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Research Article

Geometric Degree Reduction of Wang–Ball Curves

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There are substantial methods of degree reduction in the literature. Existing methods share some common limitations, such as lack of geometric continuity, complex computations, and one-degree reduction at a time. In this paper, an approximate geometric multidegree reduction algorithm of Wang–Ball curves is proposed. G^0-, G^1-, and G^2-continuity conditions are applied in the degree reduction process to preserve the boundary control points. The general equation for high-order (G^2 and above) multidegree reduction algorithms is nonlinear, and the solutions of these nonlinear systems are quite expensive. In this paper, C^1-continuity conditions are imposed besides the G^2-continuity conditions. While some existing methods only achieve the multidegree reduction by repeating the one-degree reduction method recursively, our proposed method achieves multidegree reduction at once. The distance between the original curve and the degree-reduced curve is measured with the L_2-norm. Numerical example and figures are presented to state the adequacy of the algorithm. The proposed method not only outperforms the existing method of degree reduction of Wang–Ball curves but also guarantees geometric continuity conditions at the boundary points, which is very important in CAD and geometric modeling.

1. Introduction

The Wang–Ball curve was proposed by Wang [1] as an extension of the cubic Ball basis pioneered by Ball [2–4]. A. Ball introduced the basis for a cubic polynomial in the CONSURF system at the British Aircraft Corporation [2]. The basis was generalized independently by Wang [1] and Said [5] in 1987 and 1989, respectively. Since then, the generalization by Wang and Said is, respectively, called the Wang–Ball curve and the Said–Ball curve (or simply the generalized Ball curves).

Degree reduction plays a vital role in computer-aided geometric design (CAGD). Given a curve W_n of a degree n, degree reduction is a process of approximating the given curve W_n by a lower degree curve R_m of a degree m, (where $m < n$). If geometric continuity conditions are applied in the degree reduction process, such a degree reduction is called geometric degree reduction. In geometric degree reduction, the information at both boundaries of the given curve is preserved using the geometric continuity conditions.

The highest degree of a polynomial that the CAD system handles varies from one CAD system to another. The problem of degree reduction mainly occurs if data are exchanged between diverse CAD systems. Thus, due to the variation of the highest degree allowed between different CAD systems, higher degree Wang–Ball curves are approximated by lower degree Wang–Ball curves. Wang–Ball curves are much more suited to degree reduction than the classical Bézier curves. Degree reduction is an important issue in CAD, and it facilitates data transfer, exchange, and compression.

There are substantial methods of degree reduction in the literature, and most of the existing methods share some common limitations, such as lack of geometric continuity and complex computations; some existing methods only achieve the multidegree reduction by repeating the one-degree reduction method recursively. Our proposed method uses the geometric continuity conditions to preserve the end points of the given curve; i.e., the boundary control points of the degree reduced curve are obtained using geometric continuity. The interior control points are obtained by solving a linear system. The general equation for high-order (G^2 and above) multidegree reduction algorithms is nonlinear, and the solutions of these nonlinear systems are quite expensive. To overcome this obstacle, C^1-continuity and G^2-continuity are imposed at the endpoints. While some existing methods only achieve the multidegree reduction by repeating the one-degree reduction method recursively, the proposed method can achieve multidegree reduction at once. Moreover, the proposed method can be extended to high-order geometric continuity.

The problem of degree reduction was tackled by some authors using metaheuristic algorithms. Hu et al. [6] proposed a method of degree reduction of the SG–Bézier curve using the grey wolf optimizer algorithm. Lu and Qin [7] transformed the degree reduction problem into the function optimization problem and applied the genetic simulated annealing algorithm to solve the problem. Liu et al. [8] presented a method of degree reduction of Q–Bézier curves using the swarm intelligence-based squirrel search algorithm. Moreover, Cao et al. [9] solved the problem of multidegree reduction of the Ball–Bézier curve using an improved squirrel search algorithm. Hu et al. [10] applied an enhanced hybrid chameleon swarm algorithm to solve the issue of degree reduction of disk Wang–Ball curves. The problem to approximate multidegree reduction of the Said–Ball curve was tackled by Hu et al. [11, 12] using an enhanced chimp optimization algorithm and an improved chimp optimization algorithm, respectively. Some researchers proposed different techniques for degree reduction of Bézier curves and its variants. Chen and Wang [13] proposed an algorithm of degree reduction of the Bézier curve with endpoint constraints. Woźny and Lewanowicz [14] developed a method of degree reduction of the Bézier curve based on dual constrained Bernstein basis polynomials. Hu et al. [15] presented an approximate degree reduction scheme of the λ-Bézier curve using C^0 and C^1 constraint conditions.

Shi [16] solved the problem of degree reduction of rational Bézier curves using weighted least squares and quadratic programming. The issue of degree reduction for the disk Bézier curve was considered in [17] by Chen and Yang. If the geometric continuity condition is required in degree reduction of the disk Bézier curve, the approximation error could be high at the center of the degree-reduced disk Bézier curve. Rababah and Hamza [18] developed weighted degree reduction of disk Bézier curves to tackle this problem. Generalized Ball curves are much more suited to degree reduction than the classical Bézier curves [19]. Degree reduction of Said–Ball curves was presented in [20] by Hu and Wang. Dong et al. [21] presented a multidegree reduction method for Wang–Ball curves using dual basis polynomials. Geometric degree reduction is an important issue in CAGD, but most existing methods of geometric degree reduction are nonlinear. The main contributions of this paper are as follows:

(i) The proposed method introduced explicit linear methods of reduction of Wang–Ball curves with G^0-, G^1-, and G^2-continuity at the endpoints

(ii) The proposed method can achieve multidegree reduction at once.

