Hindawi
Applied Computational Intelligence and Soft Computing
Volume 2022, Article ID 6280736, 15 pages
https://doi.org/10.1155/2022/6280736

Research Article

Solving Partial Integro-Differential Equations via Double Formable Transform

Bayan Ghazal,[1] Rania Saadeh ,[1] and Abdelilah K. Sedeeg[2,3]

[1]*Department of Mathematics, Faculty of Science, Zarqa University, Zarqa 13110, Jordan*
[2]*Department of Mathematics, Faculty of Education-Holy Quran and Islamic Sciences University, Sudan*
[3]*Department of Mathematics, Faculty of Sciences and Arts-Almikwah-Albaha University, Saudi Arabia*

Correspondence should be addressed to Rania Saadeh; rsaadeh@zu.edu.jo

Received 26 October 2022; Revised 29 October 2022; Accepted 8 December 2022; Published 26 December 2022

Academic Editor: V. E. Sathishkumar

Copyright © 2022 Bayan Ghazal et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this study, we present a new double integral transform called the double formable transform. Several properties and theorems related to existing conditions, partial derivatives, the double convolution theorem, and others are presented. Additionally, we use a convolution kernel to solve linear partial integro-differential equations (PIDE) by using the double formable transform. By solving numerous cases, the double formable transform's ability to turn the PIDE into an algebraic equation that is simple to solve is demonstrated.

1. Introduction

One of the most significant subjects that might be interesting in life issues is mathematics; it helps to improve mental abilities and foster logical and sound thinking. Additionally, it aids in the understanding and resolution of problems arising from a variety of natural phenomena as well as those pertaining to physical, engineering, biological, and computer science [1–5]. Recent studies have demonstrated the value of integral transforms in the solution of partial equations relating to, for example, sound, heat, static electricity, elasticity, fluid flow, and other physical and engineering phenomena [6, 7].

In addition, the integral transform helped to convert the partial and integral equations into algebraic equations that provide the problems' precise solutions in the most straightforward manner [8–11]. The double integral transforms have drawn researchers' attention recently because of their contribution to the solution of equations involving two variables and finding precise solutions in the simplest ways possible. However, with the development of science and the urgent need for mathematics to solve newly emerged problems, it was necessary to pay attention to scientists and researchers to obtain new and advanced methods to keep speed with these problems [12–14]. The most popular double integral transforms that are used to solve partial differential equations are the double Laplace transform [15, 16], the double Laplace-Sumudu transform [17], the double Sumudu transform [18], the ARA-Sumudu transform [19, 20], the double Shehu transform [21], the double Elzaki transform [12], and others [22]. Integral equations of various types played a significant role in explaining many phenomena and coming up with exact solutions when various sciences became complex as a result of their interactions with one another and scientists started studying all physical, chemical, biological, and engineering phenomena. As a result, integral equations are crucial for representing the many disciplines, which prompted scientists and researchers to look for a solution to make it possible for them to solve the integral equations.

There are several types of integral equations, including Volterra integral equations and integro-differential equations; in this paper, we will focus on these two types of integral equations and solve many examples of each [23].

In 2021, with the advent of the formable integral transform (FT) that was presented by Saadah and Ghazal [24], its ability to solve ordinary, partial differential equations, and integral equations, and its properties that could enable us to solve a wide kind of differential and integral problems, we intend to define the new double formable transform. Moreover, it is worth mentioning here that the formable transform might be considered a special case of the new integral transform presented by Jafari in [25], and the idea of double integral transforms is also generalized by the authors in [14]. The main advantage of using the double formable transform is that it preserves the values of constants, which simplifies the calculations during the solving of problems, and it transforms the target problem from two-dimensional space into four-dimensional space. In this study, we focus on the double formable transform and study the main properties and the application of solving integral equations.

Our goal in this paper is to provide a solution to two types of integral equations, the Volterra integral equations, and the partial integro-differential equations, by using the double formable transform (DFT). We have also clarified and explained some theorems and properties with their proofs, and among the most important theorems that will help us solve the examples are the double convolution, partial derivatives, and many more, as we explain in the subsequent sections.

This article is organized as follows: in Section 2, the fundamental facts of formable transform are introduced. In Section 3, we introduce a new double formable transform and present some properties and theorems. In Section 4, we apply the DFT to different types of IDEs and solve some examples. Lastly, in Section 5, the conclusion section is presented.

2. Fundamental Facts of Formable Transform [24]

A new integral transform known as the formable transform of the continuous function $h(t)$ on the interval $[0,\infty)$ is defined by

$$\mathscr{R}[h(t)] = B(s,u) = \frac{s}{u}\int_0^\infty e^{-\frac{s}{u}t} h(t)\mathrm{dt}. \tag{1}$$

The inverse formable transform is defined as

$$\mathscr{R}^{-1}[\mathscr{R}[h(t)]] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{(s/u)t} B(s,u)\mathrm{ds} = h(t). \tag{2}$$

2.1. Formable Transform of Some Basic Functions

$$\text{i. } \mathscr{R}[1] = 1. \tag{3}$$

$$\text{ii. } \mathscr{R}[t^\alpha] = \frac{u^\alpha}{s^\alpha}\Gamma(\alpha+1), \alpha>0, \tag{4}$$

$$\text{iii. } \mathscr{R}\left[e^{\mathrm{at}}\right] = \frac{s}{s-\mathrm{au}}, \tag{5}$$

$$\text{iv. } \mathscr{R}[\sin(\mathrm{at})] = \frac{\mathrm{asu}}{s^2+a^2u^2}, \tag{6}$$

$$\text{v. } \mathscr{R}[\sinh(\mathrm{at})] = \frac{\mathrm{asu}}{s^2-a^2u^2}, \tag{7}$$

$$\text{vi. } \mathscr{R}[\cos(\mathrm{at})] = \frac{s^2}{s^2+a^2u^2}, \tag{8}$$

$$\text{vii. } \mathscr{R}[\cosh(\mathrm{at})] = \frac{s^2}{s^2-a^2u^2}, \tag{9}$$

$$\text{viii. } \mathscr{R}[\mathrm{th}(t)] = \frac{u^2}{s}\frac{d(B(s,u))}{\mathrm{du}} + \frac{u}{s}B(s,u). \tag{10}$$

2.2. Formable Transform of Derivatives

$$\text{i. } \mathscr{R}[h'(t)] = \frac{s}{u}B(s,u) - \frac{s}{u}h(0), \tag{11}$$

$$\text{ii. } \mathscr{R}\left[h^{(m+1)}(t)\right] = \left(\frac{s}{u}\right)^{m+1}B(s,u) - \sum_{j=0}^{m}\left(\frac{s}{u}\right)^{m+1-j}h^{(j)}(0), \quad m = 0,1,2,\ldots. \tag{12}$$

The above results can be obtained from the definition of FT with simple calculations.

3. Double Formable Transform (DFT)

Definition 1. Let $h(x,t)$ be a continuous function of two variables $x>0$ and $t>0$. Then, the double formable transform (DFT) of a function $h(x,t)$ is defined as

$$\begin{aligned}\mathscr{R}_x\mathscr{R}_t[h(x,t)] = \mathscr{R}_x\mathscr{R}_t[h(x,t);(v,r,s,u)] &= \mathscr{R}_x[\mathscr{R}_t[h(x,t);t\longrightarrow(s,u)];x\longrightarrow(v,r)]\\ &= \frac{v}{r}\int_0^\infty e^{-(v/r)x}\left(\frac{s}{u}\int_0^\infty e^{-(s/u)t}[h(x,t)]dt\right)dx,\\ &= \frac{s}{u}\frac{v}{r}\int_0^\infty\int_0^\infty e^{-(v/rx+s/ut)}[h(x,t)]dxdt,\end{aligned} \tag{13}$$

which is equivalent to

$$\mathscr{R}_x\mathscr{R}_t[h(x,t)] = H(v,r,s,u) = \text{sv}\int_0^\infty\int_0^\infty e^{-(vx+st)}h(\text{rx},\text{ut})dxdt. \tag{14}$$

The inverse of DFT is given by

$$\mathscr{R}_x^{-1}\mathscr{R}_t^{-1}[\mathscr{R}_x\mathscr{R}_t[h(x,t)]] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\frac{1}{s}\frac{1}{r}eu^{\frac{s}{u}t+\frac{v}{r}t}H(v,r,s,u)dsdr = h(x,t). \tag{15}$$

We denote the single FT as follows:

(i) With respect to x: $\mathscr{R}_x[h(x,t)] = H(v,r,t)$.
(ii) With respect to t: $\mathscr{R}_t[h(x,t)] = H(x,s,u)$.

