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Research Article

Mathematical Model for Analysis of Uniaxial and Biaxial Reinforced Concrete Columns

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This paper presents a mathematical model for the analysis of reinforced concrete (RC) uniaxial and biaxial columns. This proposed model is a quick and faster approach for the analysis and design of reinforced concrete rectangular columns without going through the interaction charts procedure as well as other iterative methods for the computation of required axial load capacity (Pc) and moment capacity (Mc). A simplified flow chart has also been developed to find the required column capacity using this mathematical model. Eight uniaxial columns (C-1 to C-8) and seven biaxial columns (CB-1 to CB-7) are analysed in this study. Each column is analysed having different steel reinforcement ratios (ρ) with different loading conditions. In addition, the studied columns are subjected to both tension and compression failures. The detailed examples for both uniaxial and biaxial columns (one for each case) are also presented in this study. The studied columns are also analysed using computer software spColumn. The average variation of the mathematically computed values to the finite element software is not more than 10%, showing promising computational results.

1. Introduction

Columns are the vertical compression members, which transmit loads from the upper floors to the lower levels and to the soil through the foundations [1]. Based on the position of the load on the cross section, columns are classified as concentrically loaded (Figure 1) or eccentrically loaded columns (Figure 2). Eccentrically loaded columns are subjected to moments, in addition to axial force. The moments can be converted to a load P and eccentricities e_X *and* e_Y. The moments can be uniaxial, as in the case when two adjacent panels are not similarly loaded, such as columns A and B in Figure 3 [2]. A column is considered as biaxially loaded when the bending occurs about the x- and y-axis, such as in the case of corner column C in Figure 3. In a recent study [3], Al-Ansari and Afzal also presented an analytical model for generating interaction diagram charts for biaxial columns.

The strength of reinforced concrete columns is normally expressed using interaction diagrams to relate the design axial load $2\varnothing P_\mathbf{n}$ to the design bending moment $\varnothing M_\mathbf{n}$ [4, 5]. Each control point on the column interaction curve ($\varnothing P_n - \varnothing M_n$) represents one combination of design axial load, $\varnothing P_n$ and design bending moment, $\varnothing M_n$, corresponding to a neutral-axis location (Figure 4) [6].

Extensive studies have been carried out on the interaction diagrams (uniaxial and biaxial columns) of reinforced concrete (RC) rectangular columns [6–12]. Several studies have also been performed on providing numerical approaches for the analysis and design of reinforced concrete columns. Furlong et al. [13] provided an overview of the analysis and design of reinforced concrete columns subjected to biaxial bending. They reviewed several methods of analysis that use traditional design methods and compared their results with the obtained data from physical tests of normal strength concrete columns subjected to short-term axial loads and biaxial bending's. They concluded that the elliptic load contour equation [14] and the reciprocal equation [15] are the simplest to use, as they do not require complicated calculations.

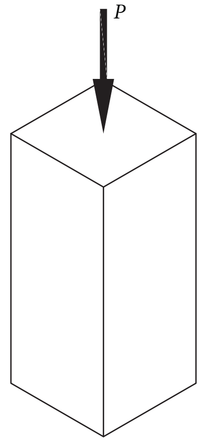

Figure 1: Concentrically loaded columns.

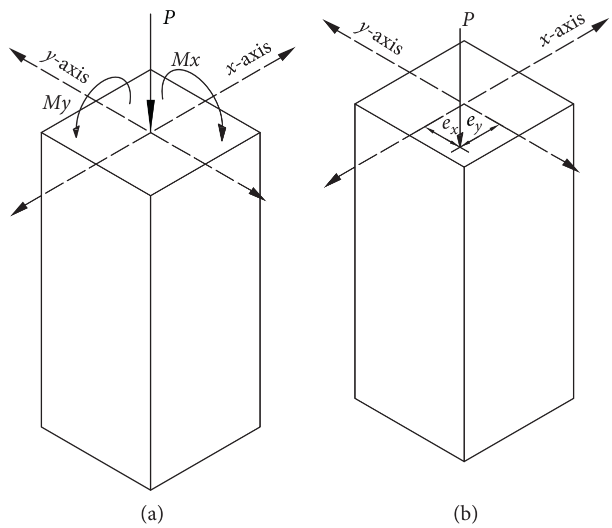

Figure 2: Eccentrically loaded column.

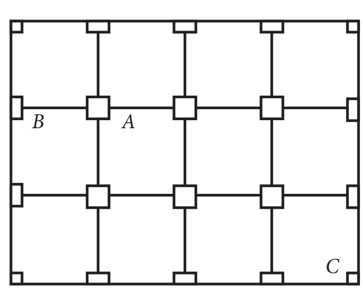

Figure 3: Uniaxially and biaxially loaded column.

Chen et al. [16] proposed an iterative numerical method for rapid section analysis and design of short concrete composite columns subjected to biaxial bending. Wang and Hsu [17] proposed the numerical method approach for the determination of load-moment curvature relationship for short and slender columns. This numerical method approach is also applicable for columns, made of different materials, and shows good agreement with the different experimental results obtained in their study.

Whitney [18] and Hsu et al. [19] provided major research studies on numerical method approaches. Whitney suggested an approximate equation to estimate the nominal compressive strength of columns subjected to compression failure. Hsu in different research projects [10, 17, 20, 21] also presented the results of experimental and analytical studies on the strength and deformation of biaxially loaded short and tied columns with *L*–, channel, and *T*-shaped cross sections. In another study, Hsu [22] suggested a general

