Research Article

Prestress Design of Tensegrity Structures Using Semidefinite Programming

Yafeng Wang[1] and Xian Xu (ORCID)[2]

[1]*Doctoral Candidate, Department of Civil Engineering, Zhejiang University, 866 Yuhangtang Road, Hangzhou, Zhejiang 310058, China*
[2]*Associate Professor, Department of Civil Engineering, Zhejiang University, 866 Yuhangtang Road, Hangzhou, Zhejiang 310058, China*

Correspondence should be addressed to Xian Xu; xian_xu@zju.edu.cn

Finding appropriate prestresses which can stabilize the system is a key step in the design of tensegrity structures. A semidefinite programming- (SDP-) based approach is developed in this paper to determine appropriate prestresses for tensegrity structures. Three different stability criteria of tensegrity structures are considered in the proposed approach. Besides, the unilateral property of members and the evenness of internal forces are taken into account. The stiffness of the whole system can also be optimized by maximizing the minimum eigenvalue of the tangent stiffness matrix. Deterministic algorithms are used to solve the semidefinite programming problem in polynomial time. The applicability of the proposed approach is verified by three typical examples. Compared to previous stochastic-based approaches, the global optimality of the solution of the proposed approach is theoretically guaranteed and the solution is exactly reproducible.

1. Introduction

Tensegrity structures are stable self-stressed pin-jointed structures. Because of the unique features and remarkable appearances of tensegrity structures, they are attracting more and more interest in the fields of science and engineering [1–3]. In the view of mathematics, in any stable prestressed pin-jointed systems such as prestressed trusses, cable net structures can be seen as tensegrity systems [4]. The above general definition of tensegrity systems is adopted in this study; that is, the term *tensegrity structure* is the same as the *stable prestressed pin-jointed structure*.

A statically and kinematically indeterminate pin-jointed assembly can be converted to a stable tensegrity structure if appropriate prestresses and gain adequate stiffness could be introduced. Therefore, finding appropriate prestresses which can stabilize the system is a key step in generating tensegrity structures from statically and kinematically indeterminate pin-jointed assemblies. There are two main stability criteria,

i.e., super stability and prestress stability, of tensegrity structures [5, 6]. Prestress stability of a prestressed pin-jointed structure can be tested by using the *Geometry force* method proposed by Calladine and Pellegrino [7]. Super stability is a stronger stability criterion for prestressed pin-jointed structures compared with prestress stability because it is irrespective of selection of materials and level of self-stresses [5, 8].

If a statically indeterminate pin-jointed assembly has only one independent self-stress state, the real prestress state introduced into the system must be in accordance with the self-stress state and only the level of prestresses needs to be designed. However, if a statically indeterminate pin-jointed assembly has more than one independent self-stress state, how to find a feasible self-stress state that can stabilize the system becomes a challenging problem, especially when the unilateral features of some specified members (i.e., they are appointed to be in tension or in compression in advance) are considered. Many studies have been carried out towards this problem. Tran and Lee [9] and Zhou et al. [10] used iterative

methods to design initial prestress of tensegrity structures, but these iterative methods can only find a feasible solution but not an optimized solution for a specified objective function. In order to find optimal initial prestress for a tensegrity system, Ohsaki et al. [11, 12] used an optimization method by enumeration of the vertices of feasible region of the prestresses in the self-equilibrium state. However, not all the three kinds of stability criteria of tensegrity structures were taken into consideration in their approach and the enumeration method may be not very effective when the structure systems become more complex. Besides these deterministic algorithms, many stochastic or heuristic algorithms have also been used to solve this problem. El-Lishani et al. [13] and Lee et al. [14] used a genetic algorithm (GA) to find feasible self-stresses of pin-jointed structures. Xu and Luo [15], Zhang and Ohsaki [16] both used a simulated annealing algorithm (SAA) to find optimal prestress of tensegrity structures. Chen et al. [17, 18] converted this problem into a modified traveling salesman problem (TSP) and used a new heuristic search method, ant colony system (ACS), to search feasible solutions. Although these approaches have been verified capable to deal with the problem, they all only considered the prestress stability criterion and did not take super stability criterion into consideration. Moreover, they are all heuristic algorithms with randomness. There is no mathematical convergence theory for these algorithms and thus no assurance of global optimality of the final solution found [19]. And due to their stochastic basis, it is difficult to reproduce results from the method. Running the same algorithm, on the same problem, may result in widely different solutions [19].

It is also worth mentioning that recently the topology optimization of tensegrity structures has been intensively studied by using mixed integer linear programming (MILP) [20–23]. The prestress design problem is also a critical sub-problem in the topology optimization of tensegrity structures. But the stability test cannot be incorporated into the current MILP-based formulation due to its nonlinear formation, not to mention incorporating the heuristic algorithms with randomness into the MILP-based formulation.

Semidefinite programming (SDP) problems are an important class of mathematical optimization problems which are applicable to a wide range of areas such as control theory, circuit design, sensor network localization, and principal component analysis [24]. In the field of graph theory, SDP was used by So and Ye [25] to realize 3-realizable graphs based on a tensegrity framework. In their approach, SDP is only used as a tool to find 3-realizable graphs which can be treated as a kind of unyielding tensegrity system, so stability conditions and evenness of prestresses of members in the tensegrity systems are not considered. Therefore, the approach is not suitable for the application of prestress design and optimization of tensegrity systems in the field of engineering application. In this paper, SDP is also adopted to solve the prestress design problem of tensegrity structures which includes both stability condition and evenness of internal forces of members. Because SDP is an optimization-based method, it can be used to find optimized solution corresponding to a specified objective function. Moreover,

SDP problems can be solved efficiently in polynomial time by implementing a suitable algorithm. Compared with heuristic search methods with randomness, there is no randomness in the process of solving SDP problems; therefore, no repetitive computation is needed. Another advantage of using SDP formulation for the prestress design problem of tensegrity structures is that it has the potential to be incorporated into or to be extended to incorporate the MILP formulation of the topology optimization of tensegrity structures.