Clearly, the DFT is a linear transformation as shown below:

$$\begin{aligned}\mathscr{R}_x\mathscr{R}_t[a\,h(x,t)+b\,m(x,t)] &= \frac{s}{u}\frac{v}{r}\int_0^\infty\int_0^\infty e^{-(v/rx+s/ut)}[a\,h(x,t)+b\,m(x,t)]dxdt\\ &= a\frac{s}{u}\frac{v}{r}\int_0^\infty\int_0^\infty e^{-(v/rx+s/ut)}[h(x,t)]dxdt + b\frac{s}{u}\frac{v}{r}\int_0^\infty\int_0^\infty e^{-(v/rx+s/ut)}[m(x,t)]dxdt\\ &= a\mathscr{R}_x\mathscr{R}_t[h(x,t)] + b\mathscr{R}_x\mathscr{R}_t[m(x,t)],\end{aligned} \tag{16}$$

where α and β are constants and $\mathscr{R}_x\mathscr{R}_t[h(x,t)], \mathscr{R}_x\mathscr{R}_t[m(x,t)]$ are exists.

Property 1. Let $h(x,t) = f(x)g(t), x>0, t>0$. Then,

$$\mathscr{R}_x\mathscr{R}_t[h(x,t)] = \mathscr{R}_x[f(x)]\mathscr{R}_t[g(t)]. \tag{17}$$

Proof.

$$\begin{aligned}\mathscr{R}_x\mathscr{R}_t[h(x,t)] = \mathscr{R}_x\mathscr{R}_t[f(x)g(t)] &= \frac{s}{u}\frac{v}{r}\int_0^\infty\int_0^\infty e^{-(v/rx+(s/u)t)}[f(x)g(t)]dxdt\\ &= \frac{v}{r}\int_0^\infty e^{-v/rx}[f(x)]dx\frac{s}{u}\int_0^\infty e^{-s/ut}[g(t)]dt = \mathscr{R}_x[f(x)]\mathscr{R}_t[g(t)].\end{aligned} \tag{18}$$

□

3.1. DFT of Some Basic Functions

(i) Let $h(x,t) = 1, x>0, t>0$. Then,

$$\mathscr{R}_x\mathscr{R}_t[1] = \frac{s}{u}\frac{v}{r}\int_0^\infty\int_0^\infty e^{-(v/rx+(s/u)t)}dxdt = \frac{v}{r}\int_0^\infty e^{-(v/r)x}dx\frac{s}{u}\int_0^\infty e^{-(s/u)t}dt = \mathscr{R}_x[1]\mathscr{R}_t[1]. \tag{19}$$

From equation(3), we get $\mathscr{R}_x\mathscr{R}_t[1] = 1$.

(ii) Let $h(x,t) = x^\alpha t^\beta, x>0, t>0$ and α,β are constants. Then,

$$\begin{aligned}\mathscr{R}_x\mathscr{R}_t[x^\alpha t^\beta] &= \frac{s}{u}\frac{v}{r}\int_0^\infty\int_0^\infty e^{-(v/rx+s/ut)}[x^\alpha t^\beta]dxdt\\ &= \frac{v}{r}\int_0^\infty e^{-v/rx}[x^\alpha]dx\frac{s}{u}\int_0^\infty e^{-s/ut}[t^\beta]dt\\ &= \mathscr{R}_x[x^\alpha]\mathscr{R}_t[t^\beta].\end{aligned} \tag{20}$$

From equation (4), we get $\mathcal{R}_x\mathcal{R}_t[x^\alpha t^\beta] = (u/s)^\alpha (r/v)^\beta \Gamma(\alpha+1)\Gamma(\beta+1)$.

(iii) Let $h(x,t) = e^{\alpha x+\beta t}$, $x>0, t>0$ and α, β are positive constants. Then,

$$\begin{aligned}\mathcal{R}_x\mathcal{R}_t\left[e^{\alpha x+\beta t}\right] &= \frac{s}{u}\frac{v}{r}\int_0^\infty\int_0^\infty e^{-(v/rx+s/ut)}\left[e^{\alpha x+\beta t}\right]dxdt\\ &= \frac{v}{r}\int_0^\infty e^{-v/rx}\left[e^{\alpha x}\right]dx\frac{s}{u}\int_0^\infty e^{-s/ut}\left[e^{\beta t}\right]dt\\ &= \mathcal{R}_x\left[e^{\alpha x}\right]\mathcal{R}_t\left[e^{\beta t}\right].\end{aligned} \tag{21}$$

From equation (5), we get $\mathcal{R}_x\mathcal{R}_t[e^{\alpha x+\beta t}] = (sv/(s-\alpha u)(v-\beta r))$.

Similarly, $\mathcal{R}_x\mathcal{R}_t[e^{i(\alpha x+\beta t)}] = sv/(s-i\alpha u)(v-i\beta r)$.

Using the property of complex analysis, we have

$$\mathcal{R}_x\mathcal{R}_t\left[e^{i(\alpha x+\beta t)}\right] = \frac{\mathrm{sv}(\mathrm{sv}-\alpha\beta\mathrm{ur}) + i\mathrm{sv}(\mathrm{sr}\beta+\mathrm{uv}\alpha)}{\left(s^2+\alpha^2u^2\right)\left(v^2+\beta^2r^2\right)}. \tag{22}$$

Using Euler's formula, $\sin x = (e^{ix}-e^{-ix}/2i)$, $\cos x = (e^{ix}+e^{-ix}/2)$,

and the formula, $\sinh x = (e^{x}-e^{-x}/2i)$, $\cosh x = (e^{x}+e^{-x}/2)$.

Thus, we find the DFT of the following functions:

$$\begin{aligned}\mathcal{R}_x\mathcal{R}_t[\sin(\alpha x+\beta t)] &= \frac{\mathrm{sv}(\mathrm{sr}\beta+\mathrm{uv}\alpha)}{\left(s^2+\alpha^2u^2\right)\left(v^2+\beta^2r^2\right)},\\ \mathcal{R}_x\mathcal{R}_t[\cos(\alpha x+\beta t)] &= \frac{\mathrm{sv}(\mathrm{sv}-\alpha\beta\mathrm{ur})}{\left(s^2+\alpha^2u^2\right)\left(v^2+\beta^2r^2\right)},\\ \mathcal{R}_x\mathcal{R}_t[\sinh(\alpha x+\beta t)] &= \frac{\mathrm{sv}(\mathrm{sr}\beta+\mathrm{uv}\alpha)}{\left(s^2-\alpha^2u^2\right)\left(v^2-\beta^2r^2\right)},\\ \mathcal{R}_x\mathcal{R}_t[\cosh(\alpha x+\beta t)] &= \frac{\mathrm{sv}(\mathrm{sv}+\alpha\beta ur)}{\left(s^2-\alpha^2u^2\right)\left(v^2-\beta^2r^2\right)},\end{aligned} \tag{23}$$

$$\begin{aligned}\mathcal{R}_x\mathcal{R}_t[J_0(\beth\sqrt{xt})] &== \frac{s}{u}\frac{v}{r}\int_0^\infty\int_0^\infty e^{-(v/rx+s/ut)}[J_0(\beth\sqrt{xt})]dxdt = \frac{v}{r}\int_0^\infty [J_0(\beth\sqrt{xt})])e^{-(v/r)x}\mathrm{dx}\frac{s}{u}\int_0^\infty e^{-(s/u)t}dt\\ &= \frac{s}{u}\int_0^\infty e^{-\left(\beth^2r/4v\right)t}e^{-(s/ut)}dt.\end{aligned}$$

