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Research Article

Explicit Form of Exact Analytical Solution for Calculating Ground Displacement and Stress Induced by Shallow Tunneling and Its Application

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In urban environment, it is often unavoidable for shallow tunnels to be constructed adjacent to existing pile foundations. To obtain the ground displacements and stresses induced by shallow tunneling and existing pile foundation loads, the key procedure involves superimposing the analytical solution for shallow tunneling in green-field with the analytical solution for existing structure loads. In green-field, the complex variable method provides exact analytical solutions of ground displacements and stresses caused by shallow tunneling. However, the exact analytical solutions are not directly expressed as explicit functions of the coordinates (x, y) in the physical plane (called implicit form of exact analytical solutions), whereas the displacements and stresses induced by existing structure loads are explicit functions of the coordinates (x, y) in the physical plane, which makes it difficult to superpose the displacements and stresses induced by existing structure loads. In this paper, explicit form of exact analytical solutions of displacements and stresses induced by shallow tunneling in green-field is obtained by using the inverse conformal transformation and the Cauchy–Riemann equations. Comparison with implicit form of exact analytical solutions shows that the explicit form of exact analytical solutions is intuitional and easily used by engineers, and moreover, the calculation amount is much smaller than that for the implicit form of exact analytical solutions. Then, an application involving superimposing the explicit form of exact analytical solutions with Mindlin's solution is implemented to analyze the secondary stress field and the related potential plastic zone caused by shallow tunneling adjacent to pile foundations. Moreover, the influences of pile foundation parameters on the ranges and shapes of the potential plastic zones induced by nearby tunneling are analyzed.

1. Introduction

Due to continuous expansion of cities and decreases in available land, the demand for urban tunnels (roads or metro) has increased sharply in recent years. At the same time, the foundations of nearby existing structures are unavoidably disturbed by this process and affect it, especially in densely constructed areas located adjacent to the tunnel construction site. The prediction and mitigation of damage caused by construction-induced ground movement represents a major factor in the design of tunnels. This is an especially important problem for shallow tunnels excavated in soft soils, where expensive remedial measures such as compensation grouting or structural underpinning may need to be considered prior to construction. To obtain the ground displacements and stresses induced by shallow tunneling and existing structure loads, the key procedure involves superimposing the analytical solution for tunneling in green-field with the analytical solution for existing structure loads.

To estimate the ground displacements and stresses caused by shallow tunneling in green-field, a number of empirical and analytical solutions have been developed. In engineering practice, these ground surface settlements are often described by Peck's empirical formula [1, 2], which provides a Gaussian distribution curve for ground settlement and is based on field observations and intuitive deductions, thus it is lacking in theory and ambiguous in its range of applicability. In addition, there are four main analytical methods, namely, the virtual image technique [3–5], the complex variable method [6, 7], the general series form stress function in polar coordinates [8–11], and the stochastic medium theory [12]. Pinto and Whittle [13] and Xiang [14] presented detailed reviews and comparisons of the empirical and analytical solutions of ground displacements and stresses for shallow tunnels in green-field. However, those solutions mentioned above are in general all based upon the premise that the ground is free from existing structure loads, and they cannot directly predict ground displacements and stresses induced by the combination of shallow tunneling and existing structure loads.

For the condition of shallow tunneling adjacent to an existing pile foundation, Xiang and Feng [15] proposed a superposition method for predicting the potential plastic zone of shallow tunneling adjacent to a pile foundation in soils. Although their results achieve good effects, there is still one approximation. According to Verruijt [6] and Pinto and Whittle [13], the tunneling-induced stresses adopted by Xiang and Feng [15] are an approximate solution that implicitly ignores the finite dimensions of the tunnel itself, and the results from approximate solution would lead to certain differences, especially for tunnels too close to the ground surface. This approximation would make the superposition method proposed by Xiang and Feng [15] unsuitable for tunnels too close to the ground surface. Referring to Xiang [14] and Pinto and Whittle [13], the complex variable method not only provides exact analytical solutions of ground displacements and stresses caused by shallow tunneling in green-field but also makes the exact analytical solutions suitable for various types of boundary conditions. Therefore, to obtain more accurate results for secondary stress field and the related potential plastic zone, the exact analytical solutions based on the complex variable method are used for the superposition method in this paper. However, there is still a problem that exact analytical solutions are not directly expressed as explicit functions of the coordinates (x, y) in the physical plane (called implicit form of exact analytical solutions), whereas the displacements and stresses induced by existing structure loads are explicit functions of the coordinates (x, y) in the physical plane, which makes it difficult to superpose the displacements and stresses induced by existing structure loads.

The present paper extends and partially revises past research work [16]. The implicit form of exact analytical solutions, which are based on the complex variable method, is derived into explicit form of exact analytical solutions by using the inverse conformal transformation and the Cauchy–Riemann equations. Then, an application is conducted to analyze the secondary stress field and the related potential plastic zone caused by shallow tunneling adjacent to pile foundations. Moreover, the influence of pile foundation parameters (pile length, load magnitude, and pile offsets) on the ranges and shapes of the potential plastic zones induced by nearby tunneling is analyzed.

2. Explicit Form of Exact Analytical Solutions of Ground Displacements and Stresses Induced by Shallow Tunneling

2.1. Implicit Form of Exact Analytical Solutions. The problem considers an elastic half-plane with a circular tunnel (Figure 1) [6]. The radius of the tunnel is expressed by r, the depth of its center below the free surface by h, and the cover by d. The ground surface boundary is free of stress and the boundary of the tunnel undergoes a given distribution of displacements (for instance, two typical boundary conditions: uniform radial displacement (the ground loss problem) u_0 and ovalization u_{d}). In the complex variable method used by Verruijt [6], it is assumed that the original domain in the z-plane (physical plane) is mapped conformally onto an annular region bounded by the circles $|\zeta| = 1$ and $|\zeta| = \alpha$, where $\alpha < 1$, on the auxiliary domain in the ζ-plane (mapped plane) by the following conformal transformation:

$$z = \omega(\zeta) = -\mathrm{ih}\frac{1-\alpha^2}{1+\alpha^2}\frac{1+\zeta}{1-\zeta} = -\mathrm{ia}\frac{1+\zeta}{1-\zeta}, \qquad (1)$$

where α is given by

$$\alpha = \frac{h}{r} - \sqrt{\left(\frac{h}{r}\right)^2 - 1}. \qquad (2)$$

In z-plane, the solutions are expressed in terms of two analytic functions $\varphi_1(z)$ and $\psi_1(z)$. The stresses and displacements are related to these functions as follows:

$$\sigma_{xx} + \sigma_{yy} = 2\left[\varphi_1'(z) + \overline{\varphi_1'(z)}\right] = 4\mathrm{Re}\left[\varphi_1'(z)\right], \qquad (3a)$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2\left[\bar{z}\varphi_1''(z) + \psi_1'(z)\right], \qquad (3b)$$

$$2\mu\left(u_x + iu_y\right) = \kappa\varphi_1(z) - z\overline{\varphi_1'(z)} - \overline{\psi_1(z)}, \qquad (3c)$$

where μ is the shear modulus of the elastic material, κ is related to Poisson's ratio ν by $\kappa = 3 - 4\nu$, i is the imaginary constant, and overbar is a complex conjugate.