The motivation of this study includes (a) providing an alternative approach for prestress design of tensegrity structures, (b) incorporating the different stability criteria into the prestress design approach, and (c) formulating the stability test within a frame of mathematical programming and thus potentially being able to incorporate with the topology optimization formulation.

2. Formulations for Tensegrity Structures

Equilibrium condition and stability condition are necessary conditions for a tensegrity structure. Considering a d-dimensional tensegrity structure with m members and n nodes, the equilibrium condition and stability condition of it can be derived as follows.

2.1. Equilibrium Condition. The topology of the tensegrity structure can be expressed by a connectivity matrix $\mathbf{C}^s \in \mathbb{R}^{m \times n}$ defined in the field of graph theory [26]. Suppose that member k connects node i and node j ($i < j$), then the ith and jth elements of the kth row of $\mathbf{C}^s$ are set to 1 and −1, respectively, as

$$\mathbf{C}^s_{(k,p)} = \begin{cases} 1, & \text{for } p = i, \\ -1, & \text{for } p = j, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Take partitions of matrix $\mathbf{C}^s$ so that

$$\mathbf{C}^s = (\mathbf{C}_x, \mathbf{C}_{fx}) = (\mathbf{C}_y, \mathbf{C}_{fy}) = (\mathbf{C}_z, \mathbf{C}_{fz}), \tag{2}$$

where $\mathbf{C}_x \in \mathbb{R}^{m \times r_x}$, $\mathbf{C}_y \in \mathbb{R}^{m \times r_y}$, and $\mathbf{C}_z \in \mathbb{R}^{m \times r_z}$ relate to the free DOFs and $\mathbf{C}_{fx} \in \mathbb{R}^{m \times s_x}$, $\mathbf{C}_{fy} \in \mathbb{R}^{m \times s_y}$, and $\mathbf{C}_{fz} \in \mathbb{R}^{m \times s_z}$ relate to the fixed DOFs in x direction, y direction, and z direction, respectively [27].

If all the DOFs of a node are restrained, then assume that $r_x = r_y = r_z = r$ and $s_x = s_y = s_z = s$ and that $\mathbf{C}_{fx} = \mathbf{C}_{fy} = \mathbf{C}_{fz} = \mathbf{C}_f \in \mathbb{R}^{m \times s}$ and $\mathbf{C}_x = \mathbf{C}_y = \mathbf{C}_z = \mathbf{C} \in \mathbb{R}^{m \times r}$, thus equation (2) can be simplified as

$$\mathbf{C}^s = (\mathbf{C}, \mathbf{C}_f). \tag{3}$$

Let t_k and l_k denote the internal force and length of member k, respectively. The member force vector and member length matrix are defined as $\mathbf{t} = \{t_k\}$ and $\mathbf{L} = \text{diag}(l_1, \ldots, l_k, \ldots l_m)$. Then, the coordinate difference matrices $\mathbf{U}_x$, $\mathbf{U}_y$, $\mathbf{U}_z \in \mathbb{R}^{m \times m}$ are given as

$$\mathbf{U}_x = \text{diag}(\mathbf{CX} + \mathbf{C}_f \mathbf{X}_f), \tag{4a}$$

$$\mathbf{U}_y = \text{diag}(\mathbf{CY} + \mathbf{C}_f \mathbf{Y}_f), \tag{4b}$$

$$\mathbf{U}_z = \text{diag}\left(\mathbf{CZ} + \mathbf{C}_f\mathbf{Z}_f\right), \qquad (4c)$$

where $\mathbf{X} \in \mathbb{R}^{r_x \times 1}$, $\mathbf{Y} \in \mathbb{R}^{r_y \times 1}$, and $\mathbf{Z} \in \mathbb{R}^{r_z \times 1}$ are the nodal coordinate vectors of the free DOFs in the x direction, y direction, and z direction, respectively and $\mathbf{X}_f \in \mathbb{R}^{s_x \times 1}, \mathbf{Y}_f \in \mathbb{R}^{s_y \times 1}$, and $\mathbf{Z}_f \in \mathbb{R}^{s_z \times 1}$ are the nodal coordinate vectors of the fixed DOFs in the x direction, y direction, and z direction, respectively. r_x, r_y, and r_z represent the number of free DOFs in the x direction, y direction, and z direction, respectively. Similarly, s_x, s_y, and s_z represent the number of fixed DOFs in the x direction, y direction, and z direction, respectively. There exists $r_x + s_x = r_y + s_y = r_z + s_z = n$.

The equilibrium matrix $\mathbf{A} \in \mathbb{R}^{3r \times m}$ can be written as

$$\mathbf{A} = \begin{pmatrix} \mathbf{C}^T\mathbf{U}_x\mathbf{L}^{-1} \\ \mathbf{C}^T\mathbf{U}_y\mathbf{L}^{-1} \\ \mathbf{C}^T\mathbf{U}_z\mathbf{L}^{-1} \end{pmatrix}. \qquad (5)$$

The equilibrium equations for a tensegrity structure without external loads can be written as [28]

$$\mathbf{At} = 0. \qquad (6)$$

Note that the internal forces of a tensegrity system can also be expressed as the combination of its independent self-stress modes [28]. For symmetrical systems, symmetry representations can also be used to simplify the equilibrium equation [29]. In this paper, the symmetry of systems is not considered; therefore, only equation (6) is used to express equilibrium equations.