Thus, we get $\mathcal{R}_x\mathcal{R}_t[J_0(\beth\sqrt{xt})] = (4sv/4\mathrm{vs}+\beth^2\mathrm{ru})$.

$$\begin{aligned}\mathcal{R}_x\mathcal{R}_t\left[\mathrm{xte}^{x+t}\right] &= \frac{s}{u}\frac{v}{r}\int_0^\infty\int_0^\infty e^{-(v/rx+s/ut)}\left[\mathrm{xte}^{x+t}\right]dxdt\\ &= \frac{v}{r}\int_0^\infty\left[\mathrm{xe}^{x}\right]e^{-(\mathrm{v/r})x}dt\frac{s}{u}\int_0^\infty\left[\mathrm{te}^{t}\right]e^{-(s/u)t}dt.\end{aligned} \tag{24}$$

From equations (5) and (6), we get

$$\mathcal{R}_x\mathcal{R}_t\left[\mathrm{xte}^{x+t}\right] = \left(\frac{u^2}{s}\frac{d(s/s-u)}{\mathrm{du}}+\frac{u}{s}\left(\frac{s}{s-u}\right)\right)\left(\frac{r^2}{v}\frac{d(v/v-r)}{\mathrm{dr}}+\frac{r}{v}\left(\frac{v}{v-r}\right)\right) = \frac{\mathrm{suvr}}{(v-r)^2(s-u)^2}. \tag{25}$$

3.2. Existence Conditions for DFT. If $h(x,t)$ is a function of exponential orders α and β as $x \longrightarrow \infty$ and $t \longrightarrow \infty$, and if there exists a positive constant M such that $\forall x > X$ and $t > T$, we have

$$|h(x,t)| \le \mathrm{M}e^{\alpha x+\beta t}, \tag{26}$$

we can write $h(x,t) = O(e^{\alpha x+\beta t})$ as $x \longrightarrow \infty$ and $t \longrightarrow \infty$, $(v/r) > \alpha$ and $(s/u) > \beta$.

Theorem 1. *Let $h(x,t)$ be a continuous function on the region $[0,X)\times t[0,T)$ of exponential orders α and β. Then, $\mathscr{R}_x\mathscr{R}_t[h(x,t)]$ exists for (v/r) and (s/u) provided $Re(v/r)>\alpha$ and $Re(s/u)>\beta$.*

Proof. Using the definition of DFT, we get

$$\begin{aligned}\left|\mathscr{R}_x\mathscr{R}_t[h(x,t)]\right| &= \left|\frac{s}{u}\frac{v}{r}\int_0^\infty\int_0^\infty e^{-(v/rx+s/ut)}[h(x,t)]dxdt\right| \le \frac{s}{u}\frac{v}{r}\int_0^\infty\int_0^\infty e^{-(v/rx+s/ut)}|h(x,t)|dxdt\\ &\le M\frac{v}{r}\int_0^\infty e^{-(v/r-\alpha)x}dx\frac{s}{u}\int_0^\infty e^{-(s/u-\beta)t}dx\\ &= \frac{\mathrm{Mvs}}{(v-\alpha r)(s-\beta u)}, \mathrm{Re}\left(\frac{v}{r}\right)>\alpha\end{aligned} \tag{27}$$

$$Re\left(\frac{s}{u}\right)>\beta.$$

□

3.3. Some Basic Theorems of DFT

Theorem 2 (Shifting Property). *Let $h(x,t)$ be a continuous function, and $\mathscr{R}^2[h(x,t)]=H(v,r,s,u)$. Then,*

$$\mathscr{R}_x\mathscr{R}_t\left[e^{\alpha x+\beta t}h(x,t)\right] = \frac{\mathrm{sv}}{(v-\alpha r)(s-\beta u)}H\left(v,\frac{\mathrm{vr}}{v-\alpha r},s,\frac{\mathrm{su}}{s-\beta u}\right). \tag{28}$$

Proof. Using the definition in equation (14),

$$\begin{aligned}\mathscr{R}_x\mathscr{R}_t\left[e^{\alpha x+\beta t}h(x,t)\right] &= \mathrm{sv}\int_0^\infty\int_0^\infty e^{-(\mathrm{vx+st})}\left[e^{\alpha rx+\beta ut}h(\mathrm{rx},\mathrm{ut})\right]dxdt\\ &= \mathrm{sv}\int_0^\infty\int_0^\infty e^{-((v-\alpha r)x+(s-\beta u)t)}h(\mathrm{rx},\mathrm{ut})dxdt.\end{aligned} \tag{29}$$

By letting $(v-\alpha r)x=vy$, and $\mathrm{dx}=(v/v-\alpha r)\mathrm{dy}$ and $(s-\beta u)t=s\tau$, and $\mathrm{dt}=(s/s-\beta u)d\tau$ in (29), we have

$$\begin{aligned}\mathscr{R}_x\mathscr{R}_t\left[e^{\alpha x+\beta t}h(x,t)\right] &= \frac{\mathrm{sv}}{(v-\alpha r)(s-\beta u)}\int_0^\infty\int_0^\infty e^{-(vy+s\tau)}h\left(\frac{\mathrm{vr}}{v-\alpha r}y,\frac{\mathrm{su}}{s-\beta u}\tau\right)dyd\tau\\ &= \frac{\mathrm{sv}}{(v-\alpha r)(s-\beta u)}H\left(v,\frac{\mathrm{vr}}{v-\alpha r},s,\frac{\mathrm{su}}{s-\beta u}\right).\end{aligned} \tag{30}$$

□

Theorem 3 (Periodic function). *Let $\mathscr{R}_x\mathscr{R}_t[h(x,t)]$ be exists, where $h(x,t)$ is a periodic function of periods α and β such that,*

$$h(x+\alpha,t+\beta)=h(x,t),\forall x,y. \tag{31}$$

Then,

$$\mathscr{R}_x\mathscr{R}_t[h(x,t)] = \left(1-e^{-\left(\frac{v}{r}\beta+\frac{s}{u}\alpha\right)}\right)^{-1}\left(\frac{s}{u}\frac{v}{r}\int_0^\alpha\int_0^\beta e^{-\left(\frac{v}{r}x+\frac{s}{u}t\right)}[h(x,t)]\mathrm{dxdt}\right). \tag{32}$$

Table 1: DFT for some functions.

$h(x,t)$	$\mathscr{R}_x\mathscr{R}_t[h(x,t)]$
1	1
$x^{\alpha}t^{\beta}$	$(u/s)^{\alpha}(r/v)^{\beta}\Gamma(\alpha+1)\Gamma(\beta+1)$.
$e^{\alpha x+\beta t}$	$sv/(s-\alpha u)(v-\beta r)$
$\sin(\alpha x+\beta t)$	$sv(vu\beta+sr\alpha)/(s^2+\alpha^2u^2)(v^2+\beta^2r^2)$
$\cos(\alpha x+\beta t)$	$sv(sv-\alpha\beta ur)/(s^2+\alpha^2u^2)(v^2+\beta^2r^2)$
$\sinh(\alpha x+\beta t)$	$sv(vu\beta+sr\alpha)/(s^2-\alpha^2u^2)(v^2-\beta^2r^2)$
$\cosh(\alpha x+\beta t)$	$sv(sv+\alpha\beta ur)/(s^2-\alpha^2u^2)(v^2-\beta^2r^2)$
xte^{x+t}	$\mathrm{suvr}/(v-r)^2(s-u)^2$
$J_0(\beth\sqrt{xt})$	$4vs/4vs+\beth^2\mathrm{ru}$
$e^{\alpha x+\beta t}h(x,t)$	$sv/(v-\alpha r)(s-\beta u)H(v,\mathrm{vr}/v-\alpha r,s,su/s-\beta u)$
$h(x-\delta,t-\sigma)H(x-\delta,t-\sigma)$	$e^{-v/r\delta-s/u\sigma}\mathscr{R}_x\mathscr{R}_t[h(x,t)]$
$h(x,t)**k(x,t)$	$\mathrm{ur/sv}\mathscr{R}_x\mathscr{R}_t[h(x,t)]\mathscr{R}_x\mathscr{R}_t[k(x,t)]$
$\partial h(x,t)/\partial t$	$s/u\mathscr{R}_x\mathscr{R}_t[h(x,t)]-s/u\mathscr{R}_x[h(x,0)]$
$\partial h(x,t)/\partial x$	$v/r\mathscr{R}_x\mathscr{R}_t[h(x,t)]-v/r\mathscr{R}_t[h(0,t)]$
$\partial^2h(x,t)/\partial t^2$	$s^2/u^2\mathscr{R}_x\mathscr{R}_t[h(x,t)]-(s^2/u^2)\mathscr{R}_x[h(x,0)]-(s/u)\mathscr{R}_x[\partial h(x,0)/\partial t]$
$\partial^2h(x,t)/\partial x^2$	$v^2/r^2\mathscr{R}_x\mathscr{R}_t[h(x,t)]-v^2/r^2\mathscr{R}_t[h(0,t)]-v/r\mathscr{R}_t[\partial h(0,t)/\partial x]$
$\partial h(x,t)/\partial x\,\partial t$	$vs/ru(\mathscr{R}_x\mathscr{R}_t[h(x,t)]-\mathscr{R}_x[h(x,0)]-\mathscr{R}_t[h(0,t)]+h(0,0))$

