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Research Article

Boolean Algebra of Soft *Q*-Sets in Soft Topological Spaces

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We define soft Q-sets as soft sets whose soft closure and soft interior are commutative. We show that the soft complement, soft closure, and soft interior of a soft Q-set are all soft Q-sets. We show that a soft subset K of a given soft topological space is a soft Q-set if and only if K is a soft symmetric difference between a soft clopen set and a soft nowhere dense set. And as a corollary, the class of soft Q-sets contains simultaneously the classes of soft clopen sets and soft nowhere dense sets. Also, we prove that the class of soft Q-sets is closed under finite soft intersections and finite soft unions, and as a main result, we prove that the class of soft Q-sets forms a Boolean algebra. Furthermore, via soft Q-sets, we characterize soft sets whose soft boundaries and soft interiors are commutative. In addition, we investigate the correspondence between Q-sets in topological spaces and soft Q-sets in soft topological spaces.

1. Introduction and Preliminaries

Some problems in medicine, engineering, the environment, economics, sociology, and other fields have their own doubts. Therefore, we are unable to deal with these problems by conventional methods. For more than thirty years, fuzzy set theory [1], rough set theory [2], and vague set theory [3] have played an essential role in dealing with these problems. Molodtsov [4] argues that each of these theories has its own set of problems. These difficulties mainly come from the inadequacy of the parameterization tool for the theories. Research through soft set theory has included almost all branches of science. Soft set theory has been applied to solve problems using Riemann integral, Beron's integral, game theory, function smoothness, operations research, measure theory, probability, and decision-making problems [4–6].

General topology, as one of the main branches of mathematics, is the branch of topology that deals with the basic definitions of set theory and structures used in topology. It is the foundation of most other branches of topology, including algebraic topology, geometric topology, and differential topology. Shabir and Naz [7] initiated soft topology, which is a new branch of topology that combines soft set theory and topology. Since then, numerous studies have appeared in soft topology [8–24] and others, and substantial contributions can still be made. A subset S of a given topological space is said to be a Q-set if the interior and closure operators of this subset are commute. Levine [25] discovered that a subset of a topological space is a Q-set if and only if it is the symmetric difference of a set that is clopen and a set that is nowhere dense. As an important application of Levine's characterization of Q-sets, the authors in [26] have proved that a topological space is compact, Hausdorff, and metastonean if and only if its classes of Borel sets and Q-sets are equal. The aim of this paper is to extend the concept of Q-sets and their related properties and results to include soft topological spaces. In this paper, We define soft Q-sets as soft sets whose soft closure and soft interior are commutative. We show that the soft complement, soft closure, and soft interior of a soft Q-set are all soft Q-sets. We show that a soft subset K of a given soft topological space is a soft Q-set if and only if K is a soft symmetric difference between a soft clopen set and a soft nowhere dense set. In addition, as a corollary, the class of soft Q-sets contains simultaneously the classes of soft clopen sets and soft nowhere dense sets. Also, we prove that the class of soft Q-sets is closed under finite soft intersections and finite soft unions, and as a main result, we prove that the class of soft Q-sets forms a Boolean algebra. Furthermore, via soft Q-sets, we characterize soft sets whose soft boundaries and soft interiors are commutative. In addition, we investigate the correspondence between Q-sets in topological spaces and soft Q-sets in soft topological spaces.

Boolean algebra constitutes the basis for the design of circuits used in electronic digital computers. In addition, it is of significance to the theory of probability, the geometry of sets, and information theory. So, this paper not only forms the theoretical basis for further applications of soft topology, but it also leads to the development of the theory of probability, the geometry of sets, and information theory.

In this paper, we follow the notions and terminologies that appeared in [27, 28]. Throughout this paper, ST and STS will denote topological space and soft topological space, respectively. Let (Y, σ, B) be a STS, (Y, μ) be a TS, $H \in SS(Y, B)$, and $U \subseteq Y$. $Cl_\sigma(H)$, $\mathrm{Int}_\sigma(H)$, $Bd_\sigma(H)$, $Cl_\mu(U)$, and $\mathrm{Int}_\mu(U)$ will be used throughout this paper to denote the soft closure of H in (Y, σ, B), the soft interior of H in (Y, σ, B), the soft boundary of H in (Y, σ, B), the closure of U in $(Y, \aleph)$, and the interior of U in $(Y, \aleph)$, respectively.

The following definitions will be used in the sequel:

Definition 1 (see [25]). *Let (Y, μ) be a TS and let $U \subseteq Y$. Then U is called a Q-set in (Y, μ) if $\mathrm{Int}_\mu(Cl_\mu(U)) = Cl_\mu(\mathrm{Int}_\mu(U))$.*

Definition 2 (see [29]). *A TS (Y, μ) is called extremally disconnected if $Cl_\mu(U) \in \mu$ for each $U \in \mu$.*

Definition 3 (see [27]). *Let Y be a universal set and B be a set of parameters. Then $G \in SS(Y, B)$ defined by*

(a) $G(b) = \begin{cases} Z & \text{if } b = e \\ \varnothing & \text{if } b \neq e \end{cases}$ will be denoted by e_Z.

(b) $G(b) = Z$ for all $b \in B$ will be denoted by C_Z.

Theorem 1 (see [7]). *Let (Y, σ, B) be a STS. Then the collection $\{G(b): G \in \sigma\}$ defines a topology on Y for every $b \in B$. This topology will be denoted by σ_b.*

Theorem 2 (see [27]). *Let Y be an initial universe and let B be a set of parameters. Let $\{\mu_b: b \in B\}$ be an indexed family of topologies on Y and let*

$$\sigma = \{G \in SS(Y, B): G(b) \in \mu_b, \quad \text{for all } b \in B\}. \tag{1}$$

Then σ defines a soft topology on Y relative to B. This soft topology will be denoted by $\oplus_{b\in B}\mu_b$.

Definition 4 (see [30]). *Let (Y, σ, B) be a STS and let $M \in SS(Y, B)$. Then M is called soft nowhere dense in (Y, σ, B) if $\mathrm{Int}_\sigma(Cl_\sigma(M)) = 0_B$.*

Definition 5 (see [31]). *A TS (Y, σ, B) is called soft extremally disconnected if $Cl_\mu(M) \in \sigma$ for each $M \in \sigma$.*

Theorem 3 (see [32]). *Let (Y, σ, B) be a STS and let $M \in SS(Y, B)$. Then*

(a) $Int_\sigma(1_B - M) = 1_B - Cl_\sigma(M)$.

(b) $Cl_\sigma(1_B - M) = 1_B - Int_\sigma(M)$.

Theorem 4. *Let (Y, σ, B) be a STS. If F and G are soft nowhere dense sets in (Y, σ, B), then $F \tilde{\cup} G$ is soft nowhere dense in (Y, σ, B).*

Proof. Similar to that used in Theorem 3.8 of [33]. □

2. Soft Q-Sets

Definition 6. Let (Y, σ, B) be a STS and let $M \in SS(Y, B)$. Then M is called a soft Q-set in (Y, σ, B) if $\mathrm{Int}_\sigma(Cl_\sigma(M)) = Cl_\sigma(\mathrm{Int}_\sigma(M))$.

Theorem 5. *Let (Y, σ, B) be a STS and let $M \in SS(Y, B)$. Then M is a soft Q-set in (Y, σ, B) if and only if $1_B - M$ is a soft Q-set in (Y, σ, B).*

Proof. Necessity. Suppose that M is a soft Q-set in (Y, σ, B). Then $\mathrm{Int}_\sigma(Cl_\sigma(M)) = Cl_\sigma(\mathrm{Int}_\sigma(M))$, and so $1_B - \mathrm{Int}_\sigma(Cl_\sigma(M)) = 1_B - Cl_\sigma(\mathrm{Int}_\sigma(M))$. Now, by Theorem 3, we have,

$$\begin{aligned} 1_B - \mathrm{Int}_\sigma(Cl_\sigma(M)) &= Cl_\sigma(1_B - Cl_\sigma(M)) \\ &= Cl_\sigma(\mathrm{Int}_\sigma(1_B - M)), \text{ and} \\ 1_B - Cl_\sigma(\mathrm{Int}_\sigma(M)) &= \mathrm{Int}_\sigma(1_B - \mathrm{Int}_\sigma(M)) \\ &= \mathrm{Int}_\sigma(Cl_\sigma(1_B - M)). \end{aligned} \tag{2}$$

Therefore, $\mathrm{Int}_\sigma(Cl_\sigma(1_B - M)) = Cl_\sigma(\mathrm{Int}_\sigma(1_B - M))$. Hence, $1_B - M$ is a soft Q-set in (Y, σ, B).

