

Research Article

Analytical Solution of Three-Dimensional Heaving Displacement of Ground Surface during Tunnel Freezing Construction Based on Stochastic Medium Theory

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Many studies indicate that the heaving displacement during tunnel freezing construction can be regarded as a plane strain problem. However, due to the fact that many of the frozen pipes are placed obliquely in practical engineering, it should be a three-dimensional problem. Therefore, considering stochastic medium theory, the analytical solution of three-dimensional heaving displacement of ground surface during tunnel freezing construction is established, and the analytical solution is applied to the project case in Beijing. The calculation results show that, considering the inclination angle between the freezing pipe and the center axis of the tunnel, the maximum heaving displacement of the ground surface occurs at the tunnel center. When the inclination angle is equal to 0°, the analytical solution of three-dimensional ground surface heaving displacement is in good agreement with that of the two-dimensional prediction model, which further verified the reliability of the established analytical solution of the three-dimensional frost heave prediction model.

1. Introduction

On the basis of the sandbox model test, the Polish scholar Litwiniszyn [1] proposed a stochastic medium theory to investigate the displacement of rock strata and ground surface during underground mining in the 1950s; in this theory, rock formation is regarded as a kind of “stochastic medium”, and the deformation of rock formation and surface caused by underground mining is regarded as a stochastic process and the prediction of rock formation and surface deformation can be realized by using probability and statistics method. After the development and improvement of Chinese scholar Liu et al. [2, 3], the application field of stochastic medium theory has developed from the analysis of surface deformation caused by underground mining to the analysis of surface deformation caused by open-pit mining, near surface excavation, and stratum drainage. Yang et al. [4] and Yang et al. [5] analyzed and discussed in detail the surface movement and deformation caused by urban tunnel

construction by using stochastic medium theory, further expanding the application field of stochastic medium theory. Due to the deformation of the surrounding rock caused by shield construction of subway tunnel in soft stratum environment, surface subsidence and shield segment rupture problems occur frequently [6]. Shi et al. [7, 8] considered the time-space development process of ground movement and deformation in tunnel excavation and, based on the stochastic medium theory, systematically researched the issues related to ground movement and deformation caused by mining method and shield method construction of urban tunnel. At present, the theory has become one of the practical methods to predict the surface deformation in urban metro tunnel engineering in China.

If ground frost heave induced by soil freezing is considered the inverse process of ground settlement [9] caused by soil excavation, then ground frost heave during tunnel freezing construction can also be predicted via the stochastic medium theory. Based on the stochastic medium theory and

shear displacement method, Zeng and Huang [10] established a calculation model of soil deformation caused by double-O-tube shield tunnel construction. Yang et al. [4], Li [11], Tao et al. [12], and Zhou et al. [13] used stochastic medium theory to analyze the surface frost heaving deformation caused by the freezing method of subway tunnel construction and obtained some meaningful conclusions. However, the common deficiency is that freezing process is not considered in the analysis process. In the study of Li et al [14], considering the frost heaving force and frost heaving deformation of noncircular tunnel with lining support, the analytical solution of frost heaving force is obtained by the theory of complex function and continuity condition. In the study of Huang et al. [15], the analytical solution of circular tunnel frost heaving force is derived on the basis of the nonuniform frost heaving of tunnel surrounding rock. In the study of Cai [16–18] based on single-pipe freezing theory, flat-panel freezing theory, and calculation equation of frost heave ratio of soil, considering the formation process of the frozen wall and using stochastic medium theory, a two-dimensional analytical solution of ground frost heave in horizontal freezing period of tunnel is established and the analytical solution is improved in this study by considering the freezing point of soil and assuming the constant surface temperature of the freezing pipe, the analytical solution methods for the radius of the freezing front and the outer radius of the expanded area are determined and applied to practical engineering cases, and then, the reliability of the analytical solution was verified by comparing with the field measurement results [19].

In the actual construction of the subway tunnel freezing method, the freezing pipe group is generally designed to be inclined and radial, which is not strictly horizontal freezing [20–23]. Therefore, the research only on the three-dimensional frost heave deformation of the ground can reflect the actual situation of the project based on the analysis of the three-dimensional freezing temperature field of the ground.

In this paper, an analytical solution method of three-dimensional frost heave deformation is established considering the inclined radial layout characteristics of the freezing pipe group in the actual tunnel construction. It is an improvement on the analytical solution of two-dimensional with the stochastic medium theory that the temperature field before the intersection of the frozen wall is calculated by single-pipe freezing theory and the temperature field after the intersection of the frozen wall is calculated by flat-panel freezing theory.

2. Analytical Solution for Ground Frost Heave

2.1. Ground Surface Settlement. Figure 1 shows that the length, width, and thickness of the microelement are $d\xi$, $d\eta$, and $d\zeta$, respectively. The distance between the center of the microelement and the surface is η . According to the stochastic medium theory, the vertical displacement of the soil at any point above the tunnel caused by the excavation of a single microelement can be obtained as

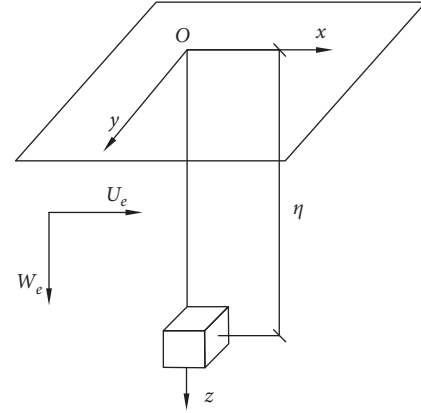


FIGURE 1: Unit volume diagram.

$$dW_e(x, y, z) = \frac{1}{r^2(z)} \exp\left[-\frac{\pi}{r^2(z)}(x^2 + y^2)\right] d\xi d\eta d\zeta, \quad (1)$$

where $r(z)$ is the convergence radius.

The global coordinate system (X, Y, Z) is transformed into cylindrical coordinate system (R, θ, z) . The equation for converting the two coordinate systems can be expressed as follows:

$$\left. \begin{aligned} Z &= z \\ X &= R \cos \theta \\ Y &= R \sin \theta \end{aligned} \right\}. \quad (2)$$

Therefore, in cylindrical coordinates, the vertical displacement of the upper soil caused by a single microelement is written as

$$W_e(R, \theta, z) = \frac{1}{r^2(z)} \exp\left[-\frac{\pi}{r^2(z)}R^2\right]. \quad (3)$$

Based on the above assumptions, the volume of the microelement remains unchanged, that is, the volume of the soil in the whole excavation area remains unchanged, then

$$\left. \begin{aligned} \varepsilon_{eR} &= \frac{\partial U_e}{\partial R} \\ \varepsilon_{e\theta} &= \frac{U_e}{R} \\ \varepsilon_{eZ} &= \frac{\partial W_e}{\partial Z} \end{aligned} \right\}, \quad (4)$$

$$\varepsilon_{eR} + \varepsilon_{e\theta} + \varepsilon_{eZ} = 0, \quad (5)$$

where ε_{eR} , $\varepsilon_{e\theta}$, and ε_{eZ} is determined by the volumetric strain of the microelement in the cylindrical coordinate system, respectively.

Substituting equations (3) and (4) into equation (5), the horizontal displacement U_e can be obtained by the boundary conditions that $R = 0$ and $R \rightarrow \pm\infty$, $U_e = 0$.

$$U_e = \frac{R}{r^3(z)} \frac{dr(z)}{dZ} \exp\left[-\frac{\pi}{r^2(z)} R^2\right]. \quad (6)$$

In the Cartesian coordinate system, the horizontal displacement of the overlying stratum after the excavation of microelements is written as follows:

$$U_e(x, y, z) = \frac{\sqrt{x^2 + y^2}}{r^3(z)} \frac{dr(z)}{dZ} \exp\left[-\frac{\pi(x^2 + y^2)}{r^2(z)}\right] d\epsilon d\xi d\eta. \quad (7)$$

Supposing $r(z)$ is a proportional function of depth, the convergence radius can be obtained as

$$r(0) = R = \frac{\eta}{\tan \beta}, \quad (8)$$

$$r(z) = R = \frac{\eta - z}{\tan \beta}, \quad (9)$$

where β is the main influence angle of the overlying soil layer and $r(z)$ is the horizontal convergence radius in depth.

In actual engineering, excavation of the soil will make the soil around the tunnel contract inward. If the tunnel is excavated in the area Ω and the area after the excavation surface is contracted and becomes Ψ , then the size of the contraction deformation is $\Delta = \Omega - \Psi$.