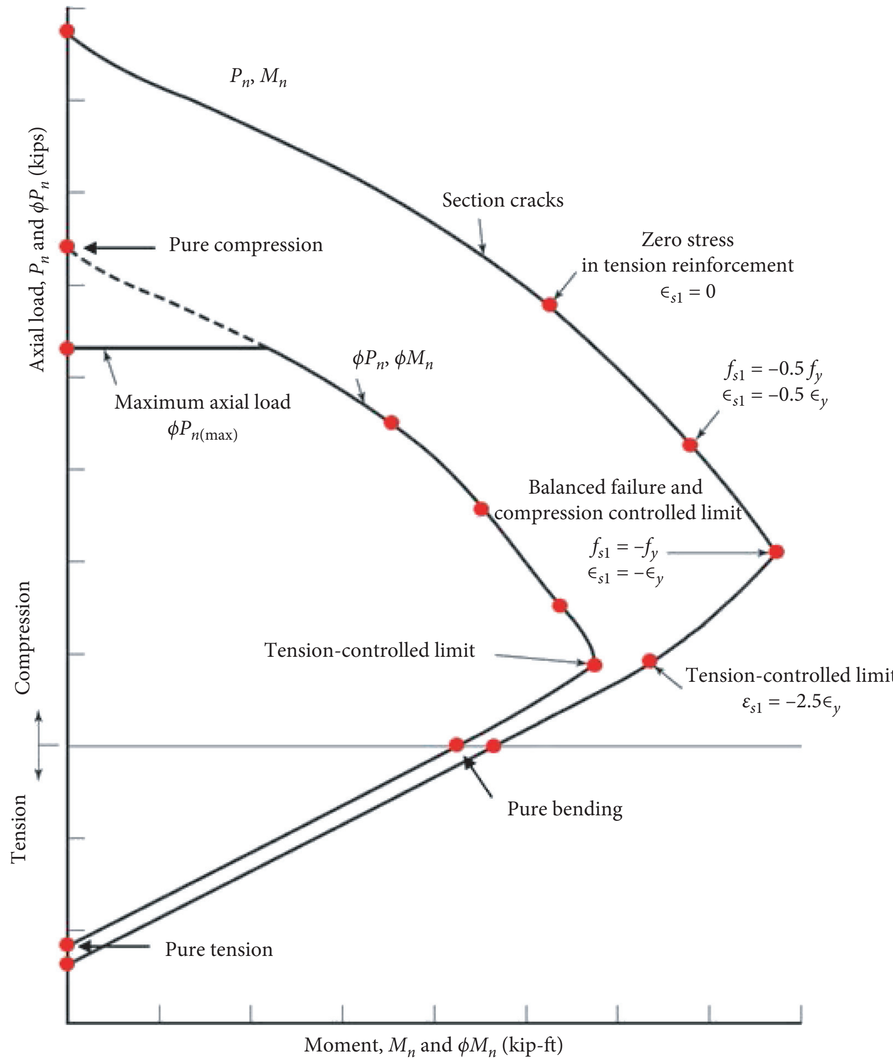

Figure 4: Control points for column interaction curve ($\varnothing P_n - \varnothing M_n$).

equation for the analysis and design of reinforced concrete short and tied rectangular columns.

This study proposes a mathematical model to analyse and design the uniaxial and biaxial columns based on ACI building code of design [23]. This model is a quick and easy approach for analysing and designing the reinforced rectangular columns without going through the interaction charts for the computation of the required axial load capacity (Pc) and moment capacity (Mcx, Mcy). A simplified flow chart has also been developed to find the required column capacity using the proposed model approach. The previous research studies of the mathematical model approach are limited to columns having compression failure only. This study includes the numerical examples of columns using the proposed mathematical model approach for both compression and tension failure cases. This relatively new approach will also be useful to the undergraduate and graduate students as well as researchers to calculate the required column capacities using this approach in their research-related activities.

Numerical examples for the selected reinforced concrete columns (uniaxial and biaxial columns) are also illustrated to check the adequacy of this proposed model. Eight uniaxial columns (C-1 to C-8) and seven biaxial columns (CB-1 to CB-7) are analysed in this study. These columns are analysed having different steel reinforcement ratios (ρ), different values of steel yield strength (fy), concrete compressive strength (f_c'), and different load capacity conditions. Moreover, the results obtained from this proposed model are compared with computer software spColumn 2016 [24].

2. Mathematical Model Formulation: ACI Code Design

The stress and strain distribution of a rectangular column section (uniaxial column) for the calculation of Pn and Mn is given in Figure 5.

The resultant force P_N is equal to the summation of all internal forces:

$$P_N = C_{\text{Con}} - T_s + C_S. \tag{1}$$

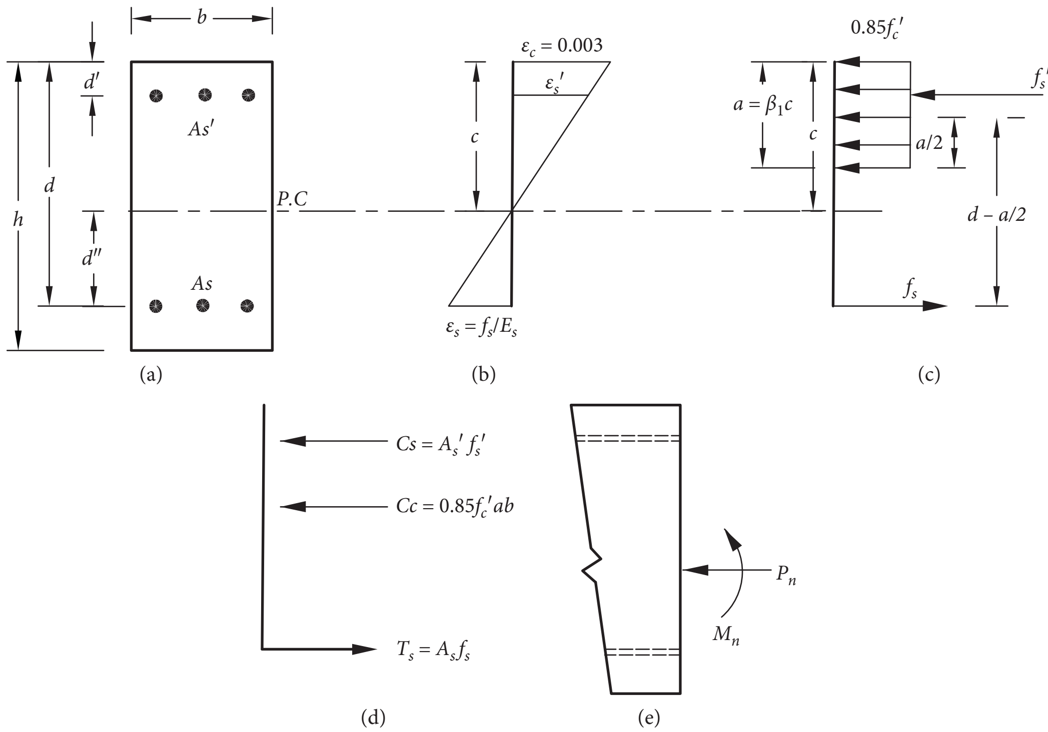

Figure 5: Calculation of Pn and Mn for a given strain distribution. (a) Section. (b) Strain. (c) Stress. (d) Internal forces. (e) Resultant forces.