Sufficiency. Suppose that $1_B - M$ is a soft Q-set in (Y, σ, B). Then, by the above, $1_B - (1_B - M) = M$ is a soft Q-set in (Y, σ, B). □

Theorem 6. *Let (Y, σ, B) be a STS and let $M \in SS(Y, B)$. If M is a soft Q-set in (Y, σ, B), then $Cl_\sigma(M)$ is a soft Q-set in (Y, σ, B).*

Proof. Suppose that M is a soft Q-set in (Y, σ, B). Then $\mathrm{Int}_\sigma(Cl_\sigma(M)) = Cl_\sigma(\mathrm{Int}_\sigma(M))$, and so

$$\begin{aligned} Cl_\sigma(\mathrm{Int}_\sigma(Cl_\sigma(M))) &= Cl_\sigma(Cl_\sigma(\mathrm{Int}_\sigma(M))) \\ &= Cl_\sigma(\mathrm{Int}_\sigma(M)). \end{aligned} \tag{3}$$

On the other hand, $\mathrm{Int}_\sigma(Cl_\sigma(Cl_\sigma(M))) = \mathrm{Int}_\sigma(Cl_\sigma(M))$. Thus, $\mathrm{Int}_\sigma(Cl_\sigma(Cl_\sigma(M))) = Cl_\sigma(\mathrm{Int}_\sigma(Cl_\sigma(M)))$, and hence $Cl_\sigma(M)$ is a soft Q-set in (Y, σ, B). □

Theorem 7. *Let (Y, σ, B) be a STS and let $M \in SS(Y, B)$. If M is a soft Q-set in (Y, σ, B), then $Int_\sigma(M)$ is a soft Q-set in (Y, σ, B).*

Proof. Suppose that M is a soft Q-set in (Y, σ, B). Then $\mathrm{Int}_\sigma(Cl_\sigma(M)) = Cl_\sigma(\mathrm{Int}_\sigma(M))$, and so

$$\begin{aligned} \mathrm{Int}_\sigma(Cl_\sigma(\mathrm{Int}_\sigma(M))) &= \mathrm{Int}_\sigma(\mathrm{Int}_\sigma(Cl_\sigma(M))) \\ &= \mathrm{Int}_\sigma(Cl_\sigma(M)). \end{aligned} \tag{4}$$

On the other hand, $Cl_\sigma(\mathrm{Int}_\sigma(\mathrm{Int}_\sigma(M))) = Cl_\sigma(\mathrm{Int}_\sigma(M))$. Thus,

$\mathrm{Int}_\sigma(Cl_\sigma(\mathrm{Int}_\sigma(M))) = Cl_\sigma(\mathrm{Int}_\sigma(\mathrm{Int}_\sigma(M)))$, and hence $\mathrm{Int}_\sigma(M)$ is a soft Q-set in (Y, σ, B).

The converse of Theorem 7 is false: □

Example 1. Let $Y = \mathbb{R}$, $B = \mathbb{N}$, and $M, G \in SS(Y, B)$ defined by

$$M(b) = \begin{cases} \mathbb{Q}, & \text{if } b \text{ is even,} \\ \mathbb{Q}^c, & \text{if } b \text{ is odd,} \end{cases} \quad \text{and}$$
$$G(b) = \begin{cases} \mathbb{Q}^c, & \text{if } b \text{ is even,} \\ \mathbb{Q}, & \text{if } b \text{ is odd.} \end{cases} \tag{5}$$

Let $\sigma = \{0_B, 1_B, G\}$. Then $\mathrm{Int}_\sigma(M) = 0_B$, and so $\mathrm{Int}_\sigma(M)$ is a soft Q-set in (Y, σ, B). On the other hand, since $\mathrm{Int}_\sigma(Cl_\sigma(M)) = \mathrm{Int}_\sigma(1_B) = 1_B$ while $Cl_\sigma(\mathrm{Int}_\sigma(M)) = Cl_\sigma(0_B) = 0_B$, then M is not a soft Q-set in (Y, σ, B).

Theorem 8. *Let (Y, σ, B) be a STS and let $M \in SS(Y, B)$. If M or $1_B - M$ is soft nowhere dense in (Y, σ, B), then M is a soft Q-set in (Y, σ, B).*

Proof. Suppose that M is soft nowhere dense. Then $\mathrm{Int}_\sigma(Cl_\sigma(M)) = 0_B$. On the other hand, since $\mathrm{Int}_\sigma(M) \tilde{\subseteq} \mathrm{Int}_\sigma(Cl_\sigma(M)) = 0_B$, then $\mathrm{Int}_\sigma(M) = 0_B$ and so $Cl_\sigma(\mathrm{Int}_\sigma(M)) = 0_B$. Therefore, $\mathrm{Int}_\sigma(Cl_\sigma(M)) = Cl_\sigma(\mathrm{Int}_\sigma(M))$, and hence M is a soft Q-set in (Y, σ, B).

Suppose that $1_B - M$ is soft nowhere dense. Then by the above, $1_B - M$ is a soft Q-set in (Y, σ, B). So, by Theorem 5, $1_B - (1_B - M) = M$ is a soft Q-set in (Y, σ, B).

The converse of Theorem 8 is false: □

Example 2. Let $Y = \{1, 2\}$, $B = \{a, b\}$, and $M \in SS(Y, B)$ defined by $M(a) = \{1\}$, $M(b) = \{2\}$. Let $\sigma = \{0_B, 1_B, M, 1_B - M\}$. Then $\mathrm{Int}_\sigma(M) = Cl_\sigma(M) = M$ and $\mathrm{Int}_\sigma(1_B - M) = Cl_\sigma(1_B - M) = 1_B - M$. Thus, $\mathrm{Int}_\sigma(Cl_\sigma(M)) = Cl_\sigma(\mathrm{Int}_\sigma(M)) = M \neq 0_B$ and $\mathrm{Int}_\sigma(Cl_\sigma(1_B - M)) = 1_B - M \neq 0_B$. Hence, M is a soft Q-set in (Y, σ, B) but neither M nor $1_B - M$ is soft nowhere dense in (Y, σ, B).

Lemma 1. *If (Y, σ, B) is a STS and $M, N, K \in SS(Y, B)$ such that M is soft clopen in (Y, σ, B), $N \tilde{\subseteq} M$ and $K \tilde{\subseteq} 1_B - M$, then*

$$\mathrm{Int}_\sigma(N \tilde{\cup} K) = \mathrm{Int}_\sigma(N) \tilde{\cup} \mathrm{Int}_\sigma(K). \tag{6}$$

Proof. $\mathrm{Int}_\sigma(N) \tilde{\cup} \mathrm{Int}_\sigma(K) \tilde{\subseteq} \mathrm{Int}_\sigma(N \tilde{\cup} K)$ is always true. To see that $\mathrm{Int}_\sigma(N \tilde{\cup} K) \tilde{\subseteq} \mathrm{Int}_\sigma(N) \tilde{\cup} \mathrm{Int}_\sigma(K)$, let $b_y \tilde{\in} \mathrm{Int}_\sigma(N \tilde{\cup} K)$.

Case 1. $b_y \tilde{\in} M$. Then $b_y \tilde{\in} (M \tilde{\cap} \mathrm{Int}_\sigma(N \tilde{\cup} K)) \in \sigma$ and

$$\begin{aligned} M \tilde{\cap} \mathrm{Int}_\sigma(N \tilde{\cup} K) &\tilde{\subseteq} M \tilde{\cap} (N \tilde{\cup} K) \\ &= (M \tilde{\cap} N) \tilde{\cup} (M \tilde{\cap} K) \\ &= N \tilde{\cup} 0_B \\ &= N. \end{aligned} \tag{7}$$

This implies that $b_y \tilde{\in} \mathrm{Int}_\sigma(N) \tilde{\subseteq} \mathrm{Int}_\sigma(N) \tilde{\cup} \mathrm{Int}_\sigma(K)$.