By virtue of the conformal transformation function $\omega(\zeta)$, the functions $\varphi_1(z)$ and $\psi_1(z)$ can be considered as functions of ζ as follows:

$$\varphi_1(z) = \varphi_1(\omega(\zeta)) = \varphi(\zeta), \qquad (4a)$$

$$\psi_1(z) = \psi_1(\omega(\zeta)) = \psi(\zeta). \qquad (4b)$$

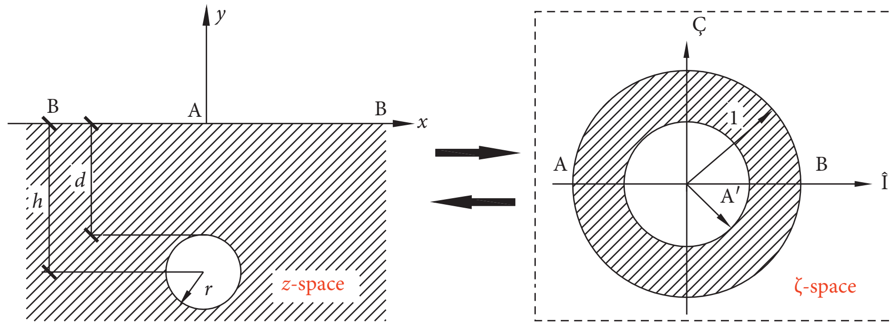


FIGURE 1: Conformal transformation for shallow tunnel, reproduced from [6].

As the functions $\varphi(\zeta)$ and $\psi(\zeta)$ are analytical, they can be expanded in Laurent series in the ζ-space as follows:

$$\varphi(\zeta) = a_0 + \sum_{k=1}^{\infty} a_k \zeta^k + \sum_{k=1}^{\infty} b_k \zeta^{-k}, \quad (5a)$$

$$\psi(\zeta) = c_0 + \sum_{k=1}^{\infty} c_k \zeta^k + \sum_{k=1}^{\infty} d_k \zeta^{-k}, \quad (5b)$$

where the coefficients a_0, a_k, b_k, c_0, c_k, and d_k are determined with recursive relations derived from the boundary conditions. First, the coefficients a_0, a_k, and b_k are found with the specific displacement boundary condition at the boundary of the tunnel (for instance, convergence and ovalization) by using related coefficients A_k in Fourier series terms.

$$\left(\left(1-\alpha^2\right)\overline{a}_1\right) - \left(\left(\kappa+\alpha^2\right)b_1\right) = A_0 - \left((\kappa+1)a_0\right), \quad (6a)$$

$$\left(\left(1+\kappa\alpha^2\right)\overline{a}_1\right) - \left(\left(1-\alpha^2\right)b_1\right) = \overline{A}_1\alpha + \left((\kappa+1)\alpha^2\overline{a}_0\right), \quad (6b)$$

$$\left(\left(1-\alpha^2\right)(k+1)\overline{a}_{k+1}\right) - \left(\left(\alpha^2+\kappa\alpha^{-2k}\right)b_{k+1}\right) = \left(\left(1-\alpha^2\right)k\overline{a}_k\right) - \left(\left(1+\kappa\alpha^{-2k}\right)b_k\right) + A_{-k}\alpha^{-k}, \quad k = 1, 2, \ldots, \quad (6c)$$

$$\left(1+\kappa\alpha^{2k+2}\right)\overline{a}_{k+1} + \left(1-\alpha^2\right)(k+1)b_{k+1} = \alpha^2\left(1+\kappa\alpha^{2k}\right)\overline{a}_k + \left(1-\alpha^2\right)kb_k + \overline{A}_{k+1}\alpha^{k+1}, \quad k = 1, 2, \ldots, \quad (6d)$$

where the related coefficients A_k are in Fourier series terms (Appendix A).

Second, the coefficients c_0, c_k, and d_k can be obtained with stress-free boundary condition at the ground surface by using the following recursive relations:

$$c_0 = -\overline{a}_0 - \frac{1}{2}a_1 - \frac{1}{2}b_1, \quad (7a)$$

$$c_k = -\overline{b}_k + \frac{1}{2}(k-1)a_{k-1} - \frac{1}{2}(k+1)a_{k+1}, \quad (7b)$$

$$d_k = -\overline{a}_k + \frac{1}{2}(k-1)b_{k-1} - \frac{1}{2}(k+1)b_{k+1}. \quad (7c)$$

The analytical solutions of ground displacements and stresses induced by shallow tunneling are obtained with the analytic functions (equations (3a)–(3c)). Moreover, Pinto [17] claims that 10–15 terms of analytic functions (equations (3a)–(3c)) are sufficient to achieve accurate solutions for both the convergence and ovalization modes of deformation.

However, the solutions proposed by Verruijt [6] (equations (3a)–(3c)) are expressed as two analytical functions ($\varphi_1(z)$ and $\psi_1(z)$) and are not directly expressed as explicit functions of the coordinates (x, y) in z-space (referred to as "implicit form of exact analytical solutions"), which makes the extensive application of implicit exact analytical solutions inconvenient. On the one hand, implicit form of exact analytical solutions is somewhat complicated and not intuitive for engineers. Implicit form of exact analytical solutions cannot be directly used by engineers like Peck empirical formulas if they have no certain professional theoretical knowledge of complex variable method. On the other hand, the superposition applicability of implicit form of exact analytical solutions is poor when solving the ground displacements and stresses induced by shallow tunneling adjacent to existing structures (surface building or pile foundation).

2.2. Explicit Form of Exact Analytical Solution. In this section, the main work is to derive the explicit form of exact analytical solution of the stress and the displacement expressed by the coordinates (x, y) in z-space (physical plane).

2.2.1. Series Form of $\varphi_1(z)$ and $\psi_1(z)$ with the Functions of Coordinate (x, y) in z-Space. The corresponding inverse conformal transformation of equation (1) is

$$\zeta = \omega^{-1}(z) = \frac{z+ia}{z-ia}. \quad (8)$$

Considering $z = x + iy$, equation (8) can be transformed into

$$\zeta = \frac{x^2+y^2-a^2+2xai}{x^2+(y-a)^2} = \xi + \eta i, \quad (9)$$

where

$$\xi = \frac{x^2 + y^2 - a^2}{x^2 + (y-a)^2}, \quad \eta = \frac{2xa}{x^2 + (y-a)^2}. \tag{10}$$

To obtain the power of ζ (i.e., ζ^k and ζ^{-k}), it is better to transform equation (9) into the triangle form of ζ as follows:

$$\zeta = \left(\sqrt{\xi^2+\eta^2}\right)\left[\cos\left(\arccos\frac{\xi}{\sqrt{\xi^2+\eta^2}}\right) + i\sin\left(\arccos\frac{\xi}{\sqrt{\xi^2+\eta^2}}\right)\right]. \tag{11}$$

According to the related operation rules of the power of complex numbers,

$$\zeta^k = \left(\sqrt{\xi^2+\eta^2}\right)^k\left[\cos\left(k\arccos\frac{\xi}{\sqrt{\xi^2+\eta^2}}\right) + i\sin\left(k\arccos\frac{\xi}{\sqrt{\xi^2+\eta^2}}\right)\right], \tag{12a}$$

$$\zeta^{-k} = \left(\sqrt{\xi^2+\eta^2}\right)^{-k}\left[\cos\left(-k\arccos\frac{\xi}{\sqrt{\xi^2+\eta^2}}\right) + i\sin\left(-k\arccos\frac{\xi}{\sqrt{\xi^2+\eta^2}}\right)\right]. \tag{12b}$$

On the basis of a consideration of symmetry, Verruijt [6] assumed that all the coefficients are purely imaginary:

$$a_0 = a_0' i, \quad a_k = a_k' i, \quad b_k = b_k' i, \quad c_0 = c_0' i, \quad c_k = c_k' i, \quad d_k = d_k' i, \tag{13}$$

where $a_0', a_k', b_k', c_0', c_k'$, and d_k' are the real parts of purely imaginary a_0, a_k, b_k, c_0, c_k, and d_k, respectively.