The remainder of this paper is arranged as follows: Basic definitions and statement of the problem are given in Section 2. G^0-, G^1-, and G^2-degree reduction of the Wang–Ball curve is demonstrated in Section 3. Numerical examples are displayed in Section 4. Finally, conclusions are given in Section 5.

2. Definitions and Statement of the Problem

In this section, some definitions and statement of the problem are given as follows:

Definition 1. The Wang–Ball curve based on the points $\{w_i\}_{i=0}^n$ of the degree n is defined as follows:

$$W_n(t) := \sum_{i=0}^{n} w_i B_i^n(t), \tag{1}$$

where $B_i^n(t), i = 0, 1, \ldots, n$ is the Wang–Ball basis defined by

$$B_i^n(t) = \begin{cases} 2^i t^i (1-t)^{i+2}, & 0 \le i \le \lfloor n/2 \rfloor - 1, \\ 2^{n-i} t^{n+2-i} (1-t)^{n-i} & \lfloor (n+1)/2 \rfloor + 1 \le i \le n, \end{cases} \tag{2}$$

in addition if n is even then

$$B_{n/2}^n(t) = 2^{n/2} t^{n/2} (1-t)^{n/2}, \tag{3}$$

and if n is odd,

$$\begin{aligned} B_{((n-1)/2)}^n(t) &= 2^{((n-1)/2)} t^{((n-1)/2)} (1-t)^{((n+1)/2)}, \\ B_{((n+1)/2)}^n(t) &= 2^{((n-1)/2)} t^{((n+1)/2)} (1-t)^{((n-1)/2)}, \end{aligned} \tag{4}$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Let $M_{m,n}$ be the Gram matrix of the dimension $(m+1) \times (n+1)$, whose elements are computed as

$$g_{ij} = \int_0^1 W_i^m(t) W_j^n(t) \mathrm{d}t, \quad i = 0, 1, \ldots, m, j = 0, 1, \ldots, n. \tag{5}$$

Computing the Wang–Ball curve in (1) at $t = 0, 1$ gives

$$\begin{aligned} W_n(0) &= w_0, \\ W_n(1) &= w_n. \end{aligned} \tag{6}$$

Evaluating the derivatives of the Wang–Ball curve in (1) yields

$$\frac{d^k}{dt^k}W_n(0) = \frac{n!}{(n-k)!}\Delta^k w_0,$$
$$\frac{d^k}{dt^k}W_n(1) = \frac{n!}{(n-k)!}\Delta^k w_{n-k}, \tag{7}$$

where

$$\Delta^0 w_i = w_i,$$
$$\Delta^k w_i = \Delta^{k-1}w_{i+1} - \Delta^{k-1}w_i, \quad i = 0,1,\ldots,n-k, k = 1,2,\ldots,n. \tag{8}$$

Definition 2. $W_n(t)$ and $R_m(t)$ are G^k-continuous at $t = 0, 1$ if a strictly increasing parametrization $\alpha(t)$: $[0,1] \longrightarrow [0,1]$ exists with $\alpha(0) = 0, \alpha(1) = 1$ and $R_m^{(l)}(t) = W_n^{(l)}(\alpha(t)), t = 0, 1, l = 0, 1, \ldots, k$.

The degree reduction of the Wang–Ball curve problem can be stated as follows.

Given control points $\{w_i\}_{i=0}^n$ of the Wang–Ball curve (1), we find control points $\{r_j\}_{j=0}^m$ of the Wang–Ball curve $(m < n)$:

$$R_m(t) := \sum_{j=0}^m r_j B_j^m(t). \tag{9}$$

Such that, the L_2-distance between $W_n(t)$ and $R_m(t)$ is minimum, and $W_n(t)$ and $R_m(t)$ satisfy G^k-continuity at the endpoints, $t = 0, 1$. For the sake of simplicity, let

$$W_n(t) = \sum_{i=0}^n w_i B_i^n(t) := B_n W_n, 0 \le t \le 1,$$
$$R_m(t) = \sum_{j=0}^m r_j B_j^m(t) := B_m R_m, 0 \le t \le 1, \tag{10}$$

where B_n and B_m are row vectors formed by the Wang–Ball basis and W_n and R_m are column vectors formed by the Wang–Ball points of the degree n and m, respectively.

3. Degree Reduction of the Wang–Ball Curve

In this section, we demonstrate the degree reduction of Wang–Ball curves with G^0-, G^1-, and G^2-continuity using the L_2-norm.

3.1. G^0-Degree Reduction of the Wang–Ball Curve. When considering G^0-degree reduction, we need to satisfy G^0-continuity between the two curves $W_n(t)$ and $R_m(t)$ at $t = 0, 1$, i.e.,

$$R_m(t) = W_n(\alpha(t)), t = 0, 1. \tag{11}$$

By evaluating (5), it is easy to obtain:

$$r_0 = w_0,$$
$$r_m = w_n. \tag{12}$$

The elements of R_m are split up into two categories: the category of boundary control points $R_m^c = [r_0, r_m]^T$ which is obtained by G^0-continuity conditions and the part of free control points $R_m^f = (R_m / R_m^c) = [r_1, \ldots, r_{m-1}]^T$. Similarly, B_m is split up into B_m^c and B_m^f.