Case 2. $b_y \tilde{\in} 1_B - M$. Then $b_y \tilde{\in} ((1_B - M) \tilde{\cap} \mathrm{Int}_\sigma(N \tilde{\cup} K)) \in \sigma$ and

$$\begin{aligned} (1_B - M) \tilde{\cap} \mathrm{Int}_\sigma(N \tilde{\cup} K) &\tilde{\subseteq} (1_B - M) \tilde{\cap} (N \tilde{\cup} K) \\ &= ((1_B - M) \tilde{\cap} N) \tilde{\cup} ((1_B - M) \tilde{\cap} K) \\ &= 0_B \tilde{\cup} K \\ &= K. \end{aligned} \tag{8}$$

This implies that $b_y \tilde{\in} \mathrm{Int}_\sigma(K) \tilde{\subseteq} \mathrm{Int}_\sigma(N) \tilde{\cup} \mathrm{Int}_\sigma(K)$. □

Lemma 2. *Let (Y, σ, B) be a STS and $K, S \in SS(Y, B)$. If K is soft clopen in (Y, σ, B), then*

$$Cl_\sigma(K \tilde{\cap} S) = K \tilde{\cap} Cl_\sigma(S). \tag{9}$$

Proof. Since K is soft clopen in (Y, σ, B), then $Cl_\sigma(K \tilde{\cap} S) \tilde{\subseteq} Cl_\sigma(K) \tilde{\cap} Cl_\sigma(S) = K \tilde{\cap} Cl_\sigma(S)$. To see that $K \tilde{\cap} Cl_\sigma(S) \tilde{\subseteq} Cl_\sigma(K \tilde{\cap} S)$, suppose to the contrary that there exists $b_y \tilde{\in} K \tilde{\cap} Cl_\sigma(S) - Cl_\sigma(K \tilde{\cap} S)$. Since $b_y \tilde{\in} 1_B - Cl_\sigma(K \tilde{\cap} S)$, then there exists $F \in \sigma$ such that $b_y \tilde{\in} F$ and $(K \tilde{\cap} S) \tilde{\cap} F = 0_B$. On the other hand, since $b_y \tilde{\in} K \tilde{\cap} F \in \sigma$ and $b_y \tilde{\in} Cl_\sigma(S)$, then $(K \tilde{\cap} F) \tilde{\cap} S = (K \tilde{\cap} S) \tilde{\cap} F \neq 0_B$. This is a contradiction.

Now we state the main result of this section. □

Theorem 9. *Let (Y, σ, B) be a STS and $M \in SS(Y, B)$. Then M is a soft Q-set in (Y, σ, B) if and only if $M = (K - N) \tilde{\cup} (N - K)$ where K is soft clopen in (Y, σ, B) and N is soft nowhere dense in (Y, σ, B).*

Proof. Necessity. Suppose that M is a soft Q-set in (Y, σ, B). Let $K = \mathrm{Int}_\sigma(Cl_\sigma(M)) = Cl_\sigma(\mathrm{Int}_\sigma(M))$. Then K is soft clopen in (Y, σ, B). Let $N = (M - K) \tilde{\cup} (K - M)$. □

Claim 1

(a) $M - K$ is soft nowhere dense in (Y, σ, B).

(b) $K - M$ is soft nowhere dense in (Y, σ, B).

(c) N is soft nowhere dense in (Y, σ, B).

(d) $M = (K - N) \tilde{\cup} (N - K)$.

Proof of Claim

(a) Since $\mathrm{Int}_\sigma(Cl_\sigma(M - K)) \tilde{\subseteq} \mathrm{Int}_\sigma(Cl_\sigma(M)) = K$ and

$$\begin{aligned} \mathrm{Int}_\sigma(Cl_\sigma(M - K)) &= \mathrm{Int}_\sigma(Cl_\sigma(M \tilde{\cap} (1_B - K))) \\ &\tilde{\subseteq} \mathrm{Int}_\sigma(Cl_\sigma(1_B - K)) \\ &= \mathrm{Int}_\sigma(1_B - K) \\ &= 1_B - K, \end{aligned} \tag{10}$$

then $\mathrm{Int}_\sigma(Cl_\sigma(M - K)) \tilde{\subseteq} K \tilde{\cap} (1_B - K) = 0_B$. Hence, $M - K$ is soft nowhere dense in (Y, σ, B).

(b) Note that $\mathrm{Int}_\sigma(Cl_\sigma(K - M)) \tilde{\subseteq} \mathrm{Int}_\sigma(Cl_\sigma(K)) = K$. Also, we have,

$$\begin{aligned}\operatorname{Int}_\sigma(Cl_\sigma(K-M)) &= \operatorname{Int}_\sigma(Cl_\sigma(K\tilde{\cap}(1_B-M)))\\ &\tilde{\subseteq}\operatorname{Int}_\sigma(Cl_\sigma(1_B-M)).\end{aligned}\tag{11}$$

But by Theorem 3,

$$\begin{aligned}\operatorname{Int}_\sigma(Cl_\sigma(1_B-M)) &= 1_B - Cl_\sigma(1_B - Cl_\sigma(1_B-M))\\ &= 1_B - Cl_\sigma(\operatorname{Int}_\sigma(M))\\ &= 1_B - K.\end{aligned}\tag{12}$$

Therefore, $\operatorname{Int}_\sigma(Cl_\sigma(K-M))\tilde{\subseteq}1_B-K$. Hence, $\operatorname{Int}_\sigma(Cl_\sigma(K-M))\tilde{\subseteq}K\tilde{\cap}(1_B-K)=0_B$. It follows that $K-M$ is soft nowhere dense in (Y,σ,B).

(c) Follows from (a) and (b) and the fact that the soft union of two soft nowhere dense sets is soft nowhere dense.

(d) $(K-N)\tilde{\cup}(N-K)=(K\tilde{\cap}(1_B-N))\tilde{\cup}(N\tilde{\cap}(1_B-K))=K\tilde{\cap}(1_B-((M\tilde{\cap}(1_B-K))\tilde{\cup}(K\tilde{\cap}(1_B-M))))\tilde{\cup}(((M\tilde{\cap}(1_B-K))\tilde{\cup}(K\tilde{\cap}(1_B-M)))\tilde{\cap}(1_B-K))=(K\tilde{\cap}(1_B-(M\tilde{\cap}(1_B-K)))\tilde{\cap}(1_B-(K\tilde{\cap}(1_B-M))))\tilde{\cup}(M\tilde{\cap}(1_B-K))=(K\tilde{\cap}((1_B-M)\tilde{\cup}K)\tilde{\cap}((1_B-K)\tilde{\cup}M))\tilde{\cup}(M\tilde{\cap}(1_B-K))=((K\tilde{\cap}M)\tilde{\cap}((1_B-M)\tilde{\cup}K))\tilde{\cup}(M\tilde{\cap}(1_B-K))=(K\tilde{\cap}M)\tilde{\cup}(M\tilde{\cap}(1_B-K))=M\tilde{\cap}(K\tilde{\cup}(1_B-K))=M\tilde{\cap}1_B=M$.

The above claim ends the proof.

Sufficiency. Suppose that $M=(K-N)\tilde{\cup}(N-K)$ where K is soft clopen in (Y,σ,B) and N is soft nowhere dense in (Y,σ,B). Since N is soft nowhere dense, then $\operatorname{Int}_\sigma(N-K)\tilde{\subseteq}\operatorname{Int}_\sigma(N)\tilde{\subseteq}\operatorname{Int}_\sigma(Cl_\sigma(N))=0_B$, and so $\operatorname{Int}_\sigma(N-K)=0_B$.