Proof. Using the definition of DFT, we get

$$\mathscr{R}_x\mathscr{R}_t[h(x,t)]=\frac{s}{u}\frac{v}{r}\int_0^{\infty}\int_0^{\infty}e^{-\left(\frac{v}{r}x+\frac{s}{u}t\right)}[h(x,t)]\mathrm{dxdt}. \tag{33}$$

Using the property of improper integral, equation (33) can be written as

$$\mathscr{R}_x\mathscr{R}_t[h(x,t)]=\frac{s}{u}\frac{v}{r}\int_0^{\alpha}\int_0^{\beta}e^{-((v/r)x+(s/u)t)}[h(x,t)]\mathrm{d}x\mathrm{d}t+\frac{s}{u}\frac{v}{r}\int_{\alpha}^{\infty}\int_{\beta}^{\infty}e^{-((v/r)x+(s/u)t)}[h(x,t)]\mathrm{d}x\mathrm{d}t. \tag{34}$$

Putting $x=\rho+\beta$ and $t=\tau+\alpha$ on the second integral in equation (34). We obtain

$$\mathscr{R}_x\mathscr{R}_t[h(x,t)]=\frac{s}{u}\frac{v}{r}\int_0^{\alpha}\int_0^{\beta}e^{-\left(\frac{v}{r}x+\frac{s}{u}t\right)}[h(x,t)]\mathrm{dxdt}+\frac{s}{u}\frac{v}{r}\int_{\alpha}^{\infty}\int_{\beta}^{\infty}e^{-\left(\frac{v}{r}(\rho+\beta)+\frac{s}{u}(\tau+\alpha)\right)}[h(\rho+\beta,\tau+\alpha)]d\rho d\tau. \tag{35}$$

Using the periodicity of the function $h(x,t)$, equation (35) can be written by

$$\mathscr{R}_x\mathscr{R}_t[h(x,t)]=\frac{s}{u}\frac{v}{r}\int_0^{\alpha}\int_0^{\beta}e^{-\left(\frac{v}{r}x+\frac{s}{u}t\right)}h(x,t)\mathrm{dxdt}+e^{-\left(\frac{v}{r}\beta+\frac{s}{u}\alpha\right)}\frac{s}{u}\frac{v}{r}\int_0^{\infty}\int_0^{\infty}e^{-\left(\frac{v}{r}\rho+\frac{s}{u}\tau\right)}h(\rho,\tau)\mathrm{d}\rho\mathrm{d}\tau. \tag{36}$$

Using the definition of DFT, we get

$$\mathscr{R}_x\mathscr{R}_t[h(x,t)]=\frac{s}{u}\frac{v}{r}\int_0^{\alpha}\int_0^{\beta}e^{-\left(\frac{v}{r}x+\frac{s}{u}t\right)}[h(x,t)]\mathrm{dxdt}+e^{-\left(\frac{v}{r}\beta+\frac{s}{u}\alpha\right)}\mathscr{R}_x[\mathscr{R}_t[h(x,t)]]. \tag{37}$$

Thus, equation (37) can be simplified into

$$\mathcal{R}_x\mathcal{R}_t[h(x,t)] = \frac{1}{\left(1-e^{-(v/r\beta+s/u\alpha)}\right)}\left(\frac{s}{u}\frac{v}{r}\int_0^{\alpha}\int_0^{\beta} e^{-\left(\frac{v}{r}x+\frac{s}{u}t\right)}[h(x,t)]\mathrm{d}x\mathrm{d}t\right). \quad (38)$$

□

Theorem 4 (Heaviside function). *Let $\mathcal{R}_x\mathcal{R}_t[h(x,t)]$ exists, then*

$$\mathcal{R}_x\mathcal{R}_t[h(x-\delta,t-\sigma)H(x-\delta,t-\sigma)] = e^{-\frac{v}{r}\delta-\frac{s}{u}\sigma}\mathcal{R}_x\mathcal{R}_t[h(x,t)], \quad (39)$$

where $H(x-\delta,t-\sigma)$ is the Heaviside unit step function defined as

$$H(x-\delta,t-\sigma) = \begin{cases} 1, & x>\delta, t>\sigma, \\ 0, & \text{Ohtherwise.} \end{cases} \quad (40)$$

Proof. Using the definition of DFT, we get

$$\begin{aligned}\mathcal{R}_x\mathcal{R}_t[h(x-\delta,t-\sigma)H(x-\delta,t-\sigma)] &= \frac{s}{u}\frac{v}{r}\int_0^{\infty}\int_0^{\infty} e^{-\left(\frac{v}{r}x+\frac{s}{u}t\right)}[h(x-\delta,t-\sigma)H(x-\delta,t-\sigma)]\mathrm{d}x\mathrm{d}t \\ &= \frac{s}{u}\frac{v}{r}\int_0^{\infty}\int_0^{\infty} e^{-\left(\frac{v}{r}x+\frac{s}{u}t\right)}[h(x-\delta,t-\sigma)]\mathrm{d}x\mathrm{d}t\end{aligned} \quad (41)$$

Putting $x-\delta=\rho$ and $t-\sigma=\tau$ in equation (41). We obtain

$$\mathcal{R}_x\mathcal{R}_t[h(x-\delta,t-\sigma)H(x-\delta,t-\sigma)] = \frac{s}{u}\frac{v}{r}\int_0^{\infty}\int_0^{\infty} e^{-\left(\frac{v}{r}(\rho+\delta)+\frac{s}{u}(\tau+\sigma)\right)}[h(\rho,\tau)]\mathrm{d}\rho\mathrm{d}\tau. \quad (42)$$

Thus, equation (42) can be simplified into

$$\begin{aligned}\mathcal{R}_x\mathcal{R}_t[h(x-\delta,t-\sigma)H(x-\delta,t-\sigma)] &= e^{-\frac{v}{r}\delta-\frac{s}{u}\sigma}\left(\frac{s}{u}\frac{v}{r}\int_0^{\infty}\int_0^{\infty} e^{-\frac{v}{r}\rho-\frac{s}{u}\tau}[h(\rho,\tau)]d\rho\, d\tau\right) \\ &= e^{-\frac{v}{r}\delta-\frac{s}{u}\sigma}\mathcal{R}_x\mathcal{R}_t[h(x,t)].\end{aligned} \quad (43)$$

□

Theorem 5 (Convolution theorem). *Let $\mathcal{R}_x\mathcal{R}_t[h(x,t)]$ and $\mathcal{R}_x\mathcal{R}_t[k(x,t)]$ are exists, then*

$$\mathcal{R}_x\mathcal{R}_t[h(x,t)**k(x,t)] = \frac{\mathrm{ur}}{\mathrm{sv}}\left(\mathcal{R}_x\mathcal{R}_t[h(x,t)]\mathcal{R}_x\mathcal{R}_t[k(x,t)]\right), \quad (44)$$

where $h(x,t)**k(x,t) = \int_0^x\int_0^t h(x-\rho,t-\tau)k(\rho,\tau)d\rho d\tau$

and the symbol $**$ denotes the double convolution with respect to x and t.

