Research Article

A New Sinusoidal Shear Deformation Theory for Static Bending Analysis of Functionally Graded Plates Resting on Winkler–Pasternak Foundations

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In this article, a new sinusoidal shear deformation theory was developed for static bending analysis of functionally graded plates resting on elastic foundations. The proposed theory used an undefined integral term to reduce the number of the unknown to four without any shear correction factors. The high accuracy and efficiency of the proposed theory were proved thanks to the comparisons of the present results with other available solutions. And then, the proposed theory was successfully applied to investigate the bending behavior of the functionally graded plates resting on Winkler–Pasternak foundations. The governing equations of motion were established by using Hamilton's principle, and the Navier's solution technique was employed to solve these equations. The effects of some factors of the geometrics, the materials properties, and the elastic foundation parameters on the bending behaviors of the FGM plates were investigated intensely. Also, some novel results and special phenomenon were carried out.

1. Introduction

Functionally graded materials (FGMs) are made from a mixture of two or more ingredients together. In such type of material, the material properties vary continuously through the thickness of the structures. Because of their exotic properties, FGMs are widely applied in many fields of engineering and industry [1–6], for example, aeronautics, nuclear engineering, advanced civil engineering, and so on. More details of the application and investigation of FGMs can be read from a state-of-the-art review by Swaminathan et al. [7]. So many scientists focused on investigating the mechanical behavior, thermal response, mechanical-electrical behavior of FGM structures such as beams, plates, and shells, especially FGM plates and shells. So, finding a simple, efficient, and suitable shear deformation to analyze these structures is one of the biggest challenges of researchers. Many plate theories were developed and applied successfully in the past, such as classical

plate theory (CPT), Mindlin plate theory or first-order shear deformation theories (FSDTs), higher-order shear deformation plate theories (HSDTs), quasi-3D deformation theories, and their variations.

Firstly, Liessa [8] researched the free vibration of rectangular plates using CPT. Javaheri et al. [9] applied CPT to investigate the buckling behavior of FGM plates subjected to in-plane compressive load. Mohammadi et al. [10] used Levy type solution based on CPT to study buckling of rectangular FGM plates. A deep investigation of the effects of some parameters on the vibration and stability of FGM plates was carried out by Hu et al. [11]. One of the most disadvantages of the CPT is that the transverse shear strains are neglected, so CPT is just suitable to investigate thin and very thin plates and cannot be applied to study thick and very thick plates.

To address the disadvantage of CPT, Reissner-Mindlin plate theory and FSDTs were developed to investigate moderate thick plates. In such type of plate theory, the transverse shear strains are considered. Croce et al. [12]

applied Reissner-Mindlin plate theory incorporated with finite element method (FEM) to investigate the mechanical behavior of FGM plates in a thermal environment. The buckling of the skew FGM plate subjected to mechanical load was investigated by Ganapathi and his coworkers [13]. Kim et al. [14] studied geometrically nonlinear behavior of FGM plates and shells using FSDT and a four-node quasiconforming shell element. Hosseini-Hashemi and his coworkers employed FSDT [15] and Reissner-Mindlin [16] to study the free vibration of rectangular FGM plates. Nguyen et al. [17] analyzed the mechanical behavior of FGM plates using FSDT. Shimpi et al. [18] developed a new FSDT to investigate the bending behavior of plates. Thai et al. [19] developed a simple FSDT with only four unknown displacement functions to examine the bending and free vibration of FGM plates. A new FSDT was developed by Thai et al. [20] to analyze FGM plates. Nguyen et al. [21] established a refined simple FSDT in which the transverse shear stresses distribution is parabolical through the thickness of the plate, so it does not need any correction factors. Yu et al. [22] used a simple FSDT and isogeometric analysis to investigate the nonlinear bending of FGM plates. Jalaei et al. [23] applied FSDT in combination with nonlocal elasticity to analyze the dynamic instability of FGM nanobeams with porosity. Vu et al. [24] developed a new FSDT to investigate static bending and vibration of two-layer plates. Senjanović et al. [25] modified the Mindlin plate theory to investigate plates with a shear locking-free rectangular plate element. Although FSDT considered the transverse shear stresses, it still needs a correction factor to avoid the shear locking phenomenon. Moreover, the shear stresses cannot be predicted correctly and are only applied to thin and moderate plates.

To overcome these drawbacks of FSDTs, many HSDTs have been developed. In comparison with CPT and FSDTs, HSDTs have many significant benefits, such as that the transverse shear stresses are parabolically distributed through the thickness and satisfy the shear-free conditions on two surfaces of the plates, so they do not need any shear correction factors. In addition, HSDT can predict very well deflections and stresses of thin to moderate and thick plates. Javaheri et al. [26] used HSDT to investigate thermal buckling of FGM plates. Yang et al. [27] studied the dynamic stability of FGM plates using HSDT. Bodaghi et al. [28] used HSDT and Levy type solution to study buckling of thick FGM plates. Ferreira et al. [29] investigated the static bending behavior of FGM plates using third-order shear deformation theory (TSDT) and the meshless method. Tran et al. [30] used HSDT in combination with isogeometric analysis to study the mechanical behavior of FGM plates. A generalized shear deformation theory (GSDT) was developed by Zenkour [31] and was applied to analyze the bending behavior of FGM plates. Shimpi [32] developed two refined plate theories (RPT) and applied them to investigate static bending of plates. Van et al. [33] modified the RPT to investigate static bending of FGM plates. A simple HSDT and a sinusoidal shear deformation theory (SSDT) were developed by Thai et al. [34, 35] to analyze static bending, free vibration, and buckling of FGM plates. Vinh et al. [36] developed a new hyperbolic shear deformation theory in conjunction with

FEM to analyze FGM Sandwich plate with porosity. Touratier [37] developed a new sinusoidal HSDT to analyze composite plates. Akgoz et al. [38] applied a SSDT in combination with strain gradient elasticity theory to analyze static bending and free vibration of microplates. Mechab et al. [39] developed a new four-variable refined plate theory with a new function to study static bending and dynamic response of FGM plates. Meiche et al. [40] developed a new hyperbolic shear deformation theory to analyze buckling and free vibration of FG Sandwich plates. Menasria et al. [41] established a new simple HSDT to study the thermal stability of FG Sandwich plates. Pandya et al. [42] applied HSDT and FEM to study the flexure of Sandwich plates. Talha et al. [43] investigated static bending behavior as well as free vibration of FGM plates. Do et al. [44] analyzed the bidirection FGM plates using FEM and new TSDT, in which the authors showed that the materials have significant roles in the behavior of the FGM plates. Vinh et al. [45] and Hoa et al. [46] developed a single variable HSDT for static bending and free vibration analysis of FGM plates and FGM nanoplates. Zenkour et al. [47] developed a simple four unknown refined theories for the analysis of static bending of FGM plates. Nguyen et al. [48] developed a new inverse trigonometric shear deformation theory for the analysis of isotropic and FGM Sandwich plates.