Similarly, the resultant moment M_N is equal to the summation of all internal moments:

$$M_N = M_{\text{conc}} + M_T + M_{C_S}. \tag{2}$$

The following steps revealed the calculation of the required internal forces and internal moments for a rectangular uniaxial RC column.

2.1. Plain Concrete Section. The Internal concrete compressive force (C_{conc}) is computed as

$$C_{\text{conc}} = (0.85 f_c' ba), \quad C_{\text{conc}} = 0.85 f_c' b\beta c, \tag{3}$$

where C_{conc} = **i**nternal concrete compression force, $f_{\mathbf{c}}'$ = **c**ompressive concrete strength, b = **c**olumn width, a = **d**epth of the compression stress block, $\beta = 0.85 - 0.008\,(f_c' - 30) \geq 0.65$, and c = ***d***istance from extreme compression fiber to the neutral axis.

Referring to Figure 5, the moment about the midpoint of the section (M_{conc}) can be computed as

$$M_{\text{conc}} = C_c\left(d - \frac{a}{2} - d''\right), \quad M_{\text{conc}} = 0.85 f_c' ba\left(d - \frac{a}{2} - d''\right), \tag{4}$$

where h = **c**olumn total depth, $d'' = ((h/2) - d')$, d = **c**olumn effective depth $(h - d')$, and d' = **d**istance from extreme compression fiber to centroid of top reinforcing steel.

2.2. Tension Steel Section. The internal tensile force T_s is computed as

$$T_s = A_s f_y, \tag{5}$$

where A_s = area of tensile steel reinforcement and f_y = **y**ield stress of reinforcing steel.The internal moment M_T is

$$M_T = A_s f_y d''. \tag{6}$$

2.3. Compression Steel Section. The internal compressive force C_s is computed as [25]

$$C_s = A_s' (f_s'), \quad C_{\mathbf{s}} = A_s' (f_s' - 0.85 f_c'), \tag{7}$$

where A_s' = area of compression steel reinforcement and $f_s' = f_y$ (if the compression steel yields).

The internal moment M_T is

$$M_T = A_s' (f_s' - 0.85 f_c')(d - d' - d''). \tag{8}$$

3. Mathematical Model Analysis

The following steps should be revealed to calculate the design axial load and moment capacity of the required rectangular RC column section. Columns may be subjected to tension failure or compression failure depends on the balanced eccentricity value (e_b):

$$e_b = \frac{M_b}{P_b}, \quad (9)$$

$$M_b = \left(C_{cb}\left(d - \frac{a_b}{2} - d''\right) + C_s(d - d' - d'') + (T_s \times d'')\right) \times 10^{-3}, \quad (10)$$

$$P_b = C_s + C_{cb} - T_s, \quad (11)$$

where $a_b = \beta \times c_b$, $c_b = (600 \times d/600 + f_y)$, $C_{cb} = (0.85 \times f'_c \times a_b \times b) \times 10^{-3}$, and $C_s = A'_s(f'_s - 0.85 f'_c) \times 10^{-3}$ (if the compression steel yields, then $f'_s = f_y$).

3.1. Tension Failure Analysis. Tension failure will occur when the balanced eccentricity value (e_b) is less than load eccentricity (e). Substituting the values of C_c, C_s, and T_s in equation (1) and solving for (a) will be a second-degree equation [14]:

$$P_N = C_{Con} - T_s + C_S \text{(Equation (1))}, \quad Aa^2 + Ba + C = 0, \quad (12)$$

where $A = (0.85 \times f'_c \times b/2)$, $B = 0.85 \times f'_c \times b \times (e' - d)$, $C = A'_s(f'_s - 0.85 f'_c)(e' - d + d') - f_y A_s e'$, and $e' = e + d''$, $(e' = e + d - (h/2)$ when $A_s = A'_s)$.

Solve for (a) to get

$$a = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \quad (13)$$

Substitute the value of a in equation (3) to calculate C_c and from equations (5) and (7) to compute T_s and C_s values. These obtained values are substituted in equation (1) for $P_N = C_{Con} - T_s + C_S$,

$$P_N = \frac{1}{e'}\left(C_c\left(d - \frac{a}{2}\right) + C_s(d - d')\right), \quad (14)$$

$$M_N = P_N \times e. \quad (15)$$

The column axial load capacity and moment capacity can therefore be computed as

$$P_c = \varnothing P_N, \quad (16)$$

$$M_c = \varnothing M_N, \quad (17)$$

(where $\varnothing$ is the column reduction factor having the value of 0.65).

3.2. Compression Failure Analysis. Compression failure will occur when the balanced eccentricity value (e_b) is bigger than the load eccentricity (e). Substituting the values of C_c, C_s, and T_s in equation (1) and solving for (a) will be a cubic equation [14]:

$$Aa^3 + Ba^2 + Ca + D = 0, \quad (18)$$

where $A = (0.85 \times f'_c \times b/2)$, $B = 0.85 \times f'_c \times b \times (e' - d)$, $C = A'_s(f'_s - 0.85 f'_c)(e' - d + d') + 600\, A_s e'$, and $D = -600\, A_s e' \beta' d$, where $e' = e + d''$ $(e' = e + d - (h/2)$ when $A_s = A'_s)$.

Once the values of $A, B, C,$ and D are calculated, the value of a can be determined by the trial method or directly by using MATLAB or any scientific calculator. Moreover, the cubic equation can also be solved using different numerical methods, for example, Newton Raphson Method. After getting the required value of (a), similar equations from (14) to (18) (as mentioned in the Tension Failure Analysis) should be used to get the required value of column axial load capacity (P_c) and moment capacity (M_c).

The following flow chart (Figure 6) can be followed to find the required capacity of the rectangular uniaxial column section.