In accordance with the superposition principle, the vertical displacement of the ground surface at any point will be the cumulative displacement of the upper soil sinking displacement, which is caused by excavation of each microunit body. It can be written as follows:

$$W(x, y, z) = \iiint_{\Delta} \frac{\tan^2 \beta}{(\eta - z)^2} \exp\left\{-\frac{\pi \tan^2 \beta}{(\eta - z)^2} [(x - \epsilon)^2 + (y - \xi)^2]\right\} d\epsilon d\xi d\eta. \quad (10)$$

The horizontal displacement of the upper soil settlement is expressed as follows:

$$\left. \begin{aligned} U_x(x, y, z) &= \iiint_{\Delta} \frac{-(x - \epsilon) \tan^2 \beta}{(\eta - z)^2} \exp\left\{-\frac{\pi \tan^2 \beta}{(\eta - z)^2} [(x - \epsilon)^2 + (y - \xi)^2]\right\} d\epsilon d\xi d\eta \\ U_y(x, y, z) &= \iiint_{\Delta} \frac{-(y - \epsilon) \tan^2 \beta}{(\eta - z)^2} \exp\left\{-\frac{\pi \tan^2 \beta}{(\eta - z)^2} [(x - \epsilon)^2 + (y - \xi)^2]\right\} d\epsilon d\xi d\eta \end{aligned} \right\}. \quad (11)$$

In accordance with equation (10), the three-dimensional model of the stochastic medium theory is utilized to predict the settlement of the upper soil after tunnel excavation. Two parameters need to be determined, namely, the main influence angle β of rock and soil layer and the deformation area Δ after tunnel excavation. Δ mainly affects the influence range of the soil layer, while β affects the shape of the settlement range.

From the above equations for calculating stratum settlement, there are many triple integral calculations involved, and the integral function of each triple integral is not integral. Therefore, when solving these triple integrals, a numerical integration method should be adopted, which can be solved by using Mathematical and Maple software.

2.2. Analytical Solution for Three-Dimensional Ground Frost Heave. Figure 2 shows that the angle α is located between the n th freezing pipe and the center axis of the tunnel. After the elapse of time t , as the cooling capacity is continuously transferred to the freezing pipe, the negative temperature in the freezing pipe will exchange heat with the surrounding soil layer, and the freezing will be diverged along the periphery of the freezing pipe, and the freezing radius of the freezing pipes gradually increases. The inner and outer freezing fronts of the frozen soil wall tend to rapidly smoothen. The frozen soil wall is assumed to expand uniformly from $R(t)$ to $R_{\Delta}(t)$ due to the frost heave characteristics of soil and the expanded area is Δ .

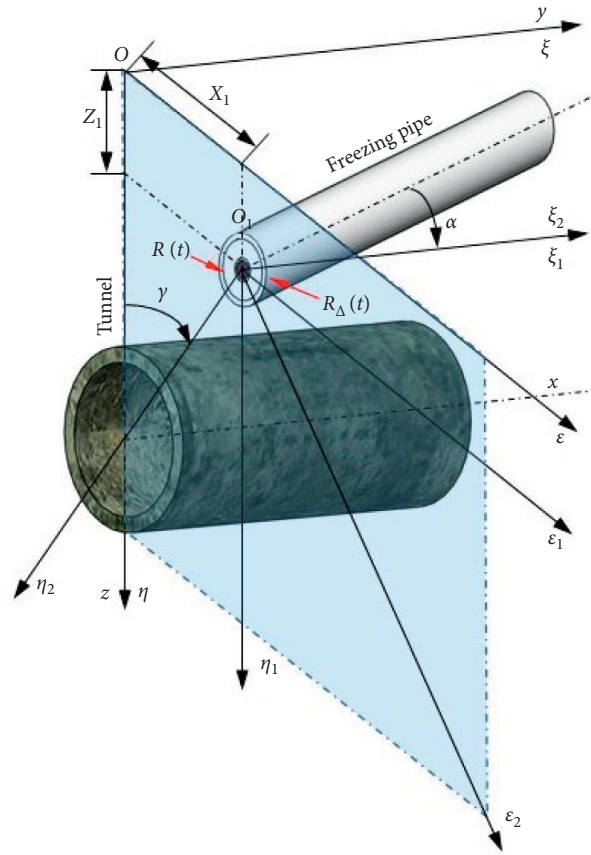


FIGURE 2: 3D layout of freezing pipe.

In accordance with the three-dimensional model of stochastic medium theory, when in the Cartesian coordinate system $\varepsilon\xi\eta$, the vertical heaving displacement of the ground

surface during the freezing time of tunnel can be expressed as follows:

$$W(x, y, z) = - \iiint_{\Delta} \frac{\tan^2 \beta}{(\eta - z)^2} \exp \left[- \frac{\pi \tan^2 \beta}{(\eta - z)^2} [(x - \varepsilon)^2 + (y - \xi)^2] \right] d\varepsilon d\xi d\eta. \quad (12)$$

Since the integral region is a hollow cylinder, the space rectangular coordinate system is transformed four times in order to solve the integral equation. Figure 3(a) shows that the Cartesian coordinate system $\varepsilon\xi\eta(xyz)$ is converted into the Cartesian coordinate system $\varepsilon_1\xi_1\eta_1$. The conversion equation of the two coordinate systems is

$$\left. \begin{aligned} \varepsilon &= \varepsilon_1 + X_1 \\ \xi &= \xi_1 \\ \eta &= \eta_1 + Z_1 \end{aligned} \right\}. \quad (13)$$

Figure 3(b) shows that the coordinate system $\varepsilon_1\xi_1\eta_1$ is converted into the coordinate system $\varepsilon_2\xi_2\eta_2$. And the center of the freezing pipe is connected with the center of the tunnel

as an axis η_2 , the tangent of the freezing circle is taken as an axis ε_2 , and the axis ξ_2 coincides with the direction of ξ_1 . The angle between the axis η and the axis η_2 is set as γ . The conversion equation of the two coordinate systems is expressed as follows:

$$(\varepsilon_1, \xi_1, \eta_1, 1) = (\varepsilon_2, \xi_2, \eta_2, 1) \begin{bmatrix} \cos \gamma & 0 & \sin \gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \gamma & 0 & \cos \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (14)$$

Figure 4(a) shows that the coordinate system $\varepsilon_3\xi_3\eta_3$ is obtained by rotating the coordinate system $\varepsilon_2\xi_2\eta_2$ around

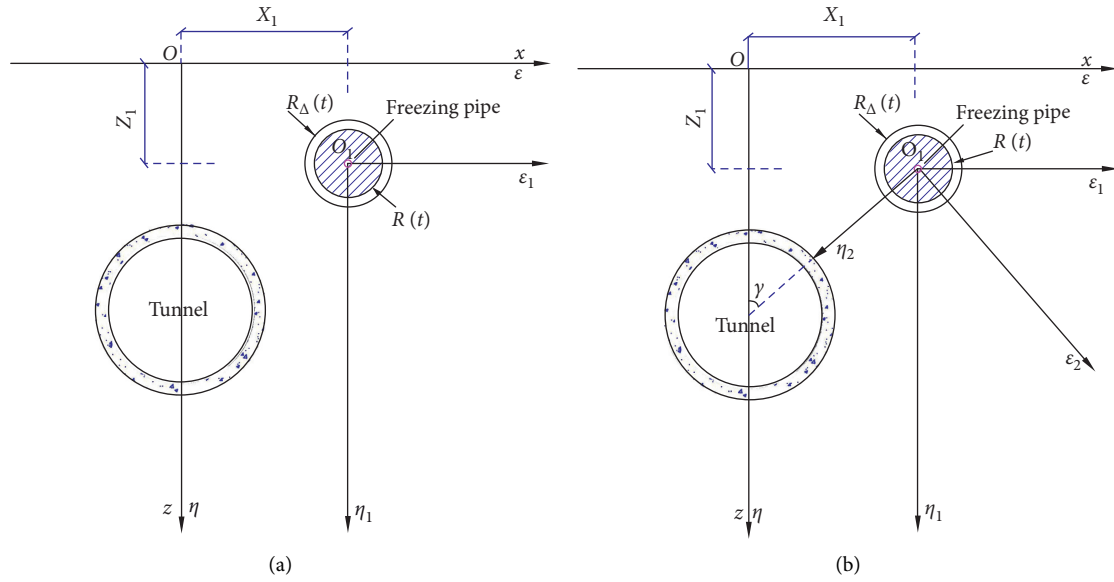


FIGURE 3: Coordinate transformation diagram: (a) Coordinate system $\varepsilon_1 \xi_1 \eta_1$. (b) Coordinate system $\varepsilon_2 \xi_2 \eta_2$.