2.2. Stability Condition

2.2.1. Stiffness Matrices.
The tangent stiffness matrix $\mathbf{K}$ of a pin-jointed structure can be described as

$$\mathbf{K} = \mathbf{K}^E + \mathbf{K}^G, \qquad (7)$$

where $\mathbf{K}^E$ and $\mathbf{K}^G$ are the linear elastic stiffness matrix and the geometrical stiffness matrix, respectively. $\mathbf{K}^E$ and $\mathbf{K}^G$ are defined as

$$\mathbf{K}^E = \mathbf{A}\overline{K}\mathbf{L}^{-1}\mathbf{A}^T,$$
$$\mathbf{K}^G = \mathbf{I} \otimes \mathbf{D}, \qquad (8)$$

where $\overline{\mathbf{K}}$ is the diagonal matrix, $\mathbf{I}$ is the identity matrix, $\otimes$ is the tensor product, and $\mathbf{D}$ is the force density matrix. Diagonal matrix $\overline{\mathbf{K}}$ and force density matrix $\mathbf{D}$ in equation (8) are defined as

$$\overline{\mathbf{K}} = \text{diag}\left(E_1A_1, \ldots, E_kA_k, \ldots, E_mA_m\right),$$

$$\mathbf{D} = \mathbf{C}^T\mathbf{QC},$$

$$\mathbf{Q} = \text{diag}\left(\frac{t_1}{l_1}, \ldots, \frac{t_k}{l_k}, \ldots, \frac{t_m}{l_m}\right), \qquad (9)$$

where E_k and A_k are Young's modulus and the cross-sectional area of member k, respectively. More details on stiffness matrices can be found in the literature [6, 30–32].

2.2.2. General Stability Criterion Based on Tangent Stiffness Matrix.
Based on the general theory of elastic stability [33], if the tangent stiffness matrix $\mathbf{K}$ is positive definite, then the structure is stable. That is, $\mathbf{d}'\mathbf{Kd} > 0$ holds for any nonzero vector $\mathbf{d}$. In order to simplify the formulation expressions, $\mathbf{N} > 0$ is used to denote that matrix $\mathbf{N}$ is positive definite in the following content. Similarly, $\mathbf{N} \geq 0$ is used to denote that matrix $\mathbf{N}$ is positive semidefinite. Therefore, positive definiteness of tangent stiffness matrix $\mathbf{K}$ can be simply expressed as

$$\mathbf{K} > 0. \qquad (10)$$

Note that if the structure is free standing, the global rigid body motions should be eliminated beforehand.

It can be seen from equations (7)–(9) that this general stability condition for a given tensegrity structure not only depends on the material properties of members but also depends on the level of prestress.

2.2.3. Prestress Stability.
If the material properties of members and the level of prestress are not considered, a reduced stability criterion, prestress stability, for prestressed pin-jointed structures was presented by Connelly [5]. For a prestress stable system, the stiffness degradation can be induced by a relatively high level of prestress [34, 35]. Prestress stability can be tested by using the *Geometry force* method proposed by Calladine and Pellegrino [7]. Let $\mathbf{M}$ denotes the inextensional mechanism modes of a statically and kinematically indeterminate pin-jointed structure. It has been proved that the matrix used by Calladine and Pellegrino [7] for the stability test is equivalent to $\mathbf{M}^T\mathbf{K}^G\mathbf{M}$, which can be seen as a reduced form of geometrical stiffness matrix $\mathbf{K}^G$ [30]. Note that if the structure is free standing, the rigid body motions are excluded from $\mathbf{M}$ beforehand. Therefore, from the definition of prestress stability, if $\mathbf{M}^T\mathbf{K}^G\mathbf{M}$ is positive definite, then the structure is prestress stable. This condition can be written as

$$\mathbf{M}^T\mathbf{K}^G\mathbf{M} > 0. \qquad (11)$$

Note that a structure may be unstable in practice even if it satisfies the prestress stability condition because the material properties and the level of prestress are not considered in the prestress stability condition. That is to say that the positive definiteness of $\mathbf{M}^T\mathbf{K}^G\mathbf{M}$ is not a sufficient condition but only a necessary condition for the stability of kinematically indeterminate structures when the material properties of members are taken into consideration. More details on this issue can be found in the literature [36].

2.2.4. Super Stability.
Super stability is a stronger stability criterion for prestressed pin-jointed structures compared with prestress stability [5]. The stability of a super stable structure is ensured irrespective of the material properties and the level of prestresses, if yielding and buckling of members are excluded. Therefore, super stability systems may be more preferred in practical applications.

Based on the super stability criterion [5], a pin-jointed structure is super stable if the geometrical stiffness matrix $\mathbf{K}^G$ satisfies equation (11) and meanwhile is positive semidefinite. The positive semidefinite condition can be written as

$$\mathbf{K}^G \geq 0. \tag{12}$$

Note that the three stability criterions mentioned above are not independent of each other. The relationship of them can be described by a diagram shown in Figure 1. The shaded part which is the intersection region of the three parts denotes super stability condition.

3. Mathematical Optimization Model

The problem of finding appropriate prestresses which can satisfy the equilibrium condition and stability condition for a pin-jointed assembly can be formulated into an optimization model. The optimization variables of the optimization model are internal forces of members, and the constraints of the optimization model are equilibrium constraint, stability constraint, and additional unilateralism constraint of member forces.

3.1. Equilibrium Constraint. By using optimization variables $\mathbf{t}$ and the equilibrium matrix $\mathbf{A}$, the equilibrium constraint can be expressed as equation (6).

3.2. Stability Constraint. Depending on which kind of stability is expected to be satisfied by the tensegrity structure under design, equations (10), (11), or (12) can be selected as the stability constraint.

Note that the strict inequalities cannot be dealt in continuous numerical optimization. Hence, if equation (10) is adopted, a transformed expression is used, that is,

$$\mathbf{K} \geq \eta \mathbf{I}, \tag{13}$$

where η is the small positive value as a margin and $\mathbf{I}$ is the identity matrix whose dimensionality is the same as the matrix $\mathbf{K}$. Note that if the minimum eigenvalue of tangent stiffness matrix $\mathbf{K}$ is needed to be optimized, η can be also treated as an optimization variable.

Similarly, if equation (11) is adopted, a transformed expression as follows is used:

$$\mathbf{M}^T \mathbf{K}^G \mathbf{M} \geq \mu \mathbf{I}, \tag{14}$$

where μ is the small positive value as a margin and $\mathbf{I}$ is the identity matrix whose dimensionality is the same as the matrix $\mathbf{M}^T \mathbf{K}^G \mathbf{M}$.