4. Numerical Examples for Uniaxial Columns

Eight reinforced rectangular columns (C-1 to C-8) having different column sizes are analysed using the numerical method approach. These columns are having different reinforcement ratios (ρ) in addition to different failure types, both tension and compression failures. The design input load data for these columns are illustrated in Table 1.

The above eight columns C1 to C8 are analysed using the mathematical model approach to find the required values of axial load capacity, Pc, and moment capacity, Mc. Moreover, these values are also compared with the computer software spColumn. The results obtained are depicted in Table 2.

These above columns are also analysed with different available methods, Whitney's 1st approximation method [18], Whitney's second approximation method [18], and the method provided by HSU [19]. These available methods are only available for the columns having the compression failure. There are no examples available for the columns with the tension failure cases. The results comparison is mentioned in Table 3.

4.1. Detailed Numerical Example for Column C-4 (400 × 400)

Input Data: Figure 7

Pu = 400 kN
Mu = 100 kN·m
f'_c = 30 MPa
fy = 415 MPa
As = 1000 mm^2
A'_s = 1000 mm^2
d' = 80 mm
ϕ = 0.65

Solution:

(1) Finding the value of $e = (M_u/P_u) = 100/400 = 250$ mm
(2) $c_b = (600 \times d/600 + f_y) = (600 \times 320/600 + 415) = 189.16$ mm
(3) $a_b = \beta \times c_b = 0.85 \times 189.16 = 160.788$ mm

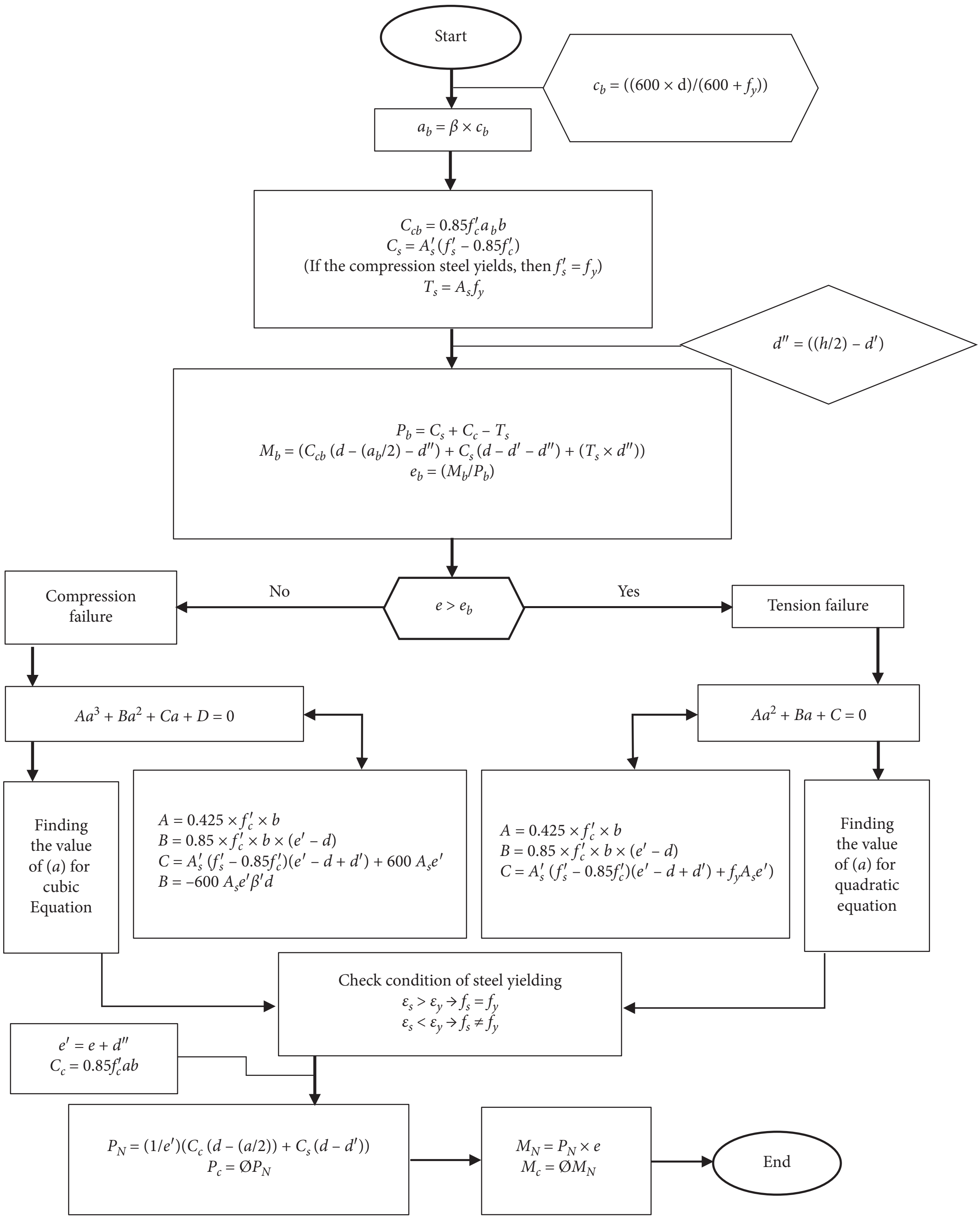

Figure 6: Flow chart of the mathematical model for the rectangular uniaxial rectangular column.

(4) $C_{\mathrm{cb}} = 0.85 f'_c a_b b = 0.85 \times 30 \times 160.788 \times 400 = 1.64 \times 10^3\ \mathrm{kN}$

(5) $C_s = A'_s(f_y - 0.85 f'_c) = 1000\ (415 - 0.85(30)) = 3.895 \times 10^2\ \mathrm{kN}$

(6) $T_s = A_s f_y = 1000 \times 415 = 4.15 \times 10^2\ \mathrm{kN}$

(7) $P_b = C_s + C_{\mathrm{cb}} - T_s = 1.615 \times 10^3\ \mathrm{kN}$

(8) $M_b = (C_c(d - (a_b/2) - d'') + C_s(d - d' - d'') + (T_s \times d''))$
$d'' = ((h/2) - d') = 120\ \mathrm{mm}$
$M_b = 2.927 \times 10^2\ \mathrm{kN \cdot m}$

(9) $e_b = (M_b/P_b) = 181.3\ \mathrm{mm} < e\ (250\ \mathrm{mm})$ (TENSION Failure)

Table 1: Uniaxial column input data.

