Hindawi
Applied Computational Intelligence and Soft Computing
Volume 2023, Article ID 6973734, 16 pages
https://doi.org/10.1155/2023/6973734

Research Article

On the Analytical Solution of Fractional SIR Epidemic Model

Ahmad Qazza and Rania Saadeh

Department of Mathematics, Faculty of Science, Zarqa University, Zarqa 13110, Jordan

Correspondence should be addressed to Rania Saadeh; rsaadeh@zu.edu.jo

Received 9 October 2022; Revised 10 December 2022; Accepted 12 January 2023; Published 2 February 2023

Academic Editor: Said El Kafhali

This article presents the solution of the fractional SIR epidemic model using the Laplace residual power series method. We introduce the fractional SIR model in the sense of Caputo's derivative; it is presented by three fractional differential equations, in which the third one depends on the first coupled equations. The Laplace residual power series method (LRPSM) is implemented in this research to solve the proposed model, in which we present the solution in a form of convergent series expansion that converges rapidly to the exact one. We analyze the results and compare the obtained approximate solutions to those obtained from other methods. Figures and tables are illustrated to show the efficiency of the LRPSM in handling the proposed SIR model.

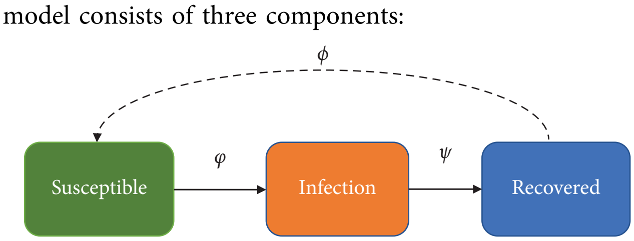
1. Introduction

Epidemiology is a discipline of biology that examines the prevalence and causes health-related problems in particular groups or communities. It may be used to manage community health concerns. Frequency, distribution, and the factors that cause diseases are the elements of epidemiology. By establishing priorities among these services, the Department of Epidemiology seeks to offer the data required for the planning, execution, and assessment of services aimed at disease prevention, control, and treatment. Epidemiology is used to investigate the historical development and decline of illnesses in the population, community diagnosis, planning and evaluation, assessment of an individual's risks and chances, identification of syndromes, and completion of the disease's natural history [1–5].

The study of fractional calculus depends on computing integrals and derivatives of noninteger orders. The importance of these studies has appeared in the various applications in physics and engineering, in which these derivatives can describe them more realistically in the fields of science. A more accurate explanation of the challenges in the actual world can be provided by the fractional application models [6–11].

Through the process of mathematical modeling, one may examine, anticipate, and offer insight into issues that arise in the actual world. It is advantageous since conventional theoretical approaches are inadequate for the study of technological, ecological, economic, and other systems examined by modern research. Fractional differential equations and systems have been used to simulate a variety of real-world issues [12–18].

The SIR model is one of the simplest fractional models, and many models are derived depending on its form. The model consists of three components:



φ: The number of susceptible individuals. When a susceptible and an infectious individual come into "infectious contact," the susceptible individual contracts the disease and transitions to the infectious side.

ψ: The number of infectious individuals. These are individuals who have been infected and are capable of infecting susceptible individuals.

ϕ: The number of removed (and immune) or deceased individuals.

The study of the SIR model was first introduced in 2009 by Ahmet and Cherruault [19], then in 2011 they present new research on analytical solutions of some related models. After that, many authors have investigated the susceptible-infected recovered models of integer fractional orders [20–23].

The integer-order SIR epidemic model is given by

$$\begin{aligned} \varphi'(\tau) &= -q_1\varphi(\tau)\psi(\tau) + q_3\psi(\tau),\\ \psi'(\tau) &= q_1\varphi(\tau)\psi(\tau) - q_3\psi(\tau) - q_2\psi(\tau),\\ \phi'(\tau) &= q_2\psi(\tau), \end{aligned} \tag{1}$$

under the initial conditions:

$$\begin{aligned} \varphi(0) &= \alpha,\\ \psi(0) &= \beta,\\ \phi(0) &= \lambda, \end{aligned} \tag{2}$$

where $\varphi(\tau)$ denotes the susceptible individuals, $\psi(\tau)$ denotes the infected individuals, $\phi(\tau)$ denotes the removed individuals, and τ denotes the time. N is the total number of the studied population such that $\varphi(\tau) + \psi(\tau) + \phi(\tau) = N$. The rate of changes between the previous and the new values are q_1, q_2 and q_3, for more details see [24, 25]. There are some constraints on the model concerning the population number N, which should be large enough, the parameters of the system should be fit, and finally, the healing does not provide immunity.

This research presents the LRPSM, which is a new analytical method that combines the Laplace transform with the residual power series method; it was first introduced in [26], and it is implemented by researchers to solve several models of fractional ordinary and partial differential equations and systems. This method shows its efficiency and applicability in solving similar problems [27–35].

The main aim of this article is to present an analytical series solution of the fractional SIR model. We use the LRPSM to get the solution in the form of a rapidly convergent series. We introduce the method and state the convergence analysis of the method, then we apply it to solve the proposed model. Numerical simulations of the results are discussed, and comparisons are made with the results obtained from other numerical methods. The strength of the LRPSM arises in the ability of solving similar models and presenting many terms of the series solutions with fewer calculations and without the need of linearization, discretization, or differentiation as with other numerical methods.

The layout of this article is as follows: in Section 2, we present some definitions and theorems related to fractional power series and the analysis of LRPSM. In Section 3, we construct the series solution of the fractional SIR model in the sense of Caputo's derivative by LRPSM. In Section 4, we introduce some numerical simulations of our results and comparisons to other numerical methods. Finally, the conclusion section is presented in Section 5.

2. Fractional Power Series

In this section of the article, we introduce some basic definitions and characteristics of the Caputo fractional derivative and the Laplace transform. Also, we present theorems about the fractional Taylor's series of expansions.

Definition 1. Caputo fractional derivative of the function $\varphi(\tau)$ of order γ is given by the following equation:

$$D^{\gamma}\varphi(\tau) = \begin{cases} J_{\tau}^{m-\gamma}\varphi^{(r)}(\tau), & r-1<\gamma<r,\ \tau\geq 0,\\ \varphi^{(r)}(\tau), & \gamma = r,\ \tau\geq 0, \end{cases} \tag{3}$$

where $r \in \mathbb{N}$ and J_{τ}^{γ} is the Riemann–Liouville integral of the fractional order γ to the function $\varphi(\tau)$, provided the integral exists.

There are many properties of Caputo's derivative, that might be found in [23, 24], and we mention some of them as follows:

(i) $D^{\gamma}c = 0,\ c \in \mathbb{R}$
(ii) $D^{\gamma}t^{\beta} = (\Gamma(\beta+1)/\Gamma(\beta+1-\gamma))\tau^{\beta-\gamma}, \tau\geq 0, \beta>-1,\ \gamma>0$
(iii) $D^{\gamma}J_{\tau}^{\gamma}\varphi(\tau) = \varphi(\tau)$
(iv) $J_{\tau}^{\gamma}D^{\gamma}\varphi(\tau) = \varphi(\tau) - \sum_{i=0}^{r-1}\varphi^{(i)}(0^{+})(\tau^{i}/i!),\ r-1<\gamma\leq r$

Definition 2. Let $\varphi(\tau)$ be a piecewise continuous function on $[0,\infty)$. If $\varphi(\tau)$ is of exponential order, then $\varphi(\tau)$ has the Laplace transform which is defined as follows:

$$\begin{aligned} \Phi(s) &= \mathscr{L}[\varphi(\tau)]\\ &= \int_0^{\infty} e^{-st}\varphi(\tau)\mathrm{d}\tau, \quad s>0. \end{aligned} \tag{4}$$

The inverse Laplace transform is given by

$$\begin{aligned} \varphi(\tau) &= \mathscr{L}^{-1}[\Phi(s)]\\ &= \int_{\alpha-i\infty}^{\alpha+i\infty} e^{st}\Phi(s)\mathrm{d}s,\\ \alpha &= Re(s) > c, \end{aligned} \tag{5}$$

for some, c provided the integral exists.

In the following arguments, we list some of the most popular properties of the Laplace transform.

