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Research Article

Assessment of Haar Wavelet-Quasilinearization Technique in Heat Convection-Radiation Equations

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We showed that solutions by the Haar wavelet-quasilinearization technique for the two problems, namely, (i) temperature distribution equation in lumped system of combined convection-radiation in a slab made of materials with variable thermal conductivity and (ii) cooling of a lumped system by combined convection and radiation are strongly reliable and also more accurate than the other numerical methods and are in good agreement with exact solution. According to the Haar wavelet-quasilinearization technique, we convert the nonlinear heat transfer equation to linear discretized equation with the help of quasilinearization technique and apply the Haar wavelet method at each iteration of quasilinearization technique to get the solution. The main aim of present work is to show the reliability of the Haar wavelet-quasilinearization technique for heat transfer equations.

1. Introduction

Haar wavelet is the lowest member of Daubechies family of wavelets and is convenient for computer implementations due to availability of explicit expression for the Haar scaling and wavelet functions [1]. The quasilinearization approach was introduced by Bellman and Kalaba [2] as a generalization of the Newton-Raphson method to solve the individual or systems of nonlinear ordinary and partial differential equations.

Haar wavelet-quasilinearization technique [3–6] is recently developed method for the nonlinear differential equation, which deals with all types of nonlinearities. Boundary value problems are considerably more difficult to deal with than initial value problems. The Haar wavelet method for boundary value problems is more complicated than for initial value problems. In the present work we deal with both initial and boundary value problems.

In this present work, our purpose to solve the nonlinear equations arising in heat transfer through Haar wavelet-quasilinearization technique and show that it is strongly reliable method for heat transfer problems than the other existing methods. Convergence of Haar wavelet-quasilinearization technique has been given in [6].

We use the cubic spline interpolation [7] to get the solution at grid points for the sake of comparison. For this purpose we use the MATLAB built-in function $y_i = \text{interp1}(x, y, x_i, \text{“spline”})$, for one-dimensional data interpolation by cubic spline interpolation.

The paper is arranged as follows: in Section 2 we review basic definition of fractional differentiation and integration, while in Section 3 we describe the Haar wavelets. In Section 4 we present the main features of the quasilinearization approach. In Section 5 we apply the Haar wavelet method with quasilinearization technique to nonlinear heat transfer problems. Finally in Section 6 we conclude our work.

2. Preliminaries

In this section, we review basic definitions of fractional differentiation and fractional integration [8].

2.1. Riemann-Liouville Fractional Integral Operator of Order α. The operator I_x^α, defined on $L_1[a, b]$ by

$$I_x^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} y(t)\, dt, \tag{1}$$

for $a \le x \le b$, where $\alpha \in \mathbb{R}^+$, is called Riemann-Liouville fractional integral of order α.

2.2. Riemann-Liouville and Caputo Fractional Derivative Operator of Order α. The operator D_x^α, defined by

$$D_x^\alpha y(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_0^x (x-t)^{n-\alpha-1} y(t)\,dt, \quad (2)$$

for $a \le x \le b$, where $\alpha \in \mathbb{R}^+$ and $n = \lceil \alpha \rceil$, is called Riemann-Liouville fractional derivative of order α.

The Caputo fractional derivative of a function $y \in L_1[a,b]$ is defined as

$$^cD_x^\alpha y(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1}\left(\frac{d}{dt}\right)^n y(t)\,dt, \quad (3)$$

for $a \le x \le b$, where $\alpha \in \mathbb{R}^+$ and $n = \lceil \alpha \rceil$.