The error of approximating $W_n(t)$ by $R_m(t)$ is given by the L_2-norm as follows:

$$\epsilon = \int_0^1 \left\| B_n W_n - B_m R_m \right\|^2 dt$$
$$= \int_0^1 \left\| B_n W_n - B_m^c R_m^c - B_m^f R_m^f \right\|^2 dt. \tag{13}$$

Differentiating with respect to the unconstrained control points R_m^f, we have

$$\frac{\partial \varepsilon}{\partial R_m^f} = 2\int_0^1 \left\| B_n W_n - B_m^c R_m^c - B_m^f R_m^f \right\| . B_m^f dt. \tag{14}$$

Let $(\partial\varepsilon / \partial R_m^f) = 0$, and evaluating the integral to obtain:

$$\frac{\partial \varepsilon}{\partial R_m^f} = M_{m,n}^w W_n - M_{m,m}^c R_m^c - M_{m,m}^f R_m^f$$
$$= 0, \tag{15}$$

where

$$M_{m,n}^w := M_{m,n}(1, \ldots, m-1; 0, 1, \ldots, n),$$
$$M_{m,m}^c := M_{m,m}(1, \ldots, m-1; 0, m), \tag{16}$$
$$M_{m,m}^f := M_{m,m}(1, \ldots, m-1; 1, \ldots, m-1),$$

and the submatrix $M_{m,n}(\ldots;\ldots)$ contains the specified rows and columns of the matrix $M_{m,n}$.

To analyze the method and to have a better insight, the points are rewritten in terms of x and y parts. The system can be expressed explicitly by rewriting the control points in the vector form as follows:

$$W_n = [x_0, \ldots, x_n, y_0, \ldots, y_n]^t,$$
$$R_m^F = [\tilde{x}_1, \ldots, \tilde{x}_{m-1}, \tilde{y}_1, \ldots, \tilde{y}_{m-1}]^t, \tag{17}$$
$$R_m^C = [\tilde{x}_0, \tilde{x}_m, \tilde{y}_0, \tilde{y}_m]^t.$$

The following matrices are defined by the direct sum $\oplus$ as follows:

$$M_{m,n}^W = M_{m,n}^w \oplus M_{m,n}^w,$$
$$M_{m,m}^C = M_{m,m}^c \oplus M_{m,m}^c, \tag{18}$$
$$M_{m,m}^F = M_{m,m}^f \oplus M_{m,m}^f.$$

Thus, the system in (15) is written as

$$M_{m,m}^F R_m^F = M_{m,n}^W W_n - M_{m,m}^C R_m^C. \tag{19}$$

The matrix $M_{m,m}^F$ has full rank; then, from (19), the unknowns can be obtained as follows:

$$R_m^F = \left(M_{m,m}^F\right)^{-1}\left(M_{m,n}^W W_n - M_{m,m}^C R_m^C\right). \tag{20}$$

The matrix $M^F_{m,m}$ is real, symmetric, and positive definite; therefore, the solution of the system always exists, and the numerical computations are numerically stable.

3.2. G^1-Degree Reduction of the Wang–Ball Curve. $R_m(t)$ and $W_n(t)$ are G^1-continuous at the points corresponding to $t = 0, 1$ if they are G^0-continuous, and the following additional conditions are satisfied:

$$R'_m(t) = \alpha'(t) W'_n(\alpha(t)), \quad \alpha'(t) > 0, t = 0, 1. \tag{21}$$

Rababah and Hamza in [22] discussed the same issue for disk Bézier curves: as in this work, they substitute $\alpha'(i) = \beta_i^2$ and $i = 0, 1$ to always have $\alpha'(i) \geq 0$ and obtain

$$R'_m(i) = \beta_i^2 W'_n(i), \quad i = 0, 1. \tag{22}$$

This method yields a system of nonlinear equations. To get rid of annoying nonlinearity, we substitute $\alpha'(i) = \beta_i$ and $i = 0, 1$. This substitution for the case of the Wang–Ball curves yields

$$R'_m(i) = \beta_i W'_n(i), \quad i = 0, 1. \tag{23}$$

Using $\alpha'(0) = \beta_0$ and $\alpha'(1) = \beta_1$ and solving (11) and (23) in terms of the control points at both the endpoints, we obtain

$$\begin{aligned} r_0 &= w_0, \\ r_m &= w_n, \\ r_1 &= w_0 + \frac{n}{m}\Delta w_0 \beta_0, \\ r_{m-1} &= w_n - \frac{n}{m}\Delta w_{n-1}\beta_1. \end{aligned} \tag{24}$$

Accordingly, we proceed by splitting R_m into two parts, R^c_m and R^f_m. Let $R^c_m = [r_0, r_1, r_{m-1}, r_m]^T$ be the part of constrained control points which is obtained by G^1-continuity and $R^f_m = (R_m/R^c_m) = [r_2, \ldots, r_{m-2}]^T$ be the part of unconstrained control points. Similarly, we split up B_m in the same way. Hence, the error term becomes

$$\begin{aligned} \varepsilon &= \int_0^1 \left\| B_n W_n - B_m R_m \right\|^2 \mathrm{d}t \\ &= \int_0^1 \left\| B_n W_n - B^c_m R^c_m - B^f_m R^f_m \right\|^2 \mathrm{d}t. \end{aligned} \tag{25}$$

The error $\varepsilon \coloneqq \varepsilon(R^f_m, \beta_0, \beta_1)$ is a function of R^f_m, β_0 and β_1. It follows that

$$\frac{\partial \varepsilon}{\partial R^f_m} = 2 \int_0^1 \left\| B_n W_n - B^c_m R^c_m - B^f_m R^f_m \right\| . B^f_m \mathrm{d}t. \tag{26}$$

Let $(\partial \varepsilon / \partial R^f_m) = 0$, and we evaluate the integral to obtain

$$\begin{aligned} \frac{\partial \varepsilon}{\partial R^f_m} &= M^w_{m,n} W_n - M^c_{m,m} R^c_m - M^f_{m,m} R^f_m \\ &= 0, \end{aligned} \tag{27}$$

where

$$\begin{aligned} M^w_{m,n} &\coloneqq M_{m,n}(2, \ldots, m-2; 0, 1, \ldots, n), \\ M^c_{m,m} &\coloneqq M_{m,m}(2, \ldots, m-2; 0, 1, m-1, m), \\ M^f_{m,m} &\coloneqq M_{m,m}(2, \ldots, m-2; 2, \ldots, m-2), \end{aligned} \tag{28}$$

and the submatrix $M_{m,n}(\ldots ; \ldots)$ contains the specified rows and columns of the matrix $M_{m,n}$.