Proof. Using the definition of DFT, we get

$$\mathscr{R}_x\mathscr{R}_t[h(x,t)**k(x,t)] = \frac{s}{u}\frac{v}{r}\int_0^{\infty}\int_0^{\infty} e^{-\left(\frac{v}{r}x+\frac{s}{u}t\right)}\left[\int_0^{x}\int_0^{t} h(x-\rho,t-\tau)k(\rho,\tau)d\rho d\tau\right]dxdt. \tag{45}$$

Using the Heaviside unit step function, equation (45) can be written as

$$\mathscr{R}_x\mathscr{R}_t[h(x,t)**k(x,t)] = \frac{s}{u}\frac{v}{r}\int_0^{\infty}\int_0^{\infty} e^{-\left(\frac{v}{r}x+\frac{s}{u}t\right)}\left[\int_0^{\infty}\int_0^{\infty} h(x-\rho,t-\tau)H(x-\rho,t-\tau)k(\rho,\tau)d\rho d\tau\right]dxdt. \tag{46}$$

Thus, equation (46) can be written as

$$\begin{aligned}\mathscr{R}_x\mathscr{R}_t[h(x,t)**k(x,t)] &= \int_0^{\infty}\int_0^{\infty} k(\rho,\tau)d\rho d\tau\left[\frac{s}{u}\frac{v}{r}\int_0^{\infty}\int_0^{\infty} e^{-\left(\frac{v}{r}x+\frac{s}{u}t\right)}h(x-\rho,t-\tau)H(x-\rho,t-\tau)dxdt\right]\\ &= \int_0^{\infty}\int_0^{\infty} k(\rho,\tau)d\rho d\tau\left(e^{-\frac{v}{r}\rho-\frac{s}{u}\tau}\mathscr{R}_x\mathscr{R}_t[h(x,t)]\right)\\ &= \mathscr{R}_x\mathscr{R}_t[h(x,t)]\int_0^{\infty}\int_0^{\infty} e^{\left(-\frac{v}{r}\rho-\frac{s}{u}\tau\right)}k(\rho,\tau)d\rho d\tau = \frac{ur}{sv}\mathscr{R}_x\mathscr{R}_t[h(x,t)]\mathscr{R}_x\mathscr{R}_t[k(x,t)].\end{aligned} \tag{47}$$

□

Theorem 6 (Derivatives properties). *Let $h(x,t)$ be a continuous function. Then, we get the following derivatives properties:*

(a) $\mathscr{R}_x\mathscr{R}_t[\partial h(x,t)/\partial t] = s/u\mathscr{R}_x\mathscr{R}_t[h(x,t)] - s/u\mathscr{R}_x[h(x,0)]$.

(b) $\mathscr{R}_x\mathscr{R}_t[\partial h(x,t)/\partial x] = v/r\mathscr{R}_x\mathscr{R}_t[h(x,t)] - v/r\mathscr{R}_t[h(0,t)]$.

(c) $\mathscr{R}_x\mathscr{R}_t[\partial^2 h(x,t)/\partial t^2] = s^2/u^2\mathscr{R}_x\mathscr{R}_t[h(x,t)] - s^2/u^2\mathscr{R}_x[h(x,0)] - s/u\mathscr{R}_x[\partial h(x,0)/\partial t]$.

(d) $\mathscr{R}_x\mathscr{R}_t[\partial^2 h(x,t)/\partial x^2] = v^2/r^2\mathscr{R}_x\mathscr{R}_t[h(x,t)] - v^2/r^2\mathscr{R}_t[h(0,t)] - v/r\mathscr{R}_t[\partial h(0,t)/\partial x]$

$$_{(e)}\mathscr{R}_x\mathscr{R}_t\left[\frac{\partial^2 h(x,t)}{\partial x\,\partial t}\right] = \frac{vs}{ru}\left(\mathscr{R}_x\mathscr{R}_t[h(x,t)] - \mathscr{R}_x[h(x,0)] - \mathscr{R}_t[h(0,t)] + h(0,0)\right). \tag{48}$$

We only present the proof of (a), (c), and (e), the proof of (b) and (d) are the same.

Proof

(a) $\mathscr{R}_x\mathscr{R}_t[\partial h(x,t)/\partial t] = s/uv/r\int_0^{\infty}\int_0^{\infty} e^{-(v/rx+s/ut)}[\partial h(x,t)/\partial t]dxdt = v/r\int_0^{\infty} e^{-v/rx}dxs/u\int_0^{\infty} e^{-s/ut}[\partial h(x,t)/\partial t]dt$

Using the integration by part, we obtain.

Let $u = e^{-(s/u)t} \Rightarrow du = -(s/u)e^{-(s/u)t}dt$, $dv = (\partial h(x,t)/\partial t)dt \Rightarrow v = h(x,t)$.

Thus, $s/u\int_0^{\infty} e^{-(s/u)t}[\partial h(x,t)/\partial t]dt = (s/u)(-h(x,0) + (s/u)\int_0^{\infty} e^{-(s/u)t}[h(x,t)]dt)$,

So, we get,

$$\mathscr{R}_x\mathscr{R}_t\left[\frac{\partial h(x,t)}{\partial t}\right] = \frac{s}{u}\mathscr{R}_x\mathscr{R}_t[h(x,t)] - \frac{s}{u}\mathscr{R}_x[h(x,0)]. \tag{49}$$

□

Proof

(c). $\mathscr{R}_x\mathscr{R}_t[\partial^2 h(x,t)/\partial t^2] = s/uv/r\int_0^{\infty}\int_0^{\infty} e^{-(v/rx+s/ut)}[\partial^2 h(x,t)/\partial t^2]dxdt = v/r\int_0^{\infty} e^{-v/rx}dxs/u\int_0^{\infty} e^{-s/ut}[\partial^2 h(x,t)/\partial t^2]dt$.

Using the integration by part, we obtain.
Let $u = e^{-(s/u)}\ t \Rightarrow du = -(s/u)e^{-(s/u)t}dt$, $dv = \partial^2 h(x,t)/\partial t^2 dt \Rightarrow v = \partial h(x,t)/\partial t$.

Thus, $s/u \int_0^\infty e^{-s/ut}[\partial^2 h(x,t)/\partial t^2]dt = s/u(-\partial h(x,0)/\partial t + s/u \int_0^\infty e^{-s/ust}[\partial h(x,t)/\partial t]dt)$.
Using equation (49), we have

$$\mathscr{R}_x\mathscr{R}_t\left[\frac{\partial^2 h(x,t)}{\partial t^2}\right] = \frac{s^2}{u^2}\mathscr{R}_x\mathscr{R}_t[h(x,t)] - \frac{s^2}{u^2}\mathscr{R}_x[h(x,0)] - \frac{s}{u}\mathscr{R}_x\left[\frac{\partial h(x,0)}{\partial t}\right]. \tag{50}$$

proof of (e) by similar (a) and (c) arguments, one can get

$$\mathscr{R}_x\mathscr{R}_t\left[\frac{\partial^2 h(x,t)}{\partial x\,\partial t}\right] = \frac{s}{u}\frac{v}{r}\int_0^\infty\int_0^\infty e^{-\left(\frac{v}{r}x + \frac{s}{u}t\right)}\left[\frac{\partial^2 h(x,t)}{\partial x\,\partial t}\right]dxdt = \frac{v}{r}\int_0^\infty e^{-\frac{v}{r}x}\left[\frac{\partial^2 h(x,t)}{\partial x\,\partial t}\right]dx\frac{s}{u}\int_0^\infty e^{-\frac{s}{u}t}dt. \tag{51}$$

Using the integration by part, we obtain

$$\frac{v}{r}\int_0^\infty e^{-\frac{v}{r}x}\left[\frac{\partial^2 h(x,t)}{\partial x\,\partial t}\right]dx = \left(-\frac{v}{r}\int_0^\infty e^{-\frac{v}{r}x}\left[\frac{\partial h(0,t)}{\partial t}\right]dt + \left(\frac{v}{r}\right)^2\frac{s}{u}\int_0^\infty\int_0^\infty e^{-\left(\frac{v}{r}x + \frac{s}{u}t\right)}\left[\frac{\partial h(x,t)}{\partial t}\right]dxdt\right). \tag{52}$$

Using equations (11) and (49), we have

$$\mathscr{R}_x\mathscr{R}_t\left[\frac{\partial^2 h(x,t)}{\partial x\,\partial t}\right] = \frac{\mathrm{vs}}{\mathrm{ru}}\left(\mathscr{R}_x\mathscr{R}_t[h(x,t)] - \mathscr{R}_x[h(x,0)] - \mathscr{R}_t[h(0,t)] + h(0,0)\right). \tag{53}$$

We summarize the previous results in the following table below Table 1: □

4. Presentation of DFT Method in Solving Integral Equations and Applications

To show the applicability and strength of DFT in solving integral equations, we use it to solve some kinds of integral equations and illustrate the method by solving several examples.

