$ with $\|\omega\| = \gamma_j$, one has the following equation:

$$\Gamma_\lambda(\omega) \ge \left(\frac{4cq\beta_j^q}{a}\right)^{\frac{2}{2-q}}\left(\frac{a}{4} - \frac{a}{4q}\right) := b_j^*, \tag{39}$$

which implies that $b_j(\lambda) = \inf_{\omega\in Z_j, \|\omega\|=\gamma_j} \Gamma_\lambda(\omega) \ge b_j^* \longrightarrow \infty$ as $j \longrightarrow \infty$. Therefore, by Theorem 4, for a.e. $\lambda \in [1, 2]$, there exists a sequence $\left\{\omega_n^j(\lambda)\right\}_{n=1}^\infty$ such that:

$$\sup_n \left\|\omega_n^j(\lambda)\right\| < \infty, \ \ \Phi'_\lambda\left(\omega_n^j(\lambda)\right) \longrightarrow 0, \tag{40}$$

and

$$\Gamma_\lambda\left(\omega_n^j(\lambda)\right) \longrightarrow c_j(\lambda) \ge b_j(\lambda) \ge b_j^*, \tag{41}$$

as $n \longrightarrow \infty$. Furthermore, using the fact that $c_j(\lambda) \le \sup_{\omega\in B_j} \Gamma_\lambda(\omega) := c_j^*$, and $H_0^1(\Omega)$ is imbedded compactly into $L^r(\Omega)$ for $2 \le r < 2^*$, by a standard argument, we infer that $\left\{\omega_n^j(\lambda)\right\}_{n=1}^\infty$ has a convergent subsequence. Consequently, there exist $z^j(\lambda)$ such that $\Gamma'_\lambda(z^j(\lambda)) = 0$ and $\Gamma_\lambda(z^j(\lambda)) \in [b_j^*, c_j^*]$. As a result, we can find $\lambda_n \longrightarrow 1$ such that $\{z_n\}$ being exactly what kind of want to happen.

Claim 2. *$\{z_n\}_{n=1}^\infty$ must be bounded in W. Suppose by contradiction that, $\|z_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$. Denote by $u_n := \frac{z_n}{\|z_n\|}$. Then up to a subsequence, we have the following equation:*

$$\begin{cases} u_n \rightharpoonup u & \text{in } H_0^1(\Omega), \\ u_n \longrightarrow u \text{ in } L^t(\Omega) & \text{for } 2 \le t < 2^*, \\ u_n \longrightarrow u & \text{a.e. } x \in \Omega. \end{cases} \tag{42}$$

There are two possible cases: (i) $u \ne 0$ in W; (ii) $u = 0$ in W.

In Case (i), it follows from $\Gamma'_{\lambda_n}(z_n) = 0$ and Lemma 1 that:

$$\lambda_n\int_\Omega \frac{f(x,z_n)z_n}{\|z_n\|^4}dx = \frac{a}{\|z_n\|^2} + b + \frac{1}{\|z_n\|^4}\int_\Omega \phi_{z_n} z_n^2 dx \le C. \tag{43}$$

On the other hand, by Fatou's lemma and condition (H_1), (H_3), we infer to the following equation:

$$\int_\Omega \frac{f(x,z_n)z_n}{\|z_n\|^4} = \int_{\{u_n(x)\ne 0\}} |u_n|^4 \frac{f(x,z_n)z_n}{|z_n|^4} \longrightarrow \infty, \tag{44}$$

which yields a contradiction.

In Case (ii), we may define the following functional as in [35]:

$$\Gamma_{\lambda_n}(t_n z_n) := \max_{t\in[0,1]} \Gamma_{\lambda_n}(t z_n). \tag{45}$$

Set $u_n^* := (4\ell)^{\frac{1}{2}} u_n$ with $\ell > 0$, then we obtain when n is large enough, that:

$$\begin{aligned}\Gamma_{\lambda_n}(t_n z_n) &= \frac{a t_n^2}{2}\|z_n\|^2 + \frac{b t_n^4}{4}\|z_n\|^4 + \frac{t_n^4}{4}\int_\Omega \phi_{z_n}|z_n|^2 dx - \lambda_n \int_\Omega F(x, t_n z_n) dx \\ &\geq \Gamma_{\lambda_n}(u_n^*) = 2a\ell + 4b\ell^2 + 4\ell^2 \int_\Omega \phi_{u_n}|u_n|^2 dx - \lambda_n \int_\Omega F\left(x, \sqrt{4\ell}u_n\right) dx \\ &\geq 2a\ell,\end{aligned} \tag{46}$$

which implies that $\lim_{n\longrightarrow\infty} \Gamma_{\lambda_n}(t_n z_n) = \infty$, since $\ell > 0$ can be large arbitrarily. Here, we have used the fact that $u_n \rightharpoonup 0$ in W and $u_n \longrightarrow 0$ in $L^p(\Omega), p\in[1,6)$, and so:

$$\begin{aligned}\int_\Omega \phi_{u_n}|u_n|^2 dx &\leq \left(\int_\Omega |\phi_{u_n}|^6 dx\right)^{\frac{1}{6}} \left(\int_\Omega |u_n|^{\frac{12}{5}} dx\right)^{\frac{5}{6}} \\ &\leq S^{-1}\|\phi_{u_n}\| |u_n|_{12/5}^2 \leq C\|u_n\|^2 |u_n|_{12/5}^2 \longrightarrow 0,\end{aligned} \tag{47}$$

together with:

$$\int_\Omega F(x, u_n) dx \leq \frac{D_\varepsilon}{q}\int_\Omega |u_n|^q dx + \frac{\varepsilon}{2}\int_\Omega |u_n|^2 dx \longrightarrow 0. \tag{48}$$