Since K is soft clopen in (Y,σ,B), $K-N\tilde{\subseteq}K$ and $N-K\tilde{\subseteq}1_B-K$, then by Lemma 1,

$$\begin{aligned}\operatorname{Int}_\sigma((K-N)\tilde{\cup}(N-K)) &= \operatorname{Int}_\sigma(K-N)\tilde{\cup}\operatorname{Int}_\sigma(N-K)\\ &= \operatorname{Int}_\sigma(K-N)\tilde{\cup}0_B\\ &= \operatorname{Int}_\sigma(K-N)\\ &= \operatorname{Int}_\sigma(K\tilde{\cap}(1_B-N))\\ &= \operatorname{Int}_\sigma(K)\tilde{\cap}\operatorname{Int}_\sigma(1_B-N)\\ &= K\tilde{\cap}\operatorname{Int}_\sigma(1_B-N).\end{aligned}\tag{13}$$

Also, by Lemma 2, $Cl_\sigma(K\tilde{\cap}\operatorname{Int}_\sigma(1_B-N))=K\tilde{\cap}Cl_\sigma(\operatorname{Int}_\sigma(1_B-N))$. Moreover, by Theorem 8, $Cl_\sigma(\operatorname{Int}_\sigma(1_B-N))=\operatorname{Int}_\sigma(Cl_\sigma(1_B-N))$. Therefore,

$$\begin{aligned}Cl_\sigma(\operatorname{Int}_\sigma(M)) &= Cl_\sigma(\operatorname{Int}_\sigma((K-N)\tilde{\cup}(N-K)))\\ &= Cl_\sigma(K\tilde{\cap}\operatorname{Int}_\sigma(1_B-N))\\ &= K\tilde{\cap}Cl_\sigma(\operatorname{Int}_\sigma(1_B-N))\\ &= K\tilde{\cap}\operatorname{Int}_\sigma(Cl_\sigma(1_B-N)).\end{aligned}\tag{14}$$

Now,

$$\begin{aligned}\operatorname{Int}_\sigma(Cl_\sigma(M)) &= \operatorname{Int}_\sigma(Cl_\sigma((K-N)\tilde{\cup}(N-K)))\\ &= \operatorname{Int}_\sigma(Cl_\sigma(K-N)\tilde{\cup}Cl_\sigma(N-K)).\end{aligned}\tag{15}$$

Since K is soft clopen in (Y,σ,B), $Cl_\sigma(K-N)\tilde{\subseteq}Cl_\sigma(K)=K$, and $Cl_\sigma(N-K)\tilde{\subseteq}Cl_\sigma(1_B-K)=1_B-K$, then by Lemma 1,

$$\begin{aligned}&\operatorname{Int}_\sigma(Cl_\sigma(K-N)\tilde{\cup}Cl_\sigma(N-K))\\ &= \operatorname{Int}_\sigma(Cl_\sigma(K-N))\tilde{\cup}\operatorname{Int}_\sigma(Cl_\sigma(N-K)).\end{aligned}\tag{16}$$

Also, since N is soft nowhere dense in (Y,σ,B), then $\operatorname{Int}_\sigma(Cl_\sigma(N-K))\tilde{\subseteq}\operatorname{Int}_\sigma(Cl_\sigma(N))=0_B$ and hence, $\operatorname{Int}_\sigma(Cl_\sigma(N-K))=0_B$. Moreover, by Lemma 2,

$$\begin{aligned}Cl_\sigma(K-N) &= Cl_\sigma(K\tilde{\cap}(1_B-K))\\ &= K\tilde{\cap}Cl_\sigma(1_B-K).\end{aligned}\tag{17}$$

Therefore,

$$\begin{aligned}\operatorname{Int}_\sigma(Cl_\sigma(M)) &= \operatorname{Int}_\sigma(Cl_\sigma(K-N)\tilde{\cup}Cl_\sigma(N-K))\\ &= \operatorname{Int}_\sigma(Cl_\sigma(K-N))\tilde{\cup}\operatorname{Int}_\sigma(Cl_\sigma(N-K))\\ &= \operatorname{Int}_\sigma(Cl_\sigma(K-N))\tilde{\cup}0_B\\ &= \operatorname{Int}_\sigma(Cl_\sigma(K-N))\\ &= \operatorname{Int}_\sigma(Cl_\sigma(K\tilde{\cap}(1_B-N)))\\ &= \operatorname{Int}_\sigma(K\tilde{\cap}Cl_\sigma(1_B-N))\\ &= \operatorname{Int}_\sigma(K)\tilde{\cap}\operatorname{Int}_\sigma(Cl_\sigma(1_B-N))\\ &= K\tilde{\cap}\operatorname{Int}_\sigma(Cl_\sigma(1_B-N))\\ &= Cl_\sigma(\operatorname{Int}_\sigma(M)).\end{aligned}\tag{18}$$

It follows that M is a soft Q-set in (Y,σ,B). □

Corollary 1. *Let (Y,σ,B) be a STS. If K is a soft clopen set in (Y,σ,B), then K is a soft Q-set in (Y,σ,B) .*

Proof. Let $N=0_B$. Then N is soft soft nowhere dense in (Y,σ,B). Since $K=(K-N)\tilde{\cup}(N-K)$, then by Theorem 9, K is a soft Q-set in (Y,σ,B). □

Corollary 2. *Let (Y,σ,B) be a soft connected STS and $M\in SS(Y,B)$. Then M is a soft Q-set in (Y,σ,B) if and only if M is soft nowhere dense in (Y,σ,B) or 1_B-M is soft nowhere dense in (Y,σ,B) .*

Proof. By Theorem 9, M can be written as $(K-N)\tilde{\cup}(N-K)$ where K is soft clopen in (Y,σ,B) and N is soft nowhere dense in (Y,σ,B). Since (Y,σ,B) is soft connected, then the only soft clopen sets in (Y,σ,B) are 0_B and 1_B . Thus, M is a soft Q-set in (Y,σ,B) if and only if $M=(0_B-N)\tilde{\cup}(N-0_B)=N$ or $M=(1_B-N)\tilde{\cup}(N-1_B)=1_B-N$. In the case $M=N$, M is soft nowhere dense in (Y,σ,B), and in the case $M=1_B-N$, $1_B-M=N$ is soft nowhere dense in (Y,σ,B). □

3. Boolean Algebra of Soft Q-Sets

Definition 7. Let Y be a universal set and B be a set of parameters. Let $S,T\in SS(Y,B)$. The soft set $(S-T)\tilde{\cup}(T-S)$ is called the soft symmetric difference of S and T and will be denoted $S\Delta T$.

Lemma 3. *Let Y be a universal set and B be a set of parameters. Let $F, G, H \in SS(Y, B)$. Then*

(1) $F\Delta G = G\Delta F$.

(2) $(F\Delta G)\Delta H = F\Delta(G\Delta H)$.

(3) $F \tilde{\cap} (G\Delta H) = (F \tilde{\cap} G)\Delta(F \tilde{\cap} H)$.

Proof. Straightforward. □

Lemma 4. *Let (Y, σ, B) be a STS. If F and G are soft nowhere dense sets in (Y, σ, B) , then $F\Delta G$ is soft nowhere dense in (Y, σ, B).*

Proof. Let F and G be two soft nowhere dense sets in (Y, σ, B). Since $F - G \tilde{\subseteq} F$ and $G - F \tilde{\subseteq} G$, then $F - G$ and $G - F$ are soft nowhere dense sets in (Y, σ, B). Thus, by Theorem 4, $F\Delta G = (F - G) \tilde{\cup} (G - F)$ is soft nowhere dense sets in (Y, σ, B). □

Theorem 10. *Let (Y, σ, B) be a STS. If M_1 and M_2 are soft Q-sets in (Y, σ, B), then $M_1 \tilde{\cap} M_2$ is a soft Q-set in (Y, σ, B).*

Proof. By Theorem 9, $M_1 = K_1 \Delta N_1$ and $M_2 = K_2 \Delta N_2$ where K_1 and K_2 are both soft clopen in (Y, σ, B), and N_1 and N_2 are both soft nowhere dense in (Y, σ, B). Then by Lemma 3,

$$\begin{aligned} M_1 \tilde{\cap} M_2 &= M_1 \tilde{\cap} (K_2 \triangle N_2) \\ &= (M_1 \tilde{\cap} K_2)\Delta(M_1 \tilde{\cap} N_2) \\ &= ((K_1 \triangle N_1) \tilde{\cap} K_2)\Delta((K_1 \triangle N_1) \tilde{\cap} N_2) \\ &= (K_1 \tilde{\cap} K_2)\Delta(((N_1 \tilde{\cap} K_2)\Delta(K_1 \tilde{\cap} N_2))\Delta(N_1 \tilde{\cap} N_2)) \\ &= K \triangle N, \end{aligned} \tag{19}$$

Where $K = K_1 \tilde{\cap} K_2$ and $N = (((N_1 \tilde{\cap} K_2)\Delta(K_1 \tilde{\cap} N_2))\Delta(N_1 \tilde{\cap} N_2))$. By Corollary 1, K is a soft Q-set in (Y, σ, B). Since N_1 and N_2 are soft nowhere dense sets in (Y, σ, B), then $N_1 \tilde{\cap} K_2$, $K_1 \tilde{\cap} N_2$, and $N_1 \tilde{\cap} N_2$ are soft nowhere dense sets in (Y, σ, B). Thus, by Lemma 4, N is soft nowhere dense in (Y, σ, B). Therefore, by Theorem 9, $M_1 \tilde{\cap} M_2$ is a soft Q-set in (Y, σ, B). □

Theorem 11. *Let (Y, σ, B) be a STS. If M and G are soft Q-sets in (Y, σ, B) , then*

(a) *$M \tilde{\cup} G$ is a soft Q-set in (Y, σ, B).*

(b) *$M - G$ is a soft Q-set in (Y, σ, B).*

(c) *$Bd_\sigma(M)$ is a soft Q-set in (Y, σ, B).*

Proof. (a) Since M and G are soft Q-sets in (Y, σ, B), then by Theorem 5, $1_B - M$ and $1_B - G$ are soft Q-sets in (Y, σ, B). Since $1_B - (M \tilde{\cup} G) = (1_B - M) \tilde{\cap} (1_B - G)$, then by Theorem 10, $1_B - (M \tilde{\cup} G)$ is a soft Q-set in (Y, σ, B). Thus, again by Theorem 5, $M \tilde{\cup} G$ is a soft Q-set in (Y, σ, B).

