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Research Article

Application of Bipolar Fuzzy Sets in Graph Structures

Muhammad Akram and Rabia Akmal

Department of Mathematics, University of the Punjab, New Campus, Lahore 54590, Pakistan

Correspondence should be addressed to Muhammad Akram; m.akram@pucit.edu.pk

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A graph structure is a useful tool in solving the combinatorial problems in different areas of computer science and computational intelligence systems. In this paper, we apply the concept of bipolar fuzzy sets to graph structures. We introduce certain notions, including bipolar fuzzy graph structure (BFGS), strong bipolar fuzzy graph structure, bipolar fuzzy N_i-cycle, bipolar fuzzy N_i-tree, bipolar fuzzy N_i-cut vertex, and bipolar fuzzy N_i-bridge, and illustrate these notions by several examples. We study ϕ-complement, self-complement, strong self-complement, and totally strong self-complement in bipolar fuzzy graph structures, and we investigate some of their interesting properties.

1. Introduction

Concepts of graph theory have applications in many areas of computer science including data mining, image segmentation, clustering, image capturing, and networking. A graph structure, introduced by Sampathkumar [1], is a generalization of undirected graph which is quite useful in studying some structures including graphs, signed graphs, and graphs in which every edge is labeled or colored. A graph structure helps to study the various relations and the corresponding edges simultaneously.

A fuzzy set, introduced by Zadeh [2], gives the degree of membership of an object in a given set. Zhang [3] initiated the concept of a bipolar fuzzy set as a generalization of a fuzzy set. A bipolar fuzzy set is an extension of fuzzy set whose membership degree range is $[-1, 1]$. In a bipolar fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree $(0, 1]$ of an element indicates that the element somewhat satisfies the property, and the membership degree $[-1, 0)$ of an element indicates that the element somewhat satisfies the implicit counterproperty. Kauffman defined in [4] a fuzzy graph. Rosenfeld [5] described the structure of fuzzy graphs obtaining analogs of several graph theoretical concepts. Bhattacharya [6] gave some remarks on fuzzy graphs. Several concepts on fuzzy graphs were introduced by Mordeson et al. [7]. Dinesh [8] introduced the notion of a fuzzy graph structure and discussed some related properties. Akram et al. [9–13] have introduced bipolar fuzzy graphs, regular bipolar fuzzy graphs, irregular bipolar fuzzy graphs, antipodal bipolar fuzzy graphs, and bipolar fuzzy hypergraphs. In this paper, we introduce the certain notions including bipolar fuzzy graph structure (BFGS), strong bipolar fuzzy graph structure, bipolar fuzzy N_i-cycle, bipolar fuzzy N_i-tree, bipolar fuzzy N_i-cut vertex, and bipolar fuzzy N_i-bridge and illustrate these notions by several examples. We present ϕ-complement, self-complement, strong self-complement, and totally strong self-complement in bipolar fuzzy graph structures, and we investigate some of their interesting properties.

We have used standard definitions and terminologies in this paper. For other notations, terminologies, and applications not mentioned in the paper, the readers are referred to [1, 5, 7, 14–18].

2. Preliminaries

In this section, we review some definitions that are necessary for this paper.

A *graph structure* $G^* = (U, E_1, E_2, \ldots, E_k)$ consists of a nonempty set U together with relations $E_1, E_2, \ldots, E_k$ on U, which are mutually disjoint such that each E_i is irreflexive and symmetric. If $(u, v) \in E_i$ for some i, $1 \leq i \leq k$, we call it an E_i-*edge* and write it as "uv." A *graph structure*

$G^* = (U, E_1, E_2, \ldots, E_k)$ is *complete*, if (i) each edge E_i, $1 \leq i \leq k$, appears at least once in G^*; (ii) between each pair of vertices uv in U, uv is an E_i-edge for some i, $1 \leq i \leq k$. A *graph structure* $G^* = (U, E_1, E_2, \ldots, E_k)$ is *connected*, if the underlying graph is *connected*. In a *graph structure*, E_i*-path* between two vertices u and v, is the path which consists of only E_i*-edges* for some i, and similarly, E_i*-cycle* is the cycle which consists of only E_i*-edges* for some i. A *graph structure* is a *tree* if it is connected and contains no cycle or equivalently the underlying graph is a tree. G^* is an E_i*-tree*, if the *subgraph structure* induced by E_i*-edges* is a *tree*. Similarly, G^* is an $E_1E_2 \cdots E_j$*-tree*, if G^* is an E_i*-tree* for each j, $1 \leq j \leq k$. A *graph structure* is an E_i*-forest*, if the *subgraph structure* induced by E_i*-edges* is a *forest*, that is, if it has no E_i*-cycles*. Let $S \subseteq U$; then the *subgraph structure* $\langle S \rangle$ induced by S has vertex set S, where two vertices u and v in $\langle S \rangle$ are joined by an E_i*-edge*, $1 \leq i \leq k$, if and only if, they are joined by an E_i*-edge* in G^*. For some i, $1 \leq i \leq k$, the E_i*-subgraph* induced by S is denoted by E_i-$\langle S \rangle$. It has only those E_i*-edges* of G^*, joining vertices in S. If T is a subset of edge set in G^*, then subgraph structure $\langle T \rangle$ induced by T has the vertex set, "the end vertices in T", whose edges are those in T. Let $G^* = (U_1, E_1, E_2, \ldots, E_m)$ and $H^* = (U_2, E'_1, E'_2, \ldots, E'_n)$ be *graph structures*. Then G^* and H^* are *isomorphic*, if (i) $m = n$, (ii) there exist a bijection $f : U_1 \to U_2$ and a bijection $\phi : \{E_1, E_2, \ldots, E_n\} \to \{E'_1, E'_2, \ldots, E'_n\}$, say $E_i \to E'_j$, $1 \leq i, j \leq n$, such that for all $u, v \in U_1$, $uv \in E_i$ implies that $f(u)f(v) \in E'_j$.

Two *graph structures* $G^* = (U, E_1, E_2, \ldots, E_k)$ and $H^* = (U, E'_1, E'_2, \ldots, E'_k)$, on the same vertex set U, are *identical*, if there exists a bijection $f : U \to U$, such that for all u and v in U, uv is an E_i*-edge* in G^*, then $f(u)f(v)$ is an E'_i*-edge* in H^*, where $1 \leq i \leq k$ and $E_i \simeq E'_i$ $\forall i$. Let ϕ be a permutation on $\{E_1, E_2, \ldots, E_k\}$. Then the ϕ*-cyclic complement* of G^*, denoted by $(G^*)^{\phi c}$, is obtained by replacing E_i by $\phi(E_i)$, $1 \leq i \leq k$. Let $G^* = (U, E_1, E_2, \ldots, E_k)$ be a *graph structure* and ϕ a permutation on $\{E_1, E_2, \ldots, E_k\}$; then

(i) G^* is ϕ*-self complementary*, if G^* is *isomorphic* to $(G^*)^{\phi c}$; the ϕ*-cyclic complement* of G^* and G^* is *self-complement*, if $\phi \neq$ identity permutation.

(ii) G^* is *strong* ϕ*-self complementary*, if G^* is *identical* to $(G^*)^{\phi c}$; the ϕ*-complement* of G^* and G^* is *strong self-complement*, if $\phi \neq$ identity permutation.

Definition 1 (see [2]). A *fuzzy subset* μ on a set X is a map $\mu : X \to [0, 1]$. A *fuzzy binary relation* on X is a fuzzy subset μ on $X \times X$. By a fuzzy relation we mean a fuzzy binary relation given by $\mu : X \times X \to [0, 1]$.

Definition 2 (see [8]). Let $G^* = (U, E_1, E_2, \ldots, E_k)$ be a graph structure and let $\nu, \rho_1, \rho_2, \ldots, \rho_k$ be the fuzzy subsets of $U, E_1, E_2, \ldots, E_k$, respectively, such that

$$0 \leq \rho_i(xy) \leq \mu(x) \wedge \mu(y) \quad \forall x, y \in U,\ i = 1, 2, \ldots, k. \tag{1}$$

Then $G = (\nu, \rho_1, \rho_2, \ldots, \rho_k)$ is a fuzzy graph structure of G^*.

Definition 3 (see [8]). Let $G = (\nu, \rho_1, \rho_2, \ldots, \rho_k)$ be a fuzzy graph structure of a graph structure $G^* = (U, E_1, E_2, \ldots, E_k)$. Then $F = (\nu, \tau_1, \tau_2, \ldots, \tau_k)$ is a partial fuzzy spanning subgraph structure of G if $\tau_i \subseteq \rho_i$ for $i = 1, 2, \ldots, k$.

Definition 4 (see [8]). Let G^* be a graph structure and let G be a fuzzy graph structure of G^*. If $xy \in \operatorname{supp}(\rho_i)$, then "$xy$" is said to be a ρ_i-edge of G.

Definition 5 (see [8]). The strength of a ρ_i-path $x_0x_1 \cdots x_n$ of a fuzzy graph structure G is $\bigwedge_{j=1}^{n} \rho_i(x_{j-1}x_j)$ for $i = 1, 2, \ldots, k$.

Definition 6 (see [8]). In a fuzzy graph structure G, $\rho_i^2(xy) = \rho_i \circ \rho_i(xy) = \bigvee_z \{\rho_i(xz) \wedge \rho_i(zy)\}$, $\rho_i^j(xy) = (\rho_i^{j-1} \circ \rho_i)(xy) = \bigvee_z \{\rho_i^{j-1}(xz) \wedge \rho_i(zy)\}$, $j = 2, 3, \ldots, m$, for any $m \geq 2$. Also $\rho_i^\infty(xy) = \bigvee \{\rho_i^j(xy),\ j = 1, 2, \ldots\}$.

Definition 7 (see [8]). Let xy be a ρ_i-edge of $G = (\nu, \rho_1, \rho_2, \ldots, \rho_n)$. Let $(\nu, \rho'_1, \rho'_2, \ldots, \rho'_n)$ be a partial fuzzy spanning subgraph structure obtained by deleting "xy" with $\rho'_i(xy) = 0$ and $\rho'_i(x_1y_1) = \rho_i(x_1y_1)$ $\forall \rho_i$-edges (x_1y_1) other than (x, y). If $\rho_i^\infty(uv) > \rho_i'^\infty(uv)$ for some $uv \in \operatorname{supp}(\rho_i)$, then xy is a ρ_i-bridge.

Definition 8 (see [8]). Let $G' = (\nu, \rho'_1, \rho'_2, \ldots, \rho'_n)$ be the partial fuzzy subgraph structure obtained by deleting vertex w of G, that is, $\nu'(w) = 0$ and $\nu'(v) = \nu(v)$ $\forall v \neq w$, $\rho'_i(vw) = 0$ $\forall v \in Ur$ and $\rho'_i(uv) = \rho_i(uv)$ $\forall uv \neq wv$, $i = 1, 2, \ldots, k$. Then a vertex w of G is a ρ_i-cut vertex if $\rho_i^\infty(uv) > \rho_i'^\infty(uv)$ for some u, v with $u, v \neq w$.