4.1. Volterra Integral Equation. Consider the following Volterra-integral equation:

$$h(x,t) = f(x,t) + \mu\int_0^x\int_0^t h(x-\rho, t-\tau)w(\rho,\tau)d\rho d\tau, \tag{54}$$

where $h(x,t)$ is the unknown function, $f(x,t)$ and $w(x,t)$ are two known functions and μ is a constant. The main idea of the method is to apply the DFT to equation (54) as the following:

$$\mathscr{R}_x\mathscr{R}_t[h(x,t)] = \mathscr{R}_x\mathscr{R}_t\left[f(x,t) + \mu\int_0^x\int_0^t h(x-\rho, t-\tau)w(\rho,\tau)d\rho d\tau\right]. \tag{55}$$

Now, using the differentiation property of the DFT on equation (55) and Theorem 5, we have

$$\mathscr{R}_x\mathscr{R}_t[h(x,t)] = F(v,r,s,u) + \mu\left(\frac{\text{ru}}{\text{sv}}\mathscr{R}_x\mathscr{R}_t[h(x,t)]\mathscr{R}_x\mathscr{R}_t[w(x,t)]\right). \quad (56)$$

Equation (56) can be simplified as follows:

$$\mathscr{R}_x\mathscr{R}_t[h(x,t)] = F(v,r,s,u)\left(1-\mu\frac{\text{ru}}{\text{sv}}\mathscr{R}_x\mathscr{R}_t[w(x,t)]\right)^{-1}, \quad (57)$$

Operating with the inverse of DFT on both sides of equation (57) gives

$$h(x,t) = \mathscr{R}_x^{-1}\mathscr{R}_t^{-1}\left[F(v,r,s,u)\left(1-\mu\frac{\text{ru}}{\text{sv}}\mathscr{R}_x\mathscr{R}_t[w(x,t)]\right)^{-1}\right], \quad (58)$$

where $h(x,t)$ represents the term arising from the known functions $f(x,t)$ and $w(x,t)$.

Example 1. Let us consider the following Volterra integral equation:

$$h(x,t) = \alpha - \mu\int_0^x\int_0^t h(\rho,\tau)d\rho d\tau. \quad (59)$$

By substituting the values of functions $F = \mathscr{R}_x\mathscr{R}_t[\alpha] = \alpha, W = \mathscr{R}_x\mathscr{R}_t[-1] = -1$ in the general formula in equation (58), we obtain

$$\mathscr{R}_x\mathscr{R}_t[h(x,t)] = \alpha\left(1+\mu\frac{\text{ru}}{\text{sv}}\right)^{-1} = \frac{\alpha\text{sv}}{sv+\mu\text{ru}}. \quad (60)$$

Now, applying inverse DFT to equation (60), then the solution of equation (59) is given by

$$h(x,t) = \mathscr{R}_x^{-1}\mathscr{R}_t^{-1}\left[\frac{\alpha\text{sv}}{sv+\mu\text{ru}}\right] = \mathscr{R}_x^{-1}\mathscr{R}_t^{-1}\left[\frac{4\alpha\text{sv}}{4\text{sv}+(2\sqrt{\mu})^2\text{ru}}\right] = \alpha J_0(2\sqrt{\mu\text{xt}}). \quad (61)$$

Example 2. Let us consider the following Volterra integral equation:

$$4t = \int_0^x\int_0^t h(x-\rho,t-\tau)h(\rho,\tau)d\rho d\tau. \quad (62)$$

Appling the DFT to equation (62), we have

$$\frac{4u}{s} = \frac{\text{ru}}{\text{sv}}\mathscr{R}_x\mathscr{R}_t[h(x,t)]\mathscr{R}_x\mathscr{R}_t[h(x,t)] = \frac{\text{ru}}{\text{sv}}(\mathscr{R}_x\mathscr{R}_t[h(x,t)])^2. \quad (63)$$

Equation (63) can be simplified as follows:

$$\mathscr{R}_x\mathscr{R}_t[h(x,t)] = 2\sqrt{\frac{v}{r}}. \quad (64)$$

Taking the inverse DFT to equation (64), we get the solution of equation (62) as follows:

$$h(x,t) = \mathscr{R}_x^{-1}\mathscr{R}_t^{-1}\left[2\sqrt{\frac{v}{r}}\right] = \frac{2}{\sqrt{\pi x}}. \quad (65)$$

Example 3. Let us consider the following Volterra integral equation:

$$xe^x\left(e^{-t}-1\right) = \int_0^x\int_0^t e^{\rho-\tau}h(x-\rho,t-\tau)d\rho d\tau. \quad (66)$$

Appling the DFT to equation (66), we have

$$\left(\frac{r^2}{v}\frac{d}{\text{dr}}\left(\frac{v}{v-r}\right)+\frac{r}{v}\left(\frac{v}{v-r}\right)\right)\left[\frac{s}{s+u}-1\right] = \frac{\text{ru}}{\text{sv}}\left[\frac{\text{sv}\mathscr{R}_x\mathscr{R}_t[h(x,t)]}{(v-r)(s+u)}\right]. \quad (67)$$

Equation (67) can be simplified as follows:

$$\mathcal{R}_x\mathcal{R}_t[h(x,t)] = \left(\frac{\mathrm{rv}}{(v-r)^2}\right)\left[\frac{-u}{s+u}\right]\left(\frac{(v-r)(s+u)}{ru}\right) = -\frac{v}{v-r}. \tag{68}$$

Applying the inverse DFT to equation (68), we get the solution of equation (66), we get

$$h(x,t) = \mathcal{R}_x^{-1}\mathcal{R}_t^{-1}\left[-\frac{v}{v-r}\right] = -e^x. \tag{69}$$

4.2. Volterra Integro-Partial Differential Equations. Consider the following Volterra integro-partial differential equations:

$$\frac{\partial h(x,t)}{\partial x} + \frac{\partial h(x,t)}{\partial t} = f(x,t) + \mu\int_0^x\int_0^t h(x-\rho,t-\tau)w(\rho,\tau)d\rho d\tau, \tag{70}$$

with the conditions

$$h(x,0) = k(x),\ h(0,t) = g(t), \tag{71}$$

where $h(x,t)$ is the unknown function, $f(x,t)$ and $w(x,t)$ are two known functions, and μ is a constant.