We differentiate equation (25) with respect to β_i and equate to zero to get

$$\begin{aligned} \frac{\partial \varepsilon}{\partial \beta_0} &= \left(M^1_{m,n} W_n - M^{1;c}_{m,m} R^c_m - M^{1;f}_{m,m} R^f_m \right) \bullet \Delta w_0 \\ &= 0, \end{aligned} \tag{29}$$

$$\begin{aligned} \frac{\partial \varepsilon}{\partial \beta_1} &= \left(M^{m-1}_{m,n} W_n - M^{m-1;c}_{m,m} R^c_m - M^{m-1;f}_{m,m} R^f_m \right) \bullet \Delta w_{n-1} \\ &= 0, \end{aligned} \tag{30}$$

where

$$\begin{aligned} M^j_{m,n} &\coloneqq M_{m,n}(j; 0, 1, \ldots, n), \\ M^{j;c}_{m,m} &\coloneqq M_{m,m}(j; 0, 1, m-1, m), \\ M^{j;f}_{m,m} &\coloneqq M_{m,m}(j; 2, \ldots, m-2), \quad j = 1, m-1. \end{aligned} \tag{31}$$

To analyze the method and to have a better insight, the points are rewritten in terms of x and y parts. Therefore, the variables of our system of equations are $\tilde{x}_k, \tilde{y}_k$, $k = 2, \ldots, m-2, \beta_0$, and β_1. Also, we have to decompose each of r_1 and r_{m-1} into two parts: a constant part and a part involving the parameters β_0 and β_1, respectively. Let u_1 and u_{m-1} be the constant parts of r_1 and r_{m-1}, respectively. Hence,

$$\begin{aligned} u_1 &= w_0, \\ u_{m-1} &= w_n. \end{aligned} \tag{32}$$

To write the linear system in compact shape, we define

$$\begin{aligned} W_n &= [x_0, \ldots, x_n, y_0, \ldots, y_n]^t, \\ R^F_m &= [\tilde{x}_2, \ldots, \tilde{x}_{m-2}, \tilde{y}_2, \ldots, \tilde{y}_{m-2}, \beta_0, \beta_1]^t, \\ R^C_m &= [\tilde{x}_0, u^x_1, u^x_{m-1}, \tilde{x}_m, \tilde{y}_0, u^y_1, u^y_{m-1}, \tilde{y}_m]^t. \end{aligned} \tag{33}$$

Also, the matrices Q, $L_{m,n}$, $L^c_{m,m}$, and $L^f_{m,m}$ are defined as follows:

$$Q = \begin{bmatrix} \Delta w_0 & 0 \\ 0 & \Delta w_{n-1} \end{bmatrix} \begin{bmatrix} M_{m,m}(1,1) & M_{m,m}(1,m-1) \\ M_{m,m}(m-1,1) & M_{m,m}(m-1,m-1) \end{bmatrix} \begin{bmatrix} \Delta w_0 & 0 \\ 0 & \Delta w_{n-1} \end{bmatrix},$$

$$L_{m,n} = \begin{bmatrix} M_{m,n}^{1} \Delta x_0 & M_{m,n}^{1} \Delta y_0 \\ M_{m,n}^{m-1} \Delta x_{n-1} & M_{m,n}^{m-1} \Delta y_{n-1} \end{bmatrix},$$

$$L_{m,m}^{c} = \begin{bmatrix} M_{m,m}^{1;c} \Delta x_0 & M_{m,m}^{1;c} \Delta y_0 \\ M_{m,m}^{m-1;c} \Delta x_{n-1} & M_{m,m}^{m-1;c} \Delta y_{n-1} \end{bmatrix}, \quad (34)$$

$$L_{m,m}^{f} = \begin{bmatrix} M_{m,m}^{1;f} \Delta x_0 & M_{m,m}^{1;f} \Delta y_0 \\ M_{m,m}^{m-1;f} \Delta x_{n-1} & M_{m,m}^{m-1;f} \Delta y_{n-1} \end{bmatrix}.$$

The following matrices are defined by the direct sum $\oplus$ as follows:

$$M_{m,n}^{w++} = M_{m,n}^{w} \oplus M_{m,n}^{w},$$
$$M_{m,m}^{c++} = M_{m,m}^{c} \oplus M_{m,m}^{c}, \quad (35)$$
$$M_{m,m}^{f++} = M_{m,m}^{f} \oplus M_{m,m}^{f}.$$

After some calculations, the system in (27) together with (29) and (30) is written as

$$M_{m,m}^{F} R_m^{F} = M_{m,n}^{W} W_n - M_{m,m}^{C} R_m^{C}, \quad (36)$$

where

$$M_{m,n}^{W} = \begin{bmatrix} M_{m,n}^{w++} \\ L_{m,n} \end{bmatrix},$$

$$M_{m,m}^{C} = \begin{bmatrix} M_{m,m}^{c++} \\ L_{m,m}^{c} \end{bmatrix}, \quad (37)$$

$$M_{m,m}^{F} = \begin{bmatrix} M_{m,m}^{f++} & \frac{n}{m}\left(L_{m,m}^{f}\right)^{t} \\ L_{m,m}^{f} & \frac{n}{m} Q \end{bmatrix}.$$

The matrix $M_{m,m}^{F}$ consists of the matrices $M_{m,m}^{f++}$, $(L_{m,m}^{f})^{t}$, $L_{m,m}^{f}$, and Q. The matrix $M_{m,m}^{f++}$ is a positive definite, and the matrix Q, excluding Δw_0, and Δw_{n-1} parts, is also positive definite. Therefore, the matrix $M_{m,m}^{F}$ is nonsingular.