Definition 9 (see [8]). $G = (\nu, \rho_1, \rho_2, \ldots, \rho_k)$ is a ρ_i-cycle if and only if $(\operatorname{supp}(\nu), \operatorname{supp}(\rho_1), \operatorname{supp}(\rho_2), \ldots, \operatorname{supp}(\rho_k))$ is a E_i-cycle.

Definition 10 (see [8]). $G = (\nu, \rho_1, \rho_2, \ldots, \rho_k)$ is a fuzzy ρ_i-cycle if and only if $(\operatorname{supp}(\nu), \operatorname{supp}(\rho_1), \operatorname{supp}(\rho_2), \ldots, \operatorname{supp}(\rho_k))$ is an E_i-cycle and there exists no unique "xy" in $\operatorname{supp}(\rho_i)$ such that $\rho_i(xy) = \bigwedge \{\rho_i(uv) \mid uv \in \operatorname{supp}(\rho_i)\}$.

Definition 11 (see [8]). $G = (\nu, \rho_1, \rho_2, \ldots, \rho_k)$ is a fuzzy ρ_i-tree if it has a partial fuzzy spanning subgraph structure, $F_i = (\nu, \tau_1, \tau_2, \ldots, \tau_k)$, which is a τ_i-tree where for all ρ_i-edges not in F_i, $\rho_i(xy) < \tau_i^\infty(xy)$.

Definition 12 (see [8]). Let $G^* = (U, E_1, E_2, \ldots, E_k)$ be a graph structure and let $\nu, \rho_1, \rho_2, \ldots, \rho_k$ be the fuzzy subsets of $U, E_1, E_2, \ldots, E_k$, respectively, such that

$$0 \leq \rho_i(xy) \leq \mu(x) \wedge \mu(y) \quad \forall x, y \in V,\ i = 1, 2, \ldots, k. \tag{2}$$

Then $G = (\nu, \rho_1, \rho_2, \ldots, \rho_k)$ is a fuzzy graph structure of G^*.

Definition 13 (see [3]). Let X be a nonempty set. A *bipolar fuzzy set* B in X is an object having the form

$$B = \left\{\left(x, \mu_B^P(x), \mu_B^N(x)\right) \mid x \in X\right\}, \tag{3}$$

where $\mu_B^P : X \to [0, 1]$ and $\mu_B^N : X \to [-1, 0]$ are mappings.

We use the positive membership degree $\mu_B^P(x)$ to denote the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set B and the negative membership degree $\mu_B^N(x)$ to denote the satisfaction degree of an element x to some implicit counterproperty corresponding to a bipolar fuzzy set B. If $\mu_B^P(x) \neq 0$ and $\mu_B^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for B. If $\mu_B^P(x) = 0$ and $\mu_B^N(x) \neq 0$, it is the situation that x does not satisfy the property of B but somewhat satisfies the counter property of B. It is possible for an element x to be such that $\mu_B^P(x) \neq 0$ and $\mu_B^N(x) \neq 0$ when the membership function of the property overlaps that of its counterproperty over some portion of X.

For the sake of simplicity, we will use the symbol $B = (\mu_B^P, \mu_B^N)$ for the bipolar fuzzy set:

$$B = \left\{\left(x, \mu_B^P(x), \mu_B^N(x)\right) \mid x \in X\right\}. \tag{4}$$

Definition 14 (see [3]). Let X be a nonempty set. Then we call a mapping $A = (\mu_A^P, \mu_A^N) : X \times X \rightarrow [0,1] \times [-1,0]$ a *bipolar fuzzy relation* on X such that $\mu_A^P(x,y) \in [0,1]$ and $\mu_A^N(x,y) \in [-1,0]$.

Definition 15 (see [9]). A bipolar fuzzy graph $G = (V, A, B)$ is a nonempty set V together with a pair of functions $A = (\mu_A^P, \mu_A^N) : V \rightarrow [0,1] \times [-1,0]$ and $B = (\mu_B^P, \mu_B^N) : V \times V \rightarrow [0,1] \times [-1,0]$ such that for all $x, y \in V$,

$$\begin{aligned} \mu_B^P(x,y) &\leq \min\left(\mu_A^P(x), \mu_A^P(y)\right), \\ \mu_B^N(x,y) &\geq \max\left(\mu_A^N(x), \mu_A^N(y)\right). \end{aligned} \tag{5}$$

Notice that $\mu_B^P(x,y) > 0$, $\mu_B^N(x,y) < 0$ for $(x,y) \in V \times V$, $\mu_B^P(x,y) = \mu_B^N(x,y) = 0$ for $(x,y) \notin V \times V$, and B is symmetric relation.

3. Bipolar Fuzzy Graph Structures

Definition 16. $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ is called a *bipolar fuzzy graph structure* (BFGS) of a graph structure (GS) $G^* = (U, E_1, E_2, \ldots, E_n)$ if $M = (\mu_M^P, \mu_M^N)$ is a *bipolar fuzzy set on* U and for each $i = 1, 2, \ldots, n$; $N_i = (\mu_{N_i}^P, \mu_{N_i}^N)$ is a *bipolar fuzzy set on* E_i such that

$$\begin{aligned} \mu_{N_i}^P(xy) &\leq \mu_M^P(x) \wedge \mu_M^P(y), \\ \mu_{N_i}^N(xy) &\geq \mu_M^N(x) \vee \mu_M^N(y) \\ &\qquad \forall xy \in E_i \subset U \times U. \end{aligned} \tag{6}$$

Note that $\mu_{N_i}^P(xy) = 0 = \mu_{N_i}^N(xy)$ for all $xy \in U \times U - E_i$ and $0 < \mu_{N_i}^P(xy) \leq 1$, $-1 \leq \mu_{N_i}^N(xy) < 0$ $\forall xy \in E_i$, where U and E_i $(i = 1, 2, \ldots, n)$ are called *underlying vertex set* and *underlying i-edge set* of $\check{G}_b$, respectively.

Definition 17. Let $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ be a *bipolar fuzzy graph structure* of a *graph structure* $G^* = (U, E_1, E_2, \ldots, E_n)$. If $\check{H}_b = (M', N_1', N_2', \ldots, N_n')$ is a *bipolar fuzzy graph structure* of G^* such that

$$\begin{aligned} \mu_{M'}^P(x) &\leq \mu_M^P(x), \\ \mu_{M'}^N(x) &\geq \mu_M^N(x) \\ &\qquad \forall x \in U, \\ \mu_{N_i'}^P(xy) &\leq \mu_{N_i}^P(xy), \\ \mu_{N_i'}^N(xy) &\geq \mu_{N_i}^N(xy) \\ &\qquad \forall xy \in E_i,\ i = 1, 2, \ldots, n, \end{aligned} \tag{7}$$

then $\check{H}_b$ is called a *bipolar fuzzy subgraph structure* of BFGS $\check{G}_b$.

BFGS $\check{H}_b = (M', N_1', N_2', \ldots, N_n')$ is a *bipolar fuzzy induced subgraph structure* of $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$, by a subset W of U if

$$\begin{aligned} \mu_{M'}^P(x) &= \mu_M^P(x), \\ \mu_{M'}^N(x) &= \mu_M^N(x) \\ &\qquad \forall x \in W, \\ \mu_{N_i'}^P(xy) &= \mu_{N_i}^P(xy), \\ \mu_{N_i'}^N(xy) &= \mu_{N_i}^N(xy) \\ &\qquad \forall x, y \in W,\ i = 1, 2, \ldots, n. \end{aligned} \tag{8}$$

Similarly, BFGS $\check{H}_b$ is a *bipolar fuzzy spanning subgraph structure* of $\check{G}_b$ if $M' = M$ and

$$\begin{aligned} \mu_{N_i'}^P &\leq \mu_{N_i}^P, \\ \mu_{N_i'}^N &\geq \mu_{N_i}^N, \\ i &= 1, 2, \ldots, n. \end{aligned} \tag{9}$$

Example 18. Consider a graph structure $G^* = (U, E_1, E_2)$ such that $U = \{a_1, a_2, a_3, a_4\}$, $E_1 = \{a_1a_2, a_2a_4\}$, and $E_2 = \{a_3a_4, a_1a_4\}$.

(i) Let M, N_1, and N_2 be bipolar fuzzy subsets of U, E_1, and E_2, respectively, such that

$$\begin{aligned} M = \{&(a_1, 0.5, -0.2), (a_2, 0.7, -0.3), (a_3, 0.4, -0.3), \\ &(a_4, 0.7, -0.3)\}, \\ N_1 = \{&(a_1a_2, 0.5, -0.2), (a_2a_4, 0.7, -0.3)\}, \\ N_2 = \{&(a_3a_4, 0.3, -0.2), (a_1a_4, 0.3, -0.1)\}. \end{aligned} \tag{10}$$

Then, by direct calculations, it is easy to see that $\check{G}_b = (M, N_1, N_2)$ is a BFGS of G^* as shown in Figure 1.

(ii) Consider $M_1 = \{(a_1, 0.4, -0.1), (a_2, 0.5, -0.3), (a_3, 0.4, -0.2), (a_4, 0.1, -0.3)\}$, $N_{11} = \{(a_1a_2, 0.4, -0.1), (a_2a_4, 0.1, -0.2)\}$, and $N_{12} = \{(a_3a_4, 0.1, -0.2), (a_1a_4, 0.1, -0.0)\}$. Then,

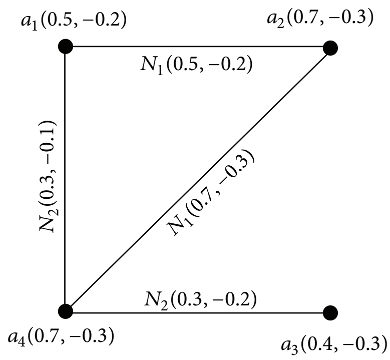


Figure 1: $\check{G}_b = (M, N_1, N_2)$.

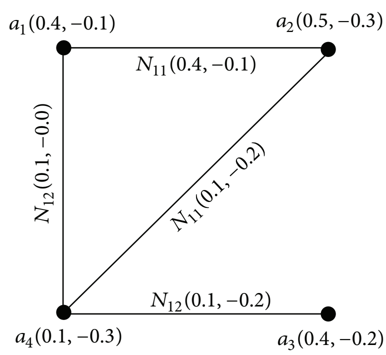


Figure 2: *Bipolar fuzzy subgraph structure* $\check{K}_b = (M_1, N_{11}, N_{12})$.

by routine calculations, it is easy to see that $\check{K}_b = (M_1, N_{11}, N_{12})$ is the bipolar fuzzy subgraph structure of $\check{G}_b$ as shown in Figure 2.