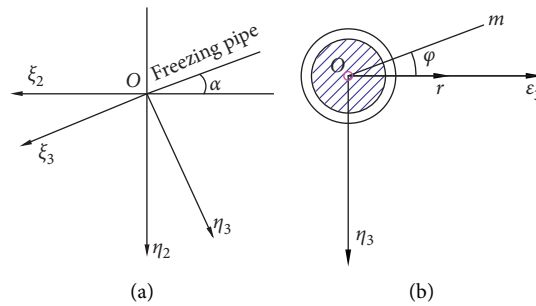


FIGURE 4: Coordinate transformation diagram: (a) Coordinate system $\varepsilon_3 \xi_3 \eta_3$. (b) Coordinate system $r \varphi \zeta$.

the axis ε_2 with angle α . And the conversion equation of the two coordinate systems can be written by

$$(\varepsilon_2, \xi_2, \eta_2, 1) = (\varepsilon_3, \xi_3, \eta_3, 1) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha & 0 \\ 0 & -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (15)$$

Figure 4(b) shows that the integral region is a hollow cylinder, φ is the angle from $o\varepsilon_3$ to om in the polar coordinate $r\varphi$, and the coordinate system $\varepsilon_3 \xi_3 \eta_3$ is converted into a cylindrical coordinate system $r\varphi\zeta$ for convenient calculation. The conversion equation of the two coordinate systems is as follows:

$$\left. \begin{aligned} \varepsilon_3 &= r \cos \varphi \\ \eta_3 &= -r \sin \varphi \\ \xi_3 &= -\zeta \end{aligned} \right\}. \quad (16)$$

From equations (12) to (16), the conversion equation from the Cartesian coordinate system $\varepsilon\xi\eta$ into the cylindrical coordinate system $r\varphi\zeta$ can be expressed as follows:

$$\begin{aligned} \varepsilon &= r \cos \varphi \cos \gamma + (\zeta \sin \alpha + r \sin \varphi \cos \alpha) \sin \gamma + X_1, \\ \xi &= r \sin \varphi \sin \alpha - \zeta \cos \alpha, \\ \eta &= r \cos \varphi \sin \gamma - (\zeta \sin \alpha + r \sin \varphi \cos \alpha) \cos \gamma + Z_1. \end{aligned} \quad (17)$$

In accordance with the substitution equation of the triple integral, equation (12) may be written as follows:

$$\begin{aligned}
W(x, y, z) = & - \iiint_{\Delta(t)} \frac{\tan^2 \beta}{[r \cos \phi \sin \gamma - (\zeta \sin \alpha + r \sin \phi \cos \alpha) \cos \gamma + Z_1 - z]^2} \\
& \exp \left\{ - \frac{\pi \tan^2 \beta}{[r \cos \phi \sin \gamma - (\zeta \sin \alpha + r \sin \phi \cos \alpha) \cos \gamma + Z_1 - z]^2} \right. \\
& \left. [(X - r \cos \phi \cos \gamma - (\zeta \sin \alpha + r \sin \phi \cos \alpha) \sin \gamma - X_1)^2 + (y - r \sin \phi \sin \alpha + \zeta \cos \alpha)^2] \right\} r dr d\phi d\zeta.
\end{aligned} \tag{18}$$

The integral area is a hollow cylinder, the radius is from $R(t)$ to $R_\Delta(t)$, the length of the freezing pipe is l , and the vertical frost heave displacement of each point on the surface

of the tunnel during the freezing period in the cylindrical coordinate system $r\phi\zeta$ can be written as follows:

$$\begin{aligned}
W(x, y, z) = & - \int_{R(t)}^{R_\Delta(t)} \int_0^{2\pi} \int_0^l \frac{\tan^2 \beta}{[r \cos \phi \sin \gamma - (\zeta \sin \alpha + r \sin \phi \cos \alpha) \cos \gamma + Z_1 - z]^2} \\
& \exp \left\{ - \frac{\pi \tan^2 \beta}{[r \cos \phi \sin \gamma - (\zeta \sin \alpha + r \sin \phi \cos \alpha) \cos \gamma + Z_1 - z]^2} \right. \\
& \left. [(X - r \cos \phi \cos \gamma - (\zeta \sin \alpha + r \sin \phi \cos \alpha) \sin \gamma - X_1)^2 + (y - r \sin \phi \sin \alpha + \zeta \cos \alpha)^2] \right\} r dr d\phi d\zeta,
\end{aligned} \tag{19}$$

where xyz are the coordinates of the points on the earth's surface in the Cartesian coordinate system (xyz) . And $X_1 Z_1$ are the abscissa and vertical coordinates of the freezing pipe center in the Cartesian system (xyz) , respectively. α is the angle between the freezing pipe and the central line of the tunnel, which is positive in elevation and negative in depression. β is the primary influence angle of ground heaving. $r\phi\zeta$ are polar radius, polar angle, and vertical coordinates in cylindrical coordinates, respectively. And γ is the angle between the connecting line and the axis η between the center of the freezing pipe and the center of the tunnel. l is the length of the freezing pipe.

2.3. Analytical Solution before the Closure of the Frozen Wall. Suppose that a circle of freezing pipes is arranged around the tunnel, and the freezing pipes are arranged uniformly. Freezing pipes are arranged horizontally and the soil properties are the same. If the number of frozen pipes is n , in accordance with the superposition principle, the displacement of ground frost heave caused by tunnel freezing before the closure of the frozen wall is equal to the sum of ground frost heave displacement caused by n single-pipe freezing. Then the freezing radius of all freezing pipes at t time is $R(t)$, and the frost heave radius is $R_\Delta(t)$.

The serial number of freezing pipes is recorded as 1, 2, ..., n . In the Cartesian coordinate system $e\xi\eta(xyz)$, the coordinate of the center of the n th freezing pipe is defined as $(x_i, z_i, 0)$, which can be obtained as

$$\begin{aligned}
X_i &= R \cos \left(i \frac{2\pi}{n} \right), \\
Z_i &= h - R_d \sin \left(i \frac{2\pi}{n} \right),
\end{aligned} \tag{20}$$

where h is the depth between the tunnel center to the ground surface and R_d is the radius of the freezing pipe layout circle.

Before calculating the frost heave of the ground surface caused by the i th freezing pipe, the Cartesian coordinate system needs to be converted to the cylindrical coordinate system $r\phi\zeta$. The conversion equation is

$$\varepsilon = r \cos \phi \cos \gamma + (\zeta \sin \alpha + r \sin \phi \cos \alpha) \sin \gamma + X_i, \tag{21}$$

$$\xi = r \sin \phi \sin \alpha - \zeta \cos \alpha, \tag{22}$$

$$\eta = r \cos \phi \sin \gamma - (\zeta \sin \alpha + r \sin \phi \cos \alpha) \cos \gamma + Z_i. \tag{23}$$

Substituting equation (20) into equations (21)–(23), respectively, then

$$\varepsilon = r \cos \phi \cos \gamma + (\zeta \sin \alpha + r \sin \phi \cos \alpha) \sin \gamma + X_i, \tag{24}$$

$$\xi = r \sin \phi \sin \alpha - \zeta \cos \alpha, \tag{25}$$

$$\eta = r \cos \phi \sin \gamma - (\zeta \sin \alpha + r \sin \phi \cos \alpha) \sin \gamma + Z_i. \tag{26}$$

Hence,

$$\gamma = i \frac{2\pi}{n}. \tag{27}$$

Substituting equation (27) into equations (23)–(25), respectively, then

$$\varepsilon = r \cos \phi \cos \left(i \frac{2\pi}{n} \right) + (\zeta \sin \alpha + r \sin \phi \cos \alpha) \sin \left(i \frac{2\pi}{n} \right) + R_d \cos \left(i \frac{2\pi}{n} \right), \quad (28)$$

$$\xi = r \sin \phi \sin \alpha - \zeta \cos \alpha, \quad (29)$$

$$\eta = r \cos \phi \sin \left(i \frac{2\pi}{n} \right) - (\zeta \sin \alpha + r \sin \phi \cos \alpha) \cos \left(i \frac{2\pi}{n} \right) + R_d \cos \left(i \frac{2\pi}{n} \right). \quad (30)$$