Column identifier	Pu (kN)	Mu (kN·m)	As (mm^2)	A_s' (mm^2)	f_c' (MPa)	f_y (MPa)	ϕ	d' (mm)	e (mm)
C1 (200 × 400)	300	60	400	400	30	300	0.7	80	200
C2 (200 × 400)	200	50	400	400	20	300	0.7	60	250
C3 (300 × 500)	800	200	1000	1000	30	415	0.7	75	250
C4 (400 × 400)	400	100	1000	1000	30	415	0.65	80	250
C5 (300 × 450)	1300	157.4	1530	1530	25	300	0.65	75	121
C6 (200 × 400)	400	20	402	402	30	300	0.7	60	50
C7 (300 × 500)	600	100	1500	1500	30	415	0.65	90	167
C8 (400 × 800)	1000	200	2000	2000	30	415	0.65	80	200

Table 2: Uniaxial column design results.

Column identifier	e_b (mm)	Failure condition	Mathematical model		spColumn	
			Pc (kN)	Mc (kN·m)	Pc (kN)	Mc (kN·m)
C1	140.7	Tension	377.6	75.5	378.3	75.6
C2	141.5	Tension	234.8	58.7	247.4	61.84
C3	233	Tension	1033	258	1036	259
C4	181.8	Tension	650.3	162.6	650	163
C5	222	Compression	1310	158	1425	172.5
C6	138	Compression	1191	59.5	1810	90.52
C7	276	Compression	1495	250	1677	279.5
C8	364	Compression	3606	722	3975	795

Table 3: Column design results comparison with different methods.

Column identifier	Failure condition	Mathematical model		Whitney 1st approximation		Whitney 2nd approximation		HSU	
		Pc (kN)	Mc (kN·m)	Pc (kN)	Mc (kN·m)	Pc (kN)	Mc (kN·m)	Pc (kN)	Mc (kN·m)
C1	Tension	377.6	75.5	—	—	—	—	—	—
C2	Tension	234.8	58.7	—	—	—	—	—	—
C3	Tension	1033	258	—	—	—	—	—	—
C4	Tension	650	162	—	—	—	—	—	—
C5	Comp.	1310	158	1580	191	1285	155	1580	191
C6	Comp.	1191	59.5	1877	94	1125	56	1878	94
C7	Comp.	1495	250	1677	280	1482	247	1677	280
C8	Comp.	3606	722	4300	860	3582	716	4300	860

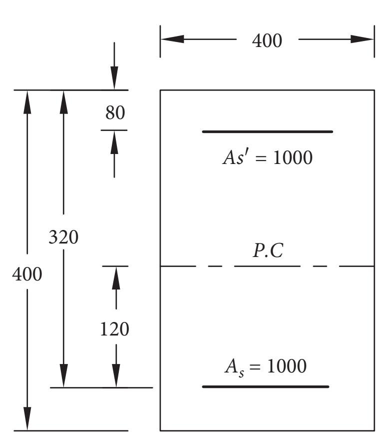

Figure 7: Column C-4.

(10) Finding the value of (a) using the Quadratic Equation
$Aa^2 + Ba + C = 0$
$A = 0.425 \times f_c' \times b = 5.1 \times 10^3$
$B = 0.85 \times f_c' \times b \times (e' - d)$, where; $(e' = e + d'' = 370\,\text{mm})$
$B = 5.1 \times 10^5$
$C = A_s'(f_y - 0.85f_c')(e' - d + d') - f_y A_s e' = -1.029 \times 10^8$
$a = (-B + \sqrt{B^2 - 4AC}/2A)$
$a = 100.59\,\text{mm}$
(11) Check if the tension steel has yielded;
$c = (a/\beta) = 118.34\,\text{mm}$
$\varepsilon_s = ((d - c/c)) \times 0.003 = 0.005$, $\varepsilon_y = (f_y/Es) = 0.002$
$\varepsilon_s > \varepsilon_y$, steel yields, $\therefore$ $(f_s = f_y)$
(12) $P_N = (1/e')(C_c(d - (a/2)) + C_s(d - d') \times 10^{-3}$
$C_c = 0.85f_c'ab = 1.026 \times 10^6\,\text{kN}$
$P_N = 1000.55\,\text{kN}$
$M_N = P_N \times e = 250.1\,\text{kN·m}$
(13) $P_c = \varnothing P_N = 650.35\,\text{kN}$
(14) $M_c = \varnothing M_N = 162.56\,\text{kN·m}$

5. Numerical Examples for Biaxial Columns

Seven reinforced biaxial rectangular columns (CB-1 to CB-7) having different column sizes are also analysed using the proposed model. These columns are also having different reinforcement ratios (ρ) in addition to different failure types, that is, tension-tension, compression-compression, and tension-compression failures. The design input load data for these columns are illustrated in Table 4. The column cross section subjected to biaxial bending is shown in Figure 8. A similar flow chart has to be adopted (as discussed in the uniaxial column sections), once for the case of eccentricity in the x-direction (ex) and later for the eccentricity in the y-direction (ey) to obtain the required values of load capacities in x- and y-direction $(\varnothing P_{\mathbf{x}}, \varnothing P_{\mathbf{y}})$. These values are later used in Bresler's formula [15] (equation (19)) to find the value of Pc. Moreover, the Mcx and Mcy values can be found by using equations (21) and (22) accordingly:

$$P_{\mathbf{c}} = \frac{1}{(1/\varnothing P_x) + \left(1/\varnothing P_y\right) - \left(1/\varnothing P_{N\,\max}\right)}, \qquad (19)$$

$$\varnothing P_{\mathbf{N}\,\max} = 0.8\phi\,(0.85f'\mathbf{c}(\text{Ag} - \text{Ast}) + f y\,\text{Ast}), \qquad (20)$$

where $\varnothing P_{\mathbf{N}\,\max}$ = **m**aximum permissible column load, Ast = **t**otal area of steel, and Ag = (Gross area of cross section) − (sectional area of concrete *member member*).