With equations (12a), (12b), and (13), equations (5a) and (5b) can be transformed into

$$\begin{aligned}\varphi(\zeta) &= a_0 + \sum_{k=1}^{\infty} a_k\zeta^k + \sum_{k=1}^{\infty} b_k\zeta^{-k} \\ &= a_0' i + \sum_{k=1}^{\infty} a_k' i\left(\sqrt{\xi^2+\eta^2}\right)^k\left[\cos\left(k\arccos\frac{\xi}{\sqrt{\xi^2+\eta^2}}\right) + i\sin\left(k\arccos\frac{\xi}{\sqrt{\xi^2+\eta^2}}\right)\right] + \sum_{k=1}^{\infty} b_k' i_k\left(\sqrt{\xi^2+\eta^2}\right)^{-k} \\ &\quad \cdot\left[\cos\left(-k\arccos\frac{\xi}{\sqrt{\xi^2+\eta^2}}\right) + i\sin\left(-k\arccos\frac{\xi}{\sqrt{\xi^2+\eta^2}}\right)\right],\end{aligned} \tag{14a}$$

$$\begin{aligned}\psi(\zeta) &= c_0 + \sum_{k=1}^{\infty} c_k\zeta^k + \sum_{k=1}^{\infty} d_k\zeta^{-k} \\ &= c_0' i + \sum_{k=1}^{\infty} c_k' i\left(\sqrt{\xi^2+\eta^2}\right)^k\left[\cos\left(k\arccos\frac{U}{\sqrt{\xi^2+\eta^2}}\right) + i\sin\left(k\arccos\frac{U}{\sqrt{\xi^2+\eta^2}}\right)\right] + \sum_{k=1}^{\infty} d_k' i\left(\sqrt{\xi^2+\eta^2}\right)^{-k} \\ &\quad \cdot\left[\cos\left(-k\arccos\frac{U}{\sqrt{\xi^2+\eta^2}}\right) + i\sin\left(-k\arccos\frac{U}{\sqrt{\xi^2+\eta^2}}\right)\right].\end{aligned} \tag{14b}$$

Equations (14a) and (14b) can be simplified to the form of real and imaginary parts of $\varphi_1(\zeta)$ and $\psi_1(\zeta)$ as follows:

$$\varphi(\zeta) = \left(\left(\left(a_0' + \sum_{k=1}^{\infty} a_k' W_1 + \sum_{k=1}^{\infty} b_k' W_2\right)i\right) - \left(\sum_{k=1}^{\infty} a_k' W_3 + \sum_{k=1}^{\infty} b_k' W_4\right)\right), \tag{15a}$$

$$\psi(\zeta) = \left(\left(\left(c_0' + \sum_{k=1}^{\infty} c_k' W_1 + \sum_{k=1}^{\infty} d_k' W_2\right)i\right) - \left(\sum_{k=1}^{\infty} c_k' W_3 + \sum_{k=1}^{\infty} d_k' W_4\right)\right), \tag{15b}$$

where

$$W_1 = \left[\frac{x^2+(y+a)^2}{x^2+(y-a)^2}\right]^{k/2}\cos\left[k\,\arccos\left(\frac{x^2+y^2-a^2}{\sqrt{x^4+y^4+a^4+2a^2x^2+2x^2y^2-2a^2y^2}}\right)\right],$$

$$W_2 = \left(\frac{x^2+(y+a)^2}{x^2+(y-a)^2}\right)^{-(k/2)}\cos\left[(-k)\arccos\left(\frac{x^2+y^2-a^2}{\sqrt{x^4+y^4+a^4+2a^2x^2+2x^2y^2-2a^2y^2}}\right)\right],$$

$$W_3 = \left(\frac{x^2+(y+a)^2}{x^2+(y-a)^2}\right)^{k/2}\sin\left[k\,\arccos\left(\frac{x^2+y^2-a^2}{\sqrt{x^4+y^4+a^4+2a^2x^2+2x^2y^2-2a^2y^2}}\right)\right],$$

$$W_4 = \left(\frac{x^2+(y+a)^2}{x^2+(y-a)^2}\right)^{-(k/2)}\sin\left[(-k)\arccos\left(\frac{x^2+y^2-a^2}{\sqrt{x^4+y^4+a^4+2a^2x^2+2x^2y^2-2a^2y^2}}\right)\right]. \quad (16)$$

With equations (4a) and (4b), equations (15a) and (15b) can be written into equations (17a) and (17b):

$$\varphi_1(z) = \left(\left(\left(a_0' + \sum_{k=1}^{\infty} a_k' W_1 + \sum_{k=1}^{\infty} b_k' W_2\right)i\right) - \left(\sum_{k=1}^{\infty} a_k' W_3 + \sum_{k=1}^{\infty} b_k' W_4\right)\right), \quad (17a)$$

$$\psi_1(z) = \left(\left(\left(c_0' + \sum_{k=1}^{\infty} c_k' W_1 + \sum_{k=1}^{\infty} d_k' W_2\right)i\right) - \left(\sum_{k=1}^{\infty} c_k' W_3 + \sum_{k=1}^{\infty} d_k' W_4\right)\right). \quad (17b)$$

2.2.2. Derivations of Analytic Functions Based on the Cauchy–Riemann Equations. if the function $[f(z) = u(x,y) + iv(x,y)]$ is analytical, the derivation of analytical function can be calculated with

$$f'(z) = \frac{\partial u}{\partial x} + \left(i\frac{\partial v}{\partial x}\right) = \frac{\partial v}{\partial y} - \left(i\frac{\partial u}{\partial y}\right). \quad (18)$$

Because $\varphi_1(z)$ and $\psi_1(z)$ are analytical, the derivations of $\varphi_1(z)$ and $\psi_1(z)$ can be obtained:

$$\varphi_1'(z) = \left(\left(\left(\sum_{k=1}^{\infty} a_k'\frac{\partial}{\partial x}W_1 + \sum_{k=1}^{\infty} b_k'\frac{\partial}{\partial x}W_2\right)i\right) - \left(\sum_{k=1}^{\infty} a_k'\frac{\partial}{\partial x}W_3 + \sum_{k=1}^{\infty} b_k'\frac{\partial}{\partial x}W_4\right)\right), \quad (19a)$$

$$\psi_1'(z) = \left(\left(\left(\sum_{k=1}^{\infty} c_k'\frac{\partial}{\partial x}W_1 + \sum_{k=1}^{\infty} d_k'\frac{\partial}{\partial x}W_2\right)i\right) - \left(\sum_{k=1}^{\infty} c_k'\frac{\partial}{\partial x}W_3 + \sum_{k=1}^{\infty} d_k'\frac{\partial}{\partial x}W_4\right)\right), \quad (19b)$$

$$\varphi_1''(z) = \left(\left(\left(\sum_{k=1}^{\infty} a_k'\frac{\partial^2}{\partial x^2}W_1 + \sum_{k=1}^{\infty} b_k'\frac{\partial^2}{\partial x^2}W_2\right)i\right) - \left(\sum_{k=1}^{\infty} a_k'\frac{\partial^2}{\partial x^2}W_3 + \sum_{k=1}^{\infty} b_k'\frac{\partial^2}{\partial x^2}W_4\right)\right). \quad (19c)$$