To get the solution of equation (70), we firstly apply the single formable transform to all conditions in equation (71) and the DFT to equation(70), as follows:

$$\begin{aligned}\mathcal{R}_x[h(x,0)] &= \mathcal{R}_x[k(x)] = K,\\ \mathcal{R}_t[h(0,t)] &= \mathcal{R}_t[g(t)] = G.\end{aligned} \tag{72}$$

Thus, we have

$$\mathcal{R}_x\mathcal{R}_t\left[\frac{\partial h(x,t)}{\partial x} + \frac{\partial h(x,t)}{\partial t}\right] = \mathcal{R}_x\mathcal{R}_t\left[f(x,t) + \mu\int_0^x\int_0^t h(x-\rho,t-\tau)w(\rho,\tau)d\rho d\tau\right]. \tag{73}$$

Using the differentiation property of the DFT to equation (73) and the double convolution theorem, we have

$$\begin{aligned}&\frac{v}{r}\mathcal{R}_x\mathcal{R}_t[h(x,t)] - \frac{v}{r}G + \frac{s}{u}\mathcal{R}_x\mathcal{R}_t[h(x,t)] - \frac{s}{u}K\\ &= F(v,r,s,u) + \mu\frac{\mathrm{ru}}{\mathrm{sv}}\mathcal{R}_x\mathcal{R}_t[h(x,t)]\mathcal{R}_x\mathcal{R}_t[w(x,t)].\end{aligned} \tag{74}$$

Equation (44) can be simplified as follows:

$$\mathcal{R}_x\mathcal{R}_t[h(x,t)] = \left(F(v,r,s,u) + \frac{v}{r}G + \frac{s}{u}K\right)\left(\frac{v}{r} + \frac{s}{u} - \mu\frac{\mathrm{ur}}{\mathrm{sv}}\mathcal{R}_x\mathcal{R}_t[w(x,t)]\right)^{-1}. \tag{75}$$

Operating the inverse of DFT to both sides of equation (75), we get

$$h(x,t) = \mathcal{R}_x^{-1}\mathcal{R}_t^{-1}\left[\left(F + \frac{v}{r}G + \frac{s}{u}K\right)\left(\frac{v}{r} + \frac{s}{u} - \mu\frac{\mathrm{ur}}{\mathrm{sv}}\mathcal{R}_x\mathcal{R}_t[w(x,t)]\right)^{-1}\right], \tag{76}$$

where $h(x,t)$ represents the term arising from the known functions $f(x,t), w(x,t), k(x)$, and $g(t)$.

Example 4. Let us consider the following Volterra integro-partial differential equations:

$$\frac{\partial h(x,t)}{\partial x}+\frac{\partial h(x,t)}{\partial t}=-1+e^{x}+e^{t}+e^{x+t}+\int_{0}^{x}\int_{0}^{t}h(x-\rho,t-\tau)d\rho d\tau. \tag{77}$$

subject to the conditions

$$h(x,0)=e^{x}, h(0,t)=e^{t}. \tag{78}$$

By substituting the values of functions

$$\begin{aligned} K &= \mathscr{R}_x[e^x]=\frac{v}{v-r},\\ G &= \mathscr{R}_t\left[e^t\right]=\frac{s}{s-u}, \mathscr{R}_x\mathscr{R}_t[w(x,t)]=\mathscr{R}_x\mathscr{R}_t[1]=1,\\ F &= \mathscr{R}_x\mathscr{R}_t\left[-1+e^x+e^t+e^{x+t}\right]=-1+\frac{v}{v-r}+\frac{s}{s-u}+\frac{\mathrm{sv}}{(s-u)(v-r)}, \end{aligned} \tag{79}$$

in the general formula in equation (76), we obtain

$$\mathscr{R}_x\mathscr{R}_t[h(x,t)]\left(\frac{v}{r}+\frac{s}{u}-\frac{\mathrm{ur}}{\mathrm{sv}}\right)=-1+\frac{v}{v-r}+\frac{s}{s-u}+\frac{\mathrm{sv}}{(s-u)(v-r)}+\frac{v}{r}\left(\frac{s}{s-u}\right)+\frac{s}{u}\left(\frac{v}{v-r}\right). \tag{80}$$

Equation (80) can be simplified as follows:

$$\begin{aligned} \mathscr{R}_x\mathscr{R}_t[h(x,t)] &= \left[\frac{r}{v-r}+\frac{s}{s-u}+\frac{sv}{(s-u)(v-r)}+\frac{v}{r}\left(\frac{s}{s-u}\right)+\frac{s}{u}\left(\frac{v}{v-r}\right)\right]\left(\frac{v}{r}+\frac{s}{u}-\frac{ur}{sv}\right)^{-1}\\ &= \mathrm{suvr}\left[\frac{-(ur/sv)+(v/r)+(s/u)}{ur(s-u)(v-r)}\right]\left(\frac{v}{r}+\frac{s}{u}-\frac{ur}{sv}\right)^{-1}. \end{aligned} \tag{81}$$

Thus,

$$\mathscr{R}_x\mathscr{R}_t[h(x,t)]=\frac{sv}{(s-u)(v-r)}. \tag{82}$$

Taking the inverse of DFT to equation (82), then the solution of equation (77) is

$$h(x,t)=\mathscr{R}_x^{-1}\mathscr{R}_t^{-1}\left[\frac{\mathrm{sv}}{(s-u)(v-r)}\right]=e^{x+t}. \tag{83}$$

4.3. Integro-Partial Differential Equations. Consider the following integro-partial differential equation:

$$\frac{\partial^2 h(x,t)}{\partial t^2}-\frac{\partial^2 h(x,t)}{\partial x^2}+h(x,t)+\int_{0}^{x}\int_{0}^{t}w(x-\rho,t-\tau)h(\rho,\tau)d\rho d\tau=f(x,t), \tag{84}$$

with the conditions

$$\begin{aligned} h(x,0) &= k_1(x),\\ \frac{\partial h(x,0)}{\partial t} &= k_2(x),\\ h(0,t) &= g_1(t),\\ \frac{\partial h(0,t)}{\partial x} &= g_2(t), \end{aligned} \tag{85}$$

where $h(x,t)$ is unknown function, $f(x,t)$ and $w(x,t)$ are two known functions.

The main idea of this method is to apply formable transform to all conditions in equation (85) and the DFT to equation (84), as the following:

The single formable transform to conditions in equation (85), we get

$$\begin{aligned} &\mathcal{R}_x[h(x,0)] = \mathcal{R}_x[k_1(x)] = K_1, \quad \mathcal{R}_x\left[\frac{\partial h(x,0)}{\partial t}\right] = \mathcal{R}_x[k_2(x)] = K_2,\\ &\mathcal{R}_t[h(0,t)] = \mathcal{R}_t[g_1(t)] = G_1, \quad \mathcal{R}_t\left[\frac{\partial h(0,t)}{\partial x}\right] = \mathcal{R}_t[g_2(t)] = G_2. \end{aligned} \tag{86}$$

The DFT on both sides of equation (84), to get

$$\mathcal{R}_x\mathcal{R}_t\left[\frac{\partial^2 h(x,t)}{\partial t^2} - \frac{\partial^2 h(x,t)}{\partial x^2} + h(x,t) + \int_0^x\int_0^t w(x-\rho,t-\tau)h(\rho,\tau)d\rho\, d\tau\right] = \mathcal{R}_x\mathcal{R}_t[f(x,t)]. \tag{87}$$

Using the differentiation property of the DFT of equation (87), above conditions and Theorem 5, we have

$$\begin{aligned} &\left[\frac{s^2}{u^2}\mathcal{R}_x\mathcal{R}_t[h(x,t)] - \frac{s^2}{u^2}K_1 - \frac{s}{u}K_2\right] - \left[\frac{v^2}{r^2}\mathcal{R}_x\mathcal{R}_t[h(x,t)] - \frac{v^2}{r^2}G_1 - \frac{v}{r}G_2\right]\\ &\quad + \mathcal{R}_x\mathcal{R}_t[h(x,t)] + \frac{\mathrm{ru}}{\mathrm{sv}}\mathcal{R}_x\mathcal{R}_t[h(x,t)]\mathcal{R}_x\mathcal{R}_t[w(x,t)] = F(v,r,s,u). \end{aligned} \tag{88}$$

Equation (88) can be simplifying as follows:

$$\mathcal{R}_x\mathcal{R}_t[h(x,t)] = \left(F(v,r,s,u) + \frac{s^2}{u^2}K_1 + \frac{s}{u}K_2 - \frac{v^2}{r^2}G_1 - \frac{v}{r}G_2\right)\left(\frac{1}{\left(s^2/u^2\right) - \left(v^2/r^2\right) + 1 + (\mathrm{ru}/\mathrm{sv})\mathcal{R}_x\mathcal{R}_t[w(x,t)]}\right). \tag{89}$$

Operating with the inverse of DFT on both sides of equation (89), gives

$$h(x,t) = \mathcal{R}_x^{-1}\mathcal{R}_t^{-1}\left[\left(F + \frac{s^2}{u^2}K_1 + \frac{s}{u}K_2 - \frac{v^2}{r^2}G_1 - \frac{v}{r}G_2\right)\left(\frac{1}{\left(s^2/u^2\right) - \left(v^2/r^2\right) + 1 + (\mathrm{ru}/\mathrm{sv})\mathcal{R}_x\mathcal{R}_t[w(x,t)]}\right)\right], \tag{90}$$

where $h(x,t)$ represents the term arising from the known functions $f(x,t), k_1(x), g_1(t), k_2(x), g_2(t)$, and $w(x,t)$.