Definition 19. Let $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ be a *bipolar fuzzy graph structure* of a graph structure $G^* = (U, E_1, E_2, \ldots, E_n)$. Then $xy \in E_i$ is called a *bipolar fuzzy N_i-edge* or simply *N_i-edge*, if

$$\mu^P_{N_i}(xy) > 0 \text{ or } \mu^N_{N_i}(xy) < 0. \tag{11}$$

Then *support* of N_i, $i = 1, 2, \ldots, n$, consequently, is

$$\operatorname{supp}(N_i) = \left\{xy \in E_i : \mu^P_{N_i}(xy) > 0,\ \mu^N_{N_i}(xy) < 0\right\}. \tag{12}$$

Definition 20. *N_i-path* in a BFGS $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ of a graph structure $G^* = (U, E_1, E_2, \ldots, E_n)$ is a sequence $a_1, a_2, \ldots, a_m$ of distinct vertices (except the choice $a_m = a_1$) in U, such that $a_{j-1}a_j$ is a *bipolar fuzzy N_i-edge* for all $j = 2, 3, \ldots, m$.

Definition 21. A BFGS $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ with underlying vertex set U is said to be *N_i-strong* for some $i \in \{1, 2, 3, \ldots, n\}$ if for all $xy \in \operatorname{supp}(N_i)$

$$\begin{aligned} \mu^P_{N_i}(xy) &= \mu^P_M(x) \wedge \mu^P_M(y), \\ \mu^N_{N_i}(xy) &= \mu^N_M(x) \vee \mu^N_M(y). \end{aligned} \tag{13}$$

A BFGS $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ is said to be *strong* if it is *N_i-strong* BFGS for all $i \in \{1, 2, 3, \ldots, n\}$.

Example 22. Consider BFGS $\check{G}_b = (M, N_1, N_2)$ as shown in Figure 3.

Then $\check{G}_b$ is a strong BFGS since it is both N_1- and N_2-strong.

Definition 23. A BFGS $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ with underlying vertex set U is said to be *complete* or $N_1N_2\cdots N_n$-*complete*, if the following are true:

(i) $\check{G}_b$ a is *strong* BFGS.
(ii) $\operatorname{supp}(N_i) \neq \emptyset\ \forall i = 1, 2, 3, \ldots, n$.
(iii) For each pair of vertices $x, y \in U$, xy is an *N_i-edge* for some i.

Example 24. Let $\check{G}_b = (M, N_1, N_2)$ be BFGS of graph structure $G^* = (U, E_1, E_2)$ such that $U = \{a_1, a_2, a_3\}$, $E_1 = \{a_2a_3\}$, and $E_2 = \{a_1a_2, a_1a_3\}$ as shown in Figure 4. By routine calculations, it is easy to see that $\check{G}_b$ is a strong BFGS.

Moreover, $\operatorname{supp}(N_1) \neq \emptyset$, $\operatorname{supp}(N_2) \neq \emptyset$, and every pair of vertices belonging to U is either an N_1-edge or an N_2-edge. So $\check{G}_b$ is a complete BFGS, that is, N_1N_2-complete BFGS.

Definition 25. Let $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ be a BFGS with underlying vertex set U. Then positive and negative strengths of a *N_i-path* "$P_{N_i} = a_1a_2\cdots a_m$" are called gain and loss of that *N_i-path* and denoted by $G.P_{N_i}$ and $L.P_{N_i}$, respectively, such that

$$\begin{aligned} G.P_{N_i} &= \bigwedge_{j=2}^{m} \left[\mu^P_{N_i}(a_{j-1}a_j)\right], \\ L.P_{N_i} &= \left|\bigvee_{j=2}^{m} \mu^N_{N_i}(a_{j-1}a_j)\right|. \end{aligned} \tag{14}$$

Example 26. Consider a BFGS $\check{G}_b = (M, N_1, N_2)$ as shown in Figure 4. We note that $P_{N_2} = a_1a_3a_4a_1$ is an N_2-path. So $G.P_{N_2} = \mu^P_{N_2}(a_3a_1) \wedge \mu^P_{N_2}(a_1a_2) = 0.5 \wedge 0.4 = 0.4$. Consider

$$\begin{aligned} L.P_{N_2} &= \left|\mu^N_{N_2}(a_3a_1) \vee \mu^N_{N_2}(a_1a_2)\right| = |-0.4 \vee -0.4| \\ &= |-0.4| = 0.4. \end{aligned} \tag{15}$$

Definition 27. Let $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ be a BFGS with *underlying vertex set U*. Then

(i) *N_i-gain of connectedness* between x and y is defined by $\mu^{\infty,+}_{N_i}(xy) = \bigvee_{j\geq 1}\{\mu^{j,+}_{N_i}(xy)\}$, such that $\mu^{j,+}_{N_i}(xy) = (\mu^{j-1,+}_{N_i} \circ \mu^{1,+}_{N_i})(xy)$ for $j \geq 2$ and $\mu^{2,+}_{N_i}(xy) = (\mu^{1,+}_{N_i} \circ \mu^{1,+}_{N_i})(xy) = \bigvee_z\{\mu^{1,+}_{N_i}(xz) \wedge \mu^{1,+}_{N_i}(zy)\}$, where $\mu^{1,+}_{N_i} = \mu^P_{N_i}$, $\forall i$.
(ii) *N_i-loss of connectedness* between x and y is defined by $\mu^{\infty,-}_{N_i}(xy) = \bigvee_{j\geq 1}\{\mu^{j,-}_{N_i}(xy)\}$, such that $\mu^{j,-}_{N_i}(xy) = (\mu^{j-1,-}_{N_i} \circ \mu^{1,-}_{N_i})(xy)$ for $j \geq 2$ and $\mu^{2,-}_{N_i}(xy) = (\mu^{1,-}_{N_i} \circ$

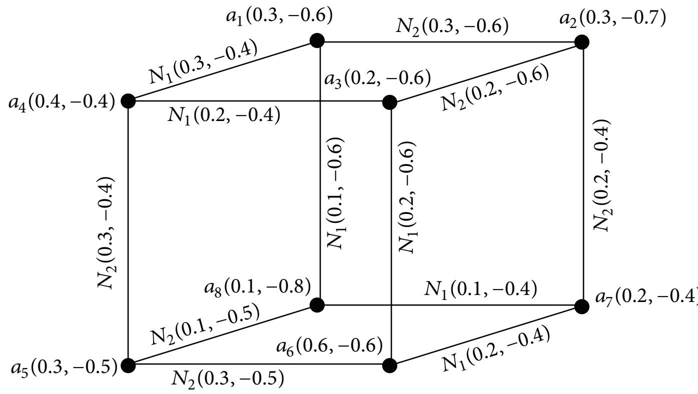


FIGURE 3: BFGS $\check{G}_b = (M, N_1, N_2)$.

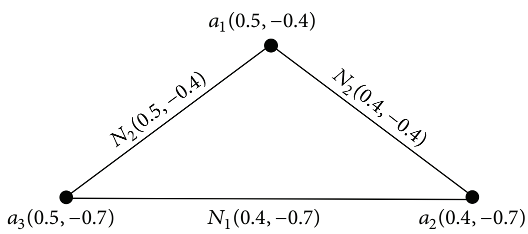


FIGURE 4: $\check{G}_b = (M, N_1, N_2)$.

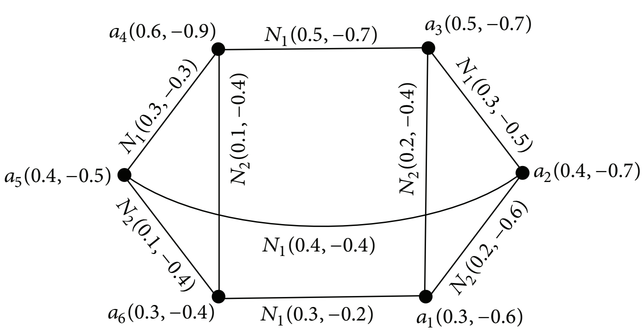


FIGURE 5: $\check{G}_b = (M, N_1, N_2)$.

$\mu^{1,-}_{N_i})(xy) = \bigvee_z \{\mu^{1,-}_{N_i}(xz) \wedge \mu^{1,-}_{N_i}(zy)\}$, where $\mu^{1,-}_{N_i} = |\mu^{N}_{N_i}|$, $\forall i$.

Example 28. Let $\check{G}_b = (M, N_1, N_2)$ be BFGS of graph structure $G = (U, E_1, E_2)$ such that $U = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, $E_1 = \{a_1a_6, a_2a_3, a_2a_5, a_3a_4, a_4a_5\}$, and $E_2 = \{a_1a_3, a_1a_2, a_4a_6, a_5a_6\}$, as is shown in Figure 5.

Since $\mu^{1,+}_{N_1}(a_2a_3) = 0.3$, $\mu^{1,+}_{N_1}(a_2a_4) = 0.0$, $\mu^{1,+}_{N_1}(a_2a_5) = 0.4$, $\mu^{1,+}_{N_1}(a_3a_4) = 0.5$, $\mu^{1,+}_{N_1}(a_5a_3) = 0.0$, $\mu^{1,+}_{N_1}(a_4a_5) = 0.3$, and $\mu^{1,+}_{N_1}(a_1a_6) = 0.3$, therefore

$$
\begin{aligned}
\mu^{2,+}_{N_1}(a_2a_3) &= \left(\mu^{1,+} n_{N_1} \circ \mu^{1,+}_{N_1}\right)(a_2a_3) \\
&= \left[\mu^{1,+}_{N_1}(a_2a_4) \wedge \mu^{1,+}_{N_1}(a_4a_3)\right] \\
&\quad \vee \left[\mu^{1,+}_{N_1}(a_2a_5) \wedge \mu^{1,+}_{N_1}(a_5a_3)\right] \\
&= [0.0 \wedge 0.5] \vee [0.4 \wedge 0.0] = 0, \\
\mu^{2,+}_{N_1}(a_2a_4) &= \left(\mu^{1,+} n_{N_1} \circ \mu^{1,+}_{N_1}\right)(a_2a_4) \\
&= \left[\mu^{2,+}_{N_1}(a_2a_3) \wedge \mu^{1,+}_{N_1}(a_3a_4)\right] \\
&\quad \vee \left[\mu^{1,+}_{N_1}(a_2a_5) \wedge \mu^{1,+}_{N_1}(a_5a_4)\right] \\
&= [0.3 \wedge 0.5] \vee [0.4 \wedge 0.3] = 0.3, \\
\mu^{2,+}_{N_1}(a_2a_5) &= \left(\mu^{1,+} n_{N_1} \circ \mu^{1,+}_{N_1}\right)(a_2a_5) \\
&= \left[\mu^{1,+}_{N_1}(a_2a_3) \wedge \mu^{1,+}_{N_1}(a_3a_5)\right] \\
&\quad \vee \left[\mu^{1,+}_{N_1}(a_2a_4) \wedge \mu^{1,+}_{N_1}(a_4a_5)\right] \\
&= [0.3 \wedge 0.0] \vee [0.0 \wedge 0.3] = 0, \\
\mu^{2,+}_{N_1}(a_3a_4) &= \left(\mu^{1,+} n_{N_1} \circ \mu^{1,+}_{N_1}\right)(a_3a_4) \\
&= \left[\mu^{1,+}_{N_1}(a_3a_2) \wedge \mu^{1,+}_{N_1}(a_2a_4)\right] \\
&\quad \vee \left[\mu^{1,+}_{N_1}(a_3a_5) \wedge \mu^{1,+}_{N_1}(a_5a_4)\right] \\
&= [0.3 \wedge 0.0] \vee [0.0 \wedge 0.3] = 0, \\
\mu^{2,+}_{N_1}(a_3a_5) &= \left(\mu^{1,+} n_{N_1} \circ \mu^{1,+}_{N_1}\right)(a_3a_5) \\
&= \left[\mu^{1,+}_{N_1}(a_3a_2) \wedge \mu^{1,+}_{N_1}(a_2a_5)\right] \\
&\quad \vee \left[\mu^{1,+}_{N_1}(a_3a_4) \wedge \mu^{1,+}_{N_1}(a_4a_5)\right] \\
&= [0.3 \wedge 0.4] \vee [0.5 \wedge 0.3] = 0.3, \\
\mu^{2,+}_{N_1}(a_4a_5) &= \left(\mu^{1,+} n_{N_1} \circ \mu^{1,+}_{N_1}\right)(a_4a_5) \\
&= \left[\mu^{1,+}_{N_1}(a_4a_2) \wedge \mu^{1,+}_{N_1}(a_2a_5)\right] \\
&\quad \vee \left[\mu^{1,+}_{N_1}(a_4a_3) \wedge \mu^{1,+}_{N_1}(a_3a_5)\right] \\
&= [0.0 \wedge 0.4] \vee [0.5 \wedge 0.0] = 0,
\end{aligned}
$$