2.2.3. Explicit Analytical Solutions of Ground Displacements and Stresses. By taking equations (19a)–(19c) into equations (3a)–(3c), the explicit exact analytical solutions of ground displacements and stresses can be obtained:

$$\sigma_{xx}^{\mathrm{ST}} = \sum_{k=1}^{\infty}\left[\left((-2a_k' + c_k')\frac{\partial}{\partial x}W_3\right) + \left((-2b_k' + d_k')\frac{\partial}{\partial x}W_4\right) - \left(y\left(a_k'\frac{\partial^2}{\partial x^2}W_1 + b_k'\frac{\partial^2}{\partial x^2}W_2\right)\right) + x\left(a_k'\frac{\partial^2}{\partial x^2}W_3 + b_k'\frac{\partial^2}{\partial x^2}W_4\right)\right], \quad (20a)$$

$$\sigma_{yy}^{\mathrm{ST}} = \sum_{k=1}^{\infty}\left[\left(-(2a_k' + c_k')\frac{\partial}{\partial x}W_3\right) - \left((2b_k' + d_k')\frac{\partial}{\partial x}W_4\right) + y\left(a_k'\frac{\partial^2}{\partial x^2}W_1 + b_k'\frac{\partial^2}{\partial x^2}W_2\right) - x\left(a_k'\frac{\partial^2}{\partial x^2}W_3 + b_k'\frac{\partial^2}{\partial x^2}W_4\right)\right], \quad (20b)$$

$$\sigma_{xy}^{\mathrm{ST}} = \sum_{k=1}^{\infty}\left[\left(c_k'\frac{\partial}{\partial x}W_1\right) + \left(d_k'\frac{\partial}{\partial x}W_2\right) + x\left(a_k'\frac{\partial^2}{\partial x^2}W_1 + b_k'\frac{\partial^2}{\partial x^2}W_2\right) + y\left(a_k'\frac{\partial^2}{\partial x^2}W_3 + b_k'\frac{\partial^2}{\partial x^2}W_4\right)\right], \quad (20c)$$

$$u_x^{ST} = (2\mu)^{-1}\left\{\sum_{k=1}^{\infty}\left[\left((-\kappa a_k' + c_k')W_3\right) + \left((-\kappa b_k' + d')_k W_4\right) + x\left(a_k'\frac{\partial}{\partial x}W_3 + b_k'\frac{\partial}{\partial x}W_4\right) - y\left(a_k'\frac{\partial}{\partial x}W_1 + b_k'\frac{\partial}{\partial x}W_2\right)\right]\right\}. \tag{20d}$$

$$u_y^{ST} = (2\mu)^{-1}\left\{(\kappa a_0' + c')_0 + \sum_{k=1}^{\infty}\left[(\kappa a_k' + c_k')W_1 + (\kappa b_k' + d_k')W_2 + x\left(a_k'\frac{\partial}{\partial x}W_1 + b_k'\frac{\partial}{\partial x}W_2\right) + y\left(a_k'\frac{\partial}{\partial x}W_3 + b_k'\frac{\partial}{\partial x}W_4\right)\right]\right\}. \tag{20e}$$

The explicit form of exact analytical solutions (equations (20a)–(20e)) relates to only the original domain in the z-plane and is directly expressed as functions of coordinates (x, y) in z-space. The superscript ST indicates that the reason of induced ground displacements and stresses is shallow tunneling.

3. Application of Explicit Form of Exact Analytical Solutions: Secondary Stress Field and Potential Plastic Zone of Shallow Tunneling Adjacent to a Pile Foundation

3.1. Conceptual Model and Calculation Procedure. To predict the degree and extent of tunneling effects on a pile foundation, Xiang and Feng [15] proposed a theoretical superposition method for predicting the potential plastic zone of shallow tunneling adjacent to a pile foundation in soils. The practical problem of a shallow tunneling project adjacent to a pile foundation is simplified into the mechanics model shown in Figure 2.

In this paper, to obtain more accurate results for the secondary stress field and the related potential plastic zone, a similar calculation procedure is adopted. The differences between the calculation procedure for the superposition method proposed by Xiang and Feng [15] and that for the new method used in this paper are detailed below. It is worth noting that Mindlin's solution, which is used to calculate the ground stresses due to pile foundation loads, is an exact solution and is preserved in the new superposition method. The cause of the inaccuracy of the superposition method proposed by Xiang and Feng [15] is mainly due to the approximate solution [5] that is used to calculate the ground displacements induced by shallow tunneling in green-field. Therefore, the improvement of this paper is to adopt an exact solution (explicit form of exact analytical solutions) for calculating tunneling-induced stress instead of the approximate solution used by Xiang and Feng [15].

Through the above explicit derivation, the explicit form of exact analytical solutions are expressed as functions of the coordinates (x, y), which can be directly superimposed with the stress induced by pile foundation loads. By superimposing these several equations, the envelope of the potential plastic zone induced by tunneling adjacent to the pile is easily obtained, which highlights the convenience of explicit form of exact analytical solutions. Then, by using the software MATLAB, the theoretical procedure described above is conducted successfully.

Referring to Xiang and Feng [15], the assumed parameters in all the presented calculations are as follows: silt clayey soil ($E = 10$ MPa, $\nu = 0.25$, $c = 30$ kPa, and $\varphi = 30$); a circular tunnel $R = 3$ m, $h = 8$ m, 10 m, uniform convergence $u_0 = 30$ mm, oval deformation $u_d = 10$ mm, $s = 78$ kN/m, and $P = 235$ kN.

3.2. Comparison with the Results of Xiang and Feng. To detect any differences in the plastic zone in green-field or plastic zone with pile load calculated by the different methods, a comparison between the results of this paper and those of Xiang and Feng [15] is performed for two relative tunnel depths (the ratio of cover depth to tunnel diameter) (Figures 3 and 4).

In Figure 3, comparisons of the potential plastic zones in green-field from the two superposition methods are presented for two relative tunnel depths. It is shown that the horizontal range between the two plastic zones in green-field with a relative depth of 0.83 differs by 38.7 cm in Figure 3(a), whereas the horizontal range with a relative depth of 1.17 differs by 27.8 cm in Figure 3(b).

In Figure 4, comparisons of the potential plastic zones with pile load from the two superposition methods are presented for two relative tunnel depths. There exists a more marked difference between the ranges and shapes of the two plastic zones with pile load with a relative depth of 0.83 in Figure 4(a). The difference between the range and shape of two plastic zones with pile load is greater with a relative depth of 0.83 than with a relative depth of 1.17 in Figure 4(b).

As a conclusion, the plastic zones around relatively deep tunnels adjacent to a pile foundation obtained from the two superposition methods are similar to each other, whereas the plastic zones around relatively shallow tunnels adjacent to a pile foundation from the two superposition methods are obviously different from each other. In other words, the superposition method proposed by Xiang and Feng [15] can obtain satisfactory results for deep tunnels, whereas the superposition method used in this paper is suitable for both deep tunnels and shallow tunnels because it is an exact analytical solution.