3. The Haar Wavelets

The Haar function contains just one wavelet during some subinterval of time and remains zero elsewhere and is orthogonal. The uniform Haar wavelets are useful for the treatment of solution of differential equations which have no abrupt behavior. The ith uniform Haar wavelet $h_i(x)$, $x \in [a,b]$, is defined as follows [9]:

$$h_i(x) = \begin{cases} 1, & \dfrac{k}{m} \le x < \dfrac{k+0.5}{m}; \\ -1, & \dfrac{k+0.5}{m} \le x < \dfrac{k+1}{m}; \\ 0, & \text{otherwise}, \end{cases} \quad (4)$$

where $i = 2^j + k + 1$, $j = 0, 1, 2, \ldots, J$ is dilation parameter, $m = 2^j$, and $k = 0, 1, 2, \ldots, 2^j - 1$ is translation parameter. J is maximal level of resolution and the maximal value of i is $2M$ where $M = 2^J$. In particular, $h_1(x) := \chi_{[a,b]}(x)$, where $\chi_{[a,b]}(x)$ is characteristic function on interval $[a,b]$, is the Haar scaling function. For the uniform Haar wavelet, the wavelet-collocation method is applied. The collocation points for the Haar wavelets are usually taken as $x_j = (j+0.5)/2M$, where $j = 1, 2, \ldots, 2M$.

3.1. Integral of the Haar Wavelets. Any function $y \in L_2[a,b]$ can be represented in terms of the Haar series:

$$y(x) = \sum_{l=1}^{\infty} b_l h_l(x), \quad l = 2^j + k + 1, \quad j = 0, 1, 2, \ldots, J, \quad k = 0, 1, 2, \ldots, 2^j - 1, \quad (5)$$

where b_l are the Haar wavelet coefficients given as $b_l = \int_{-\infty}^{\infty} y(x) h_l(x)\,dx$.

The Riemann-Liouville fractional integral of the Haar wavelets is given as

$$I_x^\alpha h_1(x) = \frac{(x-a)^\alpha}{\Gamma(\alpha+1)},$$

$$\begin{aligned} P_{\alpha,l}(x) &= I_x^\alpha h_l(x) \\ &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} h_l(s)\,ds \\ &= \frac{1}{\Gamma(\alpha+1)} \begin{cases} (x-a(l))^\alpha, & a(l) \le x < b(l); \\ (x-a(l))^\alpha - 2(x-b(l))^\alpha, & b(l) \le x < c(l); \\ (x-a(l))^\alpha - 2(x-b(l))^\alpha \\ \quad +(x-c(l))^\alpha, & x \ge c(l). \end{cases} \end{aligned} \quad (6)$$

4. Quasilinearization [2]

The quasilinearization approach is a generalized Newton-Raphson technique for functional equations. It converges quadratically to the exact solution, if there is convergence at all, and it has monotonic convergence.

Let us consider the nonlinear nth order differential equation

$$L^n y(x) = f\left(y(x), y'(x), \ldots, y^{n-1}(x), x\right). \quad (7)$$

Application of quasilinearization technique to (7) yields

$$\begin{aligned} L^n y_{r+1}(x) &= f\left(y_r(x), y_r'(x), \ldots, y_r^{n-1}(x), x\right) \\ &\quad + \sum_{j=0}^{n-1} \left(y_{r+1}^j(x) - y_r^j(x)\right) f_{y^j}\left(y_r(x), y_r'(x), \ldots, y_r^{n-1}(x), x\right), \end{aligned} \quad (8)$$

with the initial/boundary conditions at $(r+1)$th iteration, where n is the order of the differential equation. Equation (8) is always a linear differential equation and can be solved recursively, where $y_r(x)$ is known and one can use it to get $y_{r+1}(x)$.

5. Applications

5.1. Temperature Distribution Equation in Lumped System of Combined Convection-Radiation in a Slab Made of Materials with Variable Thermal Conductivity. Let the lumped system have volume V, surface area A, density ρ, specific heat c, initial temperature T_i, temperature of the convection environment T_a, heat transfer coefficient h, and c_a which is specific heat at temperature T_a. Consider that the mathematical model describing the temperature distribution in lumped system of combined convection-radiation in a slab made of

Table 1: Numerical results for temperature distribution equation for $\varepsilon = 0.6$: Haar wavelet-quasilinearization technique at 4th iteration and level of resolutions $J = 8$.