(b) Since G is a soft Q-set in (Y, σ, B), then by Theorem 5, $1_B - G$ is a soft Q-set in (Y, σ, B). Thus, by Theorem 10, $M - G = M \tilde{\cap} (1_B - G)$ is a soft Q-set in (Y, σ, B).

(c) Since M is a soft Q-set in (Y, σ, B), then by Theorem 5, $1_B - M$ is a soft Q-set in (Y, σ, B). Thus, by Theorem 6, $Cl_\sigma(M)$ and $Cl_\sigma(1_B - M)$ are soft Q-sets in (Y, σ, B). Hence, by Theorem 10, $Cl_\sigma(M) \tilde{\cap} Cl_\sigma(1_B - M) = Bd_\sigma(M)$ is a soft Q-set in (Y, σ, B). □

Theorem 12. *Let (Y, σ, B) be a STS and let $\mathbb{Q}$ be the class of all soft Q -sets of (Y, σ, B). Then $\mathbb{Q}$ is a Boolean algebra with respect to the distinguished elements and Boolean operations defined by*

(1) $0 = 0_B$

(2) $1 = 1_B$

(3) $M \wedge N = M \tilde{\cap} N$

(4) $M \vee N = M \tilde{\cup} N$

(5) $M' = 1_B - M$.

Proof. We need to show that the

(a) right sides of (1)–(5) are soft Q-sets; and

(b) Boolean axioms are satisfied by definition.

(a) Since $Cl_\sigma(\text{Int}_\sigma(0_B)) = \text{Int}_\sigma(Cl_\sigma(0_B)) = 0_B$ and $Cl_\sigma(\text{Int}_\sigma(1_B)) = \text{Int}_\sigma(Cl_\sigma(1_B)) = 1_B$, then the right sides of (1) and (2) are soft Q-sets. Also, by Theorems 10, 11 (a), and 5, the right sides of (3)–(5) are soft Q-sets.

(b) Straightforward.

The following example shows that the Boolean algebra of all soft Q-sets of a STS need not be complete, in general: □

Example 3. Let $Y = \mathbb{Z}$, $B = \{a\}$, and

$$\sigma = \{F \in SS(Y, B)\colon F(a) = \emptyset \text{ or } Y - F(a) \text{ is finite}\}. \tag{20}$$

It is not difficult to check that (Y, σ, B) is soft connected and a soft set M is soft nowhere dense in (Y, σ, B) if and only if $M(a)$ is finite. Thus, by Corollary 2, M is a soft Q-set in (Y, σ, B) if and only if $M(a)$ is finite or $Y - M(a)$ is finite. For each $n \in \mathbb{N}$, let $M_n \in SS(Y, B)$ such that $M_n(a) = \{n\}$. Then M_n is a soft Q-set in (Y, σ, B) for each $n \in \mathbb{N}$.

Claim 2. The set $\{M_n\colon n \in \mathbb{N}\}$ has no supremum.

Proof of Claim. Suppose to the contrary that $\{M_n\colon n \in \mathbb{N}\}$ has a supremum S. Then for each $n \in \mathbb{N}$, $M_n \tilde{\subseteq} S$ and so, $\widetilde{\cup_{n\in\mathbb{N}}} M_n \tilde{\subseteq} S$. Thus, $\mathbb{N} = (\widetilde{\cup_{n\in\mathbb{N}}} M_n)(a) \subseteq S(a)$. Hence, $\mathbb{Z} - S(a)$ is finite. Since $\mathbb{Z} - S(a) \subseteq \mathbb{Z} - \mathbb{N}$ and $\mathbb{Z} - S(a)$ is finite while $\mathbb{Z} - \mathbb{N}$ is infinite, then there exists $x \in (\mathbb{Z} - \mathbb{N}) \cap S(a)$. Let $T \in SS(Y, B)$ with $T(a) = S(a) - \{x\}$. Then for each $n \in \mathbb{N}$, $M_n \tilde{\subseteq} T$. On the other hand, $T \tilde{\subseteq} S$ and $T \neq S$, a contradiction.

4. Correspondence

We start this section with the following natural question:

Question 1. *Let (Y,σ,B) be a STS and M be a soft Q-set in (Y,σ,B). Is it true that $M(b)$ is a Q-set in (Y,σ_b) for all $b\in B$?*

The following example gives a negative answer to Question 1.

Example 4. *Let $Y=\mathbb{R}$, $B=\{a,b\}$, and $F,G,H\in SS(Y,B)$ defined by $F(a)=(-\infty,0)$, $F(b)=\varnothing$, $G(a)=G(b)=[0,\infty)$, $H(a)=\varnothing$, and $H(b)=(-\infty,0)$. Let $\sigma=\{0_B,1_B,F,G,F\tilde{\cup}G\}$. Since $H=1_B-(F\tilde{\cup}G)$, then $\mathrm{Int}_\sigma(Cl_\sigma(H))=\mathrm{Int}_\sigma(H)=0_B$ and $Cl_\sigma(\mathrm{Int}_\sigma(H))=Cl_\sigma(0_B)=0_B$. Therefore, H is a soft Q-set in (Y,σ,B). Note that $\sigma_b=\{\varnothing,Y,[0,\infty)\}$. Since $\mathrm{Int}_{\sigma_b}(Cl_{\sigma_b}(H(b)))=\mathrm{Int}_{\sigma_b}(Cl_{\sigma_b}(-\infty,0))=\mathrm{Int}_{\sigma_b}(\mathbb{R})=\mathbb{R}$, while, $Cl_{\sigma_b}(\mathrm{Int}_{\sigma_b}(H(b)))=Cl_{\sigma_b}(\mathrm{Int}_{\sigma_b}(-\infty,0))=Cl_{\sigma_b}(\varnothing)=\varnothing$, then $H(b)$ is not a Q-set in (Y,σ_b).*

Theorem 13. *Let $\{(Y,\mu_b): b\in B\}$ be a family of STSs and let $M\in SS(Y,B)$. Then M is a soft Q-set in $(Y,\oplus_{b\in B}\mu_b,B)$ if and only if $M(a)$ is a Q-set in (Y,μ_a) for all $a\in B$.*

Proof. Necessity. Suppose that M is a soft Q-set in $(Y,\oplus_{b\in B}\mu_b,B)$ and let $a\in B$. Let $\sigma=\oplus_{b\in B}\mu_b$. Since M is a soft Q-set in $(Y,\oplus_{b\in B}\mu_b,B)$, then $\mathrm{Int}_\sigma(Cl_\sigma(M))=Cl_\sigma(\mathrm{Int}_\sigma(M))$, and so $(\mathrm{Int}_\sigma(Cl_\sigma(M)))(a)=(Cl_\sigma(\mathrm{Int}_\sigma(M)))(a)$. But by Lemma 4.9 of [34],

$$\begin{aligned}(\mathrm{Int}_\sigma(Cl_\sigma(M)))(a)&=\mathrm{Int}_{\mu_a}((Cl_\sigma(M))(a))\\&=\mathrm{Int}_{\mu_a}\big(Cl_{\mu_a}(M(a))\big),\text{ and}\\(Cl_\sigma(\mathrm{Int}_\sigma(M)))(a)&=Cl_{\mu_a}((\mathrm{Int}_\sigma(M))(a))\\&=Cl_{\mu_a}\big(\mathrm{Int}_{\mu_a}(M(a))\big).\end{aligned}\tag{21}$$

Therefore, $\mathrm{Int}_{\mu_a}(Cl_{\mu_a}(M(a)))=Cl_{\mu_a}(\mathrm{Int}_{\mu_a}(M(a)))$. Hence, $M(a)$ is a Q-set in (Y,μ_a).