Although HSDT and their variations have many benefits, the normal stress in z -direction is neglected, so they cannot predict exactly the behaviors of very thick plates. In recent years, many quasi-3D theories were developed, in which the deflection and normal stress in z -direction are considered. As a consequence, these types of plate theories are appropriate for the analysis of very thick plates. Qian et al. [49] used HSDT and normal deformation theory incorporated with meshless local Petrov-Galerkin method (MLPG) to investigate dynamic and static bending behavior of thick FGM plates. Gilhooley et al. [50] analyzed thick FGM plates using HSDT and normal deformation plate theory incorporated with MLPG and radial basis functions. Mantari et al. [51–54] developed a family of quasi-3D theories for analysis static bending, free vibration and buckling of FGM plates. Nguyen et al. [55] developed an efficient beam element based on quasi-3D theory to analyze the bending behavior of FGM beams. Zenkour et al. [56, 57] developed many quasi-3D theories for the analysis of isotropic and FGM Sandwich plates. Vinh [58] developed a hybrid quasi-3D theory for deflections, stress, and free vibration analysis of bi-FGM Sandwich plates resting on elastic foundation. Thai et al. [59] developed a simple quasi-3D sinusoidal shear deformation theory for FGM plates analysis. Neves et al. [60, 61] established some efficient quasi-3D and applied to analyze static bending, free vibration, and buckling of isotropic and Sandwich FGM plates.

Analysis of beams, plates, and shells structures with elastic foundation support is an important problem in engineering. In comparison with the structures without elastic foundation support, the behavior of these structures resting on elastic foundation is completely different. Chakraverty et al. [62] studied free vibration of thin FGM plates resting on the Winkler foundation with general boundary conditions by Rayleigh-Ritz method. Mantari et al. [63] analyzed free vibration of composite plates resting on elastic

foundation. Akavci [64] developed an efficient shear deformation to analyze free vibration of FGM thick plates resting on elastic foundation. Thai et al. [65] studied bending, buckling, and free vibration of thick FGM plates resting on elastic foundations using simple, refined theory. In other works of Thai et al. [66], a closed-form solution was developed to investigate the buckling of thick FGM plates. Baferani et al. [67] developed an accurate solution for free vibration analysis of FGM thick rectangular plates. Ameer et al. [68] developed a new trigonometric shear deformation theory to analyze the bending behavior of FGM plates resting on elastic foundations with the Winkler–Pasternak type model. Al Khateeb et al. [69] used a refined four-unknown plate theory to analyze advanced plates resting on elastic foundations in the hygrothermal environment. Attia et al. [70] employed a refined four variable plate theory for thermoelastic analysis of FGM plates supported by a variable elastic foundation. Avcar et al. [71] studied free vibration of FGM plates resting on elastic foundations with the Winkler–Pasternak type model. Benyoucef et al. [72] studied the bending behavior of FGM plates resting on the Winkler–Pasternak foundation. Han et al. [73] analyzed plates resting on the two-parameter foundation using the numerical differential quadrature method and the Reissner–Mindlin plate model. Gupta et al. [74] studied free vibration and bending response of FGM plates supported by Winkler–Pasternak foundations using nonpolynomial HSDT and normal shear deformation theory. Said et al. [75] developed a new simple hyperbolic shear deformation theory to analyze FGM plates resting on elastic foundations with the Winkler–Pasternak model. Zaoui et al. [76] developed new 2D and quasi-3D shear deformation theories for free vibration of FGM plates resting on elastic foundations.

This study aims to establish a novel sinusoidal shear deformation theory for bending behavior analysis of FGM plates supported by elastic foundations with the Winkler–Pasternak type model. This theory considered the parabolical distribution of the shear stresses and satisfied the free conditions of those on top and bottom surfaces of the plates. Hamilton's principle is applied to construct the governing equations of motion and the Navier's solution technique is used to solve these equations. The accuracy and efficiency of the proposed theory are proved thanks to several validation studies. Then the proposed theory is employed to investigate the flexure behavior of the FGM plates resting on Winkler–Pasternak type foundations. Several numerical investigations on the effects of some parameters are carried out and some new results and special phenomenon are illustrated.

2. Theoretical Formulation

2.1. FGM Plates Resting on Winkler–Pasternak Foundations.

Figure 1 shows the model of FGM plates with the dimension of $a \times b$ and the thickness of h lies on the elastic foundation. The material properties of the plate are assumed to vary continuously from bottom to top surfaces of the plates by a power-law function. In this study, the elastic foundation is modelled by the Winkler–Pasternak type that consists of two components which are a Winkler foundation with the stiffness of k_w and a shear layer with the stiffness of k_s .

FIGURE 1: The model of FGM plates resting on Winkler–Pasternak foundations.

The material properties of the plates are assumed to vary continuously through the thickness of the plates as power-law functions [3, 4].

$$\begin{aligned} E(z) &= E_m + (E_c - E_m)V_f, \\ \nu(z) &= \nu_m + (\nu_c - \nu_m)V_f, \end{aligned} \quad (1)$$

where

$$V_f = \left(\frac{1}{2} + \frac{z}{h} \right)^p, \quad (2)$$

where E_c , E_m , ν_c and ν_m are Young's modulus and Poisson's ratio of the ceramic and metal, respectively, p is the power-law exponent, and h is the thickness of the plates.

2.2. New Sinusoidal Shear Deformation Theory

2.2.1. Displacement Field. The displacement field at any point in the plates using new sinusoidal shear deformation theory is written by the following:

$$\begin{aligned} u(x, y, z) &= u(x, y) - z \frac{\partial w}{\partial x} + f(z) \left(\frac{\partial \theta}{\partial x} + \frac{\partial w}{\partial x} \right), \\ v(x, y, z) &= v(x, y) - z \frac{\partial w}{\partial y} + f(z) \left(\frac{\partial \theta}{\partial y} + \frac{\partial w}{\partial y} \right), \\ w(x, y, z) &= w(x, y). \end{aligned} \quad (3)$$

By introducing two unknown derivative quantities $\partial \theta / \partial x$ and $\partial \theta / \partial y$, the proposed shear deformation theory consists of only four unknown displacement functions, it is similar to the simple FSDT of Thai et al. [19] and Ameer et al. [68]. The number of unknowns of the proposed theory is smaller than other HSDTs with five to eight unknowns, so the computation cost can be reduced. However, the transverse displacement is not separated into two parts as in simple FSDT or simple HSDT, so it is simpler and more efficient than other HSDTs. Moreover, other plate theories can be obtained easily by varying the functions $f(z)$. For example, the CPT

can be taken by setting $f(z) = 0$, the FSDT can be achieved by setting $f(z) = z$, and the HSDT of Reddy [3] can be achieved by setting $f(z) = z(1 - 4z^2/3h^2)$. In this study, a new sinusoidal shear deformation theory is obtained by setting the following:

$$f(z) = \frac{h\sqrt{15}}{3\pi} \sin\left(\frac{\pi z}{h}\right). \quad (4)$$

The new sinusoidal shear deformation theory satisfies two conditions of the shear strains and stresses of the plates.

The first condition is that the distribution of the shear stresses through the thickness of the plates is parabolical. The second condition is that the shear strains and stresses equal to zeros at any points on two free surfaces of the plates. So, the proposed theory does not need any shear correction factors as in the FSDTs.

2.2.2. Constitutive Equations. The strains fields of the plate are written as follows:

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} - z \frac{\partial^2 w}{\partial x^2} + f(z) \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} \right), \\ \varepsilon_y &= \frac{\partial v}{\partial y} - z \frac{\partial^2 w}{\partial y^2} + f(z) \left(\frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 w}{\partial y^2} \right), \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - z \left(\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y \partial x} \right) + 2f(z) \left(\frac{\partial^2 \theta}{\partial y \partial x} + \frac{\partial^2 w}{\partial x \partial y} \right), \\ \gamma_{xz} &= \frac{\partial w}{\partial x} - \frac{\partial w}{\partial x} + f'(z) \left(\frac{\partial \theta}{\partial x} + \frac{\partial w}{\partial x} \right) = g(z) \left(\frac{\partial \theta}{\partial x} + \frac{\partial w}{\partial x} \right), \\ \gamma_{yz} &= \frac{\partial w}{\partial y} - \frac{\partial w}{\partial y} + f'(z) \left(\frac{\partial \theta}{\partial y} + \frac{\partial w}{\partial y} \right) = g(z) \left(\frac{\partial \theta}{\partial y} + \frac{\partial w}{\partial y} \right), \end{aligned} \quad (5)$$

where

$$g(z) = f'(z) = \frac{\sqrt{15}}{3} \cos\left(\frac{\pi z}{h}\right). \quad (6)$$

From equations (5) and (6), it can be obvious that the free conditions of the shear strains and stresses on the top and bottom surfaces of the plates are satisfied automatically.