$$\begin{aligned}
\mu_{N_1}^{2,+}(a_1a_6) &= \left(\mu^{1,+}n_{N_1} \circ \mu_{N_1}^{1,+}\right)(a_1a_6) = 0,\\
\mu_{N_1}^{3,+}(a_2a_3) &= \left(\mu^{2,+}n_{N_1} \circ \mu_{N_1}^{1,+}\right)(a_2a_3)\\
&= \left[\mu_{N_1}^{2,+}(a_2a_4) \wedge \mu_{N_1}^{1,+}(a_4a_3)\right]\\
&\quad \vee \left[\mu_{N_1}^{2,+}(a_2a_5) \wedge \mu_{N_1}^{1,+}(a_5a_3)\right]\\
&= [0.3 \wedge 0.5] \vee [0.0 \wedge 0.0] = 0.3,\\
\mu_{N_1}^{3,+}(a_2a_4) &= \left(\mu^{2,+}n_{N_1} \circ \mu_{N_1}^{1,+}\right)(a_2a_4)\\
&= \left[\mu_{N_1}^{2,+}(a_2a_3) \wedge \mu_{N_1}^{1,+}(a_3a_4)\right]\\
&\quad \vee \left[\mu_{N_1}^{2,+}(a_2a_5) \wedge \mu_{N_1}^{1,+}(a_5a_4)\right]\\
&= [0.0 \wedge 0.5] \vee [0.0 \wedge 0.3] = 0.0,\\
\mu_{N_1}^{3,+}(a_2a_5) &= \left(\mu^{2,+}n_{N_1} \circ \mu_{N_1}^{1,+}\right)(a_2a_5)\\
&= \left[\mu_{N_1}^{2,+}(a_2a_3) \wedge \mu_{N_1}^{1,+}(a_3a_5)\right]\\
&\quad \vee \left[\mu_{N_1}^{2,+}(a_2a_4) \wedge \mu_{N_1}^{1,+}(a_4a_5)\right]\\
&= [0.0 \wedge 0.0] \vee [0.3 \wedge 0.3] = 0.3,\\
\mu_{N_1}^{3,+}(a_3a_4) &= \left(\mu^{2,+}n_{N_1} \circ \mu_{N_1}^{1,+}\right)(a_3a_4)\\
&= \left[\mu_{N_1}^{2,+}(a_3a_2) \wedge \mu_{N_1}^{1,+}(a_2a_4)\right]\\
&\quad \vee \left[\mu_{N_1}^{2,+}(a_3a_5) \wedge \mu_{N_1}^{1,+}(a_5a_4)\right]\\
&= [0.0 \wedge 0.0] \vee [0.3 \wedge 0.3] = 0.3,\\
\mu_{N_1}^{3,+}(a_3a_5) &= \left(\mu^{2,+}n_{N_1} \circ \mu_{N_1}^{1,+}\right)(a_3a_5)\\
&= \left[\mu_{N_1}^{2,+}(a_3a_2) \wedge \mu_{N_1}^{1,+}(a_2a_5)\right]\\
&\quad \vee \left[\mu_{N_1}^{2,+}(a_3a_4) \wedge \mu_{N_1}^{1,+}(a_4a_5)\right]\\
&= [0.0 \wedge 0.4] \vee [0.0 \wedge 0.3] = 0.0,\\
\mu_{N_1}^{3,+}(a_4a_5) &= \left(\mu^{2,+}n_{N_1} \circ \mu_{N_1}^{1,+}\right)(a_4a_5)\\
&= \left[\mu_{N_1}^{2,+}(a_4a_2) \wedge \mu_{N_1}^{1,+}(a_2a_5)\right]\\
&\quad \vee \left[\mu_{N_1}^{2,+}(a_4a_3) \wedge \mu_{N_1}^{1,+}(a_3a_5)\right]\\
&= [0.3 \wedge 0.4] \vee [0.0 \wedge 0.0] = 0.3,\\
\mu_{N_1}^{3,+}(a_1a_6) &= \left(\mu^{2,+}n_{N_1} \circ \mu_{N_1}^{1,+}\right)(a_1a_6) = 0.
\end{aligned} \tag{16}$$

Similarly,

$$\begin{aligned}
\mu_{N_1}^{4,+}(a_2a_3) &= \left(\mu^{1,+}n_{N_1} \circ \mu_{N_1}^{1,+}\right)(a_2a_3) = 0,\\
\mu_{N_1}^{4,+}(a_2a_4) &= \left(\mu^{1,+}n_{N_1} \circ \mu_{N_1}^{1,+}\right)(a_2a_4) = 0.3,\\
\mu_{N_1}^{4,+}(a_2a_5) &= \left(\mu^{1,+}n_{N_1} \circ \mu_{N_1}^{1,+}\right)(a_2a_5) = 0,\\
\mu_{N_1}^{4,+}(a_3a_4) &= \left(\mu^{1,+}n_{N_1} \circ \mu_{N_1}^{1,+}\right)(a_3a_4) = 0,\\
\mu_{N_1}^{4,+}(a_3a_5) &= \left(\mu^{1,+}n_{N_1} \circ \mu_{N_1}^{1,+}\right)(a_3a_5) = 0.3,\\
\mu_{N_1}^{4,+}(a_4a_5) &= \left(\mu^{1,+}n_{N_1} \circ \mu_{N_1}^{1,+}\right)(a_4a_5) = 0,\\
\mu_{N_1}^{4,+}(a_1a_6) &= \left(\mu^{1,+}n_{N_1} \circ \mu_{N_1}^{1,+}\right)(a_1a_6) = 0.
\end{aligned} \tag{17}$$

This implies that

$$\begin{aligned}
\mu_{N_1}^{\infty,+}(a_2a_3) &= \vee\{0.3, 0.0, 0.3, 0.0\} = 0.3,\\
\mu_{N_1}^{\infty,+}(a_2a_4) &= \vee\{0.0, 0.3, 0.0, 0.3\} = 0.3,\\
\mu_{N_1}^{\infty,+}(a_2a_5) &= \vee\{0.4, 0.0, 0.3, 0.0\} = 0.4,\\
\mu_{N_1}^{\infty,+}(a_3a_4) &= \vee\{0.5, 0.0, 0.3, 0.0\} = 0.5,\\
\mu_{N_1}^{\infty,+}(a_3a_5) &= \vee\{0.0, 0.3, 0.0, 0.3\} = 0.3,\\
\mu_{N_1}^{\infty,+}(a_4a_5) &= \vee\{0.3, 0.0, 0.3, 0.0\} = 0.3,\\
\mu_{N_1}^{\infty,+}(a_1a_6) &= \vee\{0.3, 0.0, 0.0, 0.0\} = 0.3.
\end{aligned} \tag{18}$$

Since

$$\begin{aligned}
\mu_{N_1}^{1,-}(a_2a_3) &= 0.5,\\
\mu_{N_1}^{1,-}(a_2a_4) &= 0.0,\\
\mu_{N_1}^{1,-}(a_2a_5) &= 0.4,\\
\mu_{N_1}^{1,-}(a_3a_4) &= 0.7,\\
\mu_{N_1}^{1,-}(a_5a_3) &= 0.0,\\
\mu_{N_1}^{1,-}(a_4a_5) &= 0.3,\\
\mu_{N_1}^{1,-}(a_1a_6) &= 0.2,
\end{aligned} \tag{19}$$

we have

$$\begin{aligned}
\mu_{N_1}^{2,-}(a_2a_3) &= \left(\mu^{1,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_2a_3)\\
&= \left[\mu_{N_1}^{1,-}(a_2a_4) \wedge \mu_{N_1}^{1,-}(a_4a_3)\right]\\
&\quad \vee \left[\mu_{N_1}^{1,-}(a_2a_5) \wedge \mu_{N_1}^{1,-}(a_5a_3)\right]\\
&= [0.0 \wedge 0.7] \vee [0.4 \wedge 0.0] = 0.0,\\
\mu_{N_1}^{2,-}(a_2a_4) &= \left(\mu^{1,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_2a_4)\\
&= \left[\mu_{N_1}^{2,-}(a_2a_3) \wedge \mu_{N_1}^{1,-}(a_3a_4)\right]\\
&\quad \vee \left[\mu_{N_1}^{1,-}(a_2a_5) \wedge \mu_{N_1}^{1,-}(a_5a_4)\right]\\
&= [0.5 \wedge 0.7] \vee [0.4 \wedge 0.3] = 0.5,
\end{aligned}$$