If $\Phi(s) = \mathscr{L}[\varphi(\tau)]$, $\Psi(s) = \mathscr{L}[\psi(\tau)]$, and c, d are constants, then

(i) $\mathscr{L}[c\varphi(\tau) + d\psi(\tau)] = c\Phi(s) + d\Psi(s)$
(ii) $\mathscr{L}^{-1}[c\Phi(s) + d\Psi(s)] = c\varphi(\tau) + d\psi(\tau)$
(iii) $\lim_{s\longrightarrow\infty} s\Phi(s) = \varphi(0)$
(iv) $\mathscr{L}[D^{\gamma}\varphi(\tau)] = s^{\gamma}\Phi(s) - \sum_{k=0}^{r-1}s^{\gamma-k-1}\varphi^{(k)}(0),\ \ r-1<\alpha\leq r$
(v) $\mathscr{L}[D^{n\gamma}\varphi(\tau)] = s^{n\gamma}\Phi(s) - \sum_{k=0}^{n-1}s^{(n-k)\gamma-1}D^{k\gamma}\varphi(0),\ 0<\alpha<1$

For the proof, readers can see [2, 4, 23].

The following theorems illustrate some results about Taylor's fractional series and the convergence analysis of the new approach.

Theorem 1 (see [23]). *Suppose the function $\varphi(\tau)$ has a power series representation at $\tau = \delta$ of the shape:*

$$\varphi(\tau) = \sum_{n=0}^{\infty} \alpha_n (\tau - \delta)^{n\gamma}, \quad 0 < r - 1 < \gamma \le r,\ 0 < \tau < \xi. \tag{6}$$

If $\varphi(\tau) \in C[\delta, \delta + \xi]$ and $D^{n\gamma}\varphi(\tau) \in C[\delta, \delta + \xi]$ for $n = 0, 1, \ldots$, then we have the following formula for the coefficient α_n in equation (6), that takes the expression as follows:

$$\alpha_n = \frac{D^{n\gamma}\varphi(\delta)}{\Gamma(n\gamma + 1)}, \quad n = 0, 1, 2, \ldots, \subset, \tag{7}$$

where $D^{n\gamma} = \underbrace{D^{\gamma} D^{\gamma} \cdots D^{\gamma}}_{n-\text{times}}$.

Note that putting $\gamma = 1$ in expression (6), we get the usual Taylor power series.

The next theorem illustrates a new form of Theorem 1 in the Laplace space considering $\delta = 0$.

Theorem 2. *The fractional power series in (6) has three possibilities for convergence:*

(i) If $\tau = \delta$, then the series is convergent and the radius of convergent is zero

(ii) If $\tau \ge \delta$, the series is convergent and the radius of convergent is ∞

(iii) If $\delta \le \tau < \delta + \zeta$, for some possible real number ζ, then the series converges, and if $\tau > \delta + \zeta$, then the series (1) diverges

Theorem 3 (see [23]). *If the fractional power series expansion of the function $\Phi(s) = \mathscr{L}[\varphi(\tau)]$ is expressed as follows:*

$$\Phi(s) = \sum_{n=0}^{\infty} \frac{\alpha_n}{s^{n\gamma+1}}, \quad s > 0,\ 0 < \gamma \le 1, \tag{8}$$

then the coefficient α_n can be obtained from the following formula:

$$\alpha_n = D^{n\gamma}\varphi(0). \tag{9}$$

Moreover, the inverse Laplace transform of the series expansion (8) in Theorem 3 has the form:

$$\varphi(\tau) = \sum_{n=1}^{\infty} \frac{D^{n\gamma}\varphi(0)}{\Gamma(n\gamma + 1)} \tau^{n\gamma}, \quad \tau \ge 0,\ 0 < \gamma \le 1. \tag{10}$$

Theorem 4 (see [23]) (Convergence analysis). *Let $\Phi(s) = \mathscr{L}[\varphi(\tau)]$ be a function that has the fractional power series in equation (2). If $s\mathscr{L}[D^{(n+1)\gamma}\varphi(\tau)] \le V$, on $0 < s \le b$, where $0 < \gamma \le 1$ and $V > 0$, then the remainder $R_n(s)$ of the series representation (2) has the following bound:*

$$|R_n(s)| \le \frac{V}{s^{(n+1)\gamma+1}}, \quad 0 < s \le b. \tag{11}$$

3. Construction of Solutions to Fractional SIR Model

3.1. Fractional SIR Epidemic Model. There are several phenomena in the real world in engineering and physics, which can be reformulated by fractional initial value problems. Not all of these problems can be solved exactly, so, they are challenging researchers around the whole world. Our aim in this section is to introduce the main idea of the LRPSM in solving systems of nonlinear fractional differential equations that might be difficult to solve by usual techniques.

Now, we present the fractional SIR model:

$$\begin{aligned} D^{\gamma_1}\varphi(\tau) &= -q_1\varphi(\tau)\psi(\tau) + q_3\psi(\tau), \\ D^{\gamma_2}\psi(\tau) &= q_1\varphi(\tau)\psi(\tau) - q_3\psi(\tau) - q_2\psi(\tau), \\ D^{\gamma_3}\phi(\tau) &= q_2\psi(\tau), \end{aligned} \tag{12}$$

where $\gamma_i \in (0, 1]$, $\forall i = 1, 2, 3$, D^{γ_i} denotes the Caputo derivative, q_1, q_2, and q_3 are real positive numbers, q_1 denotes the infection rate, q_2 denotes the removal rate, and q_3 is the recovery rate.

The given initial conditions for system (12) are as follows:

$$\begin{aligned} \varphi(0) &= \alpha, \\ \psi(0) &= \beta, \\ \phi(0) &= \lambda. \end{aligned} \tag{13}$$

Also, we have:

$$\varphi(\tau) + \psi(\tau) + \phi(\tau) = N, \tag{14}$$

and the relation:

$$D^{\gamma_1}\varphi(\tau) + D^{\gamma_2}\psi(\tau) + D^{\gamma_3}\phi(\tau) = 0. \tag{15}$$

The relation (15) gives an extra condition that enable us to solve only two equations of three variables.

3.2. LRPSM for Solving Fractional SIR Model. The basic idea of LRPSM is to apply the Laplace transform on the target equations, then define the so-called Laplace residual functions. After that, multiply each equation by $s^{k\alpha+1}$ by the truncated Laplace residual functions and take the limit at infinity to get the required values of the series coefficients recursively.

To get the series solution of the system (12) using the proposed method, we first operate the Laplace transform on both sides of each equation in system (5), to get:

$$\begin{aligned} \mathscr{L}[D^{\gamma_1}\varphi(\tau)] &= -q_1\mathscr{L}[\varphi(\tau)\psi(\tau)] + q_3\mathscr{L}[\psi(\tau)], \\ \mathscr{L}[D^{\gamma_2}\psi(\tau)] &= q_1\mathscr{L}[\varphi(\tau)\psi(\tau)] - q_3\mathscr{L}[\psi(\tau)] - q_2\mathscr{L}[\psi(\tau)], \\ \mathscr{L}[D^{\gamma_3}\phi(\tau)] &= q_2\mathscr{L}[\psi(\tau)]. \end{aligned} \tag{16}$$

Running Laplace transform on system (16), we get:

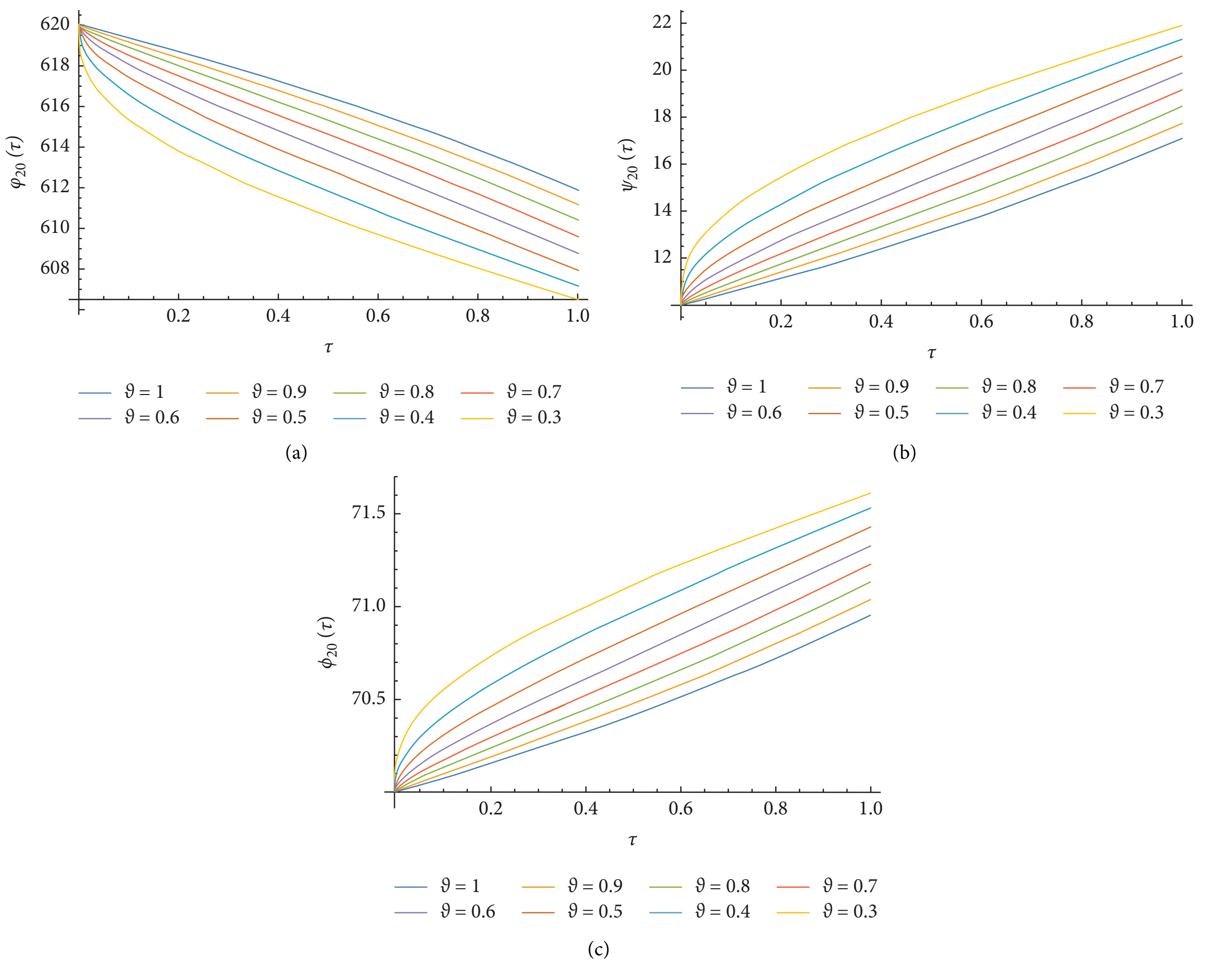


Figure 1: The values of (a) the solution of $\varphi(\tau)$, (b) the solution of $\psi(\tau)$, and (c) the solution of $\phi(\tau)$, for distinct values of $\vartheta(\vartheta = 1, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3)$ using the LRPSM.

$$
\begin{aligned}
& s^{\gamma_1}\Phi(s) - s^{\gamma_1 - 1}\varphi(0) = -q_1\mathscr{L}\left[\mathscr{L}^{-1}[\Phi(s)]\mathscr{L}^{-1}[\Psi(s)]\right] + q_3\Psi(s), \\
& s^{\gamma_2}\Psi(s) - s^{\gamma_1 - 1}\psi(0) = q_1\mathscr{L}\left[\mathscr{L}^{-1}[\Phi(s)]\mathscr{L}^{-1}[\Psi(s)]\right] - q_3\Psi(s) - q_2\Psi(s), \\
& s^{\gamma_3}\mathscr{G}(s) - s^{\gamma_1 - 1}\phi(0) = q_2\Psi(s).
\end{aligned} \tag{17}
$$

Simplifying the equations in system (17) and substituting the initial conditions (13), we get:

$$
\begin{aligned}
\Phi(s) &= \frac{\alpha}{s} - \frac{q_1}{s^{\gamma_1}}\mathscr{L}\left[\mathscr{L}^{-1}[\Phi(s)]\mathscr{L}^{-1}[\Psi(s)]\right] + \frac{q_3}{s^{\gamma_1}}\Psi(s), \\
\Psi(s) &= \frac{\beta}{s} + \frac{q_1}{s^{\gamma_2}}\mathscr{L}\left[\mathscr{L}^{-1}[\Phi(s)]\mathscr{L}^{-1}[\Psi(s)]\right] - \frac{q_3}{s^{\gamma_2}}\Psi(s) - q_2\Psi(s), \\
\mathscr{G}(s) &= \frac{\lambda}{s} + \frac{q_2}{s^{\gamma_3}}\Psi(s).
\end{aligned} \tag{18}
$$

Suppose that the solution of system (18) can be presented in the following series forms:

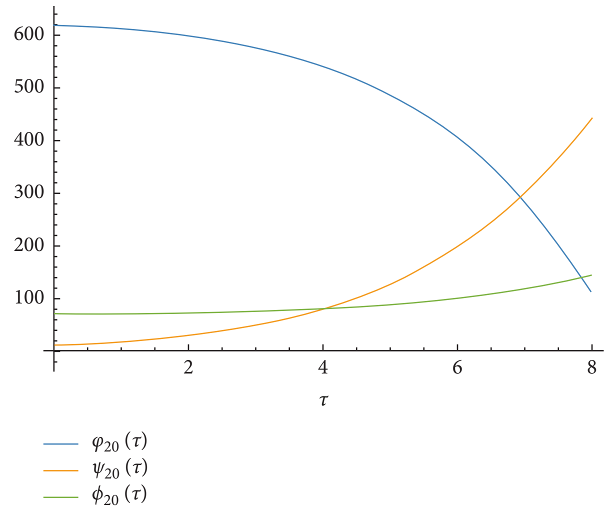


Figure 2: The values of $\varphi_{20}(\tau)$, $\psi_{20}(\tau)$, and $\phi_{20}(\tau)$ using LRPSM for $\vartheta = 1$.

$$\begin{aligned}\Phi(s) &= \sum_{n=0}^{\infty} \frac{\alpha_n}{s^{n\gamma_1+1}},\\ \Psi(s) &= \sum_{n=0}^{\infty} \frac{\beta_n}{s^{n\gamma_2+1}},\\ \mathscr{G}(s) &= \sum_{n=0}^{\infty} \frac{\lambda_n}{s^{n\gamma_3+1}}.\end{aligned} \tag{19}$$

Using the property that $\lim_{s\longrightarrow\infty} s\,\Phi(s) = \varphi(0)$, it is obvious that

$$\begin{aligned}\lim_{s\longrightarrow\infty} s\Phi(s) &= \varphi(0)\\ &= \alpha_0\\ &= \alpha,\\ \lim_{s\longrightarrow\infty} s\Psi(s) &= \psi(0)\\ &= \beta_0\\ &= \beta,\\ \lim_{s\longrightarrow\infty} s\mathscr{G}(s) &= \phi(0)\\ &= \lambda_0\\ &= \lambda.\end{aligned} \tag{20}$$

The series expansion in (19) can be written as follows:

$$\begin{aligned}\Phi(s) &= \frac{\alpha}{s} + \sum_{n=1}^{\infty} \frac{\alpha_n}{s^{n\gamma_1+1}},\\ \Psi(s) &= \frac{\beta}{s} + \sum_{n=1}^{\infty} \frac{\beta_n}{s^{n\gamma_2+1}},\\ \mathscr{G}(s) &= \frac{\lambda}{s} + \sum_{n=1}^{\infty} \frac{\lambda_n}{s^{n\gamma_3+1}}.\end{aligned} \tag{21}$$

Now, define the Laplace residual functions of system (18) as follows:

$$\begin{aligned}\mathscr{L}\mathrm{Res}\Phi(s) &= \Phi(s) - \frac{\alpha}{s} + \frac{q_1}{s^{\gamma_1}}\mathscr{L}\left[\mathscr{L}^{-1}[\Phi(s)]\mathscr{L}^{-1}[\Psi(s)]\right] - \frac{q_3}{s^{\gamma_1}}\Psi(s),\\ \mathscr{L}\mathrm{Res}\Psi(s) &= \Psi(s) - \frac{\beta}{s} - \frac{q_1}{s^{\gamma_2}}\mathscr{L}\left[\mathscr{L}^{-1}[\Phi(s)]\mathscr{L}^{-1}[\Psi(s)]\right] + \frac{q_3}{s^{\gamma_2}}\Psi(s) + q_2\Psi(s),\\ \mathscr{L}\mathrm{Res}\mathscr{G}(s) &= \mathscr{G}(s) - \frac{\lambda}{s} - \frac{q_2}{s^{\gamma_3}}\Psi(s).\end{aligned} \tag{22}$$