The strains fields of the plates can be written in compact form as follows:

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}^b + f\boldsymbol{\kappa}^s, \quad (7)$$

$$\boldsymbol{\gamma} = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = g\boldsymbol{\Upsilon}^0, \quad (8)$$

where

$$\boldsymbol{\varepsilon}^0 = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}, \boldsymbol{\kappa}^b = -\begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}, \boldsymbol{\kappa}^s = \begin{Bmatrix} \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 w}{\partial y^2} \\ 2 \left(\frac{\partial^2 \theta}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial y} \right) \end{Bmatrix}, \quad (9)$$

$$\boldsymbol{\Upsilon}^0 = \begin{Bmatrix} \frac{\partial \theta}{\partial x} + \frac{\partial w}{\partial x} \\ \frac{\partial \theta}{\partial y} + \frac{\partial w}{\partial y} \end{Bmatrix}. \quad (10)$$

The constitutive equation of the plate is as follows [3]:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}, \quad (11)$$

where

$$\begin{aligned} Q_{11} &= Q_{22} = \frac{E(z)}{1 - \nu^2(z)}, \\ Q_{12} &= \frac{\nu(z)E(z)}{1 - \nu^2(z)}, \\ Q_{66} &= Q_{55} = Q_{44} = \frac{E(z)}{2(1 + \nu(z))}. \end{aligned} \quad (12)$$

It can be written in short form as the following equation:

$$\begin{Bmatrix} \boldsymbol{\sigma} \\ \boldsymbol{\tau} \end{Bmatrix} = \begin{bmatrix} \mathbf{D}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_s \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\gamma} \end{Bmatrix}, \quad (13)$$

where

$$\begin{aligned} \boldsymbol{\sigma} &= \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}, \\ \boldsymbol{\tau} &= \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix}, \end{aligned} \quad (14)$$

$$\begin{aligned} \mathbf{D}_b &= \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}, \\ \mathbf{D}_s &= \begin{bmatrix} Q_{55} & 0 \\ 0 & Q_{44} \end{bmatrix}. \end{aligned} \quad (15)$$

2.2.3. Governing Equations. The Hamilton's principle is employed to obtain the equations of motion

$$0 = \int_0^T (\delta U - \delta V) dt, \quad (16)$$

where δU is the variation of the strain energy and δV is the variation of the work done by external forces and reaction forces of the elastic foundation. The variation of the strain energy is obtained as the following expression [64]:

$$\delta U = \int_A \int_z (\sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \tau_{xy} \delta \gamma_{xy} + \tau_{xz} \delta \gamma_{xz} + \tau_{yz} \delta \gamma_{yz}) dA dz. \quad (17)$$

After integrating through the thickness of the plates, one gets the following:

$$\delta U = \int_A [N_x \delta \varepsilon_x^0 + N_y \delta \varepsilon_y^0 + N_{xy} \delta \varepsilon_{xy}^0 + M_x \delta \kappa_x^b + M_y \delta \kappa_y^b + M_{xy} \delta \kappa_{xy}^b + P_x \delta \kappa_x^s + P_y \delta \kappa_y^s + P_{xy} \delta \kappa_{xy}^s + S_y \delta \gamma_{yz}^0 + S_x \delta \gamma_{xz}^0] dA, \quad (18)$$

where N_{ij} , M_{ij} , P_{ij} and S_{ij} are the stress resultants which are calculated by

$$\begin{aligned} (N_x, N_y, N_{xy}) &= \int_z (\sigma_x, \sigma_y, \sigma_{xy}) dz, \\ (M_x, M_y, M_{xy}) &= \int_z (\sigma_x, \sigma_y, \sigma_{xy}) z dz, \\ (P_x, P_y, P_{xy}) &= \int_z (\sigma_x, \sigma_y, \sigma_{xy}) f(z) dz, \\ (S_x, S_y) &= \int_z (\tau_{xz}, \tau_{yz}) g(z) dz. \end{aligned} \quad (19)$$

After integrating through the thickness of the plates and reorder in the matrix form

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \\ \mathbf{P} \\ \mathbf{S} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{E} & \mathbf{0} \\ \mathbf{B} & \mathbf{D} & \mathbf{F} & \mathbf{0} \\ \mathbf{E} & \mathbf{F} & \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_s \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa}^b \\ \boldsymbol{\kappa}^s \\ \boldsymbol{\gamma}^0 \end{Bmatrix}, \quad (20)$$

where

$$\begin{aligned}
\mathbf{N} &= \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix}, \\
\mathbf{M} &= \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix}, \\
\mathbf{P} &= \begin{bmatrix} P_x \\ P_y \\ P_{xy} \end{bmatrix}, \\
\mathbf{S} &= \begin{bmatrix} S_x \\ S_y \end{bmatrix},
\end{aligned} \tag{21}$$

$$(\mathbf{A}, \mathbf{B}, \mathbf{E}, \mathbf{D}, \mathbf{F}, \mathbf{H}) = \int_{-h/2}^{h/2} (1, z, f, z^2, zf, f^2) \mathbf{D}_b dz, \tag{22}$$

$$\mathbf{A} \mathbf{s} = \int_{-h/2}^{h/2} \mathbf{D}_s g^2 dz. \tag{23}$$

The reaction force of the Winkler–Pasternak is calculated by [68, 76]

$$R_f = k_w w - k_s \nabla^2 w, \tag{24}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, k_w is the stiffness of the Winkler's layer, and k_s is the stiffness of the shear layer. When $k_s = 0$, the Winkler–Pasternak's foundation model becomes the Winkler's foundation.

The variation of the work done by external distributed force and reaction force of the elastic foundations is calculated by [68, 76]

$$\delta V = \int_A (q - k_w w + k_s \nabla^2 w) \delta w dA. \tag{25}$$

Substituting equations (18) and (25) into equation (16) and integrating by parts, the equilibrium equations of the plates are obtained as follows:

$$\begin{aligned}
\delta u: \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0, \\
\delta v: \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} &= 0, \\
\delta \theta: \frac{\partial^2 P_x}{\partial x^2} + 2 \frac{\partial^2 P_{xy}}{\partial x \partial y} + \frac{\partial^2 P_y}{\partial y^2} - \frac{\partial S_x}{\partial x} - \frac{\partial S_y}{\partial y} &= 0, \\
\delta w: \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - \frac{\partial^2 P_x}{\partial x^2} - 2 \frac{\partial^2 P_{xy}}{\partial x \partial y} - \frac{\partial^2 P_y}{\partial y^2} + \frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} + (q - k_w w + k_s \nabla^2 w) &= 0.
\end{aligned} \tag{26}$$