From (36), the solution is given as follows:

$$R_m^{F} = \left(M_{m,m}^{F}\right)^{-1} \left(M_{m,n}^{W} W_n - M_{m,m}^{C} R_m^{C}\right). \quad (38)$$

The matrix $M_{m,m}^{F}$ is real, symmetric, and positive definite; therefore, the solution of the system always exists, and numerical computations are numerically stable.

3.3. G^2-Degree Reduction. $R_m(t)$ and $W_n(t)$ are G^2-continuous at the points corresponding to $t = 0, 1$ if they are G^0- and G^1-continuous (i.e. (24) fulfilled) and satisfy the following additional conditions:

$$R_m''(t) = \left(\alpha'(t)\right)^2 W_n''(\alpha(t)) + \alpha''(t) W_n'(s(t)), \quad \alpha'(t) > 0, t = 0, 1. \quad (39)$$

These equations are simplified by setting $\alpha'(i) = \beta_i^2$, $\alpha''(i) = \eta_i$, and $i = 0, 1$ to get the following system of equations:

$$R_m''(i) = \beta_i^4 W_n''(i) + \eta_i W_n'(i). \quad (40)$$

To get rid of nonlinearity, we set $\beta_i = 1, i = 0, 1$. Substituting these in (40) and applying (8) yield

$$r_2 = 2r_1 - r_0 + \frac{n(n-1)}{m(m-1)} \Delta^2 w_0 + \frac{n}{m(m-1)} \Delta w_0 \eta_0,$$

$$r_{m-2} = 2r_{m-1} - r_m + \frac{n(n-1)}{m(m-1)} \Delta^2 w_{n-2} + \frac{n}{m(m-1)} \Delta w_{n-1} \eta_1. \quad (41)$$

Thus, the two curves $R_m(t)$ and $W_n(t)$ are G^2-continuous if they satisfy (11), (23), and (40); i.e., the following are obtained:

$$r_0 = w_0,$$
$$r_m = w_n,$$
$$r_1 = w_0 + \frac{n}{m} \Delta w_0,$$
$$r_{m-1} = w_n - \frac{n}{m} \Delta w_{n-1},$$
$$r_2 = 2r_1 - r_0 + \frac{n(n-1)}{m(m-1)} \Delta^2 w_0 + \frac{n}{m(m-1)} \Delta w_0 \eta_0,$$
$$r_{m-2} = 2r_{m-1} - r_m + \frac{n(n-1)}{m(m-1)} \Delta^2 w_{n-2} + \frac{n}{m(m-1)} \Delta w_{n-1} \eta_1. \quad (42)$$

Accordingly, we proceed by splitting R_m into two parts, R_m^c and R_m^f. Let $R_m^c = [r_0, r_1, r_2, r_{m-2}, r_{m-1}, r_m]^T$ be the part of constrained control points which is obtained by G^2-continuity and $R_m^f = (R_m / R_m^c) = [r_3, \ldots, r_{m-3}]^T$ be the part of unconstrained control points. Similarly, we split up B_m in the same way. Hence, the error term becomes

$$\begin{aligned}\varepsilon &= \int_0^1 \left\|B_n W_n - B_m R_m\right\|^2 \mathrm{d}t \\ &= \int_0^1 \left\|B_n W_n - B_m^c R_m^c - B_m^f R_m^f\right\|^2 \mathrm{d}t.\end{aligned} \tag{43}$$

The error $\varepsilon \coloneqq \varepsilon(R_m^f, \eta_0, \eta_1)$ is a function of Q_m^f, η_0, and η_1. We differentiate with respect to the points R_m^f to get

$$\frac{\partial \varepsilon}{\partial (R_m)^f} = 2\int_0^1 \left\|B_n W_n - B_m^c R_m^c - B_m^f R_m^f\right\| \cdot B_m^f \mathrm{d}t. \tag{44}$$

Let $(\partial\varepsilon/\partial R_m^f) = 0$, and we evaluate the integral to obtain

$$\begin{aligned}\frac{\partial \varepsilon}{\partial R_m^f} &= M_{m,n}^w W_n - M_{m,m}^c R_m^c - M_{m,m}^f R_m^f \\ &= 0,\end{aligned} \tag{45}$$

where

$$\begin{aligned}M_{m,n}^w &\coloneqq M_{m,n}(3,\ldots,m-3;0,1,\ldots,n),\\ M_{m,m}^c &\coloneqq M_{m,m}(3,\ldots,m-3;0,1,2,m-2,m-1,m),\\ M_{m,m}^f &\coloneqq M_{m,m}(3,\ldots,m-3;3,\ldots,m-3),\end{aligned} \tag{46}$$

and the submatrix $M_{m,n}(\ldots;\ldots)$ contains the specified rows and columns of the matrix $M_{m,n}$.