Sufficiency. Suppose that $M(a)$ is a Q-set in (Y,μ_a) for all $a\in B$. Then for each $a\in B$, $\mathrm{Int}_{\mu_a}(Cl_{\mu_a}(M(a)))=Cl_{\mu_a}(\mathrm{Int}_{\mu_a}(M(a)))$. Let $\sigma=\oplus_{b\in B}\mu_b$. Then by Lemma 4.9 of [34], for every $a\in B$,

$$\begin{aligned}\mathrm{Int}_{\mu_a}\big(Cl_{\mu_a}(M(a))\big)&=\mathrm{Int}_{\mu_a}((Cl_\sigma(M))(a))\\&=(\mathrm{Int}_\sigma(Cl_\sigma(M)))(a),\text{ and}\\Cl_{\mu_a}\big(\mathrm{Int}_{\mu_a}(M(a))\big)&=Cl_{\mu_a}((\mathrm{Int}_\sigma(M))(a))\\&=(Cl_\sigma(\mathrm{Int}_\sigma(M)))(a).\end{aligned}\tag{22}$$

Thus, $\mathrm{Int}_\sigma(Cl_\sigma(M))=Cl_\sigma(\mathrm{Int}_\sigma(M))$. Hence, M is a soft Q-set in $(Y,\oplus_{b\in B}\mu_b,B)$. □

Corollary 3. *Let (Y,μ) be a TS and B be a set of parameters. Then M is a soft Q-set in $(Y,\tau(\mu),B)$ if and only if $M(b)$ is a Q-set in (Y,μ) for all $b\in B$.*

Proof. For each $b\in B$, let $\mu_b=\mu$. Then $\tau(\mu)=\oplus_{b\in B}\mu_b$. So, by Theorem 13, we get the result. □

Proposition 1. *Let $\{(Y,\mu_b): b\in B\}$ be a family of STSs and let $M\in SS(Y,B)$. Then $(Y,\oplus_{b\in B}\mu_b,B)$ is soft extremally disconnected if and only if (Y,μ_b) is extremally disconnected for all $b\in B$.*

Proof. *Necessity.* Suppose that $(Y,\oplus_{b\in B}\mu_b,B)$ is soft extremally disconnected and let $b\in B$. Let $U\in\mu_b$. Then $b_U\in\oplus_{b\in B}\mu_b$ and so $Cl_{\oplus_{b\in B}\mu_b}(b_U)=b_{Cl_{\mu_b}(U)}\in\oplus_{b\in B}\mu_b$. Hence, $(b_{Cl_{\mu_b}(U)})(b)=Cl_{\mu_b}(U)\in\mu_b$. Therefore, (Y,μ_b) is extremally disconnected.

Sufficiency. Suppose that (Y,μ_b) is extremally disconnected for all $b\in B$. Let $F\in\oplus_{b\in B}\mu_b$. Then for all $b\in B$, $F(b)\in\mu_b$ and so $Cl_{\mu_b}(F(b))\in\mu_b$. Thus, by Lemma 4.9 of [34], $(Cl_{\oplus_{b\in B}\mu_b}(F))(b)=Cl_{\mu_b}(F(b))\in\mu_b$ for all $b\in B$. Therefore, $Cl_{\oplus_{b\in B}\mu_b}(F)\in\oplus_{b\in B}\mu_b$. Hence, $(Y,\oplus_{b\in B}\mu_b,B)$ is soft extremally disconnected.

Corollary 4. *Let (Y,μ) be a TS and B be a set of parameters. Then $(Y,\tau(\mu),B)$ is soft extremally disconnected if and only if (Y,μ) is extremally disconnected.*

Proof. For each $b\in B$, let $\mu_b=\mu$. Then $\tau(\mu)=\oplus_{b\in B}\mu_b$. So, by Proposition 1, we get the result. □

Theorem 14. *Let (Y,σ,B) be a soft extremally disconnected STS. Then every soft open set in (Y,σ,B) is a soft Q-set in (Y,σ,B).*

Proof. Let $M\in\sigma$. Since (Y,σ,B) is soft extremally disconnected, then $Cl_\sigma(M)\in\sigma$ and so $\mathrm{Int}_\sigma(Cl_\sigma(M))=Cl_\sigma(M)$. On the other hand, since $M\in\sigma$, then $\mathrm{Int}_\sigma(M)=M$ and so $Cl_\sigma(\mathrm{Int}_\sigma(M))=Cl_\sigma(M)$. Therefore, $\mathrm{Int}_\sigma(Cl_\sigma(M))=Cl_\sigma(\mathrm{Int}_\sigma(M))$. Thus, M is a soft Q-set in (Y,σ,B).

The following example shows that soft Q-sets in a soft extremally disconnected STS need not be soft open sets in general: □

Example 5. *Let Y be any infinite set and B be any set of parameters. Let μ be the cofinite topology on Y. Consider the STS $(Y,\tau(\mu),B)$. Since (Y,μ) is extremally disconnected, then by Corollary4, $(Y,\tau(\mu),B)$ is soft extremally disconnected. Choose $U\subseteq Y$ such that U is a non-empty finite set. Then $\mathrm{Int}_\mu(Cl_\mu(U))=Cl_\mu(\mathrm{Int}_\mu(U))=\varnothing$, and hence U is a Q-set in (Y,μ). Since for each $b\in B$, $(C_U)(b)=U$, then by Corollary3, C_U is a soft Q-set in $(Y,\tau(\mu),B)$. On the other hand, it is clear that $C_U\notin\tau(\mu)$.*

The following example shows in Theorem 14 that the condition "soft extremally disconnected" cannot be dropped:

Example 6. *Let $Y=\mathbb{R}$ and B be any set of parameters. Let μ be the usual topology on Y. Consider the STS $(Y,\tau(\mu),B)$. Then $C_{(0,1)}\in\tau(\mu)$. On the other hand, since $\mathrm{Int}_{\tau(\mu)}(Cl_{\tau(\mu)}(C_{(0,1)}))=C_{(0,1)}$ but $Cl_{\tau(\mu)}(\mathrm{Int}_{\tau(\mu)}(C_{(0,1)}))=C_{[0,1]}$, then $C_{(0,1)}$ is not a soft Q-set in $(Y,\tau(\mu),B)$.*

5. The Commutativity of the Soft Boundary and Soft Interior of a STS

Lemma 5. *Let (Y, σ, B) be a STS. If $M \in \sigma$ and N is soft dense in (Y, σ, B), then $Cl_\sigma(M \tilde{\cap} N) = Cl_\sigma(M) \tilde{\cap} Cl_\sigma(N) = Cl_\sigma(M)$.*

Proof. We only need to show that $Cl_\sigma(M) \tilde{\cap} Cl_\sigma(N) \tilde{\subseteq} Cl_\sigma(M \tilde{\cap} N)$. Let $b_y \tilde{\in} Cl_\sigma(M) \tilde{\cap} Cl_\sigma(N)$ and let $G \in \sigma$ such that $b_y \tilde{\in} G$. Since $b_y \tilde{\in} Cl_\sigma(M)$, then $G \tilde{\cap} M \neq 0_B$. Since $G \tilde{\cap} M \in \sigma - \{0_B\}$ and N is soft dense in (Y, σ, B), then $(G \tilde{\cap} M) \tilde{\cap} N = G \tilde{\cap} (M \tilde{\cap} N) \neq 0_B$. It follows that $b_y \tilde{\in} Cl_\sigma(M \tilde{\cap} N)$. □

Lemma 6. *Let (Y, σ, B) be a STS and let $M \in SS(Y, B)$. Then $Int_\sigma(Bd_\sigma(M)) \tilde{\cap} Bd_\sigma(Int_\sigma(M)) = 0_B$.*