After inserting equation (20) into equation (26), the governing equations of the plate are obtained as follows:

$$\begin{aligned}
(E_{12} + 2E_{33} - B_{12} - 2B_{33}) \frac{\partial^3 w}{\partial x \partial y^2} + (E_{12} + 2E_{33}) \frac{\partial^3 \theta}{\partial x \partial y^2} + (-B_{11} + E_{11}) \frac{\partial^3 w}{\partial x^3} \\
+ E_{11} \frac{\partial^3 \theta}{\partial x^3} + (A_{12} + A_{33}) \frac{\partial^2 v}{\partial x \partial y} + A_{11} \frac{\partial^2 u}{\partial x^2} + A_{33} \frac{\partial^2 u}{\partial y^2} = 0,
\end{aligned} \tag{27a}$$

$$\begin{aligned}
(E_{12} + 2E_{33} - B_{12} - 2B_{33}) \frac{\partial^3 w}{\partial x^2 \partial y} + (E_{12} + 2E_{33}) \frac{\partial^3 \theta}{\partial x^2 \partial y} + (-B_{22} + E_{22}) \frac{\partial^3 w}{\partial y^3} \\
+ E_{22} \frac{\partial^3 \theta}{\partial y^3} + (A_{12} + A_{33}) \frac{\partial^2 v}{\partial x \partial y} + A_{22} \frac{\partial^2 v}{\partial y^2} + A_{33} \frac{\partial^2 v}{\partial x^2} = 0,
\end{aligned} \tag{27b}$$

TABLE 1: The deflections and stresses of the square isotropic FGM plates without elastic foundation.

p	Method	w^*	σ_x^*	σ_y^*	τ_{xy}^*	τ_{xz}^*	τ_{yz}^*
Ceramic	Zenkour [31]	0.4665	2.8932	1.9103	1.2850	0.5114	0.4429
	Present	0.4665	2.8929	1.9104	1.2844	0.5127	0.4440
1	Zenkour [31]	0.9287	4.4745	2.1692	1.1143	0.5114	0.5446
	Present	0.9287	4.4740	2.1693	1.1140	0.5127	0.5459
2	Zenkour [31]	1.1940	5.2296	2.0338	0.9907	0.4700	0.5734
	Present	1.1940	5.2289	2.0338	0.9903	0.4710	0.5746
3	Zenkour [31]	1.3200	5.6108	1.8593	1.0047	0.4367	0.5629
	Present	1.3200	5.6100	1.8594	1.0043	0.4376	0.5640
4	Zenkour [31]	1.3890	5.8915	1.7197	1.0298	0.4204	0.5346
	Present	1.3890	5.8907	1.7198	1.0293	0.4213	0.5357
5	Zenkour [31]	1.4356	6.1504	1.6104	1.0451	0.4177	0.5031
	Present	1.4356	6.1496	1.6105	1.0446	0.4185	0.5040
6	Zenkour [31]	1.4727	6.4043	1.5214	1.0536	0.4227	0.4755
	Present	1.4727	6.4034	1.5215	1.0531	0.4235	0.4764
7	Zenkour [31]	1.5049	6.6547	1.4467	1.0589	0.4310	0.4543
	Present	1.5049	6.6537	1.4468	1.0584	0.4318	0.4551
8	Zenkour [31]	1.5343	6.8999	1.3829	1.0628	0.4399	0.4392
	Present	1.5342	6.8989	1.3830	1.0622	0.4407	0.4400
9	Zenkour [31]	1.5617	7.1383	1.3283	1.0662	0.4481	0.4291
	Present	1.5616	7.1373	1.3284	1.0656	0.4490	0.4299
10	Zenkour [31]	1.5876	7.3689	1.2820	1.0694	0.4552	0.4227
	Present	1.5875	7.3679	1.2821	1.0689	0.4561	0.4235
Metal	Zenkour [31]	2.5327	2.8932	1.9103	1.2850	0.5114	0.4429
	Present	2.5326	2.8929	1.9104	1.2844	0.5127	0.4440

$$\begin{aligned}
& (2H_{12} + 4H_{33} - 2F_{12} - 4F_{33}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + (2H_{12} + 4H_{33}) \frac{\partial^4 \theta}{\partial x^2 \partial y^2} + \\
& (-F_{11} + H_{11}) \frac{\partial^4 w}{\partial x^4} + (-F_{22} + H_{22}) \frac{\partial^4 w}{\partial y^4} + H_{11} \frac{\partial^4 \theta}{\partial x^4} + H_{22} \frac{\partial^4 \theta}{\partial y^4} + \\
& (E_{12} + 2E_{33}) \frac{\partial^3 u}{\partial x \partial y^2} + (E_{12} + 2E_{33}) \frac{\partial^3 v}{\partial x^2 \partial y} - As_{22} \frac{\partial^2 w}{\partial y^2} - As_{22} \frac{\partial^2 \theta}{\partial y^2} - \\
& As_{11} \frac{\partial^2 w}{\partial x^2} - As_{11} \frac{\partial^2 \theta}{\partial x^2} + E_{11} \frac{\partial^3 u}{\partial x^3} + E_{22} \frac{\partial^3 v}{\partial y^3} = 0,
\end{aligned} \tag{27c}$$

$$\begin{aligned}
& -(2H_{12} + 4H_{33} + 2D_{12} + 4D_{33} - 4F_{12} - 8F_{33}) \frac{\partial^4 w}{\partial x^2 \partial y^2} - (2H_{12} + 4H_{33} - \\
& 2F_{12} - 4F_{33}) \frac{\partial^4 \theta}{\partial x^2 \partial y^2} - (H_{11} + D_{11} - 2F_{11}) \frac{\partial^4 w}{\partial x^4} - (D_{22} - 2F_{22} + H_{22}) \frac{\partial^4 w}{\partial y^4} - \\
& (-F_{11} + H_{11}) \frac{\partial^4 \theta}{\partial x^4} - (-F_{22} + H_{22}) \frac{\partial^4 \theta}{\partial y^4} - (E_{12} + 2E_{33} - B_{12} - 2B_{33}) \frac{\partial^3 u}{\partial x \partial y^2} - \\
& (E_{12} + 2E_{33} - B_{12} - 2B_{33}) \frac{\partial^3 v}{\partial x^2 \partial y} - (-B_{11} + E_{11}) \frac{\partial^3 u}{\partial x^3} - (E_{22} - B_{22}) \frac{\partial^3 v}{\partial y^3} - \\
& (-As_{11} - k_s) \frac{\partial^2 w}{\partial x^2} - (As_{22} - k_s) \frac{\partial^2 w}{\partial y^2} + As_{22} \frac{\partial^2 \theta}{\partial y^2} + As_{11} \frac{\partial^2 \theta}{\partial x^2} + q - k_w w = 0.
\end{aligned} \tag{27d}$$

2.2.4. Navier's Solution. In this study, a fully simple supported FGM plate subjected to a distributed transverse load is considered. The Navier's solution technique is

employed to solve the equations of motion, the unknown displacement functions of the plates are assumed as in the following formulae:

TABLE 2: The deflections and stresses of the square FGM plates with different values of the side-to-thickness ratio.