From equation (19), under the condition of space problem, the vertical frost heave displacement of each point

on the ground surface caused by the i th freezing pipe can be expressed as

$$W(x, y, z) = - \int_{R(t)}^{R_\Delta(t)} \int_0^{2\pi} \int_0^l \frac{\tan^2 \beta}{[r \cos \phi \sin(i2\pi/n) - (\zeta_i \sin \alpha + r_i \sin \phi_i \cos \alpha) \cos(i2\pi/n) + Z_i - z]^2} \exp \left\{ \frac{-\pi \tan^2 \beta}{[r_i \cos \phi_i \sin(i2\pi/n) - (\zeta_i \sin \alpha + r_i \sin \phi_i \cos \alpha) \cos(i2\pi/n) + Z_i - z]^2} \right\} [(X - r_i \cos \phi \cos(i2\pi/n) - (\zeta \sin \alpha + r_i \sin \phi_i \cos \alpha) \sin(i2\pi/n) - X_i)^2 + (y - r_i \sin \phi_i \sin \alpha + \zeta_i \cos \alpha)^2] r dr d\phi d\zeta. \quad (31)$$

Substituting equations (28)–(30) into equation (31), respectively, then

$$W(x, y, z) = - \int_{R(t)}^{R_\Delta(t)} \int_0^{2\pi} \int_0^l \frac{\tan^2 \beta}{[r \cos \phi \sin(i2\pi/n) - (\zeta_i \sin \alpha + r_i \sin \phi_i \cos \alpha) \cos(i2\pi/n) + h - R_d \sin(i2\pi/n) - z]^2} \exp \left\{ \frac{-\pi \tan^2 \beta}{[r_i \cos \phi_i \sin(i2\pi/n) - (\zeta_i \sin \alpha + r_i \sin \phi_i \cos \alpha) \cos(i2\pi/n) + h - R_d \sin(i2\pi/n) - z]^2} \right\} \left[\left(X - r_i \cos \phi \cos(i2\pi/n) - (\zeta \sin \alpha + r_i \sin \phi_i \cos \alpha) \sin(i2\pi/n) - R_d \cos \left(i \frac{2\pi}{n} \right) \right)^2 + (y - r_i \sin \phi_i \sin \alpha + \zeta_i \cos \alpha)^2 \right] r dr d\phi d\zeta. \quad (32)$$

In accordance with the superposition principle, the displacement of ground frost heave before the closure of the frozen wall during tunnel freezing is equal to the sum of

heaving displacement caused by n single pipes, which can be obtained by

$$W(x, y, z) = - \sum_{i=1}^n \int_{R(t)}^{R_\Delta(t)} \int_0^{2\pi} \int_0^l \frac{\tan^2 \beta}{[r \cos \phi \sin(i2\pi/n) - (\zeta_i \sin \alpha + r_i \sin \phi_i \cos \alpha) \cos(i2\pi/n) + h - R_d \sin(i2\pi/n) - z]^2} \exp \left\{ \frac{-\pi \tan^2 \beta}{[r_i \cos \phi_i \sin(i2\pi/n) - (\zeta_i \sin \alpha + r_i \sin \phi_i \cos \alpha) \cos(i2\pi/n) + h - R_d \sin(i2\pi/n) - z]^2} \right\} \left[\left(X - r_i \cos \phi \cos \left(i \frac{2\pi}{n} \right) - (\zeta \sin \alpha + r_i \sin \phi_i \cos \alpha) \sin \left(i \frac{2\pi}{n} \right) - R_d \cos \left(i \frac{2\pi}{n} \right) \right)^2 + (y - r_i \sin \phi_i \sin \alpha + \zeta_i \cos \alpha)^2 \right] r dr d\phi d\zeta. \quad (33)$$

Similarly, the horizontal frost heave displacement of the ground surface before the closure of frozen wall during tunnel freezing construction can be expressed by

$$\begin{aligned}
 U(x, y, z) = & - \sum_{i=1}^n \int_0^{2\pi} \int_{R(t)}^{R_\Delta(t)} \int_0^l \left(x - r_i \cos \varphi_i \cos \left(i \frac{2\pi}{n} \right) - (\zeta_i \sin \alpha + r_i \sin \varphi_i \cos \alpha) \sin \left(i \frac{2\pi}{n} \right) - R_d \cos \left(i \frac{2\pi}{n} \right) \right) \\
 & \frac{\tan^2 \beta}{(r_i \cos \varphi_i \sin(i2\pi/n) - (\zeta_i \sin \alpha + r_i \sin \varphi_i \cos \alpha) \cos(i2\pi/n) + R_d \sin(i2\pi/n) - z)^3} \\
 & \exp \left\{ -\pi \tan^2 \beta \left[r_i \cos \varphi_i \sin \left(i \frac{2\pi}{n} \right) - (\zeta_i \sin \alpha + r_i \sin \varphi_i \cos \alpha) \cos \left(i \frac{2\pi}{n} \right) + h - R_d \sin \left(i \frac{2\pi}{n} \right) - z \right]^2 \right. \\
 & \left. \left[\left(x - r_i \cos \varphi_i \cos \left(i \frac{2\pi}{n} \right) - (\zeta_i \sin \alpha + r_i \sin \varphi_i \cos \alpha) \sin \left(i \frac{2\pi}{n} \right) - R_d \cos \left(i \frac{2\pi}{n} \right) \right)^2 + (y - r_i \sin \varphi_i \sin \alpha + \zeta_i \cos \alpha)^2 \right] \right\} r dr d\varphi d\zeta.
 \end{aligned} \tag{34}$$

$$\begin{aligned}
 U(x, y, z) = & - \sum_{i=1}^n \int_0^{2\pi} \int_{R(t)}^{R_\Delta(t)} \int_0^l -(y - r_i \sin \varphi_i \sin \alpha + \zeta_i \cos \alpha) \\
 & \frac{\tan^2 \beta}{(r_i \cos \varphi_i \sin(i2\pi/n) - (\zeta_i \sin \alpha + r_i \sin \varphi_i \cos \alpha) \cos(i2\pi/n) + R_d \sin(i2\pi/n) - z)^3} \\
 & \exp \left\{ -\pi \tan^2 \beta \left[r_i \cos \varphi_i \sin \left(i \frac{2\pi}{n} \right) - (\zeta_i \sin \alpha + r_i \sin \varphi_i \cos \alpha) \cos \left(i \frac{2\pi}{n} \right) + h - R_d \sin \left(i \frac{2\pi}{n} \right) - z \right]^2 \right. \\
 & \left. \left[\left(x - r_i \cos \varphi_i \cos \left(i \frac{2\pi}{n} \right) - (\zeta_i \sin \alpha + r_i \sin \varphi_i \cos \alpha) \sin \left(i \frac{2\pi}{n} \right) - R_d \cos \left(i \frac{2\pi}{n} \right) \right)^2 \right. \right. \\
 & \left. \left. + (y - r_i \sin \varphi_i \sin \alpha + \zeta_i \cos \alpha)^2 \right] \right\} r dr d\varphi d\zeta.
 \end{aligned} \tag{35}$$

2.4. Analytical Solution after the Closure of the Frozen Wall. As the frozen soil columns gradually enlarge with prolonged freezing time, the adjacent frozen soil columns interconnect to form a continuous frozen soil wall around the tunnel. The inner and outer freezing fronts of the frozen soil wall tend to rapidly smoothen. Figure 5 shows that the frozen soil wall is assumed to expand uniformly from $R(t)$ to $R_\Delta(t)$ due to the frost heave characteristics of soil and the expanded area is Δ . Ground frost heave is induced by the expansion effect of the entire frozen wall after the closure of the frozen wall.