The moments in the x- and y-direction can be found as

$$M_{\text{cx}} = P_C \times e_{uy}, \qquad (21)$$

$$M_{\text{cy}} = P_C \times e_{ux}. \qquad (22)$$

The above seven columns (CB-1 to CB-7) are analysed with mathematical model approach to find the required values of axial load capacity Pc, using reciprocal formula. Moreover, the values of Pc are also compared with the computer software spColumn. The results obtained are depicted in Table 5.

5.1. Detailed Numerical Example for Column CB-7 (400 × 1200)

Input Data: Figure 9

Pu = 1500 kN
Mux = 300 kN·m
Muy = 300 kN·m
$f_c' = 20\,\text{MPa}$
$fy = 300\,\text{MPa}$
$\text{As} = 3080\,\text{mm}^2$
$A_s' = 3080\,\text{mm}^2$
$d' = 60\,\text{mm}$
$\phi = 0.65$

Solution: (Solving for the X-direction)

(1) Finding the value of $e_y = (M_{ux}/P_u) = 200\,\text{mm}$
(2) $c_b = (600 \times d/600 + f_y) = (600 \times 1140/600 + 300) = 760\,\text{mm}$
(3) $a_b = \beta \times c_b = 0.93 \times 760 = 706.8\,\text{mm}$
(4) $C_{\text{cb}} = 0.85f_c'a_b b = 0.85 \times 20 \times 706.8 \times 400 = 4.81 \times 10^3\,\text{kN}$
(5) $C_s = A_s'(f_y - 0.85f_c') = 3080\ (300 - 0.85(20)) = 8.71 \times 10^2\,\text{kN}$
(6) $T_s = A_s f_y = 3080 \times 300 = 9.24 \times 10^2\,\text{kN}$
(7) $P_{bx} = C_s + C_{\text{cb}} - T_s = 4{,}754\,\text{kN}$
(8) $M_{bx} = (C_c(d - (a_b/2) - d'') + C_s(d - d' - d'') + (T_s \times d''))$
$d'' = ((h/2) - d') = 540\,\text{mm}$
$M_{bx} = 2.155 \times 10^3\,\text{kN·m}$
(9) $e_{by} = (M_{bx}/P_{bx}) = 453\,\text{mm} > ey\ (200\,\text{mm})$ (COMPRESSION Failure)
(10) Finding the value of (a) using the Cubic Equation
$Aa^3 + Ba^2 + Ca + D = 0$
$A = 0.425 \times f_c' \times b \times 10^{-3} = 3.4$
$B = 0.85 \times f_c' \times b \times (e' - d) \times 10^5$, where; $(e' = e + d'' = 740\,\text{mm})$
$B = -2.72 \times 10^3$
$C = A_s'(f_s' - 0.85f_c')(e' - d + d') + 600\,A_s e' = 1.071 \times 10^6$
$D = -600\,A_s e'^{\beta'} d = -1.45 \times 10^9$
$a = 944.47\,\text{mm}$
(11) Check if the tension steel has yielded;
$c = (a/\beta) = 1016\,\text{mm}$
$\varepsilon_s = (d - c/c) \times 0.003 = 0.0028$, $\varepsilon_y = (f_y/Es) = 0.0015$
$\varepsilon_s > \varepsilon_y$, steel yields, $\therefore$ $(f_s = f_y)$
(12) $P_{NX} = (1/e')(C_c(d - (a/2)) + C_s(d - d') \times 10^{-3}$
$C_c = 0.85f_c'ab = 6.422 \times 10^3\,\text{kN}$
$P_{NX} = 7068\,\text{kN}$
$M_{NX} = P_{NX} \times ex = 1414\,\text{kN·m}$
(13) $P_{CX} = \varnothing P_{NX} = 4594\,\text{kN}$

Table 4: Biaxial column input data.

Column identifier	Pu (kN)	Mux (kN·m)	Muy (kN·m)	As (mm^2)	A_s' (mm^2)	f_c' (MPa)	f_y (MPa)	ϕ	d' (mm)	e_x (mm)	e_y (mm)
CB-1 (300 × 600)	300	100	80	1232	1232	30	400	0.65	80	267	333
CB-2 (200 × 400)	200	40	20	628.4	628.4	20	300	0.7	40	100	200
CB-3 (300 × 300)	2500	250	120	1225	1225	30	400	0.65	70	48	100
CB-4 (375 × 500)	1700	200	100	2100	2100	30	415	0.65	60	59	118
CB-5 (400 × 500)	800	200	50	1413.8	1413.8	30	415	0.65	60	62.5	250
CB-6 (350 × 700)	400	60	40	1638	1638	20	300	0.65	45	100	150
CB-7 (400 × 1200)	1500	300	300	3080	3080	20	300	0.65	60	200	200

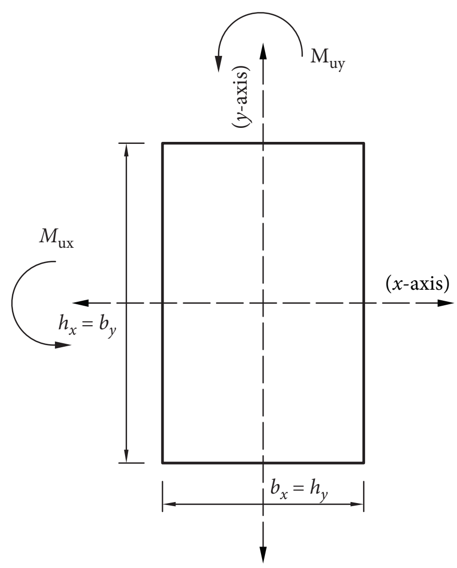

Figure 8: Biaxial column cross section.

Table 5: Biaxial column design results.