a/h	K_w	K_s	Method	w^*	σ_x^*	τ_{xy}^*	τ_{xz}^*
5	0	0	Ameur [68]	0.3431864	1.0272580	0.3491766	0.2455716
			Present	0.3431833	1.0272483	0.3491731	0.2455712
	100	0	Ameur [68]	0.2611226	0.7816172	0.2656804	0.1868497
			Present	0.2611208	0.7816111	0.2656783	0.1868498
	0	10	Ameur [68]	0.2117976	0.6339728	0.2154944	0.1515546
			Present	0.2117964	0.6339689	0.2154931	0.1515549
	100	10	Ameur [68]	0.1773918	0.5309862	0.1804881	0.1269350
			Present	0.1773910	0.5309832	0.1804871	0.1269354
	10	10	Ameur [68]	0.1638971	1.1048040	0.3911616	0.1362969
			Present	0.1638962	1.1047974	0.3911595	0.1362973
	20	10	Ameur [68]	0.1602066	2.2330340	0.7988330	0.1388572
			Present	0.1602057	2.2330216	0.7988292	0.1388576
50	100	10	Ameur [68]	0.1591483	5.5993760	2.0089200	0.1395914
			Present	0.1591473	5.5993491	2.0089096	0.1395918
100	100	10	Ameur [68]	0.1589961	11.2035900	4.0212420	0.1396969
			Present	0.1589952	11.2035259	4.0212215	0.1396974

TABLE 3: The deflections and stresses of the square FGM plates with different values of the power-law exponent.

p	K_w	K_s	Method	w^*	σ_x^*	τ_{xy}^*	τ_{xz}^*
Ceramic	0	0	Ameur [68]	0.2960316	1.9955010	0.7065175	0.2461800
			Present	0.2960287	1.9954804	0.7065107	0.2461796
	100	0	Ameur [68]	0.2328956	1.5699110	0.5558353	0.1936761
			Present	0.2328938	1.5698988	0.5558312	0.1936762
	0	10	Ameur [68]	0.1928403	1.2999050	0.4602384	0.1603661
			Present	0.1928391	1.2998964	0.4602354	0.1603664
	100	10	Ameur [68]	0.1638971	1.1048040	0.3911616	0.1362969
			Present	0.1638962	1.1047974	0.3911595	0.1362973
	0	0	Ameur [68]	0.5889103	3.0869970	0.6110370	0.2461801
			Present	0.5889055	3.0869638	0.6110329	0.2461796
	100	0	Ameur [68]	0.3825844	2.0054610	0.3969590	0.1599303
			Present	0.3825824	2.0054456	0.3969574	0.1599306
1	0	10	Ameur [68]	0.2852521	1.4952560	0.2959696	0.1192429
			Present	0.2852509	1.4952473	0.2959689	0.1192432
	100	10	Ameur [68]	0.2261716	1.1855640	0.2346694	0.0945457
			Present	0.2261709	1.1855574	0.2346690	0.0945460
	0	0	Ameur [68]	0.9118358	4.2488300	0.5754613	0.2016656
			Present	0.9118252	4.2487616	0.5754556	0.2016651
	100	0	Ameur [68]	0.4969093	2.3154170	0.3136003	0.1098986
			Present	0.4969062	2.3153954	0.3135990	0.1098990
	0	10	Ameur [68]	0.3443160	1.6043900	0.2172985	0.0761504
			Present	0.3443145	1.6043758	0.2172979	0.0761508
	100	10	Ameur [68]	0.2617760	1.2197820	0.1652073	0.0578955
			Present	0.2617752	1.2197735	0.1652071	0.0578959
5	0	0	Ameur [68]	1.6070280	1.9955010	0.7065178	0.2461801
			Present	1.6069441	1.9954804	0.7065107	0.2461796
	100	0	Ameur [68]	0.6501876	0.8073596	0.2858501	0.0996020
			Present	0.6501739	0.8073528	0.2858497	0.0996041
	0	10	Ameur [68]	0.4115419	0.5110251	0.1809312	0.0630440
			Present	0.4115363	0.5110249	0.1809324	0.0630458
	100	10	Ameur [68]	0.2988967	0.3711501	0.1314078	0.0457879
			Present	0.2988938	0.3711511	0.1314090	0.0457894
	0	0	Ameur [68]	1.6070280	1.9955010	0.7065178	0.2461801
			Present	1.6069441	1.9954804	0.7065107	0.2461796
	100	0	Ameur [68]	0.6501876	0.8073596	0.2858501	0.0996020
			Present	0.6501739	0.8073528	0.2858497	0.0996041
Metal	0	10	Ameur [68]	0.4115419	0.5110251	0.1809312	0.0630440
			Present	0.4115363	0.5110249	0.1809324	0.0630458
	100	10	Ameur [68]	0.2988967	0.3711501	0.1314078	0.0457879
			Present	0.2988938	0.3711511	0.1314090	0.0457894

TABLE 4: The effects of the power-law exponent on the deflection and stresses of the plates.

K_w	K_s	p	w^*	σ_x^*	σ_y^*	τ_{xy}^*	τ_{xz}^*	τ_{yz}^*
100	0	Ceramic	0.364386	2.208369	1.460092	1.038796	0.426826	0.369642
		1	0.594965	2.736732	1.330657	0.764806	0.371460	0.395564
		2	0.692440	2.857032	1.116230	0.626650	0.319020	0.389185
		3	0.732030	2.913056	0.971222	0.612666	0.287642	0.370742
		4	0.751997	2.976602	0.875132	0.615773	0.272492	0.346511
		5	0.764895	3.051605	0.805419	0.616827	0.267846	0.322594
		6	0.774849	3.132943	0.750606	0.615567	0.268848	0.302425
		7	0.783344	3.216196	0.705330	0.613308	0.272220	0.286927
		8	0.790969	3.298541	0.667022	0.610812	0.276128	0.275686
		9	0.797986	3.378347	0.634339	0.608415	0.279716	0.267830
		10	0.804524	3.454744	0.606471	0.606240	0.282672	0.262467
0	10	Metal	0.986867	0.994403	0.661150	0.595750	0.271301	0.234954
		Ceramic	0.300884	1.808923	1.196394	0.871845	0.365029	0.316124
		1	0.441970	2.011672	0.978500	0.582873	0.291983	0.310930
		2	0.493291	2.016733	0.788144	0.458342	0.241784	0.294962
		3	0.512922	2.026102	0.675587	0.440051	0.214279	0.276184
		4	0.522575	2.056313	0.604537	0.437884	0.200941	0.255524
		5	0.528729	2.099420	0.554003	0.435699	0.196133	0.236222
		6	0.533440	2.148575	0.514607	0.432584	0.195820	0.220276
		7	0.537434	2.199645	0.482197	0.429174	0.197427	0.208094
		8	0.540999	2.250323	0.454831	0.425867	0.199545	0.199225
		9	0.544262	2.299377	0.431505	0.422818	0.201513	0.192950
		10	0.547287	2.346189	0.411616	0.420069	0.203088	0.188571
100	10	Metal	0.625946	0.639555	0.424581	0.382653	0.182711	0.158232
		Ceramic	0.254115	1.496843	0.991060	0.758839	0.325433	0.281833
		1	0.346649	1.519772	0.740904	0.482092	0.251036	0.267326
		2	0.376608	1.471052	0.576833	0.372580	0.205713	0.250958
		3	0.387463	1.457653	0.488145	0.355217	0.181565	0.234018
		4	0.392635	1.469414	0.434167	0.352157	0.169877	0.216022
		5	0.395878	1.493847	0.396368	0.349550	0.165565	0.199406
		6	0.398347	1.523896	0.367099	0.346423	0.165125	0.185748
		7	0.400441	1.555859	0.343091	0.343192	0.166349	0.175336
		8	0.402313	1.587809	0.322849	0.340128	0.168029	0.167760
		9	0.404028	1.618756	0.305603	0.337333	0.169602	0.162395
		10	0.405618	1.648222	0.290894	0.334822	0.170858	0.158645
		Metal	0.445203	0.422254	0.281421	0.299906	0.153335	0.132792

$$u(x, y, t) = \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} U_{\alpha\beta} \cos \eta_{\alpha} x \sin \vartheta_{\beta} y, \quad \begin{cases} \Omega_{\alpha\beta} = q_0, & \alpha = \beta = 1, \\ \Omega_{\alpha\beta} = 0, & \alpha \neq 1, \beta \neq 1. \end{cases} \quad (30)$$

$$v(x, y, t) = \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} V_{\alpha\beta} \sin \eta_{\alpha} x \cos \vartheta_{\beta} y, \quad (28)$$

$$\theta(x, y, t) = \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \Theta_{\alpha\beta} \sin \eta_{\alpha} x \sin \vartheta_{\beta} y,$$

$$w(x, y, t) = \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} W_{\alpha\beta} \sin \eta_{\alpha} x \sin \vartheta_{\beta} y,$$

where $\eta_{\alpha} = \alpha\pi/a$ and $\vartheta_{\beta} = \beta\pi/b$.