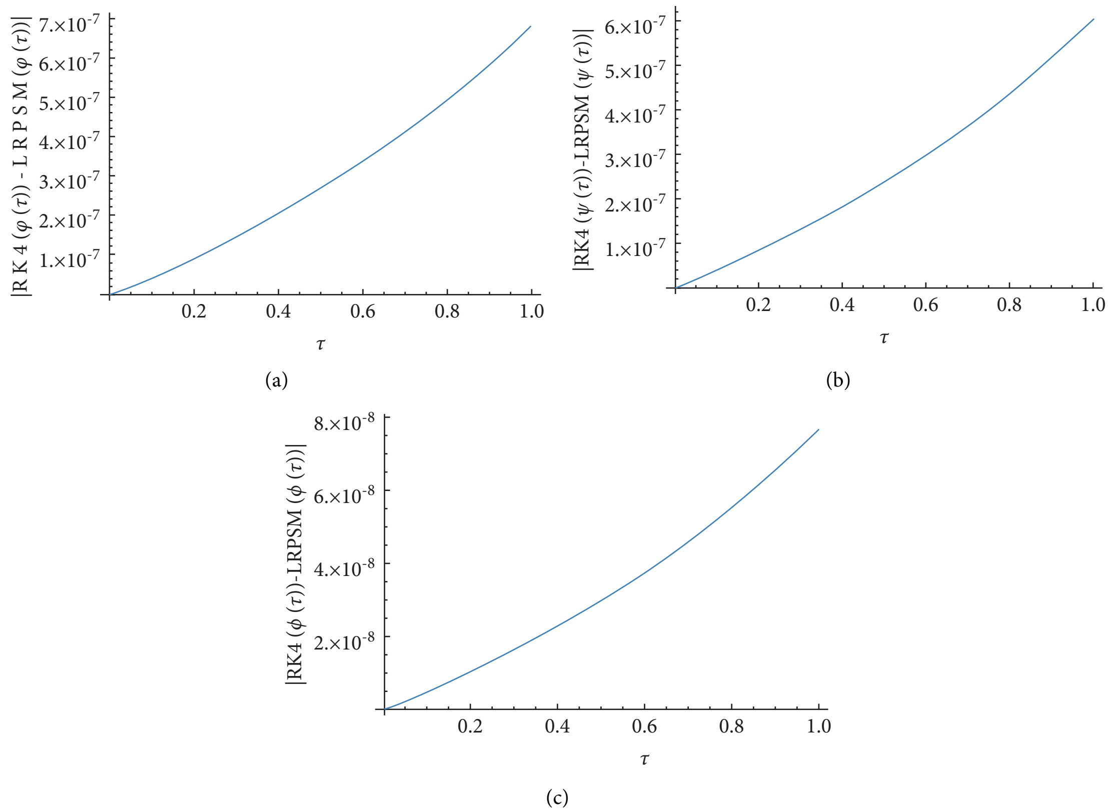


FIGURE 3: Absolute error between the Runge–Kutta method solution and the LRPSM solutions for $\vartheta = 1$.

The kth truncated series of (21) has the form:

$$\Phi_k(s) = \frac{\alpha}{s} + \sum_{n=1}^{k} \frac{\alpha_n}{s^{n\gamma_1+1}},$$
$$\Psi_k(s) = \frac{\beta}{s} + \sum_{n=1}^{k} \frac{\beta_n}{s^{n\gamma_2+1}}, \quad (23)$$
$$\mathcal{G}_k(s) = \frac{\lambda}{s} + \sum_{n=1}^{k} \frac{\lambda_n}{s^{n\gamma_3+1}}.$$

and the kth Laplace residual functions as follows:

$$\mathscr{L}\mathrm{Res}_k\Phi(s) = \Phi_k(s) - \frac{\alpha}{s} + \frac{q_1}{s^{\gamma_1}}\mathscr{L}\left[\mathscr{L}^{-1}[\Phi_k(s)]\mathscr{L}^{-1}[\Psi_k(s)]\right] - \frac{q_3}{s^{\gamma_1}}\Psi_k(s),$$
$$\mathscr{L}\mathrm{Res}_k\Psi(s) = \Psi_k(s) - \frac{\beta}{s} - \frac{q_1}{s^{\gamma_2}}\mathscr{L}\left[\mathscr{L}^{-1}[\Phi_k(s)]\mathscr{L}^{-1}[\Psi_k(s)]\right] + \frac{q_3}{s^{\gamma_2}}\Psi_k(s) + q_2\Psi_k(s), \quad (24)$$
$$\mathscr{L}\mathrm{Res}_k\mathcal{G}(s) = \mathcal{G}_k(s) - \frac{\lambda}{s} - \frac{q_2}{s^{\gamma_3}}\Psi_k(s).$$

Now, the first truncated series of (23) are as follows:

Table 1: The value of $\varphi_{20}(\tau)$ for distinct values of ϑ ($\vartheta = 1, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3$) using the LRPSM.

τ	$\vartheta=1$	$\vartheta=0.9$	$\vartheta=0.8$	$\vartheta=0.7$	$\vartheta=0.6$	$\vartheta=0.5$	$\vartheta=0.4$	$\vartheta=0.3$
0	620	620	620	620	620	620	620	620
0.1	619.368	619.163	618.894	618.538	618.066	617.433	616.568	615.36
0.2	618.702	618.384	617.991	617.501	616.886	616.107	615.109	613.811
0.3	618	617.596	617.112	616.529	615.823	614.961	613.901	612.594
0.4	617.259	616.785	616.23	615.577	614.807	613.895	612.812	611.538
0.5	616.478	615.945	615.332	614.626	613.811	612.871	611.793	610.579
0.6	615.655	615.071	614.411	613.664	612.821	611.872	610.818	609.686
0.7	614.788	614.161	613.463	612.687	611.827	610.884	609.873	608.84
0.8	613.874	613.211	612.483	611.688	610.824	609.902	608.949	608.032
0.9	612.911	612.218	611.469	610.663	609.808	608.919	608.04	607.251
1	611.897	611.181	610.418	609.612	608.774	607.933	607.14	606.494

$$\begin{aligned}\Phi_1(s) &= \frac{\alpha}{s} + \frac{\alpha_1}{s^{\gamma_1+1}},\\ \Psi_1(s) &= \frac{\beta}{s} + \frac{\beta_1}{s^{\gamma_2+1}},\\ \mathcal{G}_1(s) &= \frac{\lambda}{s} + \frac{\lambda_1}{s^{\gamma_3+1}}.\end{aligned} \tag{25}$$

Substituting the above values in the first Laplace residual functions to get the following:

$$\begin{aligned}\mathscr{L}\mathrm{Res}_1\Phi(s) &= \frac{\alpha_1}{s^{\gamma_1+1}} + \frac{q_1}{s^{\gamma_1}}\mathscr{L}\left[\mathscr{L}^{-1}\left[\frac{\alpha}{s} + \frac{\alpha_1}{s^{\gamma_1+1}}\right]\mathscr{L}^{-1}\left[\frac{\beta}{s} + \frac{\beta_1}{s^{\gamma_2+1}}\right]\right] - \frac{q_3}{s^{\gamma_1}}\left(\frac{\lambda}{s} + \frac{\lambda_1}{s^{\gamma_3+1}}\right),\\ \mathscr{L}\mathrm{Res}_1\Psi(s) &= \frac{\beta_1}{s^{\gamma_2+1}} - \frac{q_1}{s^{\gamma_2}}\mathscr{L}\left[\mathscr{L}^{-1}\left[\frac{\alpha}{s} + \frac{\alpha_1}{s^{\gamma_1+1}}\right]\mathscr{L}^{-1}\left[\frac{\beta}{s} + \frac{\beta_1}{s^{\gamma_2+1}}\right]\right] + \frac{q_3}{s^{\gamma_2}}\left(\frac{\beta}{s} + \frac{\beta_1}{s^{\gamma_2+1}}\right) + q_2\left(\frac{\beta}{s} + \frac{\beta_1}{s^{\gamma_2+1}}\right),\\ \mathscr{L}\mathrm{Res}_1\mathcal{G}(s) &= \frac{\lambda_1}{s^{\gamma_3+1}} - \frac{q_2}{s^{\gamma_3}}\left(\frac{\beta}{s} + \frac{\beta_1}{s^{\gamma_2+1}}\right).\end{aligned} \tag{26}$$

Using the following facts of the LRPSM, which can be found in [23],

(i) $\mathscr{L}\mathrm{Res}\Phi(s) = 0$

(ii) $\lim_{k\longrightarrow\infty}\mathscr{L}\mathrm{Res}_k\Phi(s) = \mathscr{L}\mathrm{Res}\Phi(s)$, for each $s > 0$

(iii) $\lim_{s\longrightarrow\infty} s\,\mathscr{L}\mathrm{Res}\Phi(s) = 0$ and $\lim_{s\longrightarrow\infty}\mathscr{L}\mathrm{Res}_k\Phi(s) = 0$

(iv) $\lim_{s\longrightarrow\infty}\mathscr{L}\mathrm{Res}_k\Phi(s) = 0$, $k = 1, 2, \ldots$ and $0 < \alpha \leq 1$

Now, multiplying each equation in (26) by s^{γ_1+1}, s^{γ_2+1}, and s^{γ_3+1}, respectively, and then taking the limit as $s \longrightarrow \infty$, we get the first coefficients of the series expansion (23) as follows:

$$\begin{aligned}\alpha_1 &= -q_1\alpha_0\beta_0 + q_3\beta_0\\ &= -q_1\alpha\beta + q_3\beta,\\ \beta_1 &= -\alpha_1 - q_2\beta_0\\ &= q_1\alpha\beta - q_3\beta - q_2\beta,\\ \lambda_1 &= q_2\beta_1.\end{aligned} \tag{27}$$

Repeating the previous steps, we can get other coefficients.