3.3. Unilateralism Constraints of Member Forces. If there are some members that are appointed in advance to be in tension or compression, unilateralism constraints should be added. Suppose that the set of labels of members appointed to be in tension is denoted as P, the set of labels of members
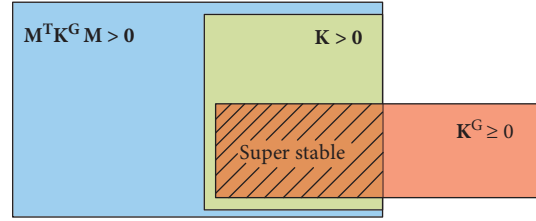

Figure 1: Relationship of stability criteria.

appointed to be in compression is denoted as Q, and the set of labels of the rest of members is denoted as N. Then, the unilateralism constraints can be expressed as

$$
\begin{aligned}
t_k \geq \varepsilon, \quad & \forall k \in P \text{ and } P \neq \varnothing, \\
t_k \leq -\varepsilon, \quad & \forall k \in Q \text{ and } Q \neq \varnothing,
\end{aligned} \tag{15}
$$

where ε is the small positive value as a margin.

In practical design, the evenness of internal forces of members in tension or compression is sometimes needed to be considered. For this purpose, equation (15) can be converted to another form by using additional optimization variables:

$$
\begin{aligned}
t_p \leq t_k \leq t_p + e_p, \quad & \forall k \in P \text{ and } P \neq \varnothing, \\
& t_p \geq \varepsilon, \ e_p \geq 0, \\
t_q - e_q \leq t_k \leq t_q, \quad & \forall k \in Q \text{ and } Q \neq \varnothing, \\
& t_q \leq -\varepsilon, \ e_q \geq 0,
\end{aligned} \tag{16}
$$

where e_p, e_q, t_p, and t_q are the continuous variables. In this way, smaller e_p or e_q means more even tension forces or more even compression forces, respectively.

3.4. Objective Function. If the objective function is not adopted in the optimization model, the solution which satisfies above constraints is a feasible solution. If optimization objective is considered, different objective functions can be used to realize different design intentions. For example, if the evenness of the member internal forces is required to be optimized, the corresponding objective function f_1 can be expressed as

$$f_1 = e_p + e_q. \tag{17}$$

If the minimum eigenvalue of the tangent stiffness matrix $\mathbf{K}$ is needed to be optimized, it is equivalent to maximize variable η in equation (13) in the optimization process. To realize this purpose, η in equation (13) can be treated as an optimization variable, and the objective function f_2 can be expressed as

$$f_2 = -\eta. \tag{18}$$

3.5. Optimization Model. The optimization model of the problem can be expressed as

$$\min_{W} \quad f, \tag{19a}$$

$$\text{s.t.} \quad \mathbf{At} = 0, \tag{19b}$$

Stability constraint,

$$t_p \le t_k \le t_p + e_p, \quad \forall k \in P \text{ and } P \ne \varnothing, \tag{19c}$$

$$t_q - e_q \le t_k \le t_q, \quad \forall k \in Q \text{ and } Q \ne \varnothing, \tag{19d}$$

$$t_p \ge \varepsilon, \; e_p \ge 0, \quad \text{if } P \ne \varnothing, \tag{19e}$$

$$t_q \le -\varepsilon, \; e_q \ge 0, \quad \text{if } Q \ne \varnothing, \tag{19f}$$

where W denotes the set of optimization variables, f denotes objective function which can be chosen as f_1 or f_2 by designers according to actual needs and constraints considered, and *stability constraint* denotes constraint on stability which is chosen from equations (10)–(12) by designers. If $P = \varnothing$, then constraints (19c) and (19e) are not considered. Similarly, constraints (19d) and (19f) are not considered if $Q = \varnothing$.

4. Semidefinite Programming

Semidefinite programming (SDP) refers to the class of optimization problems where a linear function of a symmetric matrix variable $\mathbf{Y}$ is optimized subject to linear constraints on the elements of $\mathbf{Y}$ and the additional constraint that $\mathbf{Y}$ must be positive semidefinite. The linear programming (LP) problems can be deemed as a special case of SDP, namely, when all the matrices involved are diagonal. Similar to LP problems, SDP problems also come in pairs. One of the problems is referred to as the *primal* problem, and the other one is the *dual* problem. Let $\mathbf{R} \cdot \mathbf{S}$ denotes the standard inner product of matrices $\mathbf{R}$ and $\mathbf{S}$ in the linear space S^n, i.e., $\mathbf{R} \cdot \mathbf{S} = \mathrm{tr}(\mathbf{R}^T\mathbf{S}) = \sum_{i=1}^{n}\sum_{j=1}^{n} R_{ij}S_{ij}$. The most common standard formulation of the SDP primal problem is given as follows:

$$(P_{\text{SDP}}): \min \quad \mathbf{B} \cdot \mathbf{Y},$$
$$\text{s.t.} \quad \mathbf{E}_i \cdot \mathbf{Y} = g_i \; (i = 1, \dots, m), \tag{20}$$
$$\mathbf{Y} \ge 0,$$

where $\mathbf{Y}$ is the variable vector, $\mathbf{E}_i \in S^n (i = 1, \dots, m)$ and $\mathbf{B} \in S^n$ are the symmetric matrices, and $\mathbf{g} = (g_i) \in R^m$ is the column vector. The dual problem of P_{SDP} is formulated in variables $\mathbf{z} \in R^m$ as

$$(D_{\text{SDP}}): \max \quad \mathbf{g}^T \cdot \mathbf{z},$$
$$\text{s.t.} \quad \mathbf{B} - \sum_{i=1}^{m} \mathbf{E}_i z_i \ge 0. \tag{21}$$

More details on the *primal* problem and the *dual* problem can be found in the literature [24].