$$\begin{aligned}
\mu_{N_1}^{2,-}(a_2a_5) &= \left(\mu^{1,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_2a_5) \\
&= \left[\mu_{N_1}^{1,-}(a_2a_3) \wedge \mu_{N_1}^{1,-}(a_3a_5)\right] \\
&\quad \vee \left[\mu_{N_1}^{1,-}(a_2a_4) \wedge \mu_{N_1}^{1,-}(a_4a_5)\right] \\
&= [0.5 \wedge 0.0] \vee [0.0 \wedge 0.3] = 0.0, \\
\mu_{N_1}^{2,-}(a_3a_4) &= \left(\mu^{1,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_3a_4) \\
&= \left[\mu_{N_1}^{1,-}(a_3a_2) \wedge \mu_{N_1}^{1,-}(a_2a_4)\right] \\
&\quad \vee \left[\mu_{N_1}^{1,-}(a_3a_5) \wedge \mu_{N_1}^{1,-}(a_5a_4)\right] \\
&= [0.5 \wedge 0.0] \vee [0.0 \wedge 0.3] = 0.0, \\
\mu_{N_1}^{2,-}(a_3a_5) &= \left(\mu^{1,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_3a_5) \\
&= \left[\mu_{N_1}^{1,-}(a_3a_2) \wedge \mu_{N_1}^{1,-}(a_2a_5)\right] \\
&\quad \vee \left[\mu_{N_1}^{1,-}(a_3a_4) \wedge \mu_{N_1}^{1,-}(a_4a_5)\right] \\
&= [0.5 \wedge 0.4] \vee [0.7 \wedge 0.3] = 0.4, \\
\mu_{N_1}^{2,-}(a_4a_5) &= \left(\mu^{1,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_4a_5) \\
&= \left[\mu_{N_1}^{1,-}(a_4a_2) \wedge \mu_{N_1}^{1,-}(a_2a_5)\right] \\
&\quad \vee \left[\mu_{N_1}^{1,-}(a_4a_3) \wedge \mu_{N_1}^{1,-}(a_3a_5)\right] \\
&= [0.0 \wedge 0.4] \vee [0.7 \wedge 0.0] = 0.0, \\
\mu_{N_1}^{2,-}(a_1a_6) &= \left(\mu^{1,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_1a_6) = 0, \\
\mu_{N_1}^{3,-}(a_2a_3) &= \left(\mu^{2,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_2a_3) \\
&= \left[\mu_{N_1}^{2,-}(a_2a_4) \wedge \mu_{N_1}^{1,-}(a_4a_3)\right] \\
&\quad \vee \left[\mu_{N_1}^{2,-}(a_2a_5) \wedge \mu_{N_1}^{1,-}(a_5a_3)\right] \\
&= [0.5 \wedge 0.7] \vee [0.0 \wedge 0.0] = 0.5, \\
\mu_{N_1}^{3,-}(a_2a_4) &= \left(\mu^{2,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_2a_4) \\
&= \left[\mu_{N_1}^{2,-}(a_2a_3) \wedge \mu_{N_1}^{1,-}(a_3a_4)\right] \\
&\quad \vee \left[\mu_{N_1}^{2,-}(a_2a_5) \wedge \mu_{N_1}^{1,-}(a_5a_4)\right] \\
&= [0.0 \wedge 0.7] \vee [0.0 \wedge 0.3] = 0.0, \\
\mu_{N_1}^{3,-}(a_2a_5) &= \left(\mu^{2,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_2a_5) \\
&= \left[\mu_{N_1}^{2,-}(a_2a_3) \wedge \mu_{N_1}^{1,-}(a_3a_5)\right] \\
&\quad \vee \left[\mu_{N_1}^{2,-}(a_2a_4) \wedge \mu_{N_1}^{1,-}(a_4a_5)\right] \\
&= [0.0 \wedge 0.0] \vee [0.5 \wedge 0.3] = 0.3, \\
\mu_{N_1}^{3,-}(a_3a_4) &= \left(\mu^{2,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_3a_4) \\
&= \left[\mu_{N_1}^{2,-}(a_3a_2) \wedge \mu_{N_1}^{1,-}(a_2a_4)\right] \\
&\quad \vee \left[\mu_{N_1}^{2,-}(a_3a_5) \wedge \mu_{N_1}^{1,-}(a_5a_4)\right] \\
&= [0.0 \wedge 0.0] \vee [0.4 \wedge 0.3] = 0.3, \\
\mu_{N_1}^{3,-}(a_3a_5) &= \left(\mu^{2,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_3a_5) \\
&= \left[\mu_{N_1}^{2,-}(a_3a_2) \wedge \mu_{N_1}^{1,-}(a_2a_5)\right] \\
&\quad \vee \left[\mu_{N_1}^{2,-}(a_3a_4) \wedge \mu_{N_1}^{1,-}(a_4a_5)\right] \\
&= [0.0 \wedge 0.4] \vee [0.0 \wedge 0.3] = 0.0, \\
\mu_{N_1}^{3,-}(a_4a_5) &= \left(\mu^{2,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_4a_5) \\
&= \left[\mu_{N_1}^{2,-}(a_4a_2) \wedge \mu_{N_1}^{1,-}(a_2a_5)\right] \\
&\quad \vee \left[\mu_{N_1}^{2,-}(a_4a_3) \wedge \mu_{N_1}^{1,-}(a_3a_5)\right] \\
&= [0.5 \wedge 0.4] \vee [0.0 \wedge 0.0] = 0.4, \\
\mu_{N_1}^{3,-}(a_1a_6) &= \left(\mu^{2,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_1a_6) = 0.
\end{aligned} \tag{20}$$

Similarly,

$$\begin{aligned}
\mu_{N_1}^{4,-}(a_2a_3) &= \left(\mu^{1,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_2a_3) = 0, \\
\mu_{N_1}^{4,-}(a_2a_4) &= \left(\mu^{1,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_2a_4) = 0.5, \\
\mu_{N_1}^{4,-}(a_2a_5) &= \left(\mu^{1,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_2a_5) = 0, \\
\mu_{N_1}^{4,-}(a_3a_4) &= \left(\mu^{1,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_3a_4) = 0, \\
\mu_{N_1}^{4,-}(a_3a_5) &= \left(\mu^{1,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_3a_5) = 0.4, \\
\mu_{N_1}^{4,-}(a_4a_5) &= \left(\mu^{1,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_4a_5) = 0, \\
\mu_{N_1}^{4,-}(a_1a_6) &= \left(\mu^{1,-}n_{N_1} \circ \mu_{N_1}^{1,-}\right)(a_1a_6) = 0.
\end{aligned} \tag{21}$$

This implies that

$$\begin{aligned}
\mu_{N_1}^{\infty,-}(a_2a_3) &= \vee\{0.5, 0.0, 0.5, 0.0\} = 0.5, \\
\mu_{N_1}^{\infty,-}(a_2a_4) &= \vee\{0.0, 0.5, 0.0, 0.5\} = 0.5, \\
\mu_{N_1}^{\infty,-}(a_2a_5) &= \vee\{0.4, 0.0, 0.3, 0.0\} = 0.4, \\
\mu_{N_1}^{\infty,-}(a_3a_4) &= \vee\{0.7, 0.0, 0.3, 0.0\} = 0.7, \\
\mu_{N_1}^{\infty,-}(a_3a_5) &= \vee\{0.0, 0.4, 0.0, 0.4\} = 0.4, \\
\mu_{N_1}^{\infty,-}(a_4a_5) &= \vee\{0.3, 0.0, 0.4, 0.0\} = 0.4, \\
\mu_{N_1}^{\infty,-}(a_1a_6) &= \vee\{0.3, 0.0, 0.0,0.0\} = 0.2.
\end{aligned} \tag{22}$$

For all the remaining pairs of vertices, N_1-loss and N_1-gain of connectedness are zero.

Definition 29. A BFGS $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ of a *graph structure* $G^* = (U, E_1, E_2, \ldots, E_n)$ is an N_i*-cycle* if $(\mathrm{supp}(M), \mathrm{supp}(N_1), \mathrm{supp}(N_2), \ldots, \mathrm{supp}(N_n))$ is an E_i*-cycle*.

Definition 30. A BFGS $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ of a *graph structure* $G^* = (U, E_1, E_2, \ldots, E_n)$ is a *bipolar fuzzy* N_i*-cycle* for some i if

(i) $\check{G}_b$ is an N_i*-cycle*;

(ii) there is no unique N_i*-edge* uv in $\check{G}_b$ such that $\mu^P_{N_i}(uv) = \min\{\mu^P_{N_i}(xy) : xy \in E_i = \mathrm{supp}(N_i)\}$ or $\mu^N_{N_i}(uv) = \max\{\mu^N_{N_i}(xy) : xy \in E_i = \mathrm{supp}(N_i)\}$.

Example 31. Consider BFGS $\check{G}_b = (M, N_1, N_2)$ as shown in Figure 3. Then $\check{G}_b$ is an N_1-cycle as well as bipolar fuzzy N_1-cycle, since $(\mathrm{supp}(M), \mathrm{supp}(N_1), \mathrm{supp}(N_2))$ is an E_1-cycle and there are two N_1-edges with minimum positive degree and more than one N_1-edge with maximum negative degree of all N_1-edges.

Definition 32. Let $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ be a BFGS of a graph structure $G^* = (U, E_1, E_2, \ldots, E_n)$ and x a vertex of $\check{G}_b$. Let $(M', N'_1, N'_2, \ldots, N'_n)$ be a *bipolar fuzy subgraph structure* of $\check{G}_b$ induced by $U \setminus \{x\}$ such that

$$\begin{gathered} \mu^P_{M'}(x) = 0 = \mu^N_{M'}(x), \\ \mu^P_{N'_i}(xv) = 0 = \mu^N_{N'_i}(xv) \\ \forall \text{edges } xv \in \check{G}_b, \\ \mu^P_{M'}(v) = \mu^P_{M}(v), \\ \mu^N_{M'}(v) = \mu^N_{M}(v), \\ \forall v \neq x, \\ \mu^P_{N'_i}(uv) = \mu^P_{N_i}(uv), \\ \mu^N_{N'_i}(uv) = \mu^N_{N_i}(uv) \\ \forall i, \text{ such that } u \neq x,\ v \neq x. \end{gathered} \tag{23}$$

Then x is a *bipolar fuzzy* N_i*-cut vertex* for some i, if

$$\begin{gathered} \mu^{\infty,+}_{N_i}(uv) > \mu^{\infty,+}_{N'_i}(uv), \\ \mu^{\infty,-}_{N_i}(uv) > \mu^{\infty,-}_{N'_i}(uv) \\ \text{for some } u, v \in U \setminus \{x\}. \end{gathered} \tag{24}$$

And, x is an N_i*-P bipolar fuzzy cut vertex* if only the first condition holds and a N_i*-N bipolar fuzzy cut vertex* if only the second condition holds.

Example 33. Consider BFSG $\check{G}_b = (M, N_1, N_2)$ as considered in Example 28 and shown in Figure 5; after deleting vertex a_2, the resulting bipolar fuzzy subgraph structure will be as shown in Figure 6.

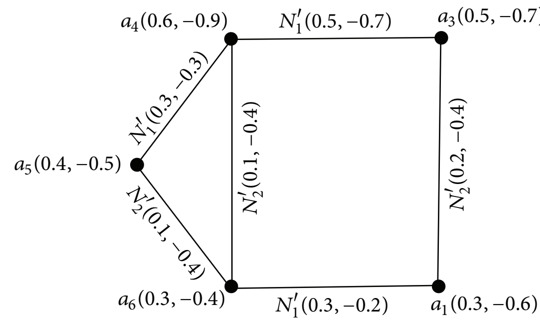


Figure 6: *Bipolar fuzzy subgraph structure* $(M \setminus \{a_2\}, N'_1, N'_2)$.