In addition, if we take $\gamma_1 = \gamma_2 = \gamma_3 = \vartheta$, one can get the coefficients of the series solution as follows:

Table 2: The value of $\psi_{20}(\tau)$ for distinct values of $\vartheta(\vartheta = 1, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3)$ using the LRPSM.

τ	$\vartheta=1$	$\vartheta=0.9$	$\vartheta=0.8$	$\vartheta=0.7$	$\vartheta=0.6$	$\vartheta=0.5$	$\vartheta=0.4$	$\vartheta=0.3$
0	10	10	10	10	10	10	10	10
0.1	10.5577	10.7388	10.9768	11.2905	11.7069	12.2658	13.0284	14.0934
0.2	11.1457	11.4261	11.7735	12.206	12.7482	13.4348	14.3149	15.4584
0.3	11.7658	12.122	12.549	13.0631	13.6858	14.4455	15.379	16.5297
0.4	12.4194	12.8378	13.3274	13.9026	14.5813	15.3851	16.3381	17.4595
0.5	13.1083	13.579	14.1194	14.7415	15.459	16.2866	17.2358	18.3036
0.6	13.8343	14.3494	14.9312	15.5888	16.3317	17.167	18.094	19.0894
0.7	14.5993	15.1521	15.7671	16.4505	17.207	18.0363	18.9256	19.8329
0.8	15.4052	15.9896	16.6304	17.3307	18.0902	18.901	19.7385	20.5441
0.9	16.2541	16.8642	17.5238	18.2326	18.9851	19.7656	20.5383	21.2299
1	17.1481	17.7782	18.4498	19.1588	19.8945	20.6334	21.329	21.8954

$$
\begin{aligned}
\alpha_1 &= \beta(q_3 - \alpha q_1),\\
\alpha_2 &= \beta(\alpha q_1 - q_3)(q_1(\beta - \alpha) + q_2 + q_3),\\
\alpha_3 &= \frac{\beta(\alpha q_1 - q_3)\left(q_1^2\left((\alpha-\beta)^2\Gamma^2(\vartheta+1) - \alpha\beta\,\Gamma(2\vartheta+1)\right)\right)}{\Gamma^2(\vartheta+1)}\\
&+ \frac{\beta(\alpha q_1 - q_3)\left(q_1\left(q_2\left((\beta-2\alpha)\Gamma^2(\vartheta+1) + \beta\Gamma(2\vartheta+1)\right) + q_3\left(\beta\Gamma(2\vartheta+1) - 2(\alpha-\beta)\Gamma^2(\vartheta+1)\right)\right)\right)}{\Gamma^2(\vartheta+1)}\\
&+ \frac{\beta(\alpha q_1 - q_3)(q_2+q_3)^2\Gamma^2(\vartheta+1)}{\Gamma^2(\vartheta+1)}, \ldots,\\
\beta_1 &= \beta(\alpha q_1 - q_2 - q_3),\\
\beta_2 &= \beta\left(\alpha q_1^2(\alpha-\beta) + q_1(q_3(\beta-2\alpha) - 2\alpha q_2) + (q_2+q_3)^2\right),\\
\beta_3 &= \frac{q_1^2\beta\left(q_3\left(2\alpha\beta\Gamma(2\vartheta+1) - \left(3\alpha^2 - 4\alpha\beta + \beta^2\right)\Gamma^2(\vartheta+1)\right) + \alpha q_2\left((2\beta-3\alpha)\Gamma^2(\vartheta+1) + \beta\Gamma(2\vartheta+1)\right)\right)}{\Gamma^2(\vartheta+1)}\\
&+ \frac{\alpha\beta q_1^3\left((\alpha-\beta)^2\Gamma^2(\vartheta+1) - \alpha\beta\Gamma(2\vartheta+1)\right)}{\Gamma^2(\vartheta+1)}\\
&+ \frac{(q_2+q_3)q_1\left(q_3\left((3\alpha-2\beta)\Gamma^2(\vartheta+1) - \beta\Gamma(2\vartheta+1)\right)3\alpha\beta q_2\Gamma^2(\vartheta+1)\right)}{\Gamma^2(\vartheta+1)}\\
&- \frac{(q_2+q_3)^3\Gamma^2(\vartheta+1)}{\Gamma^2(\vartheta+1)}, \ldots,\\
\lambda_1 &= \beta q_2,\\
\lambda_2 &= -\beta q_2(-\alpha q_1 + q_2 + q_3),\\
\lambda_3 &= \beta q_2\left(\alpha q_1^2(\alpha-\beta) + q_1(q_3(\beta-2\alpha) - 2\alpha q_2) + (q_2+q_3)^2\right), \ldots.
\end{aligned}
\tag{28}
$$

The k th coefficients of the series solutions (21) have the form:

Table 3: The value of $\phi_{20}(\tau)$ for distinct values of $\vartheta(\vartheta = 1, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3)$ using the LRPSM.

τ	$\vartheta=1$	$\vartheta=0.9$	$\vartheta=0.8$	$\vartheta=0.7$	$\vartheta=0.6$	$\vartheta=0.5$	$\vartheta=0.4$	$\vartheta=0.3$
0	70	70	70	70	70	70	70	70
0.1	70.074	70.098	70.1297	70.1714	70.2269	70.3015	70.4037	70.5469
0.2	70.1521	70.1894	70.2357	70.2934	70.3659	70.458	70.5764	70.7311
0.3	70.2346	70.2821	70.3391	70.4079	70.4914	70.5937	70.7199	70.8762
0.4	70.3216	70.3775	70.4431	70.5203	70.6116	70.7203	70.8496	71.0026
0.5	70.4135	70.4765	70.5491	70.6328	70.7297	70.842	70.9714	71.1177
0.6	70.5105	70.5796	70.6579	70.7467	70.8475	70.9613	71.0881	71.225
0.7	70.6128	70.6872	70.7703	70.8629	70.9659	71.0793	71.2015	71.3268
0.8	70.7208	70.7997	70.8865	70.9818	71.0856	71.197	71.3126	71.4244
0.9	70.8347	70.9174	71.0071	71.1039	71.2072	71.315	71.4222	71.5187
1	70.955	71.0406	71.1323	71.2296	71.3311	71.4336	71.5307	71.6103

$$\begin{aligned}
\alpha_{k+1} &= -q_1 \sum_{i=0}^{k} \frac{\alpha_i \beta_{k-i} \Gamma(1+n\vartheta)}{\Gamma(1+i\vartheta)\Gamma(1+(n-i)\vartheta)} + q_3 \beta_k, \\
\beta_{k+1} &= -\alpha_{k+1} - q_2 \beta_k, \\
\lambda_{k+1} &= q_2 \beta_k, \\
&\quad k = 0, 1, 2, \ldots.
\end{aligned} \tag{29}$$