The Cartesian coordinate system $\varepsilon\xi\eta$ is converted to the cylindrical coordinate system $r\theta\zeta$ and θ is the angle of polar coordinate $ro\theta$. The axis ξ is perpendicular to the outside of

the paper, and the axis ζ is in the same direction as the axis ξ . Then the conversion equation is

$$\varepsilon = r \cos \theta, \tag{36}$$

$$\eta = h - r \sin \theta, \tag{37}$$

$$\xi = \zeta. \tag{38}$$

In accordance with the substitution equation of the triple integral, equation (12) may be written in the cylindrical coordinate system $r\theta\zeta$ as

$$W(x, y, z) = - \iiint_{\Delta} \frac{\tan^2 \beta}{(h - r \sin \theta - z)^2} \exp \left\{ -\frac{\pi \tan^2 \beta}{(h - r \sin \theta - z)^2} \left[(x - r \cos \theta)^2 + (y - \zeta)^2 \right] \right\} r dr d\theta d\zeta. \tag{39}$$

When the angle between the freezing pipe and the horizontal direction is α , the integral region is approximately

a hollow frustum, and the vertical heaving displacement of the ground surface is written by

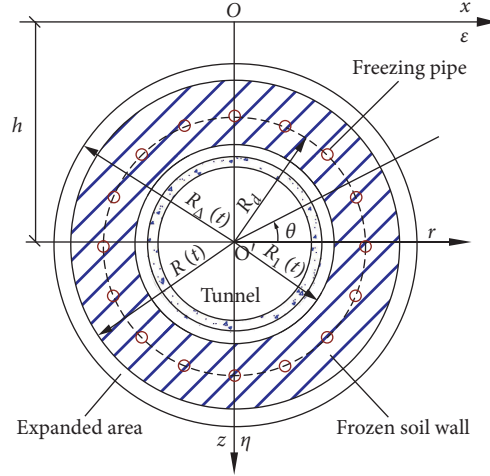


FIGURE 5: Coordinate system transformation after frozen wall intersection.

$$W(x, y, z) = \int_{R(t)}^{R_A(t)} \int_0^{2\pi} \int_{-1/2L \cos \alpha}^{1/2L \cos \alpha} f(r, \theta, \zeta) r dr d\theta d\zeta. \quad (40)$$

$$f(r, \theta, \zeta) = -\frac{\tan^2 \beta}{(h - r \sin \theta - z)^2} \exp \left\{ -\frac{\pi \tan^2 \beta}{(h - r \sin \theta - z)^2} [(x - r \cos \theta)^2 + (y - \zeta)^2] \right\}. \quad (41)$$

2.5. *Solution for the Radius of the Freezing Front.* According to the paper written by Cai and Jiang et al. [17, 24, 25], it can be known that the single-pipe freezing theory can be applied to two-dimensional heat conduction problems, while the flat-panel freezing theory is applied to one-dimensional semi-infinite heat conduction problems.

Therefore, the radius of the freezing front is the radius of a single frozen soil column before the closure of the frozen wall, which can be solved using the single-pipe freezing theory.

After the closure of the frozen wall, the radii of the freezing front are the inner and outer radii of the frozen wall, which can be solved using the flat-panel freezing theory.

2.5.1. *Single-Pipe Freezing Theory.* Figure 6 shows that the temperature field can be divided into frozen and unfrozen areas according to the location of the freezing front. The temperature of the soil in the frozen area is T_f and that in the unfrozen area is T_u , both of which are functions of time t and radial coordinate r . The radius of the frozen soil column is $r(t)$, and soil temperature at the freezing front is freezing point T_d . The initial temperature of soil is T_0 and the outer radius of the freezing pipe is r_0 . The surface temperature of the freezing pipe is assumed constant and its value is T_c .

For this two-dimensional heat conduction problem, the differential equation of heat transmission in frozen and unfrozen areas can be written as follows:

$$\frac{\partial T_f}{\partial t} = \alpha_f \left(\frac{\partial^2 T_f}{\partial r^2} + \frac{1}{r} \frac{\partial T_f}{\partial r} \right) \quad (r_0 \leq r \leq r(t)), \quad (42)$$

$$\frac{\partial T_u}{\partial t} = \alpha_u \left(\frac{\partial^2 T_u}{\partial r^2} + \frac{1}{r} \frac{\partial T_u}{\partial r} \right) \quad (r(t) \leq r < \infty), \quad (43)$$

where α_f and α_u are the thermal diffusivity of frozen and unfrozen soils, respectively:

$$\alpha_f = \frac{k_f}{c_f \rho_f}, \quad \alpha_u = \frac{k_u}{c_u \rho_u}, \quad (44)$$

where k_f and k_u are the thermal conductivity of frozen and unfrozen soils, respectively. c_f and c_u are the specific heat of frozen and unfrozen soils. ρ_f and ρ_u are the density of frozen and unfrozen soils.

The initial conditions for the differential equations are as follows:

$$t = 0, T_u = T_0. \quad (45)$$

The boundary conditions of the differential equations are as follows:

$$r = r_0, T_f = T_c, \quad (46)$$

$$r = r(t), T_f = T_u = T_d, \quad (47)$$

$$r = \infty, T_u = T_0. \quad (48)$$

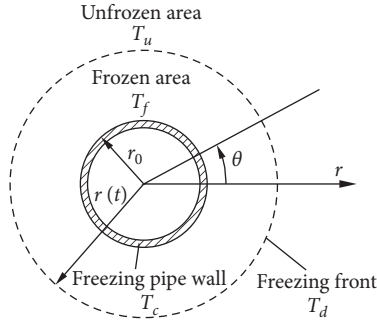


FIGURE 6: Schematic diagram of single tube freezing.

The surface temperature of the freezing pipe is easily measured using a temperature sensor. Therefore, the surface temperature of the freezing pipe is assumed constant in this study. Equation (47) is selected as the boundary condition of the differential equation.

The heat balance equation at the phase transition boundary is given by

$$\left(k_f \frac{\partial T_f}{\partial r} - k_u \frac{\partial T_u}{\partial r} \right)_{r=r(t)} = L \frac{dr(t)}{dt}, \quad (49)$$

where L is the volumetric latent heat of soil, which is given by

$$L = L_w \rho_d (w_0 - w_u), \quad (50)$$

where L_w is the latent heat of water, ρ_d is the dry density of the soil, w_0 is the initial water content in the soil, and w_u is the unfrozen water content in the frozen soil.

Combining the above initial and boundary conditions to solve the partial differential equations, the temperature field distributions in frozen and unfrozen areas can be obtained as follows:

$$T_f = T_c + (T_d - T_c) \frac{E_i(r_0^2/4\alpha_f t) - E_i(r^2/4\alpha_f t)}{E_i(r_0^2/4\alpha_f t) - E_i(r(t)^2/4\alpha_f t)} \quad (r_0 \leq r \leq r(t)), \quad (51)$$

$$T_u = T_0 + (T_d - T_0) \frac{E_i(r^2/4\alpha_u t)}{E_i(r(t)^2/4\alpha_u t)} \quad (r(t) \leq r < \infty), \quad (52)$$

where $E_i(x)$ is the exponential integral function, which is given by

$$E_i(x) = \int_x^\infty \frac{e^{-\eta}}{\eta} d\eta. \quad (53)$$

Substituting equations (51) and (52) into equation (49) yields

$$\frac{k_f (T_d - T_c) e^{-A^2/4\alpha_f}}{E_i(r_0^2/4\alpha_f t) - E_i(A^2/4\alpha_f)} + \frac{k_u (T_d - T_0) e^{-A^2/4\alpha_u}}{E_i(A^2/4\alpha_u)} = \frac{A^2}{4} L, \quad (54)$$

where A is an undetermined constant that varies with time, which can be

$$r(t) = A\sqrt{t}. \quad (55)$$

In accordance with single-tube freezing theory, the radius of the freezing front (i.e., the radius of the frozen soil column) may be solved by using equations (54) and (55) before the closure of the frozen wall when the surface temperature of the freezing pipe is constant.

The adjacent frozen soil columns begin to interconnect, which indicates that the frozen wall has just closed, and the closure time of the frozen wall can be calculated by the following equation:

$$t_i = \left(\frac{l}{2A} \right)^2, \quad (56)$$

where t_i is the closure time of the frozen wall and l is the space between adjacent freezing pipes.

2.5.2. Flat-Panel Freezing Theory. Figure 7 shows that the flat panel is assumed to generate heat exchange only with the soil on the right side, the temperature of the flat panel is T'_c , and the temperature of the soil in the freezing and unfrozen areas is T_f and T'_u , respectively, both of which are functions of time t and coordinate x . The distance between the freezing front and the flat panel is $X(t)$, and the temperature of soil at the freezing front is the freezing point T_d . The initial temperature of soil is T_0 .

For the flat-panel freezing problem, when $0 \leq x \leq X(t)$, the differential equations of the frozen area can be written as

$$\frac{\partial T_f}{\partial t} = \alpha_f \frac{\partial^2 T_f}{\partial x^2}. \quad (57)$$

Similarly, when $X(t) \leq x < \infty$, the differential equation is

$$\frac{\partial T_u}{\partial t} = \alpha_u \frac{\partial^2 T_u}{\partial x^2}. \quad (58)$$

The initial conditions for the differential equations are as follows:

$$t = 0, T_u = T_0. \quad (59)$$

The boundary conditions of the differential equations are as follows:

$$x = 0, T_f = T'_h, \quad (60)$$

$$x = X(t), T_f = T_u = T_d, \quad (61)$$

$$x = \infty, T_u = T_0. \quad (62)$$

In addition to the above boundary conditions, the heat balance equation at the phase transition boundary $X(t)$ can be expressed as

$$\left(k_f \frac{\partial T_f}{\partial x} - k_u \frac{\partial T_u}{\partial x} \right)_{x=X(t)} = L \frac{dX(t)}{dt}. \quad (63)$$

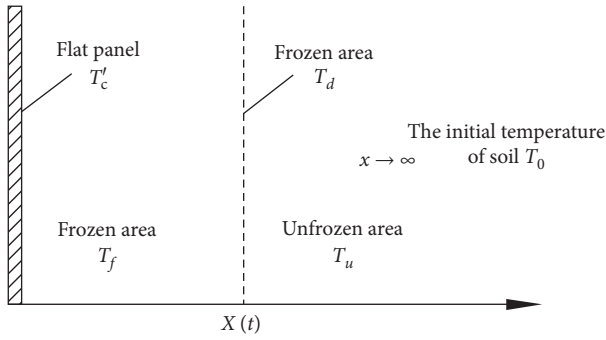


FIGURE 7: Plate freezing schematic.