3.3. Influences of Different Pile Foundation Parameters. The influences of different pile foundation parameters (pile length, load magnitude, and pile offsets) on the ranges and shapes of the potential plastic zones induced by nearby tunneling are analyzed for a relative depth of 1.17 in Figure 5. The results indicate that the whole potential plastic zones induced by tunneling and pile foundation loads would coalesce when the pile is located close enough to the tunnel, whereas the two plastic zones are separated from each other.

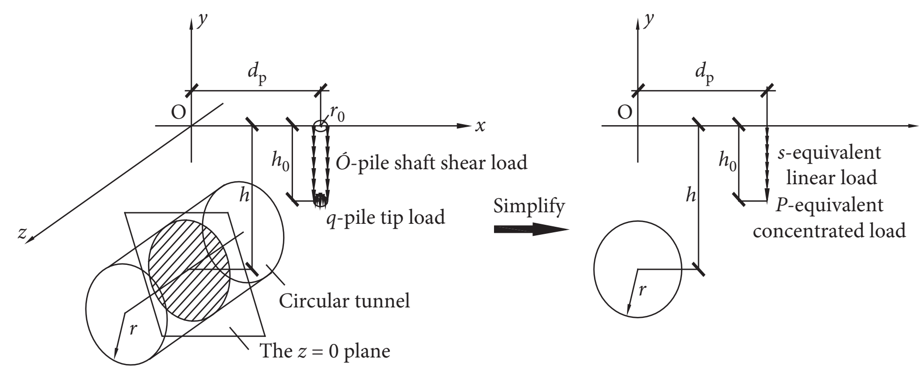


Figure 2: Conceptual model of the theoretical procedure, reproduced from [15].

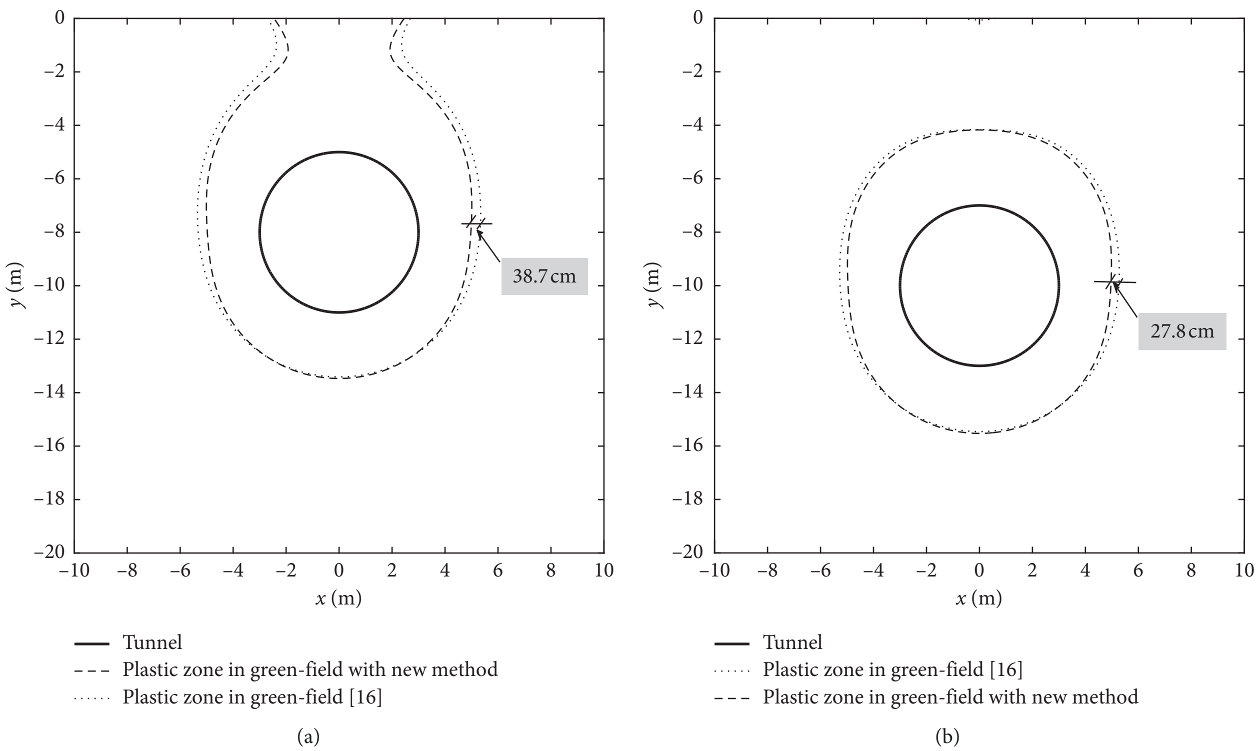


Figure 3: Comparisons of potential plastic zones in green-field. (a) Relative depth = 0.83. (b) Relative depth = 1.17.

3.4. Influences of Different Tunnel Boundary Conditions. It should be noted that the tunnel boundary conditions determine the calculation results of the ranges and shapes of the potential plastic zones caused by shallow tunneling adjacent to a pile foundation in soils. To compare the calculation results with those by Xiang and Feng [15] under the same conditions, the same boundary conditions of a shallow tunnel are adopted, which are a combination of uniform convergence and ovalization proposed by Verruijt and Booker [5] and shown in Figure 6. Pinto and Whittle [13] summarized three types of shallow tunnel boundary conditions. They claimed that the vertical translation of a shallow tunnel should be incorporated with uniform convergence and ovalization when considering the buoyancy effect, usually induced because the weight of the tunnel is usually less than the weight of the excavated soil. And by incorporating the vertical translation with the uniform convergence and ovalization of the tunnel, Park [10] introduced four types of complex boundary conditions, and those complex boundary conditions are more in accordance with engineering practice, as shown in Figures 7 and 8. In fact, the boundary condition B.C.-1 is the same as the boundary condition discussed by Verruijt and Booker [5], whereas the boundary conditions B.C.-2, B.C.-3, and B.C.-4 are different and contain vertical translation of a shallow tunnel.

As mentioned above, the part of the superposition method proposed by Xiang and Feng [15] that is used to calculate the ground displacement induced by tunneling in green-field adopts the approximate solution proposed by Verruijt and Booker [5]; thus, the superposition method proposed by Xiang and Feng [15] can adopt only the