Thus, the kth solution of systems (12) and (13) in Laplace space can be expressed as follows:

$$\begin{aligned}
\Phi_k(s) &= \frac{\alpha}{s} + \frac{\beta(q_3 - \alpha q_1)}{s^{\vartheta+1}} + \frac{\beta(\alpha q_1 - q_3)(q_1(\beta - \alpha) + q_2 + q_3)}{s^{2\vartheta+1}} + \cdots \\
&\quad + \frac{1}{s^{k\vartheta+1}} \left(-q_1 \sum_{i=0}^{k} \frac{\alpha_i \beta_{n-i} \Gamma(1+n\vartheta)}{\Gamma(1+i\vartheta)\Gamma(1+(n-i)\vartheta)} + q_3 \beta_k \right), \\
\Psi_k(s) &= \frac{\beta}{s} + \frac{\beta(\alpha q_1 - q_2 - q_3)}{s^{\vartheta+1}} \\
&\quad + \frac{\beta\left(\alpha q_1^2(\alpha - \beta) + q_1(q_3(\beta - 2\alpha) - 2\alpha q_2) + (q_2 + q_3)^2\right)}{s^{2\vartheta+1}} + \cdots \\
&\quad + \frac{-\alpha_{k+1} - q_2 \beta_k}{s^{k\vartheta+1}}, \\
\mathcal{G}_k(s) &= \frac{\lambda}{s} + \frac{q_2 \beta_1}{s^{\vartheta+1}} + \frac{q_2 \beta_2}{s^{2\vartheta+1}} + \cdots + \frac{q_2 \beta_k}{s^{k\vartheta+1}}.
\end{aligned} \tag{30}$$

Operating the inverse Laplace transform on each equation in (32), we obtain the kth solution of system (12) as follows:

Table 4: The value of $\varphi_{20}(\tau)$ using RK4 and LRPSM for $\vartheta = 1$.

τ_i	RK4	LRPSM	$\lvert RK4(\varphi(\tau)) - LRPSM(\varphi(\tau))\rvert$
0	620.000000000000000	620.000000000000000	0
0.1	619.368327701129830	619.368327657510350	4.36195×10^{-8}
0.2	618.702153816378770	618.702153724652930	9.17258×10^{-8}
0.3	617.999682039910570	617.999681895280220	1.4463×10^{-7}
0.4	617.259032603941480	617.259032401282640	2.02659×10^{-7}
0.5	616.478239561945540	616.478239295794650	2.66151×10^{-7}
0.6	615.655248125904110	615.655247790443130	3.35461×10^{-7}
0.7	614.787912080138200	614.787911669181200	4.10957×10^{-7}
0.8	613.873991297021010	613.873990803999960	4.93021×10^{-7}
0.9	612.911149382834760	612.911148800787370	5.82047×10^{-7}
1	611.896951485213090	611.896950806769950	6.78443×10^{-7}

$$\begin{aligned}
\varphi_k(\tau) &= \alpha + \frac{\beta(q_3 - \alpha q_1)}{\Gamma(\vartheta+1)}\tau^{\vartheta} + \frac{\beta(\alpha q_1 - q_3)(q_1(\beta-\alpha) + q_2 + q_3)}{\Gamma(2\vartheta+1)}\tau^{2\vartheta} + \cdots \\
&\quad + \frac{\tau^{k\vartheta}}{\Gamma(k\vartheta+1)}\left(-q_1\sum_{i=0}^{k}\frac{\alpha_i\beta_{n-i}\Gamma(1+n\vartheta)}{\Gamma(1+i\vartheta)\Gamma(1+(n-i)\vartheta)} + q_3\beta_k\right), \\
\psi(\tau) &= \beta + \frac{\beta(\alpha q_1 - q_2 - q_3)}{\Gamma(\vartheta+1)}\tau^{\vartheta} \\
&\quad + \frac{\beta\left(\alpha q_1^2(\alpha-\beta) + q_1(q_3(\beta-2\alpha) - 2\alpha q_2) + (q_2+q_3)^2\right)}{\Gamma(2\vartheta+1)}\tau^{2\vartheta} + \cdots \\
&\quad + \frac{-\alpha_{k+1} - q_2\beta_k}{\Gamma(k\vartheta+1)}\tau^{k\vartheta}, \\
\phi_k(\tau) &= \lambda + \frac{q_2\beta_1}{\Gamma(\vartheta+1)}\tau^{\vartheta} + \frac{q_2\beta_2}{\Gamma(2\vartheta+1)}\tau^{2\vartheta} + \cdots + \frac{q_2\beta_k}{\Gamma(k\vartheta+1)}\tau^{k\vartheta}.
\end{aligned} \tag{31}$$

As $k \longrightarrow \infty$, the kth truncated solutions converge to the exact solutions in the integer orders:

$$\begin{aligned}
\varphi_k(\tau) &\longrightarrow \varphi(\tau), \\
\psi_k(\tau) &\longrightarrow \psi(\tau), \\
\phi_k(\tau) &\longrightarrow \phi(\tau).
\end{aligned} \tag{32}$$

Hence, we get the required solution.

4. Numerical Simulation

This section presents the solution of a numerical example of a SIR epidemic system using the LRPSM, and it introduces numerical simulations and figures to show the effectiveness of the suggested approach. The outcomes demonstrate the dependability and strength of LRPSM in solving such problems.

We consider the epidemic SIR model as follows:

$$\begin{aligned}
D^{\vartheta}\varphi(\tau) &= -0.001\varphi(\tau)\psi(\tau) + 0.005\psi(\tau), \\
D^{\vartheta}\psi(\tau) &= 0.001\varphi(\tau)\psi(\tau) - 0.005\psi(\tau) - 0.072\psi(\tau), \\
D^{\vartheta}\phi(\tau) &= 0.072\psi(\tau), \quad 0 < \vartheta \le 1,
\end{aligned} \tag{33}$$

subject to the initial conditions:

$$\begin{aligned}
\varphi(0) &= 620, \\
\psi(0) &= 10, \\
\phi(0) &= 70.
\end{aligned} \tag{34}$$

The total number of populations is $N = 700$.

To solve system (33) by the proposed method, we apply the Laplace transform on each equation in the system, and using the initial conditions (36), we get:

Table 5: The value of $\psi_{20}(\tau)$ using RK4 and LRPSM for $\vartheta = 1$.

τ_i	RK4	LRPSM	$\lvert RK4(\psi(\tau)) - LRPSM(\psi(\tau))\rvert$
0	10.000000000000000	10.000000000000000	0
0.1	10.557682415377140	10.557682454124135	3.8747×10^{-8}
0.2	11.145742665297318	11.145742746766953	8.14696×10^{-8}
0.3	11.765752718571312	11.765752847013921	1.28443×10^{-7}
0.4	12.419356430322715	12.419356610275818	1.79953×10^{-7}
0.5	13.108271733277171	13.108271969577906	2.36301×10^{-7}
0.6	13.834292759467655	13.834293057265034	2.97797×10^{-7}
0.7	14.599291870396474	14.599292235163372	3.64767×10^{-7}
0.8	14.599291870396474	15.405222008647254	4.37545×10^{-7}
0.9	16.254116280801693	16.254116797279011	5.16477×10^{-7}
1	17.148093929799238	17.148094531720652	6.01921×10^{-7}

$$
\begin{aligned}
\Phi(s) &= \frac{620}{s} - \frac{0.001}{s^{\vartheta}}\mathscr{L}\left[\mathscr{L}^{-1}[\Phi(s)]\mathscr{L}^{-1}[\Psi(s)]\right] + \frac{0.005}{s^{\vartheta}}\Psi(s),\\
\Psi(s) &= \frac{10}{s} + \frac{0.001}{s^{\vartheta}}\mathscr{L}\left[\mathscr{L}^{-1}[\Phi(s)]\mathscr{L}^{-1}[\Psi(s)]\right] - \frac{0.005}{s^{\vartheta}}\Psi(s) - \frac{0.072}{s^{\vartheta}}\Psi(s),\\
\mathscr{G}(s) &= \frac{70}{s} + \frac{0.072}{s^{\vartheta}}\Psi(s).
\end{aligned}
\tag{35}
$$

Now, the kth truncated expansion of the solution of system (35) is as follows:

$$
\begin{aligned}
\Phi_k(s) &= \frac{620}{s} + \sum_{n=1}^{k}\frac{\alpha_n}{s^{n\vartheta+1}},\\
\Psi_k(s) &= \frac{10}{s} + \sum_{n=1}^{k}\frac{\beta_n}{s^{n\vartheta+1}},\\
\mathscr{G}_k(s) &= \frac{70}{s} + \sum_{n=1}^{k}\frac{\lambda_n}{s^{n\vartheta+1}}.
\end{aligned}
\tag{36}
$$

We define the kth Laplace residual functions as follows:

$$
\begin{aligned}
\mathscr{L}\mathrm{Res}_k\Phi(s) &= \Phi_k(s) - \frac{620}{s} + \frac{0.001}{s^{\vartheta}}\mathscr{L}\left[\mathscr{L}^{-1}[\Phi(s)]\mathscr{L}^{-1}[\Psi(s)]\right] - \frac{0.005}{s^{\vartheta}}\Psi_k(s),\\
\mathscr{L}\mathrm{Res}_k\Psi(s) &= \Psi_k(s) - \frac{10}{s} - \frac{0.001}{s^{\vartheta}}\mathscr{L}\left[\mathscr{L}^{-1}[\Phi(s)]\mathscr{L}^{-1}[\Psi(s)]\right] + \frac{0.005}{s^{\vartheta}}\Psi_k(s) + \frac{0.072}{s^{\vartheta}}\Psi_k(s),\\
\mathscr{L}\mathrm{Res}_k\mathscr{G}(s) &= \mathscr{G}_k(s) - \frac{70}{s} - \frac{0.072}{s^{\vartheta}}\Psi_k(s).
\end{aligned}
\tag{37}
$$

Table 6: The value of $\phi_{20}(\tau)$ using RK4 and LRPSM for $\vartheta = 1$.