The transverse distributed load is expanded as follows [3]:

$$q(x, y) = \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \Omega_{\alpha\beta} \sin \eta_{\alpha} x \sin \vartheta_{\beta} y, \quad (29)$$

where $\Omega_{\alpha\beta}$ depends on the types of the load. In the case of double sinusoidal load, $\Omega_{\alpha\beta}$ are calculated as follows:

In the case of uniform load, $\Omega_{\alpha\beta}$ are calculated by

$$\begin{cases} \Omega_{\alpha\beta} = \frac{16q_0}{\alpha\beta\pi^2}, & \alpha, \beta \text{ even}, \\ 0, & \alpha, \beta \text{ odd}. \end{cases} \quad (31)$$

Substituting equations (28) and (29) into equations (27a)–(27d), one gets

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ & k_{22} & k_{23} & k_{24} \\ & & k_{33} & k_{34} \\ sys & & & k_{44} \end{bmatrix} \begin{bmatrix} U_{\alpha\beta} \\ V_{\alpha\beta} \\ \Theta_{\alpha\beta} \\ W_{\alpha\beta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \Omega_{\alpha\beta} \end{bmatrix}, \quad (32)$$

where

FIGURE 2: The variation of deflection and stresses as the functions of the power-law exponent.

$$\begin{aligned}
 k_{11} &= A_{11}\eta^2 + A_{33}\vartheta^2; k_{12} = (A_{12} + A_{33})\eta\vartheta, \\
 k_{13} &= E_{11}\eta^3 + (E_{12} + 2E_{33})\vartheta^2\eta, \\
 k_{14} &= (-B_{11} + E_{11})\eta^3 + (E_{12} + 2E_{33} - B_{12} - 2B_{33})\vartheta^2\eta, \\
 k_{22} &= A_{22}\vartheta^2 + A_{33}\eta^2; k_{23} = E_{22}\vartheta^3 + (E_{12} + 2E_{33})\eta^2\vartheta, \\
 k_{24} &= (E_{22} - B_{22})\vartheta^3 + (E_{12} + 2E_{33} - B_{12} - 2B_{33})\eta^2\vartheta, \\
 k_{33} &= H_{11}\eta^4 + ((2H_{12} + 4H_{33})\vartheta^2 + As_{11})\eta^2 + (H_{22}\vartheta^2 + As_{22})\vartheta^2, \\
 k_{34} &= (-F_{11} + H_{11})\eta^4 + ((2H_{12} + 4H_{33} - 2F_{12} - 4F_{33})\vartheta^2 + As_{11})\eta^2 + ((H_{22} - F_{22})\vartheta^2 + As_{22})\vartheta^2, \\
 k_{44} &= (-2F_{11} + H_{11} + D_{11})\eta^4 + ((4D_{33} - 4F_{12} - 8F_{33} + 2H_{12} + 4H_{33} + 2D_{12})\vartheta^2 + k_s + As_{11})\eta^2 \\
 &\quad + (-2F_{22} + H_{22} + D_{22})\vartheta^4 + (k_s + As_{22})\vartheta^2 + k_w.
 \end{aligned} \tag{33}$$

FIGURE 3: The distribution of the stresses through the thickness of the FGM plates.

3. Numerical Results

3.1. Validation Study

3.1.1. Static Bending of FGM Plates without Elastic Foundation. Firstly, the proposed plate theory is applied to analyze a square isotropic FGM plate of $\text{Al}/\text{Al}_2\text{O}_3$ subjected to uniform load. In which, Young's modulus of

Al is 70 GPa and it of Al_2O_3 is 380 GPa , while the Poisson's ratios are constant and equal to 0.3, the side-to-thickness ratio is $a/h = 10$. The nondimensional deflection and stresses of the plates using proposed plate theory are compared with those of Zenkour [31], they are given in Table 1. The nondimensional quantities are calculated by [31].

TABLE 5: The effects of side-to-thickness ratio on the deflection and stresses of the plates.

K_w	K_s	a/h	w^*	σ_x^*	σ_y^*	τ_{xy}^*	τ_{xz}^*	τ_{yz}^*
0	0	5	0.130586	1.143516	0.536764	0.273727	0.251337	0.267646
		10	0.928727	4.473980	2.169274	1.114001	0.512669	0.545936
		20	7.197403	17.795845	8.699313	4.478998	1.035194	1.102368
		50	111.442044	111.049014	54.409561	28.041098	2.601379	2.770184
		100	890.373479	444.096299	217.660390	112.195139	5.208993	5.547007
100	0	5	0.120983	1.052603	0.494826	0.256145	0.237046	0.252428
		10	0.594965	2.736732	1.330657	0.764806	0.371460	0.395564
		20	1.276133	2.153282	1.059513	1.225915	0.374607	0.398915
		50	1.215731	-0.429615	-0.210124	1.892050	0.418211	0.445349
		100	1.220230	0.027689	0.013682	2.682102	0.497889	0.530197
0	10	5	0.113048	0.981569	0.461617	0.240292	0.223360	0.237854
		10	0.441970	2.011672	0.978500	0.582873	0.291983	0.310930
		20	0.737072	1.488802	0.729602	0.669769	0.224426	0.238989
		50	0.800137	0.600756	0.294553	0.467580	0.129841	0.138266
		100	0.803947	0.300253	0.147265	0.305612	0.078395	0.083482
100	10	5	0.105735	0.912435	0.429714	0.226881	0.212451	0.226237
		10	0.346649	1.519772	0.740904	0.482092	0.251036	0.267326
		20	0.492518	0.870810	0.427556	0.527452	0.194735	0.207371
		50	0.511561	0.320786	0.157370	0.392936	0.123098	0.131086
		100	0.512479	0.159363	0.078210	0.267070	0.076574	0.081543

FIGURE 4: The variation of the deflection and stresses as the functions of side-to-thickness ratio without elastic foundation ($K_w = 0$, $K_s = 0$).

FIGURE 5: The variation of the deflection and stresses as the functions of the side-to-thickness ratio.

FIGURE 6: The deflection shapes of the plate in two case (a) $K_w = 100, K_s = 0, p = 1, a/h = 33$ and (b) $K_w = 100, K_s = 0, p = 1, a/h = 46$.

TABLE 6: The effects of aspect ratio a/b on the deflection and stresses of the plates.