Then a_2 is a bipolar fuzzy N_1-N cut vertex since

$$\begin{gathered} \mu^{\infty,-}_{N_1}(a_3a_4) = 0.7 = \mu^{\infty,-}_{N'_1}(a_3a_4), \\ \mu^{\infty,-}_{N_1}(a_3a_5) = 0.4 > 0.3 = \mu^{\infty,-}_{N_1}(a_3a_5), \\ \mu^{\infty,-}_{N_1}(a_4a_5) = 0.4 > 0.3 = \mu^{\infty,-}_{N'_1}(a_4a_5), \\ \mu^{\infty,-}_{N_1}(a_1a_6) = 0.2 = \mu^{\infty,-}_{N_1}(a_1a_6). \end{gathered} \tag{25}$$

Definition 34. Let $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ be a BFGS of a graph structure $G^* = (U, E_1, E_2, \ldots, E_n)$ and let xy be an N_i*-edge*. Let $(M, N'_1, N'_2, \ldots, N'_n)$ be a *bipolar fuzzy spanning subgraph structure* of $\check{G}_b$, obtained by taking

$$\begin{gathered} \mu^P_{N'_i}(xy) = 0 = \mu^N_{N'_i}(xy), \\ \mu^P_{N'_i}(uv) = \mu^P_{N_i}(uv), \\ \mu^N_{N'_i}(uv) = \mu^N_{N_i}(uv) \\ \forall \text{edges } uv \neq xy. \end{gathered} \tag{26}$$

Then xy is a *bipolar fuzzy* N_i*-bridge* if

$$\begin{gathered} \mu^{\infty,+}_{N_i}(uv) > \mu^{\infty,+}_{N'_i}(uv), \\ \mu^{\infty,-}_{N_i}(uv) > \mu^{\infty,-}_{N'_i}(uv) \\ \text{for some } u, v \in U. \end{gathered} \tag{27}$$

Edge xy is an N_i*-P bipolar fuzzy bridge* if only the first condition holds and an N_i*-N bipolar fuzzy bridge* if only the second condition holds.

Example 35. Consider the BFGS $\check{G}_b = (M, N_1, N_2)$ as shown in Figure 6 and let $\check{G}'_b = (M, N'_1, N'_2)$ be bipolar fuzzy spanning subgraph structure of $\check{G}_b$ obtained by deleting N_1-edge (a_2a_5). Then a_2a_5 is a bipolar fuzzy N_1-bridge, since $\mu^{\infty,+}_{N_1}(a_2a_5) = 0.4 > 0.3 = \mu^{\infty,+}_{N'_1}(a_2a_5)$ and $\mu^{\infty,-}_{N_1}(a_2a_5) = 0.4 > 0.3 = \mu^{\infty,-}_{N'_1}(a_2a_5)$, and also *abipolar fuzzy* N_1*-N bridge*, since $\mu^{\infty,-}_{N_1}(a_3a_5) = 0.4 > 0.3 = \mu^{\infty,-}_{N'_1}(a_3a_5)$ and $\mu^{\infty,-}_{N_1}(a_4a_5) = 0.4 > 0.3 = \mu^{\infty,-}_{N'_1}(a_4a_5)$.

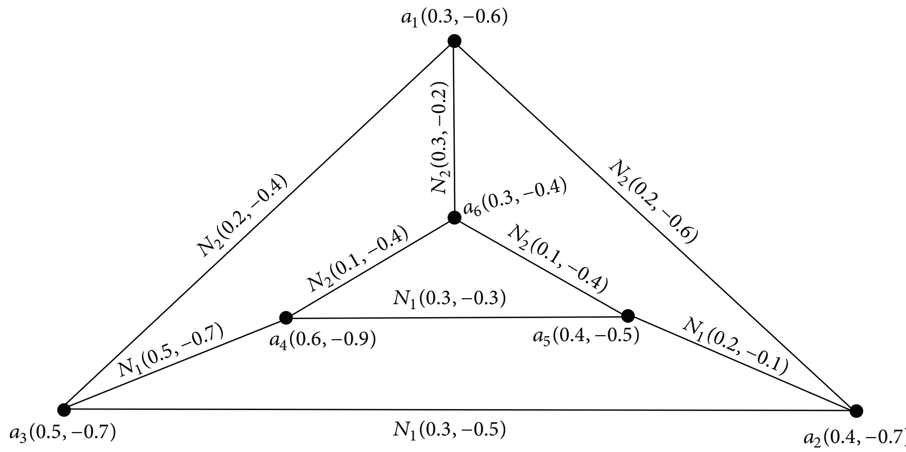


Figure 7: $\check{G}_b = (M, N_1, N_2)$.

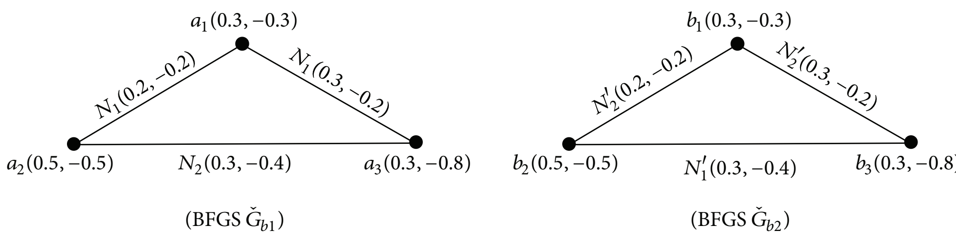


Figure 8: Isomorphic bipolar fuzzy graph structures.

Definition 36. A BFGS $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ of a *graph structure* $G^* = (U, E_1, E_2, \ldots, E_n)$ is an N_i*-tree* if $(\mathrm{supp}(A), \mathrm{supp}(N_1), \mathrm{supp}(N_2), \ldots, \mathrm{supp}(N_n))$ is an E_i*-tree*. In other words, $\check{G}_b$ is an N_i*-tree* if a subgraph of $\check{G}_b$, induced by $\mathrm{supp}(N_i)$, forms a tree.

Definition 37. A BFGS $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ of a *graph structure* $G^* = (U, E_1, E_2, \ldots, E_n)$ is a *bipolar fuzzy* N_i*-tree* if $\check{G}_b$ has a *bipolar fuzzy spanning subgraph structure* $\check{H}_b = (A, C_1, C_2, \ldots, C_n)$ such that $\check{H}_b$ is a C_i*-tree* and $\mu^P_{N_i}(xy) < \mu^{\infty,+}_{C_i}(xy)$ and $|\mu^N_{N_i}(xy)| < \mu^{\infty,-}_{C_i}(xy)$ $\forall N_i$*-edges* not in $\check{H}_b$.

In more concerned view, $\check{G}_b$ is a *bipolar fuzzy* N_i*-P tree* if only the first condition holds and a *bipolar fuzzy* N_i*-N tree* if only the second condition holds.

Example 38. Consider BFGS $\check{G}_b = (M, N_1, N_2)$ as shown in Figure 7, which is an N_2-tree. It is not an N_1-tree but a *bipolar fuzzy* N_1-tree since it has a bipolar fuzzy spanning subgraph structure (M, N_1', N_2') as an N_1-tree, which is obtained by deleting N_1-edge a_2a_5 from $\check{G}_b$ and

$$\mu^P_{N_1}(a_2a_5) = 0.2 < 0.3 = \mu^{\infty,+}_{N_1'}(a_2a_5),$$
$$\left|\mu^N_{N_1}(a_2a_5)\right| = 0.1 < 0.3 = \mu^{\infty,-}_{N_1'}(a_2a_5). \tag{28}$$

Definition 39. A BFGS $\check{G}_{s1} = (M_1, N_{11}, N_{12}, \ldots, N_{1n})$ of *graph structure* $G_1^* = (U_1, E_{11}, E_{12}, \ldots, E_{1n})$ is *isomorphic* to a BFGS $\check{G}_{s2} = (M_2, N_{21}, N_{22}, \ldots, N_{2n})$ of $G_2^* = (U_2, E_{21}, E_{22}, \ldots, E_{2n})$ if there exists a bijection $f : U_1 \to U_2$ and a permutation ϕ on the set $\{1, 2, \ldots, n\}$ such that

$$\mu^P_{M_1}(u_1) = \mu^P_{M_2}(f(u_1)),$$
$$\mu^N_{M_1}(u_1) = \mu^N_{M_2}(f(u_1)) \tag{29}$$
$$\forall u_1 \in U_1$$

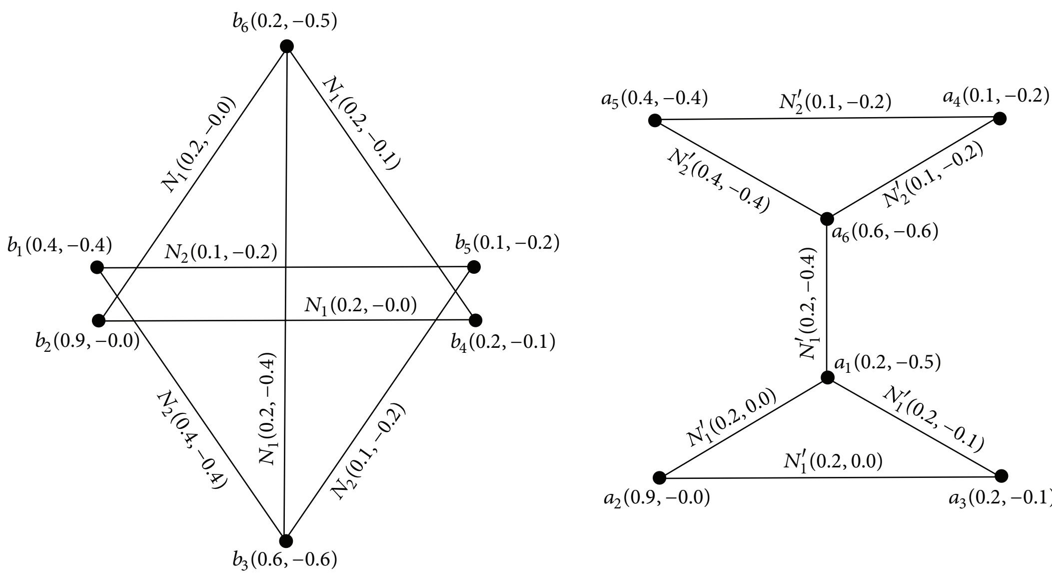
and for $\phi(i) = j$

$$\mu^P_{N_{1i}}(u_1u_2) = \mu^P_{N_{2j}}(f(u_1)f(u_2)),$$
$$\mu^N_{N_{1i}}(u_1u_2) = \mu^N_{N_{2j}}(f(u_1)f(u_2)) \tag{30}$$
$$\forall u_1u_2 \in E_{1i},\ i = 1, 2, \ldots, n.$$

Example 40. Let $\check{G}_{b1} = (M, N_1, N_2)$ and $\check{G}_{b2} = (M', N_1', N_2')$ be two BFGSs of graph structures $G_1^* = (U, E_1, E_2)$ and $G_2^* = (U', E_1', E_2')$, respectively, as shown in Figure 8.