Example 5. Let us consider the following integro-partial differential equation:

$$\frac{\partial^2 h(x,t)}{\partial t^2} - \frac{\partial^2 h(x,t)}{\partial x^2} + h(x,t) + \int_0^X \int_0^t e^{x-\rho+t-\tau} h(\rho,\tau) d\rho d\tau = e^{x+t} + xte^{x+t}, \tag{91}$$

with the conditions

$$\begin{aligned} h(x,0) &= \frac{\partial h(x,0)}{\partial t} = e^x, \\ h(0,t) &= \frac{\partial h(0,t)}{\partial x} = e^t. \end{aligned} \tag{92}$$

By substituting the values of functions

$$\begin{aligned} K_1 &= K_2 = \mathscr{R}_x\left[e^x\right] = \frac{v}{v-r}, \\ G_1 &= G_2 = \mathscr{R}_t\left[e^t\right] = \frac{s}{s-u}, \\ F &= \mathscr{R}_x\mathscr{R}_t\left[e^{x+t} + \mathrm{xte}^{x+t}\right] = \frac{\mathrm{sv}}{(v-r)(s-u)} + \frac{\mathrm{suvr}}{(v-r)^2(s-u)^2}, \end{aligned} \tag{93}$$

$\mathscr{R}_x\mathscr{R}_t[w(x,t)] = \mathscr{R}_x\mathscr{R}_t[e^{x+t}] = (\mathrm{sv}/(v-r)(s-u))$, in the general formula in equation (89), we obtain

$$\mathscr{R}_x\mathscr{R}_t[h(x,t)] = \frac{\mathrm{sv}}{(s-u)(v-r)}. \tag{94}$$

Taking the inverse DFT to equation (94), then the solution of equation (91) is

$$h(x,t) = \mathscr{R}_x^{-1}\mathscr{R}_t^{-1}\left[\frac{\mathrm{sv}}{(v-r)(s-u)}\right] = e^{x+t}. \tag{95}$$

5. Conclusions

Different phenomena are modeled using partial integro-differential equations in sciences, engineering, and social sciences. Solving partial integral equations using the double transform technique is effective. Convolution kernels are used to solve similar problems. In this study, we use the DFT technique to solve integral equations of different kinds. We present the definition of DFT and prove some basic properties and theorems related to its existence and derivatives. Finally, we apply DFT to solve partial integral equations and partial integro-differential equations. The outcomes show the simplicity and applicability of DFT in solving such problems.

Data Availability

No underlying data was collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] I. Kusmaryono, "The importance of mathematical power in mathematics learning," in *Proceedings of the International Conference on Mathematics*, pp. 35–40, Science, and Education, semarang state university, indonesia, September 2014.

[2] M. Abu Ghuwaleh, R. Saadeh, and A. Burqan, "New theorems in solving families of improper integrals," *Axioms*, vol. 11, no. 7, p. 301, 2022 Jun 21.

[3] R. Saadeh, M. Abu-Ghuwaleh, A. Qazza, and E. Kuffi, "A fundamental criteria to establish general formulas of integrals," *Journal of Applied Mathematics*, vol. 2022, Article ID 6049367, 16 pages, 2022.

[4] S. A. Davydov, A. V. Zemskov, and E. R. Akhmetova, "Thermoelastic diffusion multicomponent half-space under the effect of surface and Bulk unsteady perturbations," *Mathematical and Computational Applications*, vol. 24, no. 1, p. 26, 2019.

[5] H. Matevossian, "On the mixed Dirichlet–Steklov-type and Steklov-type biharmonic problems in weighted spaces," *Mathematical and Computational Applications*, vol. 24, no. 1, p. 25, 2019.

[6] A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics*, Courier Corporation, þNorth Chelmsford, Chelmsford, Massachusetts, United States, 2013.

[7] W. M. Elsasser, "The hydromagnetic equations," *Physical Review*, vol. 79, no. 1, p. 183, 1950.

[8] H. A. Agwa, F. M. Ali, and A. Kılıçman, "A new integral transform on time scales and its applications," *Advances in Difference Equations*, vol. 2012, Article ID 2012-60, 14 pages, 2012.

[9] D. Albayrak, S. D. Purohit, and U. Ç. A. R. Faruk, "Certain inversion and representation formulas for q-Sumudu transforms," *Hacettepe Journal of Mathematics and Statistics*, vol. 43, no. 5, pp. 699–713, 2014.

[10] X. Yang, Y. Yang, C. Cattani, and M. Zhu, "A new technique for solving the 1-D Burgers equation," *Thermal Science*, vol. 21, no. 1, pp. 129–136, 2017.

[11] S. A. Ahmed, T. M. Elzaki, M. Elbadri, and M. Z. Mohamed, "Solution of partial differential equations by new double integral transform (Laplace-Sumudu transform)," *Ain Shams Engineering Journal*, vol. 12, no. 4, pp. 4045–4049, 2021.

[12] S. A. Ahmed, A. Qazza, and R. Saadeh, "Exact solutions of nonlinear partial differential equations via the new double integral transform combined with iterative method," *Axioms*, vol. 11, no. 6, p. 247, 2022.

[13] A. Atangana and B. S. T. Alkahtani, "A novel double integral transform and its applications," *The Journal of Nonlinear Science and Applications*, vol. 09, no. 02, pp. 424–434, 2016.

[14] M. Meddahi, H. Jafari, and X. J. Yang, "Towards new general double integral transform and its applications to differential equations," *Mathematical Methods in the Applied Sciences*, vol. 45, no. 4, pp. 1916–1933, 2022.

[15] A. M. O. Anwar, F. Jarad, D. Baleanu, and F. Ayaz, "Fractional caputo heat equation within the double Laplace transform," *Romanian Journal of Physics*, vol. 58, 2013.

[16] L. Debnath, "The double Laplace transforms and their properties with applications to functional, integral and partial differential equations," *International Journal of Applied and Computational Mathematics*, vol. 2, no. 2, pp. 223–241, 2016.

[17] R. Saadeh, "Applications of double ARA integral transform," *Computation*, vol. 10, no. 12, p. 216, 2022.

[18] J. A. Ganie, A. Ahmad, and R. R. A. Jain, "Basic analogue of double Sumudu transform and its applicability in population dynamics," *Asian Journal of Mathematics*, vol. 11, pp. 12–17, 2018.

[19] R. Saadeh, A. Qazza, and A. Burqan, "On the double ARA-sumudu transform and its applications," *Mathematics*, vol. 10, no. 15, p. 2581, 2022.

[20] A. Qazza, A. Burqan, R. Saadeh, and R. Khalil, "Applications on double ARA–Sumudu transform in solving fractional partial differential equations," *Symmetry*, vol. 14, no. 9, p. 1817, 2022.

[21] S. Alfaqeih and E. Misirli, "On double Shehu transform and its properties with applications," *International Journal of Analysis and Applications*, vol. 18, no. 3, pp. 381–395, 2020.

[22] C. U. Nwoji, H. N. Onah, B. O. Mama, C. C. Ike, and E. U. Ikwueze, "Elastic buckling analysis of simply supported thin plates using the double finite Fourier sine integral transform method," *Explorematics Journal of Innovative Engineering and Technology*, vol. 1, no. 1, pp. 37–47, 2017.

[23] J. Prüss, *Evolutionary Integral Equations and Applications*, Birkhäuser, Basel, Switzerland, 2013.

[24] R. Z. Saadeh and B. f. Ghazal, "A new approach on transforms: formable integral transform and its applications," *Axioms*, vol. 10, no. 4, p. 332, 2021.

[25] H. Jafari, "A new general integral transform for solving integral equations," *Journal of Advanced Research*, vol. 32, pp. 133–138, 2021.