	4th iteration ($J = 8$)			
x	Maple	y_{GA} [10]	y_{HPM} [10]	y_{Haar}
0.0	0.834542	0.963536	0.640000	0.834543
0.2	0.840390	0.964009	0.652096	0.840391
0.4	0.858269	0.965742	0.689536	0.858269
0.6	0.889247	0.969893	0.755776	0.889248
0.8	0.935346	0.979233	0.866576	0.935346

materials with variable thermal conductivity is given by the following nonlinear boundary value problem:

$$\frac{d^2 y(x)}{dx^2} - \varepsilon y^4(x) = 0, \quad 0 \le x \le 1, \qquad \frac{dy(0)}{dx} = 0, \quad y(1) = 1, \tag{9}$$

where $y = (T - T_a)/(T_i - T_a)$ is dimensionless temperature, $x = t/(\rho V c_a/hA)$ is dimensionless time, and $\varepsilon = \beta(T - T_a)$.

5.1.1. Haar Wavelet-Quasilinearization Technique. Applying the quasilinearization technique to (9), we get

$$\frac{d^2_{n+1} y(x)}{dx^2} - 4\varepsilon y_n^3(x)\, y_{n+1}(x) = -3\varepsilon y_n^4(x), \quad 0 \le x \le 1, \qquad \frac{dy_{n+1}(0)}{dx} = 0, \quad y_{n+1}(1) = 1. \tag{10}$$

Now we implement the Haar wavelet method to (10); we approximate the higher-order derivative term by the Haar wavelet series as

$$\frac{d^2 y_{n+1}(x)}{dx^2} = \sum_{l=1}^{2M} b_l h_l(x). \tag{11}$$

Lower-order derivatives are obtained by integrating (11) and using the boundary conditions:

$$y_{n+1}(x) = \sum_{l=1}^{2M} b_l p_{2,l}(x) - C_{2,l} + 1, \tag{12}$$

where $C_{2,l} = \int_0^1 p_{2,l}(x)dx$.

Substitute (11) and (12) in (10) to obtain

$$\sum_{l=1}^{2M} b_l \left[h_l(x) - 4\varepsilon y_n^3(x)\, p_{2,l}(x) + 4\varepsilon y_n^3(x)\, C_{2,l}\right] = -3\varepsilon y_n^4(x) + 4\varepsilon y_n^3(x), \tag{13}$$

with the initial approximation $y_0(x) = 0$.

Figure 1 shows the temperature y_{Haar} by Haar wavelet-quasilinearization technique for different ε at $J = 5$ and at 4th iteration. According to Figure 1 and Tables 1 and 2, temperature increases with decreasing ε; also temperature varies with time x. Tables 1 and 2 show that the obtained solutions are in good agreement with the numerical solution provided by Maple and are better than generalized approximation method y_{GA} [10] and homotopy perturbation method y_{HPM} [10].

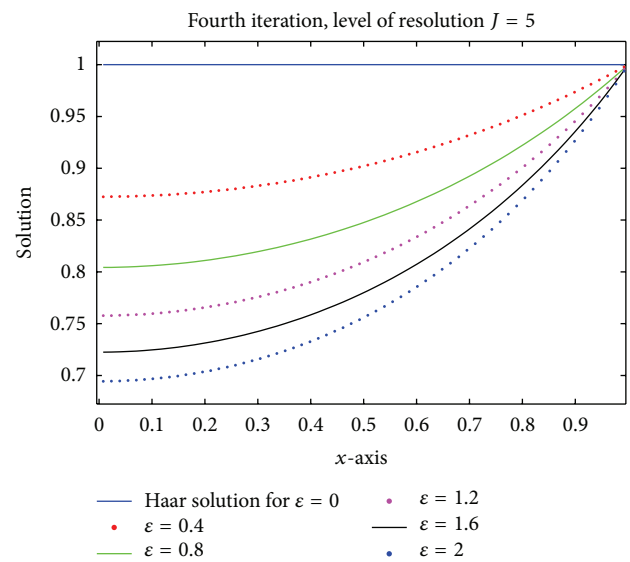

Figure 1: Solutions by Haar wavelet-quasilinearization technique for different ε at $J = 5$ and $n = 4$.