Proof. By Theorem 3, we have $\mathrm{Int}_\sigma(Cl_\sigma(1_B - M)) = \mathrm{Int}_\sigma(1_B - \mathrm{Int}_\sigma(M)) = 1_B - Cl_\sigma(\mathrm{Int}_\sigma(M))$. Thus,

$$\begin{aligned}\mathrm{Int}_\sigma(Bd_\sigma(M)) \tilde{\cap} Bd_\sigma(\mathrm{Int}_\sigma(M)) &= \mathrm{Int}_\sigma(Cl_\sigma(M) \tilde{\cap} Cl_\sigma(1_B - M)) \tilde{\cap} (Cl_\sigma(\mathrm{Int}_\sigma(M)) \tilde{\cap} Cl_\sigma(1_B - \mathrm{Int}_\sigma(M))) \\ &= \mathrm{Int}_\sigma(Cl_\sigma(M)) \tilde{\cap} \mathrm{Int}_\sigma(Cl_\sigma(1_B - M)) \tilde{\cap} (Cl_\sigma(\mathrm{Int}_\sigma(M)) \tilde{\cap} Cl_\sigma(1_B - \mathrm{Int}_\sigma(M))) \\ &= \mathrm{Int}_\sigma(Cl_\sigma(M)) \tilde{\cap} (1_B - Cl_\sigma(\mathrm{Int}_\sigma(M))) \tilde{\cap} (Cl_\sigma(\mathrm{Int}_\sigma(M)) \tilde{\cap} Cl_\sigma(1_B - \mathrm{Int}_\sigma(M))) \\ &= 0_B.\end{aligned} \tag{23}$$

□

Theorem 15. *Let (Y, σ, B) be a STS and let $M \in SS(Y, B)$. Then $Int_\sigma(Bd_\sigma(M)) = Bd_\sigma(Int_\sigma(M))$ if and only if $Int_\sigma(Cl_\sigma(M)) = Cl_\sigma(Int_\sigma(M)) = Int_\sigma(M)$.*

Proof. Necessity. Suppose that $\mathrm{Int}_\sigma(Bd_\sigma(M)) = Bd_\sigma(\mathrm{Int}_\sigma(M))$. Then by Lemma 6, $\mathrm{Int}_\sigma(Bd_\sigma(M)) = Bd_\sigma(\mathrm{Int}_\sigma(M)) = 0_B$. Since

$$\mathrm{Int}_\sigma(Bd_\sigma(M)) = \mathrm{Int}_\sigma(Cl_\sigma(M)) \tilde{\cap} (1_B - Cl_\sigma(\mathrm{Int}_\sigma(M))) \tag{24}$$

then $\mathrm{Int}_\sigma(Cl_\sigma(M)) \tilde{\subseteq} Cl_\sigma(\mathrm{Int}_\sigma(M))$.

Also, applying Theorem 3, we have

$$\begin{aligned}Bd_\sigma(\mathrm{Int}_\sigma(M)) &= Cl_\sigma(\mathrm{Int}_\sigma(M)) \tilde{\cap} Cl_\sigma(1_B - \mathrm{Int}_\sigma(M)) \\ &= Cl_\sigma(\mathrm{Int}_\sigma(M)) \tilde{\cap} Cl_\sigma(Cl_\sigma(1_B - M)) \\ &= Cl_\sigma(\mathrm{Int}_\sigma(M)) \tilde{\cap} Cl_\sigma(Cl_\sigma(1_B - M)) \\ &= Cl_\sigma(\mathrm{Int}_\sigma(M)) \tilde{\cap} Cl_\sigma(1_B - M \\ &= 0_B,\end{aligned} \tag{25}$$

and hence, $Cl_\sigma(\mathrm{Int}_\sigma(M)) \tilde{\subseteq} 1_B - Cl_\sigma(1_B - M) = \mathrm{Int}_\sigma(M)$. Therefore, $\mathrm{Int}_\sigma(Cl_\sigma(M)) \tilde{\subseteq} Cl_\sigma(\mathrm{Int}_\sigma(M)) \tilde{\subseteq} \mathrm{Int}_\sigma(M)$. But $\mathrm{Int}_\sigma(M) \tilde{\subseteq} \mathrm{Int}_\sigma(Cl_\sigma(M))$ is always true. Therefore, $\mathrm{Int}_\sigma(M) = \mathrm{Int}_\sigma(Cl_\sigma(M)) = Cl_\sigma(\mathrm{Int}_\sigma(M))$.

Sufficiency. Suppose that $\mathrm{Int}_\sigma(Cl_\sigma(M)) = Cl_\sigma(\mathrm{Int}_\sigma(M)) = \mathrm{Int}_\sigma(M)$. Then

$$\begin{aligned}\mathrm{Int}_\sigma(Bd_\sigma(M)) &= \mathrm{Int}_\sigma(Cl_\sigma(M) \tilde{\cap} Cl_\sigma(1_B - M)) \\ &= \mathrm{Int}_\sigma(Cl_\sigma(M)) \tilde{\cap} \mathrm{Int}_\sigma(Cl_\sigma(1_B - M)) \\ &= \mathrm{Int}_\sigma(M) \tilde{\cap} \mathrm{Int}_\sigma(1_B - \mathrm{Int}_\sigma(M)) \\ &\tilde{\subseteq} \mathrm{Int}_\sigma(M) \tilde{\cap} (1_B - \mathrm{Int}_\sigma(M)) \\ &= 0_B, \text{ and} \\ Bd_\sigma(\mathrm{Int}_\sigma(M)) &= Cl_\sigma(\mathrm{Int}_\sigma(M)) \tilde{\cap} Cl_\sigma(1_B - \mathrm{Int}_\sigma(M)) \\ &= Cl_\sigma(\mathrm{Int}_\sigma(M)) \tilde{\cap} Cl_\sigma(Cl_\sigma(1_B - M)) \\ &= \mathrm{Int}_\sigma(M) \tilde{\cap} Cl_\sigma(1_B - M) \\ &= \mathrm{Int}_\sigma(M) \tilde{\cap} (1_B - \mathrm{Int}_\sigma(M)) \\ &= 0_B.\end{aligned} \tag{26}$$

Thus, $\mathrm{Int}_\sigma(Bd_\sigma(M)) = Bd_\sigma(\mathrm{Int}_\sigma(M))$. □

Theorem 16. *Let (Y, σ, B) be a STS and let $M \in SS(Y, B)$. Then $Int_\sigma(Bd_\sigma(M)) = Bd_\sigma(Int_\sigma(M))$ if and only if $M = K \tilde{\cup} N$, where K is soft clopen in (Y, σ, B), N is soft nowhere dense in (Y, σ, B), and $K \tilde{\cap} N = 0_B$.*

Proof. By Theorem 15, it is sufficient to show that $\mathrm{Int}_\sigma(Cl_\sigma(M)) = Cl_\sigma(\mathrm{Int}_\sigma(M)) = \mathrm{Int}_\sigma(M)$ if and only if $M = K \tilde{\cup} N$, where K is soft clopen in (Y, σ, B), N is soft nowhere dense in (Y, σ, B), and $K \tilde{\cap} N = 0_B$.

Necessity. Suppose that $\mathrm{Int}_\sigma(Cl_\sigma(M)) = Cl_\sigma(\mathrm{Int}_\sigma(M)) = \mathrm{Int}_\sigma(M)$. Let $K = \mathrm{Int}_\sigma(M)$ and $N = M - K$. Then K is soft clopen in (Y, σ, B), $M = K \tilde{\cup} N$, and $K \tilde{\cap} N = 0_B$. The proof that N is soft nowhere dense in (Y, σ, B) is similar to that used in the proof of Claim (a) in Theorem 9.