Combining the above initial and boundary conditions to solve the partial differential equations, the temperature field distributions in the frozen area can be obtained as follows ($0 \leq x \leq X(t)$):

$$T_f = T'_c + (T_d - T'_c) \frac{\Phi(x/2\sqrt{\alpha_f t})}{\Phi(X(t)/2\sqrt{\alpha_f t})}, \quad (64)$$

The temperature field distributions in the unfrozen area can be expressed as follows ($X(t) \leq x < \infty$):

$$T_u = T_0 + (T_d - T_0) \frac{1 - \Phi(x/2\sqrt{\alpha_u t})}{1 - \Phi(X(t)/2\sqrt{\alpha_u t})}, \quad (65)$$

where $\Phi(y)$ is Gaussian integral function, which is given by

$$\Phi(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-\eta^2} d\eta. \quad (66)$$

Substituting equations (64) and (65) into equation (63) yields

$$\frac{k_f(T_d - T'_c)e^{-B^2/4\alpha_f}}{\sqrt{\alpha_f}\Phi(B/2\sqrt{\alpha_f})} + \frac{k_u(T_d - T_0)e^{-B^2/4\alpha_u}}{\sqrt{\alpha_u}[1 - \Phi(B/2\sqrt{\alpha_u})]} = \frac{\sqrt{\pi}}{2}BL, \quad (67)$$

where B is a constant to be determined, and the following condition exists:

$$X(t) = B\sqrt{t}. \quad (68)$$

Therefore, the flat panel is regarded as the freezing pipe circle after the closure of the frozen wall. The radius of the freezing fronts is calculated as follows:

$$R(t) = R_d + B\sqrt{t}, \quad (69)$$

$$R_c(t) = R_d - B\sqrt{t}, \quad (70)$$

where $R(d)$ is the radius of the freezing pipe circle, $R(t)$ is the radius of the outer freezing front, and $R_c(t)$ is the radius of the inner freezing front.

The thickness of the frozen wall $E(t)$ is obtained by

$$E(t) = R(t) - R_c(t) = 2B\sqrt{t}. \quad (71)$$

In accordance with the flat-panel freezing theory, the radii of the freezing fronts after the closure of the frozen wall can be solved by using equations (67) to (70) when the thermophysical parameters of the frozen and unfrozen soils are determined. It is generally considered that the flat-pane cooling temperature T'_c is equal to the freezing tube wall temperature T_c .

2.6. Solution for the Outer Radius of the Expanded Area. The frost heave rate in engineering is generally expressed as

$$\varepsilon_f = \frac{\Delta h_f}{h} \times 100\%, \quad (72)$$

where ε_f is the frost heave ratio of the soil, Δh_f is the longitudinal frost heave of the sample, and h is the original height of the sample.

Liu [26] thinks that the frost heave at a certain point during the freezing process of the soil is the integral of the frost heave ratio of the frozen soil layer along the thickness of the frozen layer, which is not directly related to the frost heave force. According to its principle, the frozen wall expands evenly outward, then

$$R_\Delta(t) = R_1(t) + \int_{R_1(t)}^{R(t)} (1 + \varepsilon_f) dr = R_1(t) + [R(t) - R_1(t)](1 + \varepsilon_f). \quad (73)$$

Many experimental studies on frost heave of soils show that the frost heave of soils is restrained by loads, and the frost heave ratio of soils decreases with the increase of overlying loads in exponential function.

$$\varepsilon_f = \varepsilon_{f0} \exp(-bP), \quad (74)$$

where ε_{f0} is the frost heave ratio of soil without load, P is the load, kPa, and b is empirical constant, kPa^{-1} .

Substituting equation (74) into equation (73) yields

$$R_\Delta(t) = R_1(t) + [R(t) - R_1(t)] [1 + \varepsilon_{f0} \exp(-bP)]. \quad (75)$$

3. Case Analysis

The return line tunnel of a passenger transport station in Beijing, China, with a length of 40.0 m, a width of 6.0 m, and a buried depth of 13.0 m, is a shallow-buried tunnel with a large cross section. Given that tunnel will pass beneath existing pipelines, artificial ground freezing and undermining methods, instead of the open cut method, are selected for tunnel construction. Figure 8 shows that the sand layer is frozen and reinforced, and the lower clay layer is closed, and the frozen wall is semicircular arched along the tunnel [27].

After the active freezing period of the tunnel for 35 days, a nearly semicircular arched horizontal frozen wall with a thickness of 1.2–1.6 m and an arch span of 6.0 m was finally formed. The frozen wall was a nonclosed structure, and the average temperature of the frozen wall was -10°C . The layout of the freezing pipes in the tunnel is shown in Figure 8.

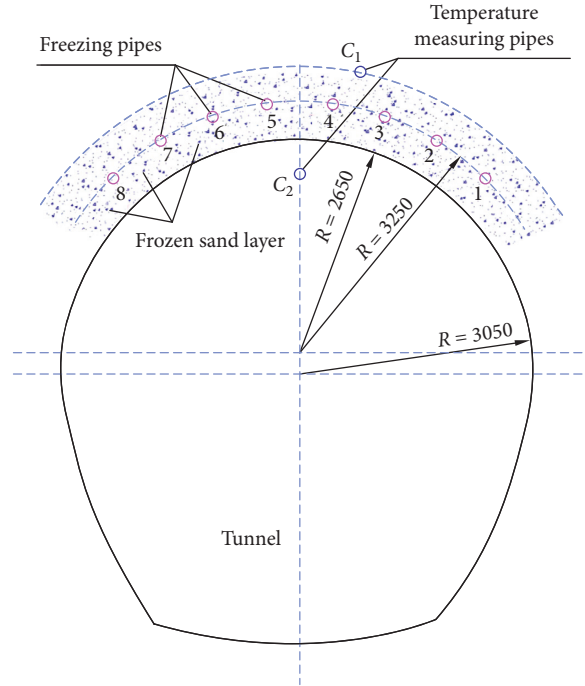


FIGURE 8: Layout of the freezing pipes in the tunnel.

For the convenience of calculation, the simplified thermal and physical parameters of the tunnel are shown in Tables 1 and 2.

The tangent value of the primary influence angle of the tunnel is $\beta = 0.8$, and the convergence radius can be obtained by

$$r(\eta) = \eta \cot \beta = 18.75 \text{ m.} \quad (76)$$

Therefore, $-20 \text{ m} \leq x \leq 20 \text{ m}$ and $-10 \text{ m} \leq y \leq 10 \text{ m}$ are taken as the calculation area. In accordance with equation (54), the thermal diffusivity of frozen and unfrozen soils may be obtained by

$$\alpha_f = \frac{k_f}{c_f \rho_f} = 0.534 \text{ m}^2/\text{d}, \alpha_u = \frac{k_u}{c_u \rho_u} = 0.0292 \text{ m}^2/\text{d}. \quad (77)$$

In accordance with equation (50), the volumetric latent heat of soil is

$$L = 79.6 \times 1489 \times (0.23 - 0.01) = 26082 \text{ Kcal/m}^3. \quad (78)$$

The axis x is perpendicular to the freezing direction and the axis y is the direction along the freezing wall. And the frost heave radius and related parameters of different freezing time were introduced into the three-dimensional frost heave prediction model.

Figure 9 shows that the inclination angle α exists between the freezing pipe and the center axis of the tunnel. Because of the horizontal freezing, the inclination angle is equal to 0° , and the length of the frozen area is 20 m. In accordance with the Legendre–Gauss integration method, the numerical programs for the displacement of the ground surface are compiled using Maple software, and the results are presented in MATLAB mathematical software.

TABLE 1: Parameters of the tunnel.