Here $\check{G}_{b1}$ is isomorphic (not identical) to $\check{G}_{b2}$ under the mapping $f : U \to U'$, defined by $f(a_1) = b_1$, $f(a_2) = b_2$, and $f(a_3) = b_3$, and a permutation ϕ given by $\phi(1) = 2$, $\phi(2) = 1$, such that

$$\mu^P_M(a_i) = \mu^P_{M'}(f(a_i)),$$
$$\mu^N_M(a_i) = \mu^N_{M'}(f(a_i))$$
$$\forall a_i \in U,$$



($\check{G}_{b1}$) ($\check{G}_{b2}$)

Figure 9: Identical bipolar fuzzy graph structures.

$$\mu^P_{N_k}(a_ia_j) = \mu^P_{N_{\phi(k)}}(f(a_i)f(a_j)),$$
$$\mu^N_{N_k}(a_ia_j) = \mu^N_{N_{\phi(k)}}(f(a_i)f(a_j))$$
$$\forall a_ia_j \in E_k,\ k = 1,2. \tag{31}$$

Definition 41. A BFGS $\check{G}_{s1} = (M_1, N_{11}, N_{12}, \ldots, N_{1n})$ of GS $G_1^* = (U, E_{11}, E_{12}, \ldots, E_{1n})$ is *identical* to a BFGS $\check{G}_{s2} = (M_2, N_{21}, N_{22}, \ldots, N_{2n})$ of GS $G_2^* = (U, E_{21}, E_{22}, \ldots, E_{2n})$ if there exist a bijection $f : U \to U$, such that

$$\mu^P_{M_1}(u) = \mu^P_{M_2}(f(u)),$$
$$\mu^N_{M_1}(u) = \mu^N_{M_2}(f(u))$$
$$\forall u \in U, \tag{32}$$
$$\mu^P_{N_{1i}}(u_1u_2) = \mu^P_{N_{2i}}(f(u_1)f(u_2)),$$
$$\mu^N_{N_{1i}}(u_1u_2) = \mu^N_{N_{2i}}(f(u_1)f(u_2))$$
$$\forall u_1u_2 \in E_{1i},\ i = 1,2,\ldots,n.$$

Example 42. Let $\check{G}_{b1} = (M, N_1, N_2)$ and $\check{G}_{b2} = (M', N_1', N_2')$ be two BFGSs of graph structures $G_1^* = (U, E_1, E_2)$ and $G_2^* = (U', E_1', E_2')$, respectively, as shown in Figure 9.

Here $\check{G}_{b1}$ is identical with $\check{G}_{b2}$ under the mapping $f : U \to U'$, defined by $f(a_1) = b_6, f(a_2) = b_2, f(a_3) = b_4, f(a_4) = b_5, f(a_5) = b_1$, and $f(a_6) = b_3$, such that

$$\mu^P_M(a_i) = \mu^P_{M'}(f(a_i)),$$
$$\mu^N_M(a_i) = \mu^N_{M'}(f(a_i))$$
$$\forall a_i \in U,$$
$$\mu^P_{N_k}(a_ia_j) = \mu^P_{N_k'}(f(a_i)f(a_j)),$$
$$\mu^N_{N_k}(a_ia_j) = \mu^N_{N_k'}(f(a_i)f(a_j))$$
$$\forall a_ia_j \in E_k,\ k = 1,2. \tag{33}$$

Definition 43. Let $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ be a BFGS of a GS $G^* = (U, E_1, E_2, \ldots, E_n)$. Let ϕ be any permutation on the set $\{E_1, E_2, \ldots, E_n\}$ and the corresponding permutation on $\{N_1, N_2, \ldots, N_n\}$; that is, $\phi(N_i) = N_j$ if and only if $\phi(E_i) = E_j$ $\forall i$.

If $xy \in N_r$ for some r and

$$\mu^P_{N_i^\phi}(xy) = \mu^P_M(x) \wedge \mu^P_M(y) - \bigvee_{j\neq i} \mu^P_{\phi N_j}(xy),$$
$$\mu^N_{N_i^\phi}(xy) = \mu^N_M(x) \vee \mu^N_M(y) - \bigwedge_{j\neq i} \mu^N_{\phi N_j}(xy), \tag{34}$$
$$i = 1,2,\ldots,n,$$

then $xy \in B_m^\phi$, while m is chosen such that $\mu^P_{N_m^\phi}(xy) \geq \mu^P_{N_i^\phi}(xy)$ and $\mu^N_{N_m^\phi}(xy) \leq \mu^N_{N_i^\phi}(xy)$ $\forall i$.

And BFGS $(M, N_1^\phi, N_2^\phi, \ldots, N_n^\phi)$, denoted by $\check{G}_b^{\phi c}$, is called the *ϕ-complement of* BFGS $\check{G}_b$.

Example 44. Let $M = \{(a_1, 0.3, -0.7), (a_2, 0.5, -0.4), (a_3, 0.7, -0.3)\}$, $N_1 = \{(a_1a_3, 0.3, -0.3), (a_2a_3, 0.5, -0.3)\}$, and $N_2 = \{(a_1a_2, 0.3, -0.4)\}$ be bipolar fuzzy subsets of U, E_1, and E_2, respectively, so that $\check{G}_b = (M, N_1, N_2)$ is a BFGS of graph structure $G^* = (U, E_1, E_2)$. Let ϕ be a permutation on the set $\{N_1, N_2\}$ such that $\phi(N_1) = N_2$ and $\phi(N_2) = N_1$.

Now for $a_2a_3 \in N_1$,

$$\begin{aligned}
\mu^P_{N_1^\phi}(a_2a_3) &= \mu^P_M(a_2) \wedge \mu^P_M(a_3) - \bigvee_{j\neq 1}\left[\mu^P_{\phi N_j}(a_2a_3)\right] \\
&= 0.5 \wedge 0.7 - \left[\mu^P_{\phi N_2}(a_2a_3)\right] \\
&= 0.5 - \mu^P_{N_1}(a_2a_3) = 0.5 - 0.5 = 0, \\
\mu^N_{N_1^\phi}(a_2a_3) &= \mu^N_M(a_2) \vee \mu^N_M(a_3) - \bigwedge_{j\neq 1}\left[\mu^N_{\phi N_j}(a_2a_3)\right] \\
&= -0.4 \vee -0.3 - \left[\mu^N_{\phi N_2}(a_2a_3)\right] \\
&= -0.3 - \mu^N_{N_1}(a_2a_3) = -0.3 + 0.3 = 0, \\
\mu^P_{N_2^\phi}(a_2a_3) &= \mu^P_M(a_2) \wedge \mu^P_M(a_3) - \bigvee_{j\neq 2}\left[\mu^P_{\phi N_j}(a_2a_3)\right] \\
&= 0.5 \wedge 0.7 - \left[\mu^P_{\phi N_1}(a_2a_3)\right] \\
&= 0.5 - \mu^P_{N_2}(a_2a_3) = 0.5 - 0 = 0.5, \\
\mu^N_{N_2^\phi}(a_2a_3) &= \mu^N_M(a_2) \vee \mu^N_M(a_3) - \bigwedge_{j\neq 2}\left[\mu^N_{\phi N_j}(a_2a_3)\right] \\
&= -0.4 \vee -0.3 - \left[\mu^N_{\phi N_1}(a_2a_3)\right] \\
&= -0.3 - \mu^N_{N_2}(a_2a_3) = -0.3 - 0 = -0.3.
\end{aligned} \tag{35}$$

Clearly, $\mu^P_{\phi N_2}(a_2a_3) = 0.5 > 0 = \mu^P_{\phi N_1}(a_2a_3)$ and $\mu^N_{\phi N_2}(a_2a_3) = -0.3 < 0 = \mu^N_{\phi N_1}(a_2a_3)$. So $a_2a_3 \in N_2^\phi$.

Similarly for $a_1a_3 \in N_1$, $\mu^P_{N_1^\phi}(a_1a_3) = 0$, $\mu^N_{N_1^\phi}(a_1a_3) = 0$, $\mu^P_{N_2^\phi}(a_1a_3) = 0.3$, and $\mu^N_{N_2^\phi}(a_1a_3) = -0.3$.

$\Rightarrow \mu^P_{N_2^\phi}(a_1a_3) = 0.3 > 0 = \mu^P_{N_1^\phi}(a_1a_3)$ and $\mu^N_{N_2^\phi}(a_1a_3) = -0.3 < 0 = \mu^N_{N_1^\phi}(a_1a_3)$. So $a_1a_3 \in N_2^\phi$.

And for $a_1a_2 \in N_2$, $\mu^P_{N_1^\phi}(a_1a_2) = 0.3$, $\mu^N_{N_1^\phi}(a_1a_2) = -0.4$, $\mu^P_{N_2^\phi}(a_1a_2) = 0$, and $\mu^N_{N_2^\phi}(a_1a_2) = 0$.

$\Rightarrow \mu^P_{N_1^\phi}(a_1a_2) = 0.3 > 0 = \mu^P_{N_2^\phi}(a_1a_2)$ and $\mu^N_{N_1^\phi}(a_1a_2) = -0.4 < 0 = \mu^N_{N_2^\phi}(a_1a_2)$. So $a_1a_2 \in N_1^\phi$.

This implies that

$$\begin{aligned}
N_1^\phi &= \{(a_1a_2, 0.3, -0.4)\}, \\
N_2^\phi &= \{(a_2a_3, 0.5, -0.3), (a_1a_3, 0.3, -0.3)\}
\end{aligned} \tag{36}$$

and $\check{G}_b^{\phi c} = (M, N_1^\phi, N_2^\phi)$ is the ϕ-complement of $\check{G}_b$.

Theorem 45. *A ϕ-complement of a bipolar fuzzy graph structure is always a strong BFGS. Moreover, if $\phi(i) = r$ for $r, i \in \{1, 2, \ldots, n\}$, then all N_r-edges in BFGS $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ become B_i^ϕ-edges in $\check{G}_b^{\phi c} = (A, B_1^\phi, B_2^\phi, \ldots, B_n^\phi)$.*

Proof. From the definition of ϕ-complement $\check{G}_b^{\phi c}$,

$$\mu^P_{N_i^\phi}(xy) = \mu^P_M(x) \wedge \mu^P_M(y) - \bigvee_{j\neq i}\mu^P_{\phi N_j}(xy), \tag{37}$$

$$\mu^N_{N_i^\phi}(xy) = \mu^N_M(x) \vee \mu^N_M(y) - \bigwedge_{j\neq i}\mu^N_{\phi N_j}(xy), \tag{38}$$

for $i = 1, 2, \ldots, n$.

Let us consider expression (37) first.

Since $\mu^N_M(x) \vee \mu^N_M(y) \le 0$ and $\bigwedge_{j\neq i}\mu^N_{\phi N_j}(xy) \le 0$, we can write

$$\mu^N_{N_i^\phi}(xy) = -\left|\mu^N_M(x) \vee \mu^N_M(y)\right| + \left|\bigwedge_{j\neq i}\mu^N_{\phi N_j}(xy)\right|. \tag{39}$$

Also from the definition of a BFGS $\mu^N_{N_j}(xy) \ge \mu^N_M(x) \vee \mu^N_M(y)\ \forall N_j$

$$\begin{gathered}
\Longrightarrow \bigwedge_{j\neq i}\mu^N_{\phi N_j}(xy) \ge \mu^N_M(x) \vee \mu^N_M(y) \Longrightarrow \\
\left|\bigwedge_{j\neq i}\mu^N_{\phi N_j}(xy)\right| \le \left|\mu^N_M(x) \vee \mu^N_M(y)\right| \Longrightarrow \\
-\left|\mu^N_M(x) \vee \mu^N_M(y)\right| + \left|\bigwedge_{j\neq i}\mu^N_{\phi N_j}(xy)\right| \le 0.
\end{gathered} \tag{40}$$

Therefore, $\mu^N_{N_i^\phi}(xy) \le 0\ \forall i$.