Sufficiency. Suppose that $M = K \tilde{\cup} N$, where K is soft clopen in (Y, σ, B), N is soft nowhere dense in (Y, σ, B), and $K \tilde{\cap} N = 0_B$. By Theorem 8 and Corollary 1, we have N and K are soft Q-sets in (Y, σ, B). So, by Theorem 11 (a), $M = K \tilde{\cup} N$ is a soft Q-set in (Y, σ, B). Hence, $\mathrm{Int}_\sigma(Cl_\sigma(M)) = Cl_\sigma(\mathrm{Int}_\sigma(M))$. Since N is soft nowhere dense in (Y, σ, B), then $1_B - Cl_\sigma(N)$ is soft dense in (Y, σ, B), and so $1_B - N$ is soft dense. Thus, by Lemma 5, $Cl_\sigma((1_B - K) \tilde{\cap} (1_B - N)) = Cl_\sigma(1_B - K) = 1_B - K$. Therefore, by Theorem 3, we have

$$\begin{aligned}\mathrm{Int}_\sigma(M) &= \mathrm{Int}_\sigma(K \tilde{\cup} N)\\ &= 1_B - Cl_\sigma\left(1_B - (K \tilde{\cup} N)\right)\\ &= 1_B - Cl_\sigma\left(\left(1_B - K\right) \tilde{\cap} \left(1_B - N\right)\right)\\ &= 1_B - \left(1_B - K\right)\\ &= K.\end{aligned} \tag{27}$$

Hence, $\mathrm{Int}_\sigma(M)$ is soft closed in (Y, σ, B). Thus,

$$\mathrm{Int}_\sigma(M) \tilde{\subseteq} \mathrm{Int}_\sigma\left(Cl_\sigma(M)\right) = Cl_\sigma\left(\mathrm{Int}_\sigma(M)\right) = \mathrm{Int}_\sigma(M). \tag{28}$$

This shows that $\mathrm{Int}_\sigma(Cl_\sigma(M)) = Cl_\sigma(\mathrm{Int}_\sigma(M)) = \mathrm{Int}_\sigma(M)$. □

6. Conclusion

In this paper, soft Q-sets as a new class of sets which contains both soft clopen and soft nowhere dense sets are introduced. Soft Q-sets have been characterized in terms of soft clopen sets and soft nowhere dense sets. It is proved that soft Q-sets form a Boolean algebra that is not complete, in general. Furthermore, soft sets whose soft boundaries and soft interiors are commutative are characterized. In addition, the correspondence between Q-sets in topological spaces and soft Q-sets in soft topological spaces is investigated.

Boolean algebra constitutes the basis for the design of circuits used in electronic digital computers. In addition, it is of significance to the theory of probability, the geometry of sets, and information theory. So, this paper not only forms the theoretical basis for further applications of soft topology, but it also leads to the development of the theory of probability, the geometry of sets, and information theory.

In the upcoming work, we plan to: (1) find sufficient condition for the Boolean algebra of soft Q-sets to be complete; and (2) investigate the behavior of soft Q-open sets under product soft topological spaces.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] Z. Pawlak, "Rough sets," *International Journal of Computer & Information Sciences*, vol. 11, no. 5, pp. 341–356, 1982.

[3] W. L. Gau and D. J. Buehrer, "Vague sets," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 23, no. 2, pp. 610–614, 1993.

[4] D. Molodtsov, "Soft set theory—first results," *Computers & Mathematics with Applications*, vol. 37, no. 4-5, pp. 19–31, 1999.

[5] D. Molodtsov, V. Y. Leonov, and D. V. Kovkov, "Soft sets technique and its application," *Nechetkie Sistemy i Myagkie Vychisleniya*, vol. 1, pp. 8–39, 2006.

[6] P. K. Maji, A. R. Roy, and R. Biswas, "An application of soft sets in a decision making problem," *Computers & Mathematics with Applications*, vol. 44, no. 8-9, pp. 1077–1083, 2002.

[7] M. Shabir and M. Naz, "On soft topological spaces," *Computers & Mathematics with Applications*, vol. 61, no. 7, pp. 1786–1799, 2011.

[8] S. Al Ghour, "Soft connectivity and soft θ-connectivity relative to a soft topological space," *Journal of Intelligent and Fuzzy Systems*, pp. 1–11. In press, 2022, https://content.iospress.com/articles/journal-of-intelligent-and-fuzzy-systems/ifs220371?resultNumber=1&totalResults=11&start=0&q=al+ghour&resultsPageSize=10&rows=10.

[9] S. Al Ghour and Z. A. Ameen, "Maximal soft compact and maximal soft connected topologies," *Applied Computational Intelligence and Soft Computing*, vol. 2022, Article ID 9860015, 7 pages, 2022.

[10] A. A. Bahredar, N. Kouhestani, and H. Passandideh, "The fundamental group of soft topological spaces," *Soft Computing*, vol. 26, no. 2, pp. 541–552, 2022.

[11] A. Mehmood, S. Al Ghour, S. Abdullah, C. Park, and J. Rye Lee, "A new approach to vague soft toplogical structures concerning soft points," *Journal of Intelligent and Fuzzy Systems*, vol. 42, no. 3, pp. 1483–1499, 2022.

[12] S. Oztunc, S. Aslan, and H. Dutta, "Categorical structures of soft groups," *Soft Computing*, vol. 25, no. 4, pp. 3059–3064, 2021.

[13] T. M. Al-Shami, "Defining and investigating new soft ordered maps by using soft semi open sets," *Acta Universitatis Sapientiae, Mathematica*, vol. 13, no. 1, pp. 145–163, 2021.

[14] T. M. Al-shami, "Bipolar soft sets: relations between them and ordinary points and their applications," *Complexity*, vol. 2021, Article ID 6621854, 14 pages, 2021.

[15] T. M. Al-shami, "On soft separation axioms and their applications on decision-making problem," *Mathematical Problems in Engineering*, vol. 2021, pp. 1–12, 2021.

[16] T. M. Al-shami, "Compactness on soft topological ordered spaces and its application on the information system," *Journal of Mathematics*, vol. 2021, Article ID 6699092, 12 pages, 2021.

[17] G. Oguz, "Soft topological transformation groups," *Mathematics*, vol. 8, no. 9, p. 1545, 2020.

[18] W. K. Min, "On soft w-structures defined by soft sets," *International Journal of Fuzzy Logic and Intelligent Systems*, vol. 20, no. 2, pp. 119–123, 2020.

[19] V. Çetkin, E. Güner, and H. Aygün, "On 2S-metric spaces," *Soft Computing*, vol. 24, no. 17, pp. 12731–12742, 2020.

[20] M. E. El-Shafei and T. M. Al-shami, "Applications of partial belong and total non-belong relations on soft separation axioms and decision-making problem," *Computational and Applied Mathematics*, vol. 39, no. 3, p. 138, 2020.

[21] J. C. R. Alcantud, "Soft open bases and a novel construction of soft topologies from bases for topologies," *Mathematics*, vol. 8, no. 5, p. 672, 2020.

[22] A. A. Bahredar and N. Kouhestani, "On ε-soft topological semigroups," *Soft Computing*, vol. 24, no. 10, pp. 7035–7046, 2020.

[23] T. M. Al-shami and M. E. El-Shafei, "Partial belong relation on soft separation axioms and decision-making problem, two birds with one stone," *Soft Computing*, vol. 24, no. 7, pp. 5377–5387, 2020.

[24] T. M. Al-shami, L. D. R. Kočinac, and B. A. Asaad, "Sum of soft topological spaces," *Mathematics*, vol. 8, no. 6, p. 990, 2020.

[25] N. Levine, "On the community of the closure and interior operators in topological spaces," *The American Mathematical Monthly*, vol. 68, no. 5, pp. 474–477, 1961.

[26] H. Choda and K. Matoba, "On a theorem of levine," *Proceedings of the Japan Academy Series A: Mathematical Sciences*, vol. 37, no. 8, pp. 462-463, 1961.

[27] S. Al Ghour and A. Bin-Saadon, "On some generated soft topological spaces and soft homogeneity," *Heliyon*, vol. 5, no. 7, Article ID e02061, 2019.

[28] S. Al Ghour and W. Hamed, "On two classes of soft sets in soft topological spaces," *Symmetry*, vol. 12, no. 2, p. 265, 2020.

[29] M. H. Stone, "Algebraic characterizations of special Boolean rings," *Fundamenta Mathematicae*, vol. 29, pp. 223–303, 1937.

[30] J. Subhashinin and C. Sekar, "Related properties of soft dense and soft pre open sets in a soft topological spaces," *International Journal of Innovative and Applied Research*, vol. 2, pp. 34–48, 2014.

[31] B. A. Asaad, "Results on soft extremally disconnectedness of soft topological spaces," *The Journal of Mathematics and Computer Science*, vol. 17, no. 04, pp. 448–464, 2017.

[32] S. Hussain and B. Ahmad, "Some properties of soft topological spaces," *Computers & Mathematics with Applications*, vol. 62, no. 11, pp. 4058–4067, 2011.

[33] M. Riaz and Z. Fatima, "Certain properties of soft metric spaces," *Journal of Fuzzy Mathematics*, vol. 25, pp. 543–560, 2017.

[34] S. Al Ghour, "Strong form of soft semiopen sets in soft topological spaces," *International Journal of Fuzzy Logic and Intelligent Systems*, vol. 21, no. 2, pp. 159–168, 2021.