We differentiate (43) with respect to η_i and equate it to zero to get

$$\begin{aligned}\frac{\partial \varepsilon}{\partial \eta_0} &= \left(M_{m,n}^2 W_n - M_{m,m}^{2;c} R_m^c - M_{m,m}^{2;f} R_m^f\right) \cdot \Delta w_0 \\ &= 0,\end{aligned} \tag{47}$$

$$\begin{aligned}\frac{\partial \varepsilon}{\partial \eta_1} &= \left(M_{m,n}^{m-2} W_n - M_{m,m}^{m-2;c} R_m^c - M_{m,m}^{m-2;f} R_m^f\right) \cdot \Delta w_{n-1} \\ &= 0,\end{aligned} \tag{48}$$

where for $j = 2, m-2$:

$$\begin{aligned}M_{m,n}^j &\coloneqq M_{m,n}(j;0,1,\ldots,n),\\ M_{m,m}^{j;c} &\coloneqq M_{m,m}(j;0,1,2,m-2,m-1,m),\\ M_{m,m}^{j;f} &\coloneqq M_{m,m}(j;3,\ldots,m-3).\end{aligned} \tag{49}$$

To facilitate the discussion, we expand the Wang–Ball curve into the x and y components. Hence, our unknowns are $\tilde{x}_k, \tilde{y}_k$, $k = 3,\ldots,m-3$, η_0, and η_1. Furthermore, we separate each of r_2 and r_{m-2} into two parts: a constant part and a part involving the parameters η_0 and η_1, respectively. Let u_2 and u_{m-2} be the constant part of r_2 and r_{m-2}, respectively. Hence,

$$\begin{aligned}u_2 &= 2r_1 - r_0 + \frac{n(n-1)}{m(m-1)}\Delta^2 w_0,\\ u_{m-2} &= 2r_{m-1} - r_m + \frac{n(n-1)}{m(m-1)}\Delta^2 w_{n-2}.\end{aligned} \tag{50}$$

Consequently, the following vectors are defined:

$$\begin{aligned}W_n &= \left[x_0,\ldots,x_n,y_0,\ldots,y_n\right]^t,\\ R_m^F &= \left[\tilde{x}_3,\ldots,\tilde{x}_{m-3},\tilde{y}_3,\ldots,\tilde{y}_{m-3},\eta_0,\eta_1\right]^t,\\ R_m^C &= \left[\tilde{x}_0,u_2^x,u_{m-2}^x,\tilde{x}_m,\tilde{y}_0,u_2^y,u_{m-2}^y,\tilde{y}_m\right]^t.\end{aligned} \tag{51}$$

We define the matrices Q, $L_{m,n}$, $L_{m,m}^c$, and $L_{m,m}^f$ as follows:

$$\begin{aligned}Q &= \begin{bmatrix}\Delta w_0 & 0\\ 0 & \Delta w_{n-1}\end{bmatrix}\begin{bmatrix}M_{m,m}(2,2) & M_{m,m}(2,m-2),\\ M_{m,m}(m-2,2) & M_{m,m}(m-2,m-2).\end{bmatrix}\begin{bmatrix}\Delta w_0 & 0\\ 0 & \Delta w_{n-1}\end{bmatrix},\\ L_{m,n} &= \begin{bmatrix}M_{m,n}^2\Delta x_0 & M_{m,n}^2\Delta y_0\\ M_{m,n}^{m-2}\Delta x_{n-1} & M_{m,n}^{m-2}\Delta y_{n-1}\end{bmatrix},\\ L_{m,m}^c &= \begin{bmatrix}M_{m,m}^{2;c}\Delta x_0 & M_{m,m}^{2;c}\Delta y_0\\ M_{m,m}^{m-2;c}\Delta x_{n-1} & M_{m,m}^{m-2;c}\Delta y_{n-1}\end{bmatrix},\\ L_{m,m}^f &= \begin{bmatrix}M_{m,m}^{2;f}\Delta x_0 & M_{m,m}^{2;f}\Delta y_0\\ M_{m,m}^{m-2;f}\Delta x_{n-1} & M_{m,m}^{m-2;f}\Delta y_{n-1}\end{bmatrix}.\end{aligned} \tag{52}$$

Input: (1) Degree of the initial curve n and its control points $\{w_0, w_1, \ldots, w_n\}$;
(2) Geometric continuity condition k;
(3) Degree of the reduced curve m;
1 Draw the initial curve and its control polygon;
2 Compute the constraint control point using:
(12), for G^0-continuity,
(24), for G^1-continuity,
(42), for G^2-continuity;
3 Compute the interior control points using:
(20), for G^0-continuity,
(38), for G^1-continuity,
(56), for G^2-continuity;
Output: Draw the degree-reduced curve.

Algorithm 1: Algorithm of geometric degree reduction of Wang–Ball curves.

The following matrices are defined by the direct sum $\oplus$ as follows:

$$\begin{aligned} M_{m,n}^{w++} &= M_{m,n}^{w} \oplus M_{m,n}^{w},\\ M_{m,m}^{c++} &= M_{m,m}^{c} \oplus M_{m,m}^{c},\\ M_{m,m}^{f++} &= M_{m,m}^{f} \oplus M_{m,m}^{f}. \end{aligned} \tag{53}$$

After some calculations, the coordinate system of the expansion of (45) together with (47) and (48) is given by

$$M_{m,m}^{F} R_m^{F} = M_{m,n}^{W} W_n - M_{m,m}^{C} R_m^{C}, \tag{54}$$

where

$$\begin{aligned} M_{m,n}^{W} &= \begin{bmatrix} M_{m,n}^{p++} \\ L_{m,n} \end{bmatrix},\\ M_{m,m}^{C} &= \begin{bmatrix} M_{m,m}^{c++} \\ L_{m,m}^{c} \end{bmatrix},\\ M_{m,m}^{F} &= \begin{bmatrix} M_{m,m}^{f++} & \dfrac{n}{m(m-1)}\left(L_{m,m}^{f}\right)^{t} \\ L_{m,m}^{f} & \dfrac{n}{m(m-1)} Q \end{bmatrix}. \end{aligned} \tag{55}$$

From (54), unknowns can be computed as.

$$R_m^{F} = \left(M_{m,m}^{F}\right)^{-1}\left(M_{m,n}^{W} W_n - M_{m,m}^{C} R_m^{C}\right). \tag{56}$$