τ_i	RK4	LRPSM	$\lvert \text{RK4}(\phi(\tau)) - \text{LRPSM}(\phi(\tau)) \rvert$
0	70.000000000000000	70.000000000000000	0
0.1	70.073989883493013	70.073989888365404	4.87239×10^{-9}
0.2	70.152103518323841	70.152103528579929	1.02561×10^{-8}
0.3	70.234565241518069	70.234565257705697	1.61876×10^{-8}
0.4	70.321610965735758	70.321610988441307	2.27055×10^{-8}
0.5	70.413488704777279	70.413488734627379	2.98501×10^{-8}
0.6	70.510459114628276	70.510459152291887	3.76636×10^{-8}
0.7	70.612796049465331	70.612796095655639	4.61903×10^{-8}
0.8	70.720787131876406	70.720787187352826	5.54764×10^{-8}
0.9	70.834734336363510	70.834734401933645	6.55701×10^{-8}
1	70.954954584987661	70.954954661509220	7.65216×10^{-8}

Multiplying each equation in (39) by $s^{k\vartheta+1}, k = 1, 2, \cdots$ recursively, and taking the limit as $s \longrightarrow \infty$, we get the coefficients of the series solutions as follows:

$$
\begin{aligned}
\alpha_1 &= -6.15, \\
\beta_1 &= 5.43, \\
\lambda_1 &= 0.72, \\
\alpha_2 &= -3.27795, \\
\beta_2 &= 2.88699, \\
\lambda_2 &= 0.39096, \\
\alpha_3 &= \frac{0.03339\Gamma(2\vartheta+1)}{\Gamma^2(\vartheta+1)} - 1.74272, \\
\beta_3 &= 1.53486 - \frac{0.0334\Gamma(2\vartheta+1)}{\Gamma^2(\vartheta+1)}, \\
\lambda_3 &= 0.20786, \\
\alpha_4 &= -0.92651 + \frac{0.02021\Gamma(2\vartheta+1)}{\Gamma^2(\vartheta+1)} + \frac{0.03555\Gamma(3\vartheta+1)}{\Gamma(\vartheta+1)\Gamma(2\vartheta+1)}, \\
\beta_4 &= 0.816 - \frac{0.0178\Gamma(2\vartheta+1)}{\Gamma^2(\vartheta+1)} - \frac{0.03556\Gamma(3\vartheta+1)}{\Gamma(\vartheta+1)\Gamma(2\vartheta+1)}, \\
\lambda_4 &= 0.11051 - \frac{0.00241\Gamma(2\vartheta+1)}{\Gamma^2(\vartheta+1)}, \\
\alpha_5 &= -0.49257 - \frac{0.00039\Gamma(4\vartheta+1)\Gamma(2\vartheta+1)}{\Gamma^3(\vartheta+1)\Gamma(3\vartheta+1)} + \frac{0.01074\Gamma(2\vartheta+1)}{\Gamma^2(\vartheta+1)} \\
&\quad + \frac{0.02151\Gamma(3\vartheta+1)}{\Gamma(2\vartheta+1)} + \frac{0.0189\Gamma(4\vartheta+1)}{\Gamma(3\vartheta+1)} + \frac{0.00946\Gamma(4\vartheta+1)}{\Gamma^2(2\vartheta+1)}, \\
\beta_5 &= 0.43382 + \frac{0.00039\Gamma(4\vartheta+1)\Gamma(2\vartheta+1)}{\Gamma^3(\vartheta+1)\Gamma(3\vartheta+1)} - \frac{0.00946\Gamma(2\vartheta+1)}{\Gamma^2(\vartheta+1)} \\
&\quad - \frac{0.01895\Gamma(3\vartheta+1)}{\Gamma(2\vartheta+1)\Gamma(\vartheta+1)} - \frac{0.018902\Gamma(4\vartheta+1)}{\Gamma(3\vartheta+1)\Gamma(\vartheta+1)} - \frac{0.009463\Gamma(4\vartheta+1)}{\Gamma^2(2\vartheta+1)}, \\
\lambda_5 &= 0.058752 - \frac{0.001282\Gamma(2\vartheta+1)}{\Gamma^2(\vartheta+1)} - \frac{0.00256\Gamma(3\vartheta+1)}{\Gamma(\vartheta+1)\Gamma(2\vartheta+1)},
\end{aligned}
\tag{38}
$$

$$\begin{aligned}\varphi_6(\tau) &= 620 - \frac{6.15\tau^{\vartheta}}{\Gamma(1+\vartheta)} - \frac{3.27795\tau^{2\vartheta}}{\Gamma(1+2\vartheta)} + \left(\frac{0.03339\Gamma(2\vartheta+1)}{\Gamma^2(\vartheta+1)} - 1.74272\right)\frac{\tau^{3\vartheta}}{\Gamma(1+3\vartheta)} \\ &+ \left(-0.92651 + \frac{0.02021\Gamma(2\vartheta+1)}{\Gamma^2(\vartheta+1)} + \frac{0.03555\Gamma(3\vartheta+1)}{\Gamma(\vartheta+1)\Gamma(2\vartheta+1)}\right)\frac{\tau^{4\vartheta}}{\Gamma(1+4\vartheta)} \\ &+ \left(-0.49257 - \frac{0.00039\Gamma(4\vartheta+1)\Gamma(2\vartheta+1)}{\Gamma^3(\vartheta+1)\Gamma(3\vartheta+1)} + \frac{0.01074\Gamma(2\vartheta+1)}{\Gamma^2(\vartheta+1)} + \frac{0.02151\Gamma(3\vartheta+1)}{\Gamma(2\vartheta+1)} + \frac{0.0189\Gamma(4\vartheta+1)}{\Gamma(3\vartheta+1)}\right. \\ &\left.+ \frac{0.00946\Gamma(4\vartheta+1)}{\Gamma^2(2\vartheta+1)}\right)\frac{\tau^{5\vartheta}}{\Gamma(1+5\vartheta)} \\ &+ \left(\begin{array}{c} -0.261876 + \dfrac{0.0057256\Gamma(4\vartheta+1)}{\Gamma^2(2\vartheta+1)} + \dfrac{0.010062\Gamma(5\vartheta+1)}{\Gamma(2\vartheta+1)\Gamma(3\vartheta+1)} \\ + \dfrac{(0.011439\Gamma(3\vartheta+1)/\Gamma(2\vartheta+1)) + (0.010049\Gamma(5\vartheta+1)/\Gamma(4\vartheta+1)) + (0.011436\Gamma(4\vartheta+1)/\Gamma(3\vartheta+1))}{\Gamma(\alpha+1)} \\ + \dfrac{0.005712\Gamma(2\vartheta+1) - (0.000209\Gamma(5\vartheta+1)/\Gamma(3\vartheta+1)) - (0.000412\Gamma(3\vartheta+1)\Gamma(5\vartheta+1)/\Gamma(4\vartheta+1)\Gamma(2\vartheta+1))}{\Gamma^2(\vartheta+1)} \\ + \dfrac{\Gamma(2\vartheta+1)\left(-\left(0.000234\Gamma^2(4\vartheta+1)/\Gamma(3\vartheta+1)\right) - 0.000219\Gamma(5\vartheta+1)\right)}{\Gamma^3(\vartheta+1)\Gamma(4\vartheta+1)} \end{array}\right) \\ &\cdot \frac{\tau^{6\vartheta}}{\Gamma(1+6\vartheta)} + \ldots,\end{aligned} \tag{39}$$