SDP has been applied in a wide range of fields [37–39]. The primal-dual interior-point methods, which were developed to solve linear programming problems [40] at first, have been widely used to solve SDP problems [41]. It is theoretically guaranteed that the pair of SDP problems (P_{SDP}) and (D_{SDP}) can be solved to the optimal solutions in polynomial time [38, 41].

Both linear constraints and matrix semidefinite constraints are included in equation (19); therefore, it is a SDP model, and primal-dual interior-point method can be applied to solve it.

5. Numerical Examples

In this paper, the SDP problems are solved by using SeDuMi [42], which implements the primal-dual interior-point method. Computations are carried out on a Dual-Processor Intel(R) Xeon(R) CPU (2.80 GHz with 32 GB RAM) computer. The parameters μ and ε are set as $\mu = \varepsilon = 1 \times 10^{-3}$. Without loss of generality, the range of prestresses is set as $[-1, 1]$, and the elastic modulus and cross-sectional areas of members are both set as unit.

5.1. Example 1: Planar Pin-Jointed Assembly I. A planar pin-jointed assembly, as shown in Figure 2, is considered in this example. The system consists of eight nodes and seven members and has three mechanism modes and two self-stress states. It has been analytically studied by Calladine and Pellegrino [7] and later used as an example in numerical prestress design by the heuristic algorithms [13, 15, 17].

The objective function f_2 is adopted in this example. Two computations using the prestress stability condition and the super stability condition, respectively, are conducted. In addition, $P = Q = \varnothing$ (i.e., $N = \{1, 2, 3, 4, 5, 6, 7\}$) is used in both computations.

The normalized computation solutions together with the analytical result given by Calladine and Pellegrino [7] are listed in Table 1. It is worth noting that no repetitive runs are needed in both computations because the primal-dual interior-point method is a deterministic algorithm rather than a stochastic algorithm.

Let C1 and C2 denote the two computation cases separately. It is found that computations C1 and C2 have the same solution, which means that the solution is not only prestress stable but also super stable. It is also shown that the solution obtained by SDP exactly agrees with the analytical result reported by Calladine and Pellegrino [7].

5.2. Example 2: Planar Pin-Jointed Assembly II. In order to demonstrate that the positive definiteness of the reduced form of the geometrical stiffness matrix $\mathbf{M}^T\mathbf{K}^G\mathbf{M}$ is not a sufficient condition for the stability of kinematically indeterminate structures, a planar pin-jointed assembly (Figure 3) used by Ohsaki and Zhang [36] is considered here. This system has one self-stress state and one mechanism mode.

No objective function was used in the computations carried out in this example. At first, the prestress stability condition was adopted in computation C1. Then, the general stability condition instead of the prestress stability condition was adopted in computation C2. The results of the two
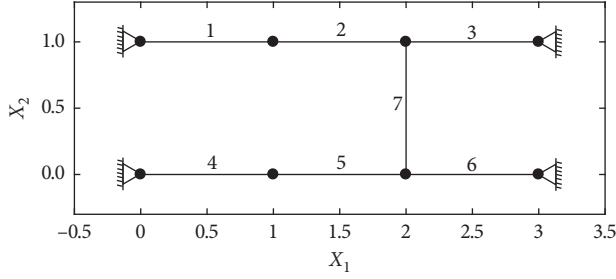
FIGURE 2: Planar pin-jointed assembly considered in example 1.

TABLE 1: Computation solutions and analytical result of example 1.

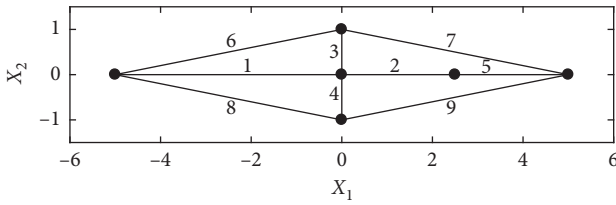| Solutions	Member prestresses						
	1	2	3	4	5	6	7
C1 and C2	1.000	1.000	1.000	1.000	1.000	1.000	0.000
Analytical result [7]	1.000	1.000	1.000	1.000	1.000	1.000	0.000

FIGURE 3: Planar pin-jointed assembly considered in example 2.

computations are listed in Table 2 where $\lambda_{\min}^{\mathbf{M}^T\mathbf{K}^G\mathbf{M}}$ and $\lambda_{\min}^{\mathbf{K}}$ denote the minimum eigenvalue of matrices $\mathbf{M}^T\mathbf{K}^G\mathbf{M}$ and $\mathbf{K}$, respectively.

It can be seen that the system found in computation C1 is prestress stable ($\lambda_{\min}^{\mathbf{M}^T\mathbf{K}^G\mathbf{M}} > 0$) but does not satisfy the general stability condition ($\lambda_{\min}^{\mathbf{K}} < 0$). This result demonstrates that a prestress stable system does not necessarily lead to a general stable system, and positive definiteness of $\mathbf{M}^T\mathbf{K}^G\mathbf{M}$ is not a sufficient condition for the stability of kinematically indeterminate structures. When the general stability condition is adopted (computation C2), the found system is a general stable system ($\lambda_{\min}^{\mathbf{K}} > 0$) and of course also a prestress stable system ($\lambda_{\min}^{\mathbf{M}^T\mathbf{K}^G\mathbf{M}} > 0$).

5.3. Example 3: Three-Dimensional Pin-Jointed Assembly. A three-dimensional pin-jointed assembly, as shown in Figure 4, is considered in this example. This system consists of 40 nodes and 132 members and has one mechanism mode and nineteen self-stress states. In this assembly, $P = \{1, 2, \ldots, 96\}$, $Q = \{97, 98, \ldots, 132\}$, and $N = \varnothing$, which means that all the members are divided into two groups: members 1–96 in tension and members 97–132 in compression.