K_w	K_s	a/b	w^*	σ_x^*	σ_y^*	τ_{xy}^*	τ_{xz}^*	τ_{yz}^*
0	0	1.0	0.928727	4.473980	2.169274	1.114001	0.512669	0.545936
		1.5	0.357803	2.078198	1.630126	0.651568	0.365627	0.455065
		2.0	0.153645	1.093877	1.144985	0.392486	0.276125	0.372635
		3.0	0.040237	0.434810	0.589102	0.175661	0.181480	0.260084
100	0	1.0	0.594965	2.736732	1.330657	0.764806	0.371460	0.395564
		1.5	0.293549	1.659870	1.322282	0.552956	0.321559	0.387350
		2.0	0.140032	0.982590	1.039558	0.365047	0.261607	0.344510
		3.0	0.039163	0.421653	0.573021	0.172374	0.179012	0.254131
0	10	1.0	0.441970	2.011672	0.978500	0.582873	0.291983	0.310930
		1.5	0.210707	1.162850	0.937033	0.412517	0.250554	0.294494
		2.0	0.104255	0.713601	0.767342	0.283341	0.211786	0.267873
		3.0	0.031862	0.338412	0.464660	0.145296	0.155178	0.212075
100	10	1.0	0.346649	1.519772	0.740904	0.482092	0.251036	0.267326
		1.5	0.186119	1.004597	0.819613	0.374334	0.233217	0.268550
		2.0	0.097677	0.660560	0.716501	0.269849	0.204507	0.254309
		3.0	0.031177	0.330115	0.454410	0.143146	0.153536	0.208284

$$\begin{aligned}
w^* &= \frac{10h^3 E_c}{q_0 a^4} w\left(\frac{a}{2}, \frac{b}{2}\right), \\
\sigma_x^* &= \frac{h}{q_0 a} \sigma_x\left(\frac{a}{2}, \frac{b}{2}, \frac{h}{2}\right), \\
\sigma_y^* &= \frac{h}{q_0 a} \sigma_y\left(\frac{a}{2}, \frac{b}{2}, \frac{h}{3}\right), \\
\tau_{xy}^* &= \frac{h}{q_0 a} \tau_{xy}\left(0, 0, -\frac{h}{3}\right), \\
\tau_{xz}^* &= \frac{h}{q_0 a} \tau_{xz}\left(0, \frac{b}{2}, 0\right), \\
\tau_{yz}^* &= \frac{h}{q_0 a} \tau_{yz}\left(\frac{a}{2}, 0, \frac{h}{6}\right).
\end{aligned} \tag{34}$$

The comparison shows that the results of the proposed plate theory agree very well with those of Zenkour [31] for all cases of the power-law exponent.

3.1.2. Static Bending of FGM Plates Resting on Winkler–Pasternak Foundation. Secondly, a comparison of the deflections and stresses of the rectangular FGM plates with $p = 0$ is demonstrated in Table 2. The plates are made of Al/Al₂O₃ with Young's modulus of Al is 70 GPa, and it of Al₂O₃ is 380 GPa, while the Poisson's ratios are constant and equal to 0.3. The plates are subjected to sinusoidal load and resting on the two parameters elastic foundation. The nondimensional quantities are calculated using equation (34), while the

nondimensional of two elastic foundation parameters are computed by [68]

$$\begin{aligned}
K_w &= \frac{a^4}{R} k_w, \\
K_s &= \frac{a^2}{R} k_s, \\
R &= \frac{E_c h^3}{12(1 - \nu^2)}.
\end{aligned} \tag{35}$$

In which, the numerical results of the plates using the proposed plate theory are compared with the results of Ameer [68] using a simple sinusoidal plate theory. For all cases of side-to-thickness ratio, the present numerical results are very close to those of Ameer [68].

Thirdly, the effects of the power-law exponent on the deflections and stresses of the square FGM plates are demonstrated in Table 3, in which $a/h = 10$. According to Table 3, the present results are in good agreement with those of Ameer [68] for all cases of the power-law exponent and elastic foundation parameters.

3.2. Parameter Study. In this section, a rectangular FGM plate made of Al/Al₂O₃ with the dimension of $a \times b$ and the thickness of h resting on Winkler–Pasternak foundation is considered. The material properties of Al/Al₂O₃ are similar to those of Zenkour [31] as follows

For Al: $E_m = 70$ GPa, $\nu_m = 0.3$.

For Al₂O₃: $E_c = 380$ GPa, $\nu_c = 0.3$.

The following nondimensional quantities are used for convenience [68]

FIGURE 7: The variation of the deflection and stresses as the functions of aspect ratio.

TABLE 7: The influence of two elastic foundation parameters on the deflection and stresses of the plates.

p	K_w	K_s	w^*	σ_x^*	σ_y^*	τ_{xy}^*	τ_{xz}^*	τ_{yz}^*
Ceramic	0	0	0.466544	2.892900	1.910362	1.284360	0.512669	0.443984
		5	0.366110	2.233322	1.476016	1.035604	0.423956	0.367156
		10	0.300884	1.808923	1.196394	0.871845	0.365029	0.316124
	50	0	0.409357	2.509476	1.658164	1.146996	0.464665	0.402412
		5	0.329621	1.989165	1.315400	0.947737	0.393215	0.340534
		10	0.275596	1.640065	1.085299	0.810796	0.343646	0.297607
	100	0	0.364386	2.208369	1.460092	1.038796	0.426826	0.369642
		5	0.299575	1.788415	1.183327	0.875256	0.367837	0.318557
		10	0.254115	1.496843	0.991060	0.758839	0.325433	0.281833
0.5	0	0	0.715351	3.797329	2.123720	1.260322	0.524196	0.507197
		5	0.503375	2.606894	1.459841	0.924420	0.398993	0.386054
		10	0.387442	1.965189	1.101672	0.736806	0.328112	0.317472
	50	0	0.589072	3.072212	1.719868	1.066647	0.453494	0.438788
		5	0.436582	2.224568	1.246861	0.821585	0.361388	0.349669
		10	0.346201	1.729868	0.970556	0.673061	0.304761	0.294878
	100	0	0.499918	2.561386	1.435324	0.929555	0.403390	0.390309
		5	0.385028	1.930177	1.082841	0.741983	0.332242	0.321468
		10	0.312649	1.538904	0.864138	0.621046	0.285681	0.276417
1.5	0	0	1.085886	4.927921	2.119488	1.021226	0.492386	0.566835
		5	0.661782	2.896759	1.248715	0.660779	0.336218	0.387054
		10	0.474174	2.018122	0.871446	0.496086	0.263344	0.303162
	50	0	0.818781	3.616826	1.558387	0.802205	0.399647	0.460073
		5	0.550194	2.351722	1.015368	0.568731	0.297138	0.342065
		10	0.413220	1.721814	0.744542	0.445510	0.241815	0.278377
	100	0	0.655151	2.816505	1.215781	0.667466	0.342489	0.394273
		5	0.469934	1.961241	0.848141	0.502217	0.268840	0.309489
		10	0.365692	1.491696	0.645955	0.405885	0.224911	0.258917
5.5	0	0	1.454942	6.276915	1.563968	1.049444	0.420340	0.489465
		5	0.781004	3.222208	0.806585	0.606118	0.259858	0.302592
		10	0.531203	2.123554	0.533247	0.434216	0.195607	0.227775
	50	0	1.010438	4.205625	1.052053	0.769002	0.321711	0.374616
		5	0.628729	2.517448	0.632253	0.509231	0.225652	0.262760
		10	0.454964	1.772825	0.446424	0.385332	0.178287	0.207606
	100	0	0.770115	3.091798	0.776578	0.616394	0.267885	0.311938
		5	0.524707	2.038749	0.513753	0.442585	0.202049	0.235276
		10	0.397176	1.508456	0.380932	0.348025	0.165027	0.192166
Metal	0	0	2.532557	2.892900	1.910362	1.284360	0.512669	0.443984
		5	1.009629	1.074906	0.712260	0.575406	0.256332	0.221990
		10	0.625946	0.639555	0.424581	0.382653	0.182711	0.158232
	50	0	1.431068	1.535900	1.017604	0.795403	0.341561	0.295800
		5	0.765990	0.778232	0.516957	0.465679	0.217676	0.188513
		10	0.521117	0.513032	0.341247	0.334892	0.165792	0.143580
	100	0	0.986867	0.994403	0.661150	0.595750	0.271301	0.234954
		5	0.614212	0.595309	0.396460	0.396463	0.193156	0.167278
		10	0.445203	0.422254	0.281421	0.299906	0.153335	0.132792