5.2. Cooling of a Lumped System by Combined Convection and Radiation. Consider that the system has volume V, surface area A, density ρ, specific heat c, emissivity E, initial temperature T_i, temperature of the convection environment T_a, heat transfer coefficient h, and c_a which is specific heat at temperature T_a. In this case system loses heat through radiation and the effective sink temperature is T_s. The mathematical model describing the cooling of a lumped system by combined convection and radiation is given by the following nonlinear initial value problem:

$$\rho V c \frac{dT(t)}{dt} + hA(T - T_a) + E\sigma A\left(T^4 - T_s^4\right) = 0, \qquad T(t = 0) = T_i. \tag{14}$$

Table 2: Numerical results for temperature distribution equation for $\varepsilon = 2.0$: Haar wavelet-quasilinearization technique at 4th iteration and level of resolutions $J = 8$.

		4th iteration ($J = 8$)		
x	Maple	y_{GA} [10]	y_{HPM} [10]	y_{Haar}
0.0	0.694318	0.968771	−0.666667	0.694362
0.2	0.703698	0.968804	−0.625600	0.703739
0.4	0.732894	0.969008	−0.489600	0.732927
0.6	0.785488	0.970024	−0.220267	0.785510
0.8	0.869161	0.975059	−0.246400	0.869176

For the solution of (14), we do the following certain changes in parameters:

$$y = \frac{T}{T_i}, \qquad y_a = \frac{T_a}{T_i}, \qquad y_s = \frac{T_s}{T_i},$$
$$x = \frac{t}{\rho V c_a / hA}, \qquad \varepsilon = \frac{E\sigma T_i^3}{h}. \tag{15}$$

Equation (14) implies after changing the parameters

$$\frac{dy(x)}{dx} + (y - y_a) + \varepsilon\left(y^4 - y_s^4\right) = 0,$$
$$y(x = 0) = 1. \tag{16}$$

For the sake of simplicity we assume that $y_a = y_s = 0$, (15) becomes

$$\frac{dy(x)}{dx} + y + \varepsilon y^4 = 0,$$
$$y(x = 0) = 1. \tag{17}$$

5.2.1. Haar Wavelet-Quasilinearization Technique. Implementation of the quasilinearization technique to (16) gives

$$\frac{dy_{n+1}(x)}{dx} + \left(1 + 4\varepsilon y_n^3\right) y_{n+1} = 3\varepsilon y_n^4,$$
$$y_{n+1}(x = 0) = 1. \tag{18}$$

According to the Haar wavelet method to (18), approximate the higher-order derivative term by the Haar wavelet series as

$$\frac{dy(x)}{dx} = \sum_{l=1}^{2M} b_l h_l(x). \tag{19}$$

Solution can be obtained by integrating (19) and using the initial condition to yield

$$y_{n+1}(x) = \sum_{l=1}^{2M} b_l p_{1,l}(x) + 1. \tag{20}$$

Substituting (19) and (20) in (18),

$$\sum_{l=1}^{2M} b_l \left[h_l(x) + \left(1 + 4\varepsilon y_n^3(x)\right) p_{1,l}(x)\right]$$
$$= 3\varepsilon y_n^4(x) - \left(1 + 4\varepsilon y_n^3(x)\right), \tag{21}$$

with the initial approximation $y_0(x) = 1$.

To get the solution on large interval, say $[0, 5]$, we divide the interval $[0, 5]$ into three subintervals $[0, 1.25]$, $[1.25, 3.75]$, and $[3.75, 5]$; let $A = 0$, $B = 1.25$, $C = 3.75$, and $D = 5$; step-size for each subinterval is

$$\Delta x_1 = \frac{(B - A)}{M/2},$$
$$\Delta x_2 = \frac{(C - B)}{M}, \tag{22}$$
$$\Delta x_3 = \frac{(D - C)}{M/2}.$$

The coordinates of the grid points are as follows.