This assembly consists of nine identical four-strut prism tensegrity units. The prism tensegrity unit has one self-stress state which can be analytically determined [9]. The analytical

solution of the unit can be extended to the whole assembly due to the symmetry of the whole system. The solution obtained based on the analytical solution of the tensegrity unit is a feasible solution and is listed in Table 3 where T_s and T_c are the number of magnitudes of compression prestress and the number of magnitudes of tension prestress, respectively. Because there is more than one self-stress state in the whole assembly, there may be other feasible prestresses besides the analytical one.

This pin-jointed assembly has been used as an example by Chen et al. [17] to search stable prestresses by using ACS algorithms. Chen et al. carried out 60 individual runs, and all of them converge to the analytical solution. The average runtime for the runs was 1.981 s. Because of the stochastic basis of ACS algorithms, the computation efficiency and the solution quality of ACS algorithms are sensitive to parameter settings such as the number of ants used [17]. Four computations using different stability conditions and objective functions are carried out in this example, as shown in Table 4 where "Yes" means that the corresponding item is adopted by the computation and "No" means that the corresponding item is not adopted by the computation. No repetitive runs are needed because the algorithm used in this paper is a deterministic algorithm. Table 5 gives the results of the four computations. $\lambda_{\min}^{\mathbf{M}^T\mathbf{K}^G\mathbf{M}}$, $\lambda_{\min}^{\mathbf{K}^G}$, and $\lambda_{\min}^{\mathbf{K}}$ denote minimum eigenvalue of matrices $\mathbf{M}^T\mathbf{K}^G\mathbf{M}$, $\mathbf{K}^G$, and $\mathbf{K}$, respectively. The runtime used by each computation is also given in the table.

When the prestress stability constraint is used (computation C1), the found system is a prestress stable system ($\lambda_{\min}^{\mathbf{M}^T\mathbf{K}^G\mathbf{M}} > 0$) but not a super stable system ($\lambda_{\min}^{\mathbf{K}^G} < 0$). When super stability constraint is used (computation C2), the minimum eigenvalue of geometrical stiffness matrix $\mathbf{K}^G$ equals to zero, which means that $\mathbf{K}^G$ is positive semidefinite, and thus, the corresponding system is a super stable system. In both solutions of computation C1 and computation C2, the numbers of prestress magnitudes in members (i.e., T_s and T_c) are large. This observation indicates that the final prestresses in members are very irregular. When objective function f_1 is used (computation C3), the found prestresses become more regular. The results of member prestresses corresponding to this computation case are listed in Table 6. It can be seen from Table 6 that the normalized solution of computation C3 is the same as the analytical result given in Table 3. The purpose of computation C4 is to maximize the minimum eigenvalue of tangent stiffness matrix $\mathbf{K}$. It can be seen that although the found prestresses in computation C4 is very irregular, the value of $\lambda_{\min}^{\mathbf{K}}$ increases a lot compared with other computations, which means that the corresponding system has higher stiffness.

It is also shown that the runtime used in each computation is less than 0.5 s. In particular, the runtime for the computation C3 which obtains the solution same to the analytical solution is 0.299 s which is considerably less than the average runtime 1.981 s needed by the ACS algorithms [17]. Noting that different computers have been used in this paper and in the previous study [17], the above comparison on the runtimes is not rigorous. But emphasis is also given to the fact that both runtime needed and solution obtained by the proposed approach are assured and reproducible.

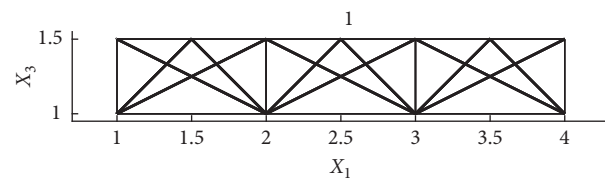TABLE 2: Results for computations carried out in example 2.

| Computations | Member prestresses | | | | | | | | | $\lambda_{min}^{M^T K^G M}$ | λ_{min}^{K} |
	1	2	3	4	5	6	7	8	9		
C1	1.000	1.000	0.200	0.200	1.000	−0.510	−0.510	−0.510	−0.510	0.8000	−0.0093
C2	0.288	0.288	0.058	0.058	0.288	−0.147	−0.147	−0.147	−0.147	0.2301	0.0077
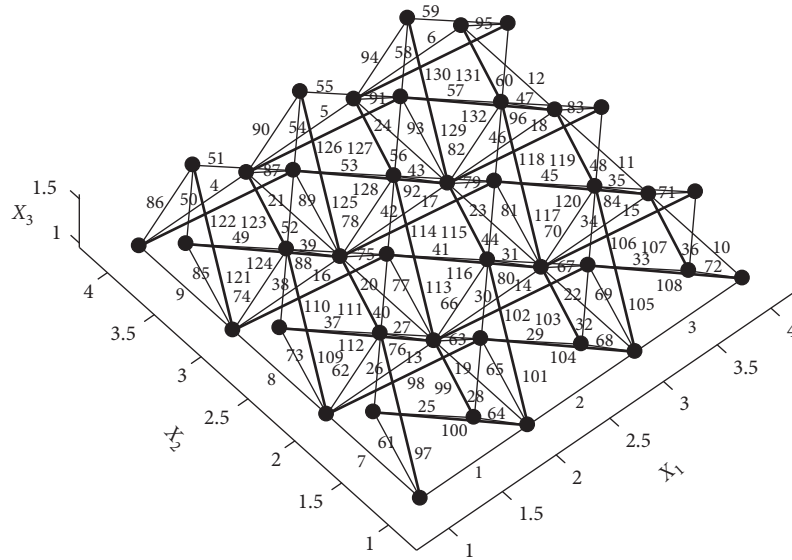
(a)
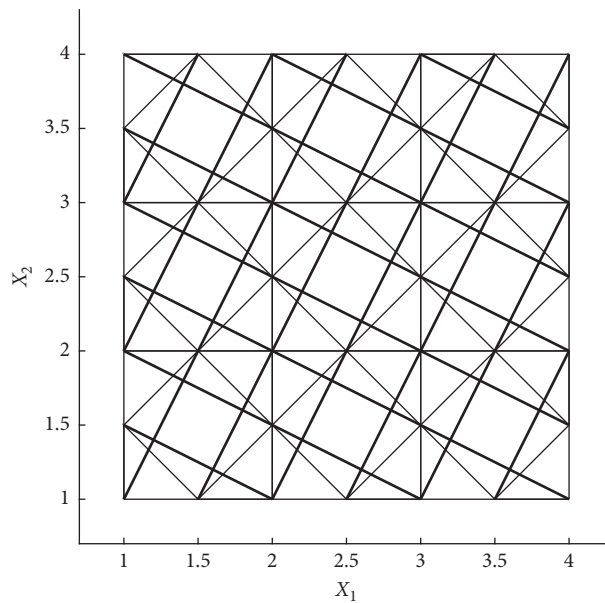
(b)