Column identifier	Failure condition	Mathematical model Pc (kN)	spColumn Pc (kN)
CB-1	Tension and tension	280	343
CB-2	Tension and tension	262	228
CB-3	Compression and compression	480	415
CB-4	Compression and compression	2132	2035
CB-5	Tension and compression	1291	1142
CB-6	Compression and compression	2852	2805
CB-7	Compression and tension	1993	2081

(14) $M_{CX} = \varnothing M_{NX} = 918.79$ kN·m

(Solving for the Y-direction) (Figure 10)

(1) Finding the value of $e_x = (M_{uy}/P_u) = 200$ mm
(2) $c_b = (600 \times d/600 + f_y) =$ $(600 \times 1140/600 + 300) = 226.67$ mm
(3) $a_b = \beta \times c_b = 0.93 \times 226.67 = 210.8$ mm
(4) $C_{cb} = 0.85 f_c' a_b b = 0.85 \times 20 \times$ $210.8 \times 1200 = 4.3 \times 103$ kN
(5) $C_s = A_s'(f_y - 0.85 f_c') = 3080$ $(300 - 0.85(20)) = 8.71 \times 102$ kN
(6) $T_s = A_s f_y = 3080 \times 300 = 9.24 \times 102$ kN
(7) $P_{by} = C_s + C_{cb} - T_s = 4.248 \times 103$ kN
(8) $M_{by} = (C_c(d - (a_b/2) - d'') + C_s(d - d' - d'') +$ $(T_s \times d''))$
$d'' = ((h/2) - d') = 140$ mm
$M_{by} = 6.582 \times 10^2$ kN·m
(9) $e_{bx} = (M_{by}/P_{by}) = 154.94$ mm $> ex$ (200 mm) (TENSION Failure)
(10) Finding the value of (a) using the Quadratic Equation
$Aa^2 + Ba + C = 0$

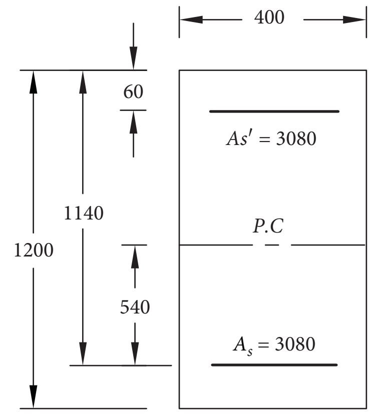

Figure 9: Column CB-7 (X-X axis).

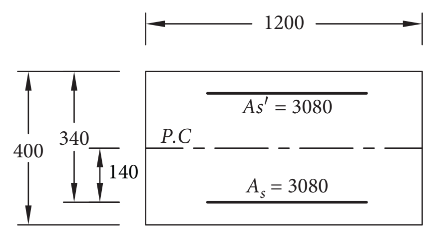

Figure 10: Column CB-7(Y-Y axis).

$A = 0.425 \times f'_c \times b \times 10^{-3} = 10200$
$B = 0.85 \times f'_c \times b \times (e' - d) \times 10^5$, where;
$(e' = e + d'' = 340\,\text{mm})$
$B = 0$
$C = A'_s(f'_s - 0.85f'_c)(e' - d + d') - f_y A_s e' = -2.62 \times 10^8$
$a = 160.227\,\text{mm}$

(11) Check if the tension steel has yielded;
$c = (a/\beta) = 172.287\,\text{mm}$
$\varepsilon_s = (d - c/c) \times 0.003 = 0.00292$, $\varepsilon_y = (f_y/Es) = 0.0015$
$\varepsilon_s > \varepsilon_y$, steel yields, $\therefore (f_s = f_y)$

(12) $P_{NY} = (1/e')(C_c(d - (a/2)) + C_s(d - d') \times 10^{-3}$
$C_c = 0.85 f'_c ab = 3.27 \times 10^6\,\text{kN}$
$P_{NY} = 3216\,\text{kN}$
$M_{NX} = P_{NY} \times ey = 643.25\ \text{kN·m}$

(13) $P_{CY} = \varnothing P_{NY} = 2091\,\text{kN}$

(14) $M_{CY} = \varnothing M_{NY} = 418.12\,\text{kN·m}$

Finding the value of Pc using Bresler's equation (19):

$$P_{\mathbf{c}} = \frac{1}{(1/\varnothing P_x) + \left(1/\varnothing P_y\right) - \left(1/\varnothing P_{N\max}\right)}, \tag{23}$$

where $\varnothing P_{N\max} = 0.8\,\phi(0.85\,f'c\,(\text{Ag} - \text{Ast}) + f y\,\text{Ast}) = 5150\,\text{kN}$, $P_{\mathbf{c}} = (1/(1/4594) + (1/2091) - (1/5150)) = 1993\,\text{kN}$.

6. Validation of the Mathematical Model

In order to validate the proposed mathematical model approach, the model is validated with the existing experimental results of columns subjected to uniaxial and biaxial loadings. The experimental results data has been extracted from the test results provided by HSU [22]. Two uniaxial columns as provided by Bresler (B-1 and B-2) and one biaxial column as provided by Anderson and Lee (SC-4) are selected from the research article [22] to compare the results with the mathematical model.

Table 6 illustrates the experimental testing data provided by HSU. The data and the results are provided in imperial units (Kips-ft) units. Therefore, they are converted to metric units accordingly to compare the values with our results.

Table 7 provides the experimental test results as well as the validation of the test data with the proposed mathematical model. The column capacity (Pc) results obtained from the experimental data are quite close to the mathematical model results, showing satisfactory computational results.

7. Results and Discussions

The results obtained from the mathematical model approach for both uniaxial and biaxial columns showed a safe and conservative column design method. The results of eight uniaxial column sections (C-1 to C-8) using the proposed model are also compared with different available mathematical models, provided by Whitney's 1st approximation method, Whitney's second approximation method, and the method provided by HSU. Columns C1 to C-4 were subjected to tension failure, whereas columns C-5 to C-8 were the compression failure cases. The other three mathematical studies (Whitney's 1st approximation, Whitney's second

Table 6: Experimental testing data [22].

Experimental investigator	Column identifier with size (in x in) b (mm x mm)	As (In^2) (mm^2)	f_c' (Ksi) (MPa)	f_y (Ksi) (MPa)	d'(In) (mm)	ex (in) (mm)	ey (in) (mm)
Bresler (uniaxial column)	B-1 (6 × 8) (152 × 203)	1.24 (800)	3.7 (25.6)	53.5 (369)	1.75 (44.5)	6 (152.4)	0
Bresler (uniaxial column)	B-2 (6 × 8) (152 × 203)	1.24 (800)	3.9 (27)	53.5 (369)	1.75 (44.5)	3 (76.2)	0
Anderson and Lee (biaxial column)	SC-4 (4 × 4) (102 × 102)	0.8 (516)	5.435 (37.5)	45.6 (314.6)	0.75 (19)	2.82 (71.63)	2.82 (71.63)

Table 7: Validation of experimental data [22].