For $j = 1, 2, \ldots, (M/2) + 1$

$$x(j) = A + (i - 1)\Delta x_1. \tag{23}$$

For $j = 1, 2, \ldots, M$

$$x\left(j + \frac{M}{2} + 1\right) = B + (i)\Delta x_2. \tag{24}$$

For $j = 1, 2, \ldots, M/2$

$$x\left(j + \frac{3M}{2} + 1\right) = C + (i)\Delta x_3. \tag{25}$$

And collocation points are as follows.

For $j = 1, 2, \ldots, 2M$

$$x_c(j) = \frac{x(j) + x(j + 1)}{2}. \tag{26}$$

Temperature y_{Haar} at higher interval, $[0, 5]$, by Haar wavelet-quasilinearization technique at $J = 5$ and iteration $n = 4$ of the cooling equation for different values of ε is shown in Figure 2. It shows that temperature decreases with increasing ε and also shows that temperature reduces to zero when time x is increasing. According to Table 3, we conclude that our results are in good agreement with exact solution and more accurate than variational iteration method y_{VIM} [11] and homotopy perturbation method y_{HPM} [11].

We can get more accurate results while increasing level of resolution J, iteration n, or both, according to convergence analysis [6].

Table 3: Numerical results for cooling equation for different ε and $x = 0.5$: Haar wavelet-quasilinearization technique at 4th iteration and level of resolutions $J = 8$.

	4th iteration ($J = 8$)			
ε	Exact	y_{VIM} [11]	y_{HPM} [11]	y_{Haar}
0.0	0.606531	0.606531	0.606531	0.606531
0.1	0.591591	0.591617	0.591638	0.591592
0.2	0.578023	0.578207	0.578371	0.578023
0.3	0.565620	0.566185	0.566732	0.565620
0.4	0.554217	0.555440	0.556720	0.554217
0.5	0.543681	0.545868	0.548335	0.543681
0.6	0.533903	0.537369	0.541576	0.533904
0.7	0.524793	0.529850	0.536445	0.524793
0.8	0.516275	0.523226	0.532940	0.516275
0.9	0.508284	0.517412	0.531062	0.508284
1.0	0.500765	0.512333	0.530812	0.500765

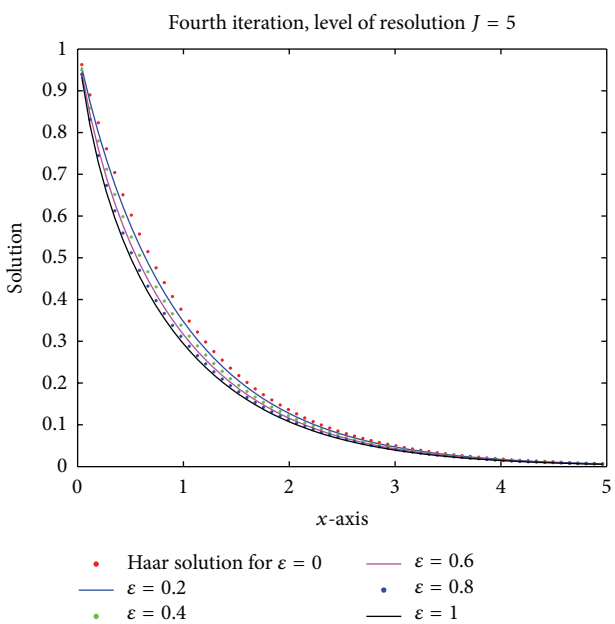

Figure 2: Solutions by Haar wavelet-quasilinearization technique for different ε at $J = 5$ and $n = 4$.

6. Conclusion

It is shown that Haar wavelet method with quasilinearization technique gives excellent results when applied to different nonlinear heat transfer problems. The results obtained from Haar wavelet-quasilinearization technique are better from the results obtained by other methods and are in good agreement with exact solutions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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