(c)

FIGURE 4: Three-dimensional pin-jointed assembly considered in example 3. (a) Perspective view. (b) Plan view. (c) Vertical view.

Table 3: Analytical result of example 3.

Member prestresses				T_s	T_c
1–12	13–24	25–96	97–132		
0.408	0.816	0.577	−1.000	1	3

Table 4: Constraints and objectives used by computations in example 3.

Compu tations	Equilibrium constraint	Stability constraint	Unilateralism constraint	Objective function
C1	Yes	Prestress stability	Yes	No
C2	Yes	Super stability	Yes	No
C3	Yes	Super stability	Yes	f_1
C4	Yes	General stability	Yes	f_2

Table 5: Results for computations carried out in example 3.

Computations	T_s	T_c	$\lambda_{min}^{M^T K^G M}$	$\lambda_{min}^{K^G}$	λ_{min}^{K}	Time (s)
C1	28	67	0.0041	−0.0068	0.0020	0.361
C2	20	41	0.0125	0.0000	0.0048	0.290
C3	1	3	0.0010	0.0000	0.0006	0.299
C4	13	43	0.0204	−0.0703	0.0071	0.476

Table 6: Result of computation C3.

	Member prestresses				T_s	T_c
	1–12	13–24	25–96	97–132		
Original result	1.000	2.000	1.414	−2.450	1	3
Normalized result	0.408	0.816	0.577	−1.000		

6. Conclusions

Finding appropriate prestresses which are able to stabilize the system is an important step in design of a tensegrity structure. In this paper, this problem has been formulated into a semidefinite programming (SDP) model. In the SDP model, different stability criteria and different objective functions can be adopted to realize specific design intentions. The proposed approach is verified by three typical examples.

As an optimization-based framework, the SDP model can be used to get more even prestress distribution and larger system stiffness. Besides, the SDP model is solvable in polynomial time, and the optimality of the solution is theoretically guaranteed. Compared to the previous heuristic and stochastic algorithm-based approaches, the approach proposed in this paper is based on a deterministic algorithm. For a given prestress design problem, the solution obtained by the proposed approach is unique and is totally reproducible. Hence, no trial or repeated computations are needed in the solving process. It is also worth noting that the SDP formulation is compatible with the MILP formulation used for topology optimization problem of tensegrity

structures. It is believed that this work will be helpful in incorporating the stability test into the topology optimization model within the frame of mathematical programming. This will be the interest of the authors' future work.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

Acknowledgments

References

[1] D. E. Ingber, "Tensegrity I. Cell structure and hierarchical systems biology," *Journal of Cell Science*, vol. 116, no. 7, pp. 1157–1173, 2003.

[2] V. Raducanu, *Architecture et système constructif: cas des systèmes de tenségrité*, Ph.D. thesis, Université de Montpellier II, Montpellier, France, 2001.

[3] A. G. Rovira and J. M. M. Tur, "Control and simulation of a tensegrity-based mobile robot," *Robotics and Autonomous Systems*, vol. 57, no. 5, pp. 526–535, 2009.

[4] R. Connelly and A. Back, "Mathematics and tensegrity," *American Scientist*, vol. 86, no. 2, pp. 142–151, 1998.

[5] R. Connelly, "Tensegrity structures: why are they stable?," in *Rigidity Theory and Applications*, pp. 47–54, Springer, New York, NY, USA, 2002.

[6] J. Y. Zhang and M. Ohsaki, "Adaptive force density method for form-finding problem of tensegrity structures," *International Journal of Solids and Structures*, vol. 43, no. 18-19, pp. 5658–5673, 2006.

[7] C. R. Calladine and S. Pellegrino, "First-order infinitesimal mechanisms," *International Journal of Solids and Structures*, vol. 27, no. 4, pp. 505–515, 1991.

[8] J. Y. Zhang and M. Ohsaki, "Stability conditions for tensegrity structures," *International Journal of Solids and Structures*, vol. 44, no. 11-12, pp. 3875–3886, 2007.

[9] H. C. Tran and J. Lee, "Initial self-stress design of tensegrity grid structures," *Computers & Structures*, vol. 88, no. 9-10, pp. 558–566, 2010.

[10] J. Zhou, W. Chen, B. Zhao, and S. Dong, "A feasible symmetric state of initial force design for cable-strut structures," *Archive of Applied Mechanics*, vol. 87, no. 8, pp. 1385–1397, 2017.

[11] M. Ohsaki, J. Y. Zhang, and Y. Ohishi, "Force design of tensegrity structures by enumeration of vertices of feasible region," *International Journal of Space Structures*, vol. 23, no. 2, pp. 117–125, 2008.

[12] M. Ohsaki, J. Zhang, and I. Elishakoff, "Multiobjective hybrid optimization-antioptimization for force design of tensegrity structures," *Journal of Applied Mechanics*, vol. 79, no. 2, article 021015, 2012.

[13] S. El-Lishani, H. Nooshin, and P. Disney, "Investigating the statical stability of pin-jointed structures using genetic algorithm," *International Journal of Space Structures*, vol. 20, no. 1, pp. 53–68, 2016.