Experimental investigator	Column identifier with size (in x in) (mm x mm)	ϕc	eb (in) (mm)	Failure condition	Experimental results Pc (kips) (kN)	Mathematical model Pc (kips) (kN)	$(P_{\text{exp}}/P_{\text{Math}})$
Bresler (uniaxial column)	B-1 (6 × 8) (152 × 203)	0.65	4.67 (118.5)	$eb < ex$ (tension)	24 (107)	29 (132)	0.83
Bresler (uniaxial column)	B-2 (6 × 8) (152 × 203)	0.65	4.6 (117)	$eb > ex$ (compression)	60 (267)	59 (263)	1.01
Anderson and Lee (biaxial column)	SC-4 (4 × 4) (102 × 102)	0.65	2.7 (69)	$eb < ex$ (tension)	13.5 (60)	11 (49)	1.22

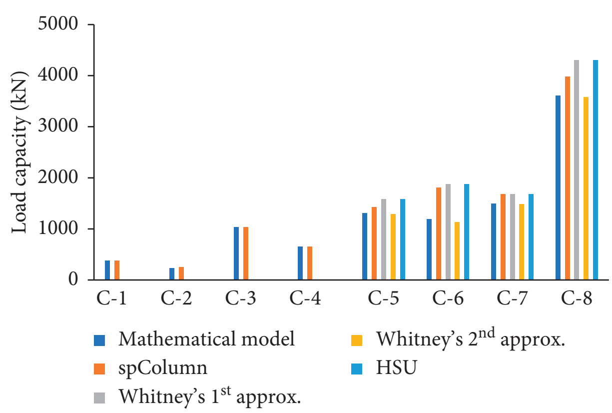

Figure 11: Axial load capacity comparison for uniaxial columns (C1–C8).

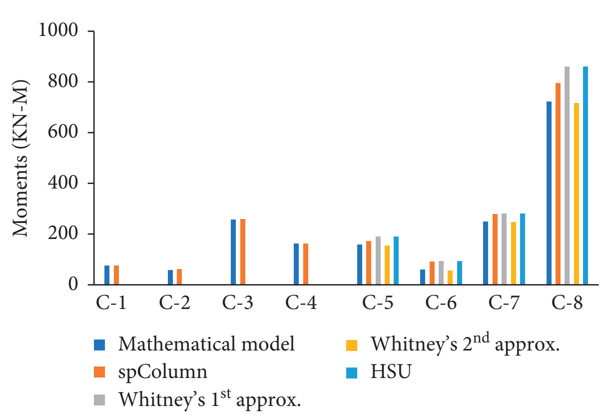

Figure 12: Moment capacity comparison for uniaxial columns (C1–C8).

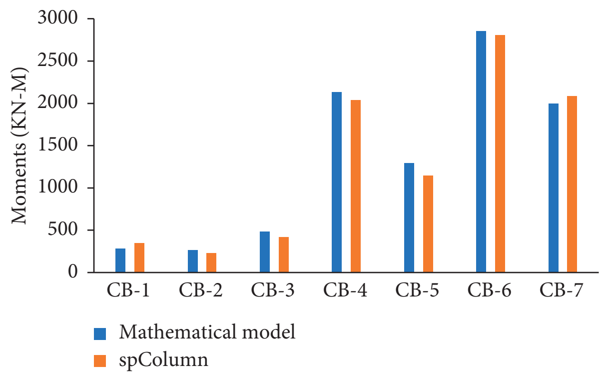

Figure 13: Axial load capacity comparison for biaxial columns (CB-1–CB-7).

approximation, and the method provided by HSU) are only limited to the case where the columns are subjected to the compression failure only.

These studied columns (C-1 to C-8) are also analysed using the computer software spColumn and the comparison results for axial load capacities (Pc) and moment capacities (Mc) are displayed in bar charts (Figures 11 and 12).

For the biaxial columns (CB-1 to CB-7), the axial load capacity results for mathematical model approach using Bresler's formula and the computer software spColumn are displayed in the bar chart (Figure 13).

The values of Pc obtained using the mathematical model are quite close to the computer software results, showing relatively satisfactory computational results.

8. Conclusion

In this study, the mathematical model is presented to analyse and design the uniaxial and biaxial columns without going through the column interaction charts to find the required axial load capacities and moment capacities. A simplified flow chart has also been developed to solve the required column section following the mathematical model steps.

Eight (RC) uniaxial columns (C-1 to C-8) and seven (RC) biaxial columns (CB-1 to CB-7) are analysed in this study. These columns are analysed having different steel reinforcement ratios (ρ), different values of steel yield strength (fy), concrete compressive strength (f_c'), and different load capacity conditions. Moreover, the studied columns are subjected to both tension and compression failures.

For the uniaxial columns, the proposed mathematical model results are also compared with the different available numerical approaches done by Whitney's 1st approximation, Whitney's 2nd approximation, and the method provided by HSU. All of these three methods were formulated based on the case of compression failure only. These uniaxial columns are also analysed using the computer software spColumn. The results obtained showed that this proposed mathematical approach showed good agreement with the computer software spColumn showing relatively satisfactory results.

The studied biaxial columns are subjected to different failure conditions, that is, tension-tension failure, compression-compression failure, and tension-compression failure. Bresler's formula was used to find the required capacity (Pc) after finding the $(Px)$and(Py) from the mathematical model approach. The biaxial columns were also analysed with the computer software. The average variation of the mathematically computed values for biaxial columns to the finite element software was not more than 10%. Moreover, the results obtained for the columns subjected to tension failure are quite close with the computer software spColumn. Moreover, this mathematical model has also been validated with the existing experimental results conducted by HSU.

In short, this newly proposed mathematical model is a good and quick approach to analyse the reinforced concrete uniaxial and biaxial columns. This model can also help the students and the academic researchers to find the column capacities without going through the column interaction charts and other long iterative approaches.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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