Parameters	Value
Tunnel radius R	2.25 m
The depth from tunnel center to ground surface h	13.0 m
Radius of freeze pipe r_0	0.054 m
Radius of freeze pipes circle Rd	3.25 m
Spacing between freeze pipes l	0.680 m
Number of freeze pipes n	8

TABLE 2: Parameters of soil.

Parameters	Value
Density ρ	1958 kg/m ³
Dry density ρ_d	1489 kg/m ³
Thermal conductivity of frozen soil k_f	31.5 kcal/(m·d·°C)
Thermal conductivity of unfrozen soil k_u	25.7 kcal/(m·d·°C)
Specific heat of frozen soil c_f	0.30 kcal/(kg·°C)
Specific heat of unfrozen soil c_u	0.45 kcal/(kg·°C)
Water content w_0	23.0%
Unfrozen water content w_u	1.0%
Freezing point T_d	0°C
Surface temperature of freezing pipe T_c	-25°C
Initial temperature of soil T_0	20°C
Frost heave ratio ε_f	0.56%

The heaving displacement of the ground surface due to freezing for 30 days, 45 days, 60 days, 75 days, and 90 days is shown in Figures 10–14, respectively.

However, if the inclination angle α is not 0° , three-dimensional frost heave prediction model will be more complex in the same case. Since the inclination angle given in reference [28] is $4^\circ \sim 14^\circ$, for the convenience of

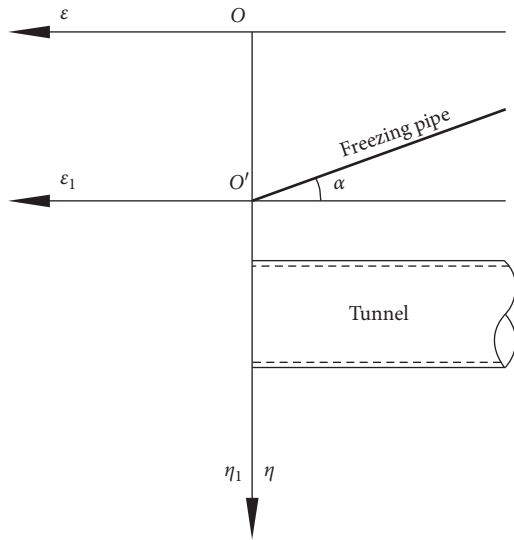


FIGURE 9: Single-tube frozen longitudinal profile.

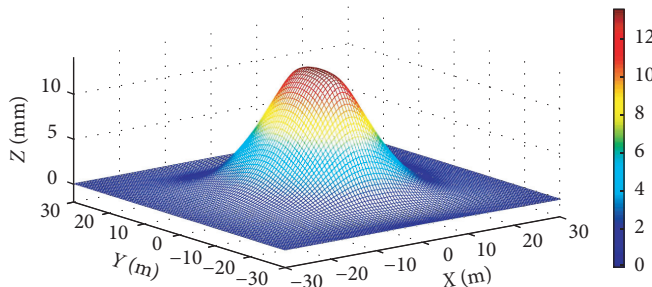


FIGURE 10: Heaving displacement of ground frost heave caused by freezing for 30 days (mm, $\alpha = 0^\circ$).

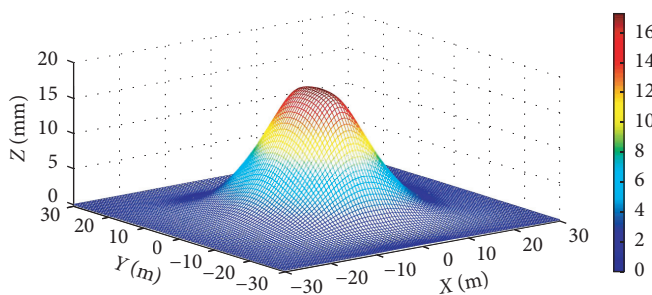


FIGURE 11: Heaving displacement of ground frost heave caused by freezing for 45 days (mm, $\alpha = 0^\circ$).

calculation, the inclination angle α is set to be 0° . Thus, the heaving displacement of the ground surface due to freezing for 30 days, 45 days, 60 days, 75 days, and 90 days is shown in Figures 15–19, respectively.

As shown in Figures 10–14, when the inclination angle α between the freezing pipe and the center axis of the tunnel is 0° , the heaving displacement of the ground surface along the x -axis is in agreement with the results of the two-dimensional heaving displacement, which gradually decreases with an increase in distance from the tunnel center. Along the y -axis

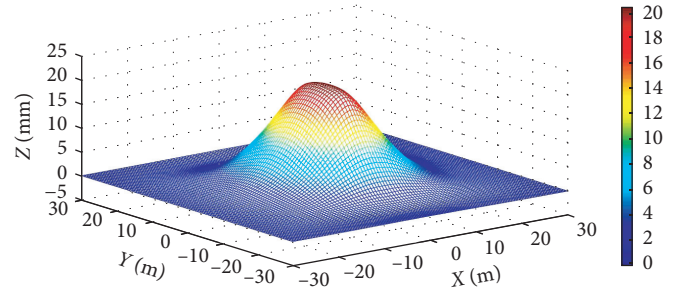


FIGURE 12: Heaving displacement of the ground surface caused by freezing for 60 days (mm, $\alpha = 0^\circ$).

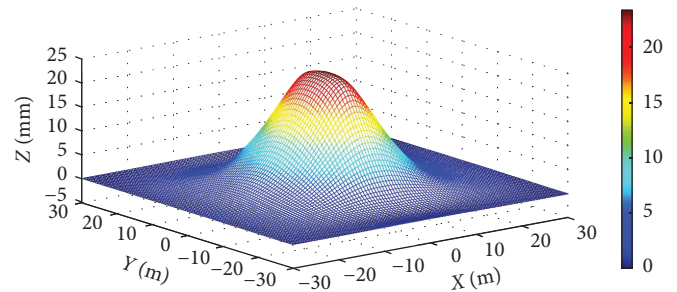


FIGURE 13: Heaving displacement of the ground surface caused by freezing for 75 days (mm, $\alpha = 0^\circ$).

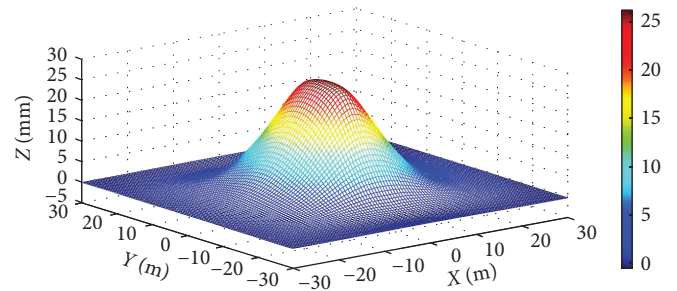


FIGURE 14: Heaving displacement of the ground surface caused by freezing for 90 days (mm, $\alpha = 0^\circ$).

direction, that is, the tunnel axis direction, the heaving displacement of the ground surface in the middle part rarely changes, basically keeps horizontal. And the heaving displacement at both ends is obviously reduced. Since the length of the frozen pipe is 20 meters, the corresponding heaving displacement is the same, which can be regarded as the result of the translation of the same section. However, in two ends of the freezing pipe, the temperature of the freezing pipe is affected by the surrounding soil, the frost heave radius changes and the heaving displacement decreases.

While the inclination angle is equal to 10° , the maximum heaving displacement of the ground surface does not occur at the central axis of the tunnel. Due to the influence of the inclination angle of the freezing pipe, one end of the freezing pipe is relatively close to the ground surface, the frost heave phenomenon will be more obvious, and the heaving displacement of the ground surface is larger than that of the two-dimensional prediction model.

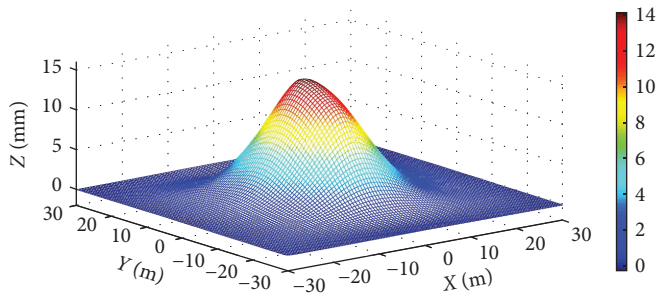


FIGURE 15: Heaving displacement of ground frost heave caused by freezing for 30 days (mm, $\alpha = 10^\circ$).