[14] S. Lee, B.-H. Woo, and J. Lee, "Self-stress design of tensegrity grid structures using genetic algorithm," *International Journal of Mechanical Sciences*, vol. 79, pp. 38–46, 2014.

[15] X. Xu and Y. Luo, "Force finding of tensegrity systems using simulated annealing algorithm," *Journal of Structural Engineering*, vol. 136, no. 8, pp. 1027–1031, 2010.

[16] J. Y. Zhang and M. Ohsaki, "Force identification of pre-stressed pin-jointed structures," *Computers & Structures*, vol. 89, no. 23-24, pp. 2361–2368, 2011.

[17] Y. Chen, J. Feng, and Y. Wu, "Prestress stability of pin-jointed assemblies using ant colony systems," *Mechanics Research Communications*, vol. 41, pp. 30–36, 2012.

[18] Y. Chen, J. Feng, and Y. Wu, "Novel form-finding of tensegrity structures using ant colony systems," *Journal of Mechanisms and Robotics*, vol. 4, no. 3, article 031001, 2012.

[19] W. Hare, J. Nutini, and S. Tesfamariam, "A survey of non-gradient optimization methods in structural engineering," *Advances in Engineering Software*, vol. 59, pp. 19–28, 2013.

[20] S. Ehara and Y. Kanno, "Topology design of tensegrity structures via mixed integer programming," *International Journal of Solids and Structures*, vol. 47, no. 5, pp. 571–579, 2010.

[21] Y. Kanno, "Exploring new tensegrity structures via mixed integer programming," *Structural and Multidisciplinary Optimization*, vol. 48, no. 1, pp. 95–114, 2013.

[22] X. Xu, Y. Wang, and Y. Luo, "General approach for topology-finding of tensegrity structures," *Journal of Structural Engineering*, vol. 142, no. 10, article 04016061, 2016.

[23] X. Xu, Y. Wang, and Y. Luo, "An improved multi-objective topology optimization approach for tensegrity structures," *Advances in Structural Engineering*, vol. 21, no. 1, pp. 59–70, 2017.

[24] M. F. Anjos and J. B. Lasserre, *Handbook on Semidefinite, Conic and Polynomial Optimization*, vol. 166 of International Series in Operations Research & Management Science, Springer, New York, NY, USA, 2012.

[25] A. M. C. So and Y. Ye, "A semidefinite programming approach to tensegrity theory and realizability of graphs," in *Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithm*, pp. 766–775, Miami, FL, USA, January 2006.

[26] A. Kaveh, "Structural mechanics: graph and matrix methods," *Electronic & Electrical Engineering Research Studies*, Macmillan International Higher Education, London, UK, 1992.

[27] N. Vassart and R. Motro, "Multiparametered form finding method: application to tensegrity systems," *International Journal of Space Structures*, vol. 14, no. 2, pp. 147–154, 2009.

[28] S. Pellegrino and C. R. Calladine, "Matrix analysis of statically and kinematically indeterminate frameworks," *International Journal of Solids and Structures*, vol. 22, no. 4, pp. 409–428, 1986.

[29] Y. Chen, J. Feng, H. Lv, and Q. Sun, "Symmetry representations and elastic redundancy for members of tensegrity structures," *Composite Structures*, vol. 203, pp. 672–680, 2018.

[30] S. Guest, "The stiffness of prestressed frameworks: a unifying approach," *International Journal of Solids and Structures*, vol. 43, no. 3, pp. 842–854, 2006.

[31] Y. Chen and J. Feng, "Generalized eigenvalue analysis of symmetric prestressed structures using group theory," *Journal of Computing in Civil Engineering*, vol. 26, no. 4, pp. 488–497, 2012.

[32] S. D. Guest, "The stiffness of tensegrity structures," *IMA Journal of Applied Mathematics*, vol. 76, no. 1, pp. 57–66, 2010.

[33] J. M. T. Thompson, "A general theory for the equilibrium and stability of discrete conservative systems," *Zeitschrift für Angewandte Mathematik und Physik ZAMP*, vol. 20, no. 6, pp. 797–846, 1969.

[34] C. Sultan and Cornel, "Stiffness formulations and necessary and sufficient conditions for exponential stability of pre-stressable structures," *International Journal of Solids and Structures*, vol. 50, no. 14-15, pp. 2180–2195, 2013.

[35] Y. Chen, Q. Sun, and J. Feng, "Stiffness degradation of pre-stressed cable-strut structures observed from variations of lower frequencies," *Acta Mechanica*, vol. 229, no. 8, pp. 3319–3332, 2018.

[36] M. Ohsaki and J. Zhang, "Stability conditions of prestressed pin-jointed structures," *International Journal of Non-Linear Mechanics*, vol. 41, no. 10, pp. 1109–1117, 2006.

[37] M. Ohsaki, K. Fujisawa, N. Katoh, and Y. Kanno, "Semi-definite programming for topology optimization of trusses under multiple eigenvalue constraints," *Computer Methods in Applied Mechanics and Engineering*, vol. 180, no. 1-2, pp. 203–217, 1999.

[38] A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2001.

[39] R. Laraki and J. B. Lasserre, "Semidefinite programming for min-max problems and games," *Mathematical Programming*, vol. 131, no. 1-2, pp. 305–332, 2010.

[40] M. Kojima, N. Megiddo, and S. Mizuno, "A primal-dual infeasible-interior-point algorithm for linear programming," *Mathematical Programming*, vol. 61, no. 1–3, pp. 263–280, 1993.

[41] M. Kojima, S. Shindoh, and S. Hara, "Interior-point methods for the monotone semidefinite linear complementarity problem in symmetric matrices," *SIAM Journal on Optimization*, vol. 7, no. 1, pp. 86–125, 1997.

[42] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optimization Methods and Software*, vol. 11, no. 1–4, pp. 625–653, 1999.