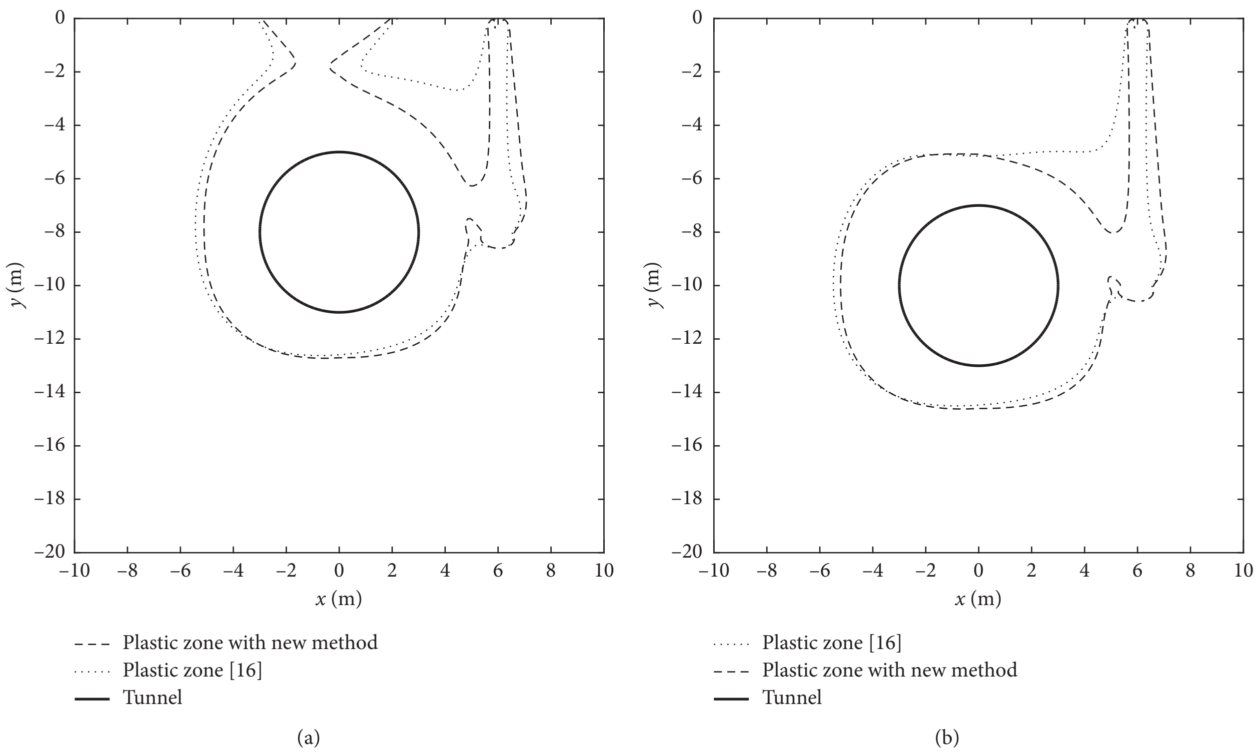


FIGURE 4: Comparisons of potential plastic zones with pile load. (a) Relative depth = 0.83. (b) Relative depth = 1.17.

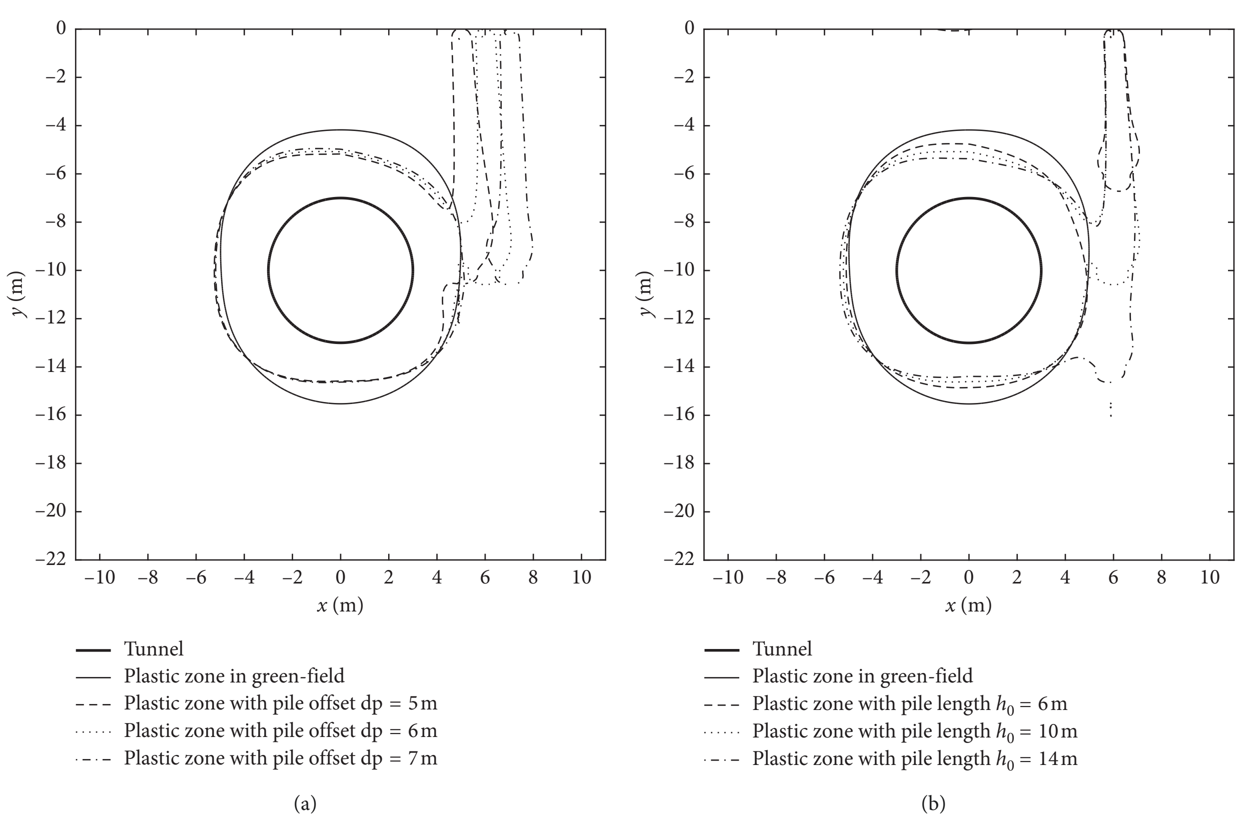
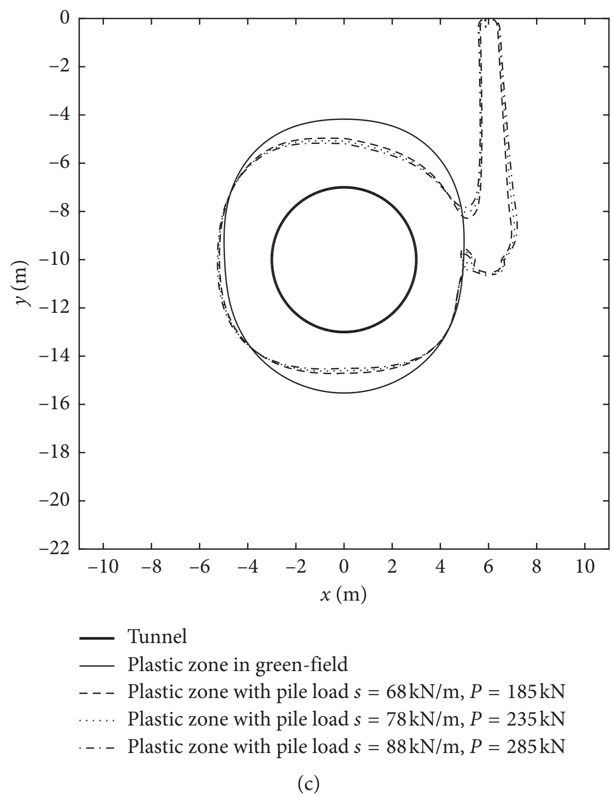


FIGURE 5: Continued.



(c)

Figure 5: Envelopes of potential plastic zones for different conditions. (a) Three different offsets of pile from tunnel. (b) Three different pile lengths. (c) Three different magnitudes of pile loads.