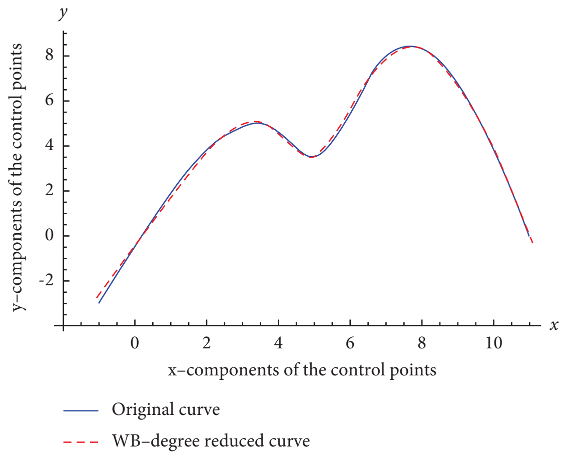
3.4. The Algorithm for Geometric Degree Reduction of Wang–Ball Curves. The step-by-step procedure of geometric degree reduction of Wang–Ball curves is presented in Algorithm 1 to guide the reader and facilitate the implementation of the numerical example. It should be clear that the end control points and the interior control points of the degree reduced curve are obtained based on the required geometric continuity condition.

4. Numerical Examples

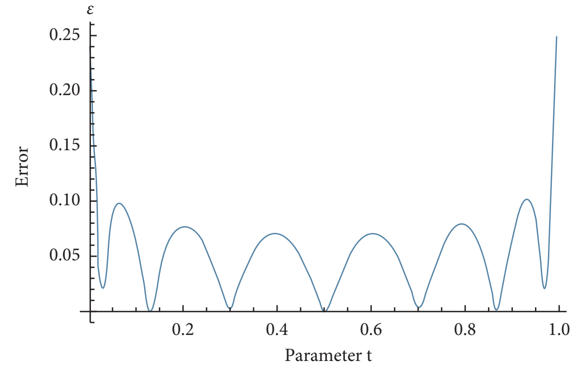
In this section, we provide an example to illustrate the effectiveness of the proposed method. Deng et al. [21] solved the problem of degree reduction of the Wang–Ball curve using basis transformation. To compare our proposed method with the existing method of degree reduction of the Wang–Ball curve, we employed an example from Deng et al. [21] and compared our result with that of Deng et al. [21]. Given a planer Wang–Ball curve of degree eight $W_n(t)$ which is an outline of a hill, whose control points are (see [21]): (−1, −3), (11, 33), (14, 1), (−6, −8), (1, −54), (11, −109), (3, 17), (−1, 48), and (11, 0), it is reduced to a planer Wang–Ball curve of degree five $R_m(t)$. WB, G^0-, G^1-, and G^2-degree reduction methods are employed to reduce the degree of $W_n(t)$, where WB stands for degree reduction without any boundary condition. The degree reduced curves by WB, G^0, G^1, and G^2 methods and the original curve are depicted, respectively, in Figures 1(a), 1(c), 1(e), and 1(g), and their corresponding error functions are shown in Figures 1(b), 1(d), 1(f), and 1(h), respectively. The original curve is depicted in the blue line, while the red dotted line represents the degree reduced curve. The error of the degree reduction is given by

$$e = \max_{0\le t\le 1}\left|W_n(t) - R_m(t)\right|. \tag{57}$$

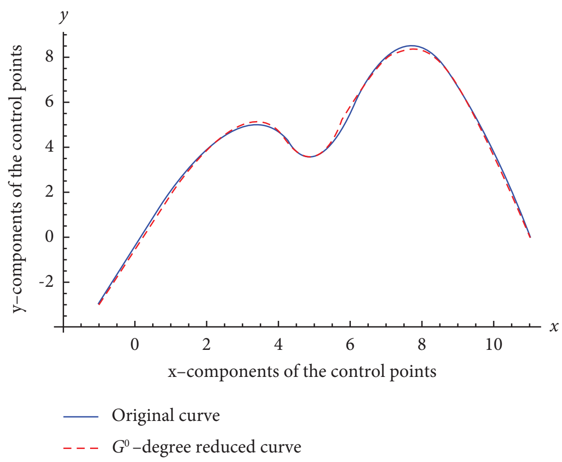
From Figure 1(b), it can be seen that the error of WB degree reduction is higher at the boundaries; this is because no boundary condition is required. Thus, the initial and last control points of the given curve do not coincide with those of the degree-reduced curve. Due to the geometric continuity conditions required at the boundaries (G^0, G^1, and G^2), the initial and last control points of the given Wang-Ball curve coincided with those of the degree-reduced Wang-Ball curve. Thus, the boundary information of the given curve is preserved. From Figures 1(d), 1(f), and 1(h), it can be seen that the error is zero at the boundaries. Note that the error in Figure 1(g) is higher than in the rest of subfigures. This occurs because the original given Wang–Ball curve is reduced to the degree five Wang–Ball curve using G^2-continuity; in this case, all the control points of the degree-



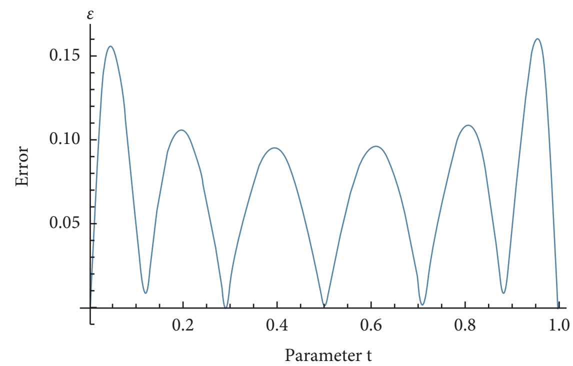
(a)