$$\begin{aligned}\psi_6(\tau) &= 10 + \frac{5.43\tau^{\vartheta}}{\Gamma(1+\vartheta)} + \frac{2.88699\tau^{2\vartheta}}{\Gamma(1+2\vartheta)} + \left(1.53486 - \frac{0.0334\Gamma(2\vartheta+1)}{\Gamma^2(\vartheta+1)}\right)\frac{\tau^{3\vartheta}}{\Gamma(1+3\vartheta)} \\ &+ \left(0.816 - \frac{0.0178\Gamma(2\vartheta+1)}{\Gamma^2(\vartheta+1)} - \frac{0.03556\Gamma(3\vartheta+1)}{\Gamma(\vartheta+1)\Gamma(2\vartheta+1)}\right)\frac{\tau^{4\vartheta}}{\Gamma(1+4\vartheta)} \\ &+ \left(0.43382 + \frac{0.00039\Gamma(4\vartheta+1)\Gamma(2\vartheta+1)}{\Gamma^3(\vartheta+1)\Gamma(3\vartheta+1)} - \frac{0.00946\Gamma(2\vartheta+1)}{\Gamma^2(\vartheta+1)} - \frac{0.01895\Gamma(3\vartheta+1)}{\Gamma(2\vartheta+1)\Gamma(\vartheta+1)} - \frac{0.018902\Gamma(4\vartheta+1)}{\Gamma(3\vartheta+1)\Gamma(\vartheta+1)}\right. \\ &\left.- \frac{0.009463\Gamma(4\vartheta+1)}{\Gamma^2(2\vartheta+1)}\right)\frac{\tau^{5\vartheta}}{\Gamma(1+5\vartheta)} \\ &+ \left(\begin{array}{c} 0.23064 - \dfrac{0.005044\Gamma(4\vartheta+1)}{\Gamma^2(2\vartheta+1)} - \dfrac{0.010062\Gamma(5\vartheta+1)}{\Gamma(2\vartheta+1)\Gamma(3\vartheta+1)} \\ + \dfrac{-(0.010075\Gamma(3\vartheta+1)/\Gamma(2\vartheta+1)) - (0.010049\Gamma(5\vartheta+1)/\Gamma(4\vartheta+1)) - (0.010075\Gamma(4\vartheta+1)/\Gamma(3\vartheta+1))}{\Gamma(\alpha+1)} \\ + \dfrac{-0.005031\Gamma(2\vartheta+1) + (0.000206\Gamma(5\vartheta+1)/\Gamma(3\vartheta+1)) + (0.000412\Gamma(3\vartheta+1)\Gamma(5\vartheta+1)/\Gamma(4\vartheta+1)\Gamma(2\vartheta+1))}{\Gamma^2(\vartheta+1)} \\ + \dfrac{\Gamma(2\vartheta+1)\left(\left(0.000206\Gamma^2(4\vartheta+1)/\Gamma(3\vartheta+1)\right) + 0.00022\Gamma(5\vartheta+1)\right)}{\Gamma^3(\vartheta+1)\Gamma(4\vartheta+1)} \end{array}\right) \\ &\cdot \frac{\tau^{6\vartheta}}{\Gamma(1+6\vartheta)} + \ldots,\end{aligned}$$

$$\begin{aligned}\phi_6(\tau) &= 70 + \frac{0.72\tau^{\vartheta}}{\Gamma(1+\vartheta)} + \frac{0.39096\tau^{2\vartheta}}{\Gamma(1+2\vartheta)} + \frac{0.20786\tau^{3\vartheta}}{\Gamma(1+3\vartheta)} + \left(0.11051 - \frac{0.00241\Gamma(2\vartheta+1)}{\Gamma^2(\vartheta+1)}\right)\frac{\tau^{4\vartheta}}{\Gamma(1+4\vartheta)} \\ &+ \left(0.058752 - \frac{0.001282\Gamma(2\vartheta+1)}{\Gamma^2(\vartheta+1)} - \frac{0.00256\Gamma(3\vartheta+1)}{\Gamma(\vartheta+1)\Gamma(2\vartheta+1)}\right)\frac{\tau^{5\vartheta}}{\Gamma(1+5\vartheta)} \\ &+ \left(\begin{array}{l} 0.031235 + \left(0.000028\Gamma(4\vartheta+1)\Gamma(2\vartheta+1)/\Gamma^3(\vartheta+1)\Gamma(3\vartheta+1)\right) - \left(0.000681\Gamma(2\vartheta+1)/\Gamma^2(\vartheta+1)\right) \\ + \dfrac{-(0.001364\Gamma(3\vartheta+1)/\Gamma(2\vartheta+1)) - (0.001361\Gamma(4\vartheta+1)/\Gamma(3\vartheta+1))}{\Gamma(\vartheta+1)} - \dfrac{0.000681\Gamma(4\vartheta+1)}{\Gamma^2(2\vartheta+1)} \end{array}\right) \\ &\cdot \frac{\tau^{6\vartheta}}{\Gamma(1+6\vartheta)} + \cdots. \end{aligned} \tag{40}$$

The 20th terms solution of systems (33) and (34) at $\vartheta = 1$ is given by the following equation:

$$\begin{aligned}
\varphi_{20}(\tau) &= 620 - 6.15\tau - 1.63898\tau^2 - 0.27932\tau^3 - 0.03248\tau^4 - 0.00231\tau^5 \\
&+ 6.631\times10^{-7}\tau^6 + 2.85675\times10^{-5}\tau^7 + 4.90414\times10^{-6}\tau^8 \\
&+ 4.94538\times10^{-7}\tau^9 + 2.5347\times10^{-8}\tau^{10} - 1.6516\times10^{-9}\tau^{11} \\
&- 5.91732\times10^{-10}\tau^{12} - 8.16453\times10^{-11}\tau^{13} - 6.84287\times10^{-12}\tau^{14} \\
&- 1.83575\times10^{-13}\tau^{15} + 5.1703\times10^{-14}\tau^{16} + 1.11303\times10^{-14}\tau^{17} \\
&+ 1.28567\times10^{-15}\tau^{18} + 8.57039\times10^{-17}\tau^{19} - 9.08862\times10^{-19}\tau^{20}, \\
\psi_{20}(\tau) &= 10 + 5.43\tau + 1.44349\tau^2 + 0.24468\tau^3 + 0.028078\tau^4 + 0.00191\tau^5 \\
&- 2.3538\times10^{-5}\tau^6 - 2.83254\times10^{-5}\tau^7 - 4.64921\times10^{-6}\tau^8 \\
&- 4.57345\times10^{-7}\tau^9 - 2.20542\times10^{-8}\tau^{10} + 1.79596\times10^{-9}\tau^{11} \\
&+ 5.80956\times10^{-10}\tau^{12} + 7.84277\times10^{-11}\tau^{13} + 6.43953\times10^{-12}\tau^{14} \\
&+ 1.52665\times10^{-13}\tau^{15} - 5.239\times10^{-14}\tau^{16} - 1.090842\times10^{-14}\tau^{17} \\
&- 1.24203\times10^{-15}\tau^{18} - 8.09972\times10^{-17}\tau^{19} + 1.20045\times10^{-18}\tau^{20}, \\
\phi_{20}(\tau) &= 70 + 0.72\tau + 0.19548\tau^2 + 0.03464\tau^3 + 0.0044\tau^4 + 0.00041\tau^5 \\
&+ 2.28749\times10^{-5}\tau^6 - 2.42105\times10^{-7}\tau^7 - 2.549283\times10^{-7}\tau^8 \\
&- 3.719366\times10^{-8}\tau^9 - 3.29288\times10^{-9}\tau^{10} - 1.443545\times10^{-10}\tau^{11} \\
&+ 1.07757\times10^{-11}\tau^{12} + 3.2176\times10^{-12}\tau^{13} + 4.0334\times10^{-13}\tau^{14} \\
&+ 3.09097\times10^{-14}\tau^{15} + 6.86992\times10^{-16}\tau^{16} - 2.21887\times10^{-16}\tau^{17} \\
&- 4.36337\times10^{-17}\tau^{18} - 4.70665\times10^{-18}\tau^{19} - 2.9159\times10^{-19}\tau^{20}.
\end{aligned} \tag{41}$$

The following figures illustrate the 20th solution of systems (33) and (34) with various values of $(\vartheta = 1, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4)$ and 0.3.

Figure 1 of $\varphi_{20}(\tau), \psi_{20}(\tau)$, and $\phi_{20}(\tau)$ shows the strength of LRPSM in solving the