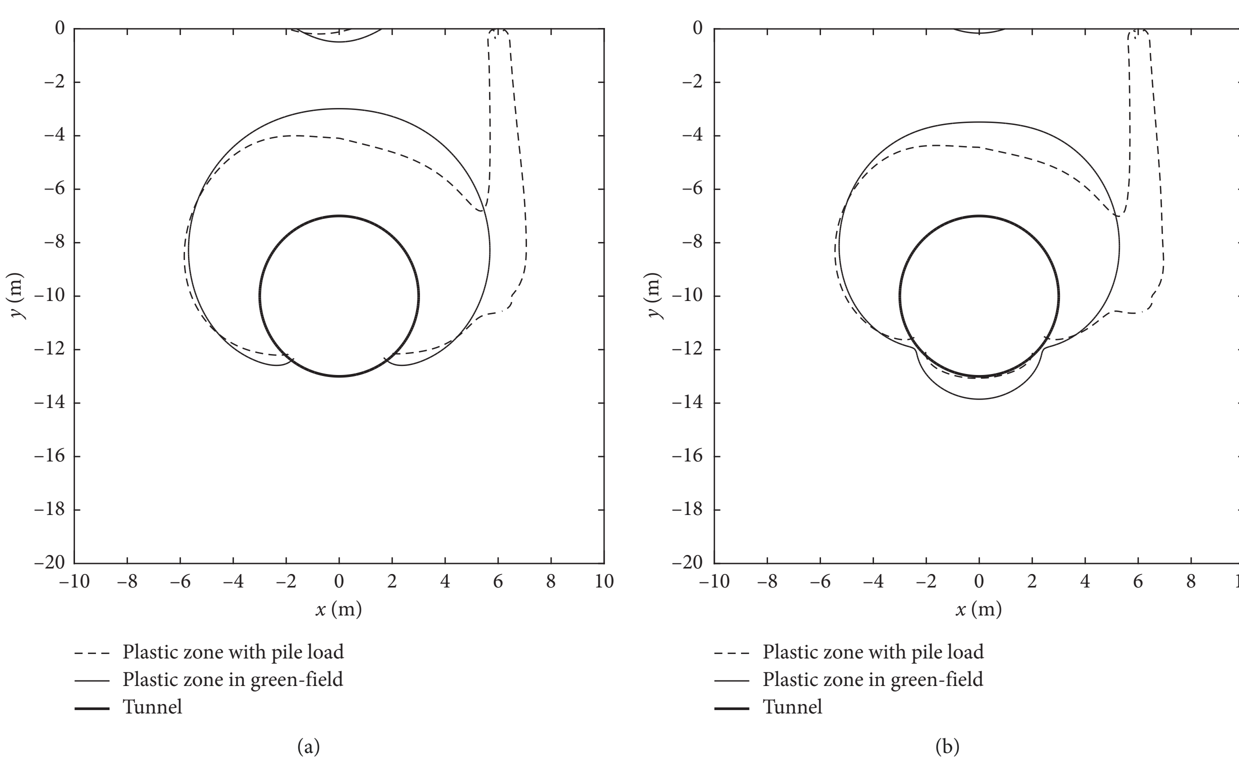


Figure 6: Envelopes of potential plastic zones for different tunnel boundary conditions. (a) B.C.-3 [10]; (b) B.C.-4 [10].

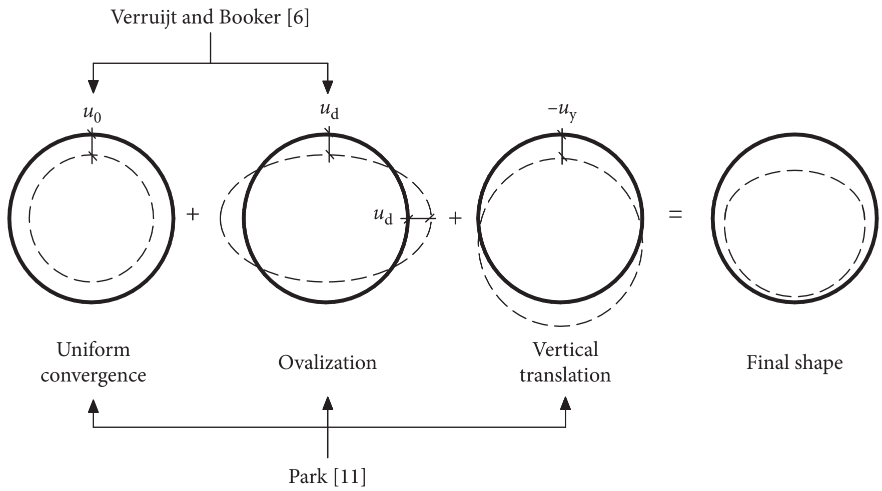


Figure 7: Boundary conditions of shallow tunnels summarized by Pinto and Whittle [13].

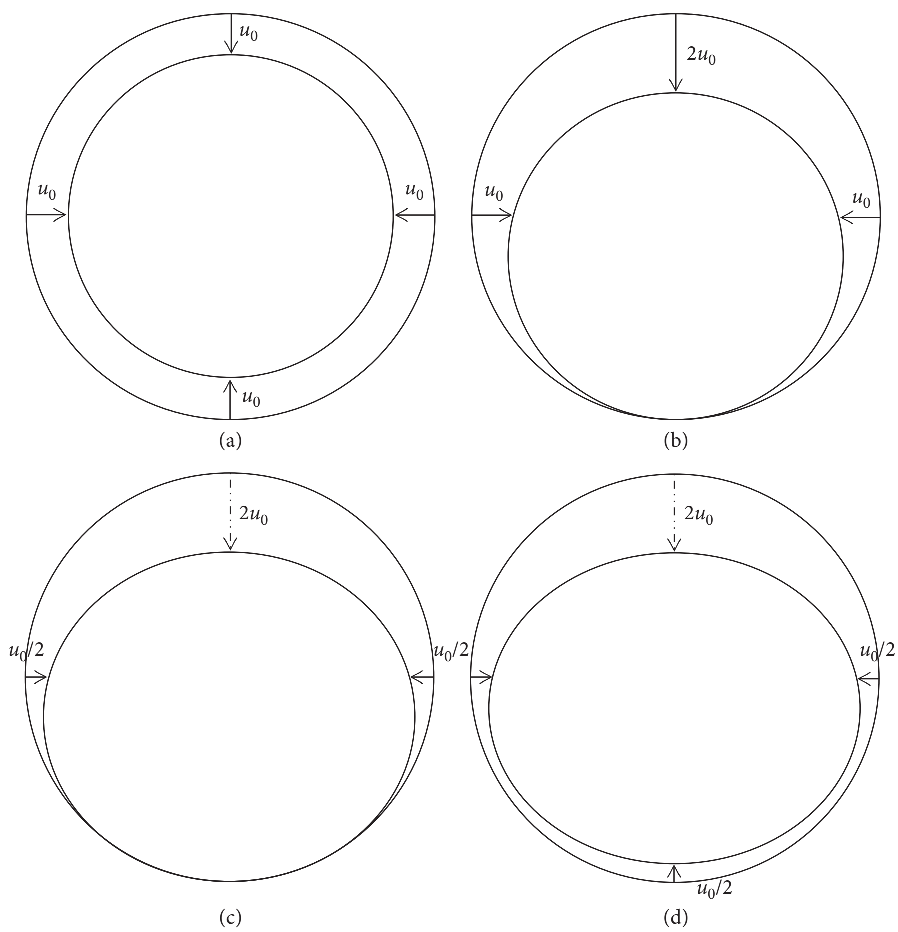


Figure 8: Boundary conditions of shallow tunnels [10]. (a) B.C.-1; (b) B.C.-2; (c) B.C.-3; (d) B.C.-4.

combination of uniform convergence and ovalization as the boundary conditions of a tunnel. However, the superposition method used in this paper is based on the explicit form of exact analytical solutions, which allows the solution to be suitable for various types of boundary conditions. In other words, the superposition method used in this paper can use not only the boundary conditions proposed by Verruijt and Booker [5] as the boundary conditions of a shallow tunnel but also the complex boundary conditions proposed by Park [10] as the boundary conditions of a shallow tunnel. The related coefficients A_k in Fourier series terms [18] in equations (6a)–(6d) are shown in Appendix A.