FIGURE 8: The influence of two parameters of the elastic foundations on the deflection of the plates with $a/h = 10$, $p = 1$.

$$\begin{aligned}
 w^* &= \frac{10h^3 E_c}{q_0 a^4} w\left(\frac{a}{2}, \frac{b}{2}\right), \\
 \sigma_x^* &= \frac{h}{q_0 a} \sigma_x\left(\frac{a}{2}, \frac{b}{2}, \frac{h}{2}\right), \\
 \sigma_y^* &= \frac{h}{q_0 a} \sigma_y\left(\frac{a}{2}, \frac{b}{2}, \frac{h}{3}\right), \\
 \tau_{xy}^* &= \frac{h}{q_0 a} \tau_{xy}\left(0, 0, -\frac{h}{3}\right), \\
 \tau_{xz}^* &= \frac{h}{q_0 a} \tau_{xz}\left(0, \frac{b}{2}, 0\right), \\
 \tau_{yz}^* &= \frac{h}{q_0 a} \tau_{yz}\left(\frac{a}{2}, 0, \frac{h}{6}\right), \\
 K_w &= \frac{a^4}{R} k_w, K_s = \frac{a^2}{R} k_s,
 \end{aligned} \tag{36}$$

where $R = E_c h^3 / 12 (1 - \nu^2)$.

3.2.1. The Effects of the Power-Law Exponent. In this subsection, the influence of the power-law exponent on the static bending behavior of the square FGM plates with $a/h = 10$ resting on the Winkler–Pasternak foundations is investigated. The numerical results of the deflections and stresses are demonstrated in Table 4 and Figure 2. It can be seen clearly that the central deflections of the FGM plates and normal stress σ_x^* , σ_y^* increase as the power-law index increase for all cases of two parameters of elastic foundations. Besides, the elastic foundations have strong effects on the bending behavior of the plates. The effects of the shear layer are stronger than the Winkler layer. However, the in-plane shear stress τ_{xy}^* and transverse shear stress τ_{xz}^* , τ_{yz}^* first

decrease and then increase with the increase of p . Especially in the case of the plate without elastic foundation, there is a small range of the power-law index in which the in-plane shear stress τ_{xy}^* and transverse shear stress τ_{xz}^* , τ_{yz}^* increase when the power-law index increases, and then they decrease as shown in Figure 2.

Figure 3 displays the distribution of the stresses through the thickness of the square FGM plates resting on Winkler–Pasternak foundations with different values of the power-law index in the case of $K_w = 100$, $K_s = 10$. It is seen that the distribution of normal stresses σ_x^* , σ_y^* and in-plane shear stress τ_{xy}^* is linear when $p = 0$ and is nonlinear when $p > 0$. Besides, the distribution of the transverse shear stress τ_{xz}^* is parabolical and symmetrical through the thickness when $p = 0$, and unsymmetrical when $p > 0$. In addition, the transverse shear stresses equal to zero at the top and bottom surfaces of the FGM plates.

3.2.2. The Effects of Side-to-thickness Ratio. Continuously, the effects of the side-to-thickness ratio on the bending behavior of the square FGM plate resting on the Winkler–Pasternak foundation are investigated. The power-law index of the material is $p = 1$ and four cases of two parameters of Winkler–Pasternak foundations are considered. The numerical results of the effects of a/h ratios are shown in Table 5, Figures 4 and 5. It is noticed that the thickness h in the nondimensional formulae is fixed and equal to $a/10$ in this subsection. As a consequence, two parameters of the Winkler–Pasternak foundations are constant as the varying of a/h ratio. It is obvious that the deflections and stresses of the FGM plates without elastic foundations increase rapidly when a/h ratio increases as shown in Table 5 and Figure 4.

According to Figure 5, it can be seen that when the FGM plates are resting on elastic foundation, the central deflections of the FGM plates increase at a lower speed in comparison with the case of FGM plates without an elastic foundation. Moreover, the elastic foundations have strong

FIGURE 9: The influence of two parameters of the elastic foundations on the distribution of the stresses with $a/h = 10$, $p = 1$.

effects on the stresses of the FGM plates. Especially in the case of $K_w = 100, K_s = 0$ (Winkler type foundation), the normal stress σ_x^* is positive when a/h ratio is small and becomes negative when a/h ratio is greater. It means that when the plate is thick, the deflection at the centre of the plate is local convex, and when the plate is very thin, the deflection is local concave at the centre of the plate. It is a special phenomenon of the plates resting on the elastic foundation in comparison with the plates without elastic foundation supported. In this numerical study, the normal stress σ_x^* approximates to zero when $a/h = 33$ (critical point) and the minimum normal stress σ_x^* (compressive stress) occurs when $a/h = 46$. Figure 6 shows the deflection shapes of the FGM plates in two cases of $K_w = 100, K_s = 0, a/h = 33, p = 1$ and $K_w = 100, K_s = 0, a/h = 46, p = 1$.

3.2.3. The Effects of Aspect Ratio. Table 6 and Figure 7 depict the influence of the aspect ratio on the bending behavior of the square FGM plate with $p = 1$ resting on Winkler–Pasternak foundations. When the aspect ratio increases, the deflections and stresses of the FGM plates resting on the Winkler–Pasternak foundation decrease. It is obvious that the elastic foundations have significant effects on the bending behavior of the square FGM plates.

3.2.4. The Effects of the Elastic Foundations. The influences of two parameters of Winkler–Pasternak foundations on the deflection and stresses of the square FGM plates are demonstrated in Table 7, Figures 8 and 9. The side-to-thickness ratio equals $a/h = 10$. According to Table 7 and Figure 8, when the values of two parameters of Winkler–Pasternak foundations increase, the deflections and stresses of the plates decrease. Besides, the distribution of the stresses through the FGM plates' thickness also depends on the varying of two foundation parameters. Figure 9 shows that the varying of these parameters leads to the change of the values of the stresses of the plates. However, the shape of the

distribution of the stresses is almost unchanged. As a consequence, the neutral surface of the FGM plate is independent of the effects of the elastic foundations.

4. Conclusions

In conclusion, the new sinusoidal shear deformation plate theory has been developed successfully with some advantages such as simpler, more efficient, and high accuracy in predicting the bending behavior of the FGM plates resting on Winkler–Pasternak foundations. The proposed theory consists of only four unknown variables and the transverse displacement does not separate into bending part and shear part, which means the computing cost and time can be reduced. Besides, the proposed theory can change into other plate theories easily. The numerical results show that the Winkler–Pasternak foundations have strong effects on the bending behavior of the FGM plates, and cause some special effects on the displacement and stresses of the FGM plates. These phenomena must be considered when designing, testing, and examining the FGM plates to avoid resonance.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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