Now a requirement is minimum value of $\mu^N_{N_i^\phi}(xy)$. Since $\mu^N_{N_i^\phi}(xy) \le 0$, that is why it is minimum when its positive part $|\bigwedge_{j\neq i}\mu^N_{\phi N_j}(xy)|$ is zero. And $|\bigwedge_{j\neq i}\mu^N_{\phi N_j}(xy)| = 0$ when $\phi N_i = N_r$ and xy is an N_r-edge. So

$$\begin{gathered}
\mu^N_{N_i^\phi}(xy) = \mu^N_M(x) \vee \mu^N_M(y), \\
\text{for } xy \in N_r,\ \phi N_i = N_r.
\end{gathered} \tag{41}$$

Similarly for expression (38), a requirement is maximum value of $\mu^P_{N_i^\phi}(xy)$. Since $\mu^P_M(x) \wedge \mu^P_M(y) \ge 0$, $\bigvee_{j\neq i}\mu^P_{\phi N_j}(xy) \ge 0$ and $\mu^P_{N_j}(xy) \le \mu^P_M(x) \wedge \mu^P_M(y)\ \forall N_j$

$$\begin{gathered}
\Longrightarrow \bigvee_{j\neq i}\mu^P_{\phi N_j}(xy) \le \mu^P_M(x) \wedge \mu^P_M(y) \Longrightarrow \\
\mu^P_M(x) \wedge \mu^P_M(y) - \bigvee_{j\neq i}\mu^P_{\phi N_j}(xy) \ge 0.
\end{gathered} \tag{42}$$

Therefore, $\mu^P_{N_i^\phi}(xy) \ge 0\ \forall i$.

Now $\mu^P_{N_i^\phi}(xy)$ will be maximum when its negative part $[-\bigvee_{j\neq i}\mu^P_{\phi N_j}(xy)]$ becomes zero. Clearly, $[-\bigvee_{j\neq i}\mu^P_{\phi N_j}(xy)] = 0$ when $\phi N_i = N_r$ and xy is an N_r-edge. So

$$\begin{gathered}
\mu^P_{N_i^\phi}(xy) = \mu^P_M(x) \wedge \mu^P_M(y), \\
\text{for } xy \in N_r,\ \phi N_i = N_r.
\end{gathered} \tag{43}$$

From (41) and (43), the conclusion is obvious. □

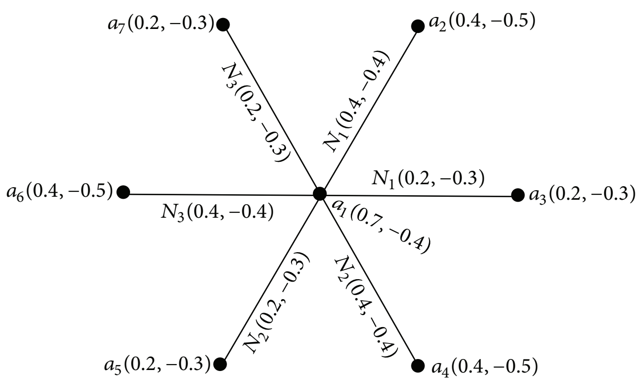


Figure 10: Totally strong self-complement BFGS.

Definition 46. Let $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ be a BFGS and let ϕ be any permutation on the set $\{1, 2, \ldots, n\}$. Then

(i) $\check{G}_b$ is *self-complement* if it is isomorphic to $\check{G}_b^{\phi c}$, the *ϕ-complement* of $\check{G}_b$;

(ii) $\check{G}_b$ is *strong self-complement* if it is *identical* to $\check{G}_b^{\phi c}$.

Definition 47. Let $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ be a BFGS. Then

(i) $\check{G}_b$ is *totally self-complement* if it is *isomorphic* to $\check{G}_b^{\phi c}$, the *ϕ-complement* of $\check{G}_b$, for all permutations ϕ on the set $\{1, 2, \ldots, n\}$;

(ii) $\check{G}_b$ is *totally strong self-complement* if it is *identical* to $\check{G}_b^{\phi c}$, the *ϕ-complement* of $\check{G}_b$, for all permutations ϕ on the set $\{1, 2, \ldots, n\}$.

Example 48. All strong BFGSs are the only examples of self-complement or totally self-complement BFGSs.

Example 49. A BFGS $\check{G}_b = (M, N_1, N_2, N_3)$ of graph structure $G^* = (U, E_1, E_2, E_3)$ as shown in Figure 10 is totally strong self-complement.

Theorem 50. *A BFGS $\check{G}_b$ is strong if and only if it is totally self-complement.*

Proof. Let $\check{G}_b$ be a strong BFGS and ϕ any permutation on the set $\{1, 2, \ldots, n\}$.

By Theorem 45, $\check{G}_b^{\phi c}$ is strong and if $\phi^{-1}(i) = j$, then all N_i-edges in $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ become N_j^{ϕ}-edges in $\check{G}_b^{\phi c} = (M, N_1^{\phi}, N_2^{\phi}, \ldots, N_n^{\phi})$

$$\begin{aligned} &\Longrightarrow \mu_{N_i}^P(a_1a_2) = \mu_M^P(a_1) \wedge \mu_M^P(a_2) \\ &= \mu_{N_j^{\phi}}^P(a_1a_2), \\ \mu_{N_i}^N(a_1a_2) &= \mu_M^N(a_1) \vee \mu_M^N(a_2) = \mu_{N_j^{\phi}}^N(a_1a_2). \end{aligned} \tag{44}$$

Hence $\check{G}_b$ is isomorphic to $\check{G}_b^{\phi c}$ under the identity mapping $f : U \to U$, such that $\mu_M^P(a) = \mu_M^P(f(a))$, $\mu_M^N(a) = \mu_M^N(f(a))$ $\forall a \in U$ and

$$\begin{aligned} \mu_{N_i}^P(a_1a_2) &= \mu_{N_j^{\phi}}^P(a_1a_2) = \mu_{N_j^{\phi}}^P(f(a_1)f(a_2)), \\ \mu_{N_i}^N(a_1a_2) &= \mu_{N_j^{\phi}}^N(a_1a_2) = \mu_{N_j^{\phi}}^N(f(a_1)f(a_2)), \\ &\forall a_1a_2 \in E_i, \end{aligned} \tag{45}$$

for $\phi^{-1}(i) = j$, $i, j = 1, 2, \ldots, n$. This holds for any permutation on the set $\{1, 2, \ldots, n\}$.

Hence $\check{G}_b$ is totally self-complement.

Conversely, let $\check{G}_b$ and $\check{G}_b^{\phi c}$ be isomorphic for any permutation ϕ on the set $\{1, 2, \ldots, n\}$. Then from the definition of ϕ-complement and isomorphism of BFGSs, we have

$$\begin{aligned} \mu_{N_i}^P(a_1a_2) &= \mu_{N_j^{\phi}}^P(f(a_1)f(a_2)) \\ &= \mu_M^P(f(a_1)) \wedge \mu_M^P(f(a_2)) \\ &= \mu_M^P(a_1) \wedge \mu_M^P(a_2), \\ \mu_{N_i}^N(a_1a_2) &= \mu_{N_j^{\phi}}^N(f(a_1)f(a_2)) \\ &= \mu_M^N(f(a_1)) \vee \mu_M^N(f(a_2)) \\ &= \mu_M^N(a_1) \vee \mu_M^N(a_2) \end{aligned} \tag{46}$$

$\forall a_1a_2 \in E_i$, $i = 1, 2, \ldots, n$.

Hence, $\check{G}_b$ is a strong BFGS. □

Remark 51. Every *self-complement* BFGS is necessarily *totally self-complement*.

Theorem 52. *If graph structure $G^* = (U, E_1, E_2, \ldots, E_n)$ is totally strong self-complement and $M = (\mu_M^P, \mu_M^N)$ is a bipolar fuzzy set of U with constant valued functions μ_M^P and μ_M^N, then a strong BFGS $\check{G}_b = (M, N_1, N_2, \ldots, N_n)$ of G^* is totally strong self-complement.*

Proof. Let $s \in [0, 1]$ and $t \in [-1, 0]$ be two constants, such that

$$\begin{aligned} \mu_M^P(u) &= s, \\ \mu_M^N(u) &= t \\ &\forall u \in U. \end{aligned} \tag{47}$$

Since G^* is totally strong self-complement, so for every permutation ϕ^{-1} on the set $\{1, 2, \ldots, n\}$, there exists a bijection $f : U \to U$, such that for every E_i-edge a_1a_2, "$f(a_1)f(a_2)$" [an E_j-edge in G^*] is an E_i-edge in $(G^*)^{\phi^{-1}c}$ and, consequently, for every N_i-edge a_1a_2, "$f(a_1)f(a_2)$" [a N_j-edge in $\check{G}_b$] is a B_i^{ϕ}-edge in $\check{G}_b^{\phi c}$. Moreover $\check{G}_b$ is strong, so we have

$$\mu_M^P(a) = s = \mu_M^P(f(a)),$$
$$\mu_M^N(a) = t = \mu_M^N(f(a)) \qquad \forall u \in U,$$

$$\begin{aligned}
\mu_{N_i}^P(a_1a_2) &= \mu_M^P(a_1) \wedge \mu_M^P(a_2) \\
&= \mu_M^P(f(a_1)) \wedge \mu_M^P(f(a_2)) \\
&= \mu_{N_j^{\phi}}^P(f(a_1)f(a_2)), \\
\mu_{N_i}^N(a_1a_2) &= \mu_M^N(a_1) \vee \mu_M^N(a_2) \\
&= \mu_M^N(f(a_1)) \vee \mu_M^N(f(a_2)) \\
&= \mu_{N_j^{\phi}}^N(f(a_1)f(a_2))
\end{aligned} \tag{48}$$

$\forall a_1a_2 \in E_i,\ i = 1, 2, \ldots, n.$

This shows that $\check{G}_b$ is strong self-complement. This holds for any permutation ϕ and ϕ^{-1} on the set $\{1, 2, \ldots, n\}$; thus $\check{G}_b$ is totally strong self-complement. This completes the proof. □

Remark 53. The converse of Theorem 52 is not necessary, since a *totally strong self-complement BFGS* $\check{G}_b = (M, N_1, N_2, N_3)$, as shown in Figure 10, is *strong* and has a *totally strong self-complement underlying graph structure*, but μ_M^P and μ_M^N are not constant valued functions.

4. Conclusions

Graph-theoretical concepts are widely used to study and model various applications in different areas. However, in many cases, some aspects of a graph-theoretical problem may be vague or uncertain. It is natural to deal with the vagueness and uncertainty using the methods of fuzzy sets. Since bipolar fuzzy set has shown advantages in handling vagueness and uncertainty than fuzzy set, we have applied the concept of bipolar fuzzy sets to graph structures. We have introduced the concept of bipolar fuzzy graph structures. We are extending our work to (1) bipolar fuzzy soft graph structures, (2) soft graph structures, (3) rough fuzzy soft graph structures, and (4) roughness in fuzzy graph structures.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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