The influences of different tunnel boundary conditions (for instance, B.C.-3 and B.C.-4) on the ranges and shapes of the potential plastic zones are indicated in Figure 6. It can be concluded from comparison with Figures 3–5 that the plastic zones with boundary conditions (B.C.-3 and B.C.-4) are different from the plastic zones with the boundary conditions proposed by Verruijt and Booker [5], especially in the lower part of the plastic zone. In other words, it would overestimate the plastic zones if the buoyancy effect of a shallow tunnel is not considered. The main reason for this difference is that vertical translation is not considered in the boundary conditions proposed by Verruijt and Booker [5].

4. Conclusion

In green-field, the complex variable method provides exact analytical solutions of ground displacements and stresses caused by shallow tunneling. However, the exact analytical solutions [6] are not directly expressed as explicit functions of the coordinates (x, y) in the physical plane (called implicit form of exact analytical solutions), whereas the displacements and stresses induced by existing structure loads are explicit functions of the coordinates (x, y) in the physical plane, which makes it difficult to superpose the displacements and stresses induced by existing structure loads. This paper transforms implicit form of exact analytical solutions into explicit form of exact analytical solutions, which improves the superposition applicability of exact analytical solution with the analytical solution for the existing structure load. With the explicit form of exact analytical solution, the secondary stress field and the related potential plastic zone caused by tunneling adjacent to pile foundations are obtained. The main conclusions are presented as follows:

(1) With the inverse conformal transformation, the series forms of complex potential functions in z-plane are obtained. By taking the derivative of the analytic functions proposed by Verruijt [5] with the Cauchy–Riemann equations, the explicit form of exact analytical solutions of displacement and stresses induced by shallow tunneling is obtained. The explicit form of exact analytical solutions is intuitional and easily used by engineers, and the calculation amount is smaller than that for the implicit analytical solutions through comparison with the implicit form of exact analytical solutions.

(2) An application involving superimposing the explicit form of exact analytical solutions with Mindlin's solution [1] is implemented to analyze the secondary stress field and the related potential plastic zone caused by tunneling adjacent to pile foundations. A comparison between the results of this paper and one of the existing approaches proposed by Xiang and Feng [15], which is an approximate solution, is performed. The plastic zones around a relatively deep tunnel adjacent to a pile foundation obtained from the two superposition methods are similar to each other, whereas the plastic zones around a relatively shallow tunnel adjacent to a pile foundation from the two superposition methods are obviously different from each other. In other words, the superposition method proposed by Xiang and Feng [15] can obtain satisfactory results for deep tunnels, whereas the superposition method used in this paper is suitable for both deep tunnels and shallow tunnels because it is an exact analytical solution.

(3) The influences of different pile foundation parameters (pile length, load magnitude, and pile offset) on the ranges and shapes of the potential plastic zones induced by nearby tunneling are also analyzed. The results indicate that the whole tunneling-induced potential plastic zones induced by tunneling and pile foundation loads around the tunnel and around the pile would coalesce when the pile is located close enough to the tunnel, whereas if the pile is far enough away from the tunnel, the two plastic zones due to tunneling-induced stress changes are separated from each other.

(4) The superposition method used in this paper can use not only the simple boundary conditions proposed by Verruijt and Booker [5] but also the complex boundary conditions proposed by Park [10]. Therefore, the explicit form of exact analytical solution for calculating ground displacement and stress induced by shallow tunneling proposed in this paper has more extensive adaptability, so it can solve the more complex problems of shallow tunnels. For example, these solutions can be used to analyze differences in the plastic zone under the influence of the buoyancy effect. The results show that it would overestimate the plastic zones if the buoyancy effect of a shallow tunnel is not considered, especially in the lower part of the plastic zone.

Appendix

A. Fourier Coefficients for Boundary Deformations of a Tunnel

(1) Uniform convergence [6]

$$\begin{aligned} A_k &= 0, \quad \forall k < 0, \\ A_0 &= -2i\mu u_0 \alpha, \\ A_1 &= 2i\mu u_0, \\ A_k &= 0, \quad \forall k > 1. \end{aligned} \tag{A.1}$$

(2) Ovalization [13]

$$\begin{aligned} A_k &= i\mu u_{\mathrm{d}} \alpha^{-k-1} \left(1-\alpha^2\right)^2, \quad \forall k < 0, \\ A_0 &= -2i\mu u_d \alpha, \\ A_1 &= 2i\mu u_{\mathrm{d}} \alpha^2 \left(2-\alpha^2\right), \\ A_k &= i\mu u_{\mathrm{d}} \alpha^{k-3} \left(1-\alpha^2\right)^2 \left[-3+(k+1)\left(1-\alpha^2\right)\right], \quad \forall k > 1. \end{aligned} \tag{A.2}$$

(3) B.C.-2 [10, 18]

$$\begin{aligned} A_k &= 0, \quad \forall k < 0, \\ A_0 &= -(1+\alpha)^2 i\mu u_0, \\ A_1 &= \left(2 + 3\alpha - \alpha^3\right) i\mu u_0, \\ A_k &= -\left(1-\alpha^2\right)^2 \alpha^{k-2} i\mu u_0, \quad \forall k > 1. \end{aligned} \tag{A.3}$$

(4) B.C.-3 [10, 18]

$$\begin{aligned} A_k &= \frac{(\alpha^2-1)^2}{4} \alpha^{k-1} i\mu u_0, \quad \forall k < 0, \\ A_0 &= -(1+\alpha)^2 i\mu u_0, \\ A_1 &= \left(\frac{3}{2} + 3\alpha + \alpha^2 - \alpha^3 - \frac{1}{2}\alpha^4\right) i\mu u_0, \\ A_k &= -\frac{(\alpha^2-1)^2(4\alpha+3) + (\alpha^2-1)^3(k+1)}{4} \alpha^{k-3} i\mu u_0, \quad \forall k > 1. \end{aligned} \tag{A.4}$$

(5) B.C.-4 [10, 18]

$$\begin{aligned} A_k &= \frac{3(\alpha^2-1)^2}{8} \alpha^{k-1} i\mu u_0, \quad \forall k < 0, \\ A_0 &= -\left(\frac{3}{4} + \frac{5}{2}\alpha + \frac{3}{4}\alpha^2\right) i\mu u_0, \\ A_1 &= \left(\frac{7}{4} + \frac{9}{4}\alpha + \frac{3}{2}\alpha^2 - \frac{3}{4}\alpha^3 - \frac{3}{4}\alpha^4\right) i\mu u_0, \\ A_k &= -\frac{3(\alpha^2-1)^2(k\alpha^2 + \alpha^2 + 2\alpha + 2 - k)}{8} \alpha^{k-3} i\mu u_0, \quad \forall k > 1. \end{aligned} \tag{A.5}$$

Data Availability

An executable file to create the potential plastic zone caused by shallow tunneling adjacent to pile foundations and the executable file used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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