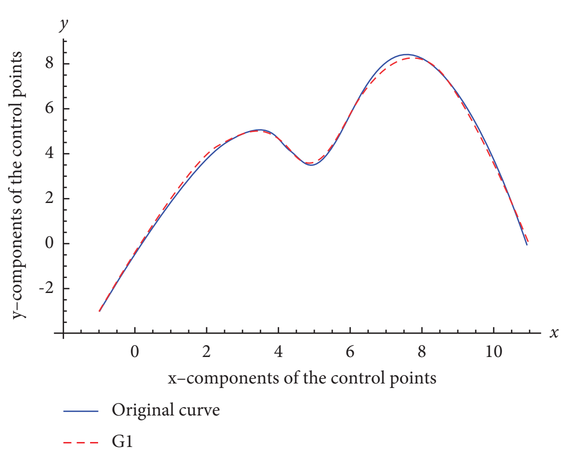
(b)



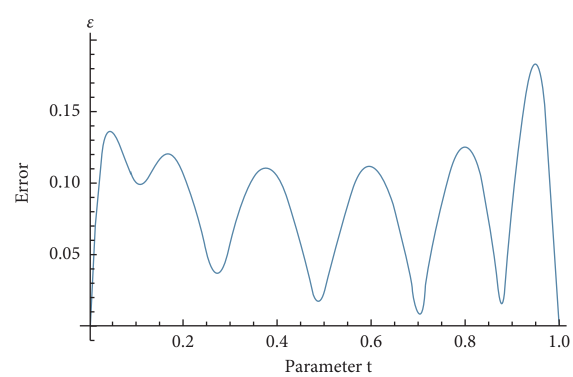
(c)



(d)

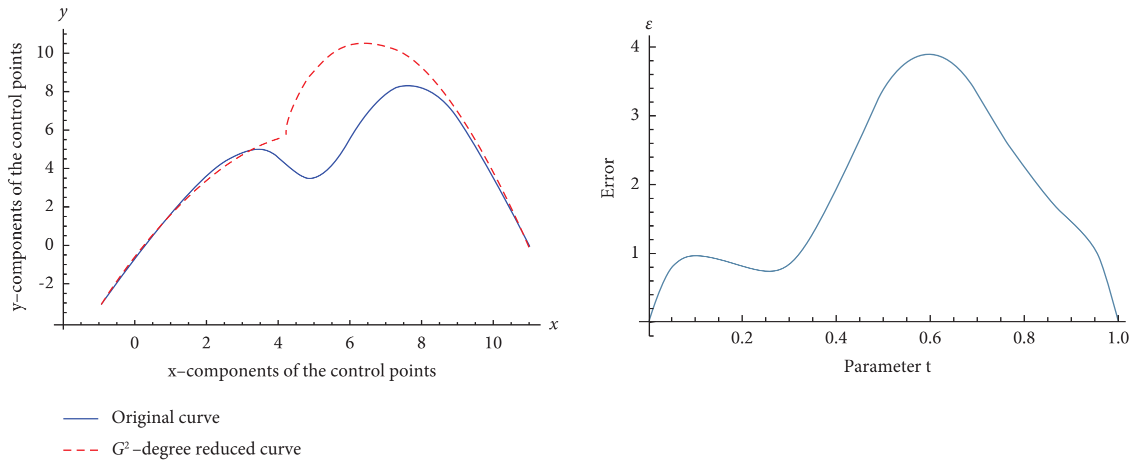


(e)



(f)

Figure 1: Continued.



(g) (h)

Figure 1: Example of degree reduction: (a) the original curve and WB-degree-reduced curve; (b) error function for WB-degree reduction; (c) the original curve and G^0-degree-reduced curve; (d) error function for G^0-degree reduction; (e) the original curve and G^1-degree-reduced curve; (f) error function for G^1-degree reduction; (g) the original curve and G^2-degree-reduced curve; (h) error function for G^2-degree reduction.

Table 1: Comparison of our method with [21].

# Points	Max. error of [21]	Max. errors of our method			
		WB	G^0	G^1	G^2
9	1.0999	0.2485	0.1601	0.1819	3.88319

reduced Wang–Ball curve are obtained using the G^2-continuity condition, and we cannot get the interior control points since all the control points of the degree-reduced curve are exhausted by the G^2-continuity condition; that is why the error is higher especially at the center. Table 1 presents the comparison of the maximum error of the proposed method with that of [21]. The maximum error produced by the proposed method is less than that of [21] except for G^2-degree reduction. Our proposed method is linear, and it does not consist of any basis transformation unlike the method presented in [21]. The computational complexity of the proposed method is cheaper than that of [21].

5. Conclusion

In this paper, we presented a new method for multidegree reduction of Wang–Ball curves with G^0-, G^1-, and G^2-continuity at the boundary points. The general equation for high-order (G^2 and above) multidegree reduction algorithms is nonlinear, and the solutions of these nonlinear systems are quite expensive. To avoid nonlinearity, we required C^1-continuity in addition to G^2-continuity. While some existing methods only achieve the multidegree reduction by repeating the one-degree reduction method recursively, our proposed method achieves multidegree reduction at once. The distance between the original curve and the degree-reduced curve is measured in the L_2-norm. A numerical example and figures are provided to demonstrate the effectiveness of the proposed method. Our proposed method not only outperforms the existing method of degree reduction of Wang–Ball curves but also guarantees geometric continuity conditions at the boundary points, which is very important in CAD and geometric modeling. Moreover, the proposed method can be extended to conditions of higher order geometric continuity.

Data Availability

The examples are given in the manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

Yusuf Fatihu Hamza and Abdellah Rababah were responsible for planning. Yusuf Fatihu Hamza, Mukhtar Fatihu Hamza, and Abdellah Rababah were responsible for design. Mukhtar Fatihu Hamza and Salisu Ibrahim were responsible for analysis. All the authors were responsible for drafting. All the authors were responsible for manuscript preparation.

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