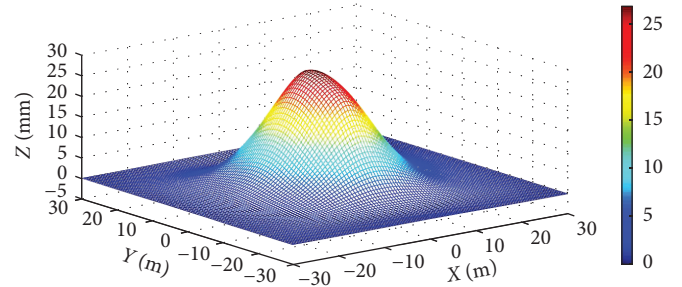


FIGURE 19: Heaving displacement of ground frost heave caused by freezing for 90 days (mm, $\alpha = 10^\circ$).

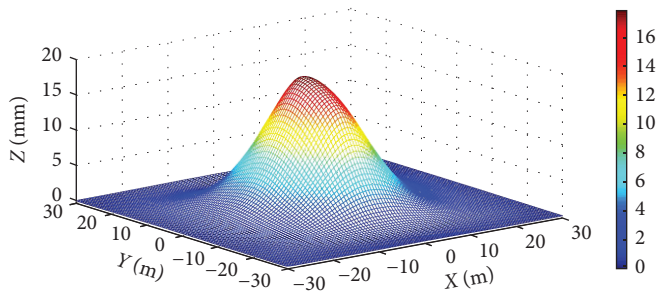


FIGURE 16: Heaving displacement of ground frost heave caused by freezing for 45 days (mm, $\alpha = 10^\circ$).

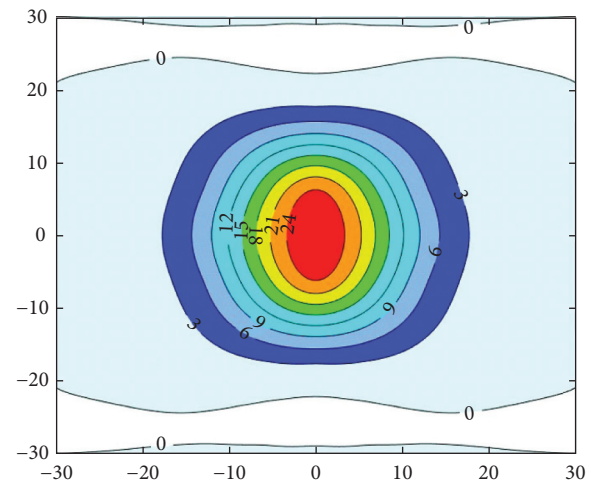


FIGURE 20: Contour figure of ground frost heave caused by freezing for 90 days (mm, $\alpha = 0^\circ$).

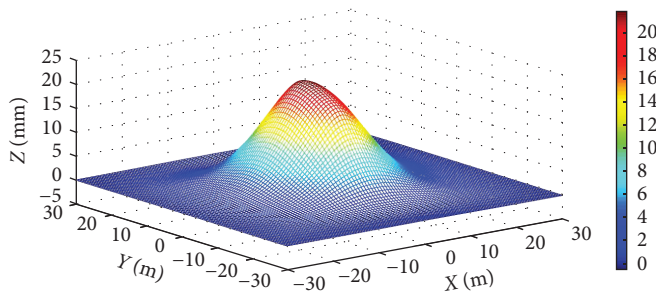


FIGURE 17: Heaving displacement of ground frost heave caused by freezing for 60 days (mm, $\alpha = 10^\circ$).

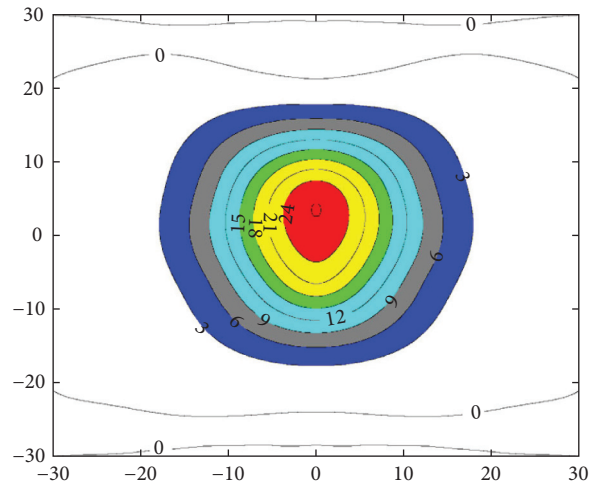


FIGURE 21: Contour figure of ground frost heave caused by freezing for 90 days (mm, $\alpha = 10^\circ$).

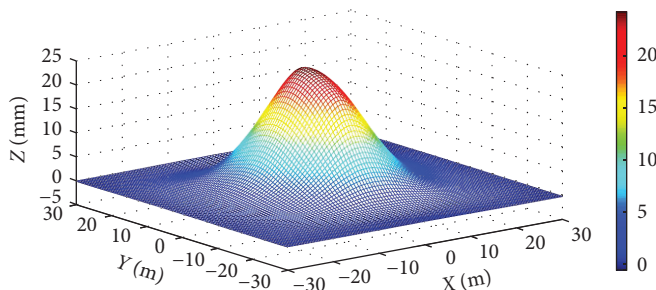


FIGURE 18: Heaving displacement of ground frost heave caused by freezing for 75 days (mm, $\alpha = 10^\circ$).

As shown in Figures 20 and 21, the heaving displacement of the ground surface by freezing for 90 days is symmetrically distributed on the x -axis and y -axis at the inclination angle of 0° . However, in the angle of 10° , the heaving displacement of the ground surface is only symmetrical about the x -axis, and the center of the maximum displacement is shifted to the y -axis.

4. Conclusion

- (1) A three-dimensional frost heave prediction model for the tunnel freezing period is established based on the stochastic medium theory.
- (2) The proposed three-dimensional frost heave prediction model is applied to an actual tunnel freezing project. Then, the three-dimensional heaving displacement of the ground surface is calculated. The results show that, considering the inclination angle α between the freezing pipe and the center axis of the tunnel, the maximum heaving displacement of the ground surface does not occur at the central axis of the tunnel. When the inclination angle is equal to 0° , the analytical solution of three-dimensional ground surface heaving displacement is in good agreement with that of the two-dimensional prediction model, which further verified the reliability of the established analytical solution of the three-dimensional frost heave prediction model.

Nomenclature

U_c :	The horizontal displacement
W_c :	The vertical displacement
$d\varepsilon$:	The length of the microelement
$d\xi$:	The width of the microelement
$d\eta$:	The thickness of the microelement
$r(z)$:	Convergence radius
$\varepsilon(eR, e\theta, ez)$:	The volumetric strain of the microelement
XYZ :	Global coordinate system
$R\theta Z$:	Cylindrical coordinate system
$\varepsilon\xi\eta$:	Cartesian coordinate system
$r\varphi\zeta$:	Cylindrical coordinate system
β :	The main influence angle
Ψ :	The contract area
Ω :	The excavation area
Δ :	The deformation area
α :	Rotation angle
γ :	Rotation angle
l :	The length of the freezing pipe
h :	The depth between the tunnel center to the ground surface
r :	Radial coordinate
$R(t)$:	The freezing radius
$R_\Delta(t)$:	The frost heave radius
Rd :	The radius of the freezing pipe layout circle
T_f :	The temperature of the soil in the frozen area
T_u :	The temperature of the soil in the unfrozen area
T_0 :	The initial temperature of soil

T_d :	The temperature of the freezing front
T_c :	The temperature of the freezing pipe wall
T'_c :	The flat-pane cooling temperature
α_f :	The thermal diffusivity of frozen soils
α_u :	The thermal diffusivity of unfrozen soils
k_f :	The thermal conductivity of frozen soils
k_u :	The thermal conductivity of unfrozen soils
c_f :	The specific heat of frozen soils
c_u :	The specific heat of unfrozen soils
ρ_f :	The density of frozen soils
ρ_u :	The density of unfrozen soils
ρ_d :	The dry density of the soil
w_0 :	The initial water content in the soil
w_u :	The unfrozen water content in the frozen soil
L_w :	The latent heat of water
L :	The volumetric latent heat of soil
A :	An undetermined constant that varies with time
B :	A constant to be determined
P :	The load (kPa)
b :	Empirical constant (kPa^{-1})
ε_f :	Frost heave ratio of soil
ε_{f0} :	Frost heave ratio of soil without load
h :	The original height of the sample
Δh_f :	The longitudinal frost heave of the sample.

Data Availability

The datasets generated and analyzed during the current study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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