Note that $t_n \in (0,1]$ and $\langle \Gamma'_{\lambda_n}(t_n z_n), t_n z_n\rangle = 0$, it follows from:

$$\begin{aligned}\Gamma_{\lambda_n}(t_n z_n) &= \Gamma_{\lambda_n}(t_n z_n) - \frac{1}{4}\left\langle \Gamma'_{\lambda_n}(t_n z_n), t_n z_n\right\rangle \\ &= \frac{a t_n^2}{4}\|z_n\|^2 + \frac{\lambda_n}{4}\int_\Omega [f(x, t_n z_n) t_n z_n - 4F(x, t_n z_n)],\end{aligned} \tag{49}$$

and $\Gamma_{\lambda_n}(0) = 0$, that $|t_n z_n|$ must tend to ∞ when $n\longrightarrow\infty$. Therefore, by (H_3) and $\lambda_n \in [1,2]$, we have the following equation:

$$\int_\Omega [f(x, t_n z_n) t_n z_n - 4F(x, t_n z_n)] \longrightarrow \infty. \tag{50}$$

On the other hand, we use the fact that $h(t) = t^4 f(x,s) s - 4F(x, ts)$ is increasing in $t\in(0,1]$, which implies that, $f(x,s)s - 4F(x,s)$ is increasing in $s>0$ by virtue of the following equation:

$$\frac{d}{ds}[f(x,s)s - 4F(x,s)] = s^4 \frac{d}{ds}\left(\frac{f(x,s)}{s^3}\right), \tag{51}$$

and $\frac{f(x,s)}{s^3}$ is increasing for $s\in(0,\infty)$. By virtue of the oddness of f, we have the following equation:

$$\int_\Omega [f(x, z_n) z_n - 4F(x, z_n)] \geq \int_\Omega [f(x, t_n z_n) t_n z_n - 4F(x, t_n z_n)] \longrightarrow \infty, \tag{52}$$

which leads to a contradiction in view of:

$$\begin{aligned}\lambda_n \int_\Omega \widetilde{F}(x, z_n) &= \Gamma_{\lambda_n}(z_n) - \frac{1}{4}\left\langle \Gamma'_{\lambda_n}(z_n), z_n\right\rangle \\ &\quad - \frac{a}{4}\|z_n\|^2 \leq \Gamma_{\lambda_n}(z_n) \in \left[b_j^*, c_j^*\right].\end{aligned} \tag{53}$$

So far, we have proved that the solution z^j satisfies $\Gamma(z^j) \in [b_j^*, c_j^*]$. Since $b_j^* \longrightarrow \infty$ as $j\longrightarrow\infty$, we see that there exist a sequence of solutions $\{z^j\}_{j=1}^\infty$ of Problem (1) such that $\Gamma_1(z^j) \longrightarrow \infty$, $j\longrightarrow\infty$.

Finally, we present an example to explain that there is a nonlinear f which satisfies all the conditions of Theorem 2, but does not satisfy the conditions of Theorem 1, especially condition (F_2).

Example 1. Let $f(x,u) = u^3 \ln(1+|u|)$. Integrating by parts, a simple computation yields that for $u\geq 0$, we have the following equation:

$$\begin{aligned}F(x,u) &= \int_0^u f(x,t) dt = \int_0^u [t^3(4 + \ln(1+t)) dt \\ &= \frac{u^4}{4}\ln(1+u) - \frac{1}{4}\left(\frac{u^4}{4} - \frac{u^3}{3} + \frac{u^2}{2} - u + \ln(1+u)\right),\end{aligned} \tag{54}$$

and

$$\begin{aligned}\widetilde{F}(x,u) &:= \frac{1}{4} f(x,u)u - F(x,u) \\ &= \frac{1}{4}\left(\frac{u^4}{4} - \frac{|u|^3}{3} + \frac{u^2}{2} - |u| + \ln(1+|u|)\right) \longrightarrow \infty,\end{aligned} \tag{55}$$

as $|u|\longrightarrow\infty$, which means that condition (H_3) is satisfy. Moreover, it is easy to see that f satisfies conditions (H_1), (H_2).

However, f does not satisfy condition (F_2). Indeed, suppose that there exists some $\mu > 4$ fulfilling $\theta F(x,u) \le f(x,u)u$ for $|u|$ large. Consequently, we have the following equation:

$$u^4 \ln(1+|u|)\left(\frac{\theta}{4} - \frac{\theta}{16\ln(1+|u|)}\right) - \frac{\theta}{4}\left(-\frac{|u|^3}{3} + \frac{u^2}{2} - |u| + \ln(1+|u|)\right) \le u^4 \ln(1+|u|), \tag{56}$$

for $|u|$ large, which contradicts to the fact $\theta > 4$.

Data Availability

Data availability is not applicable to this article as no new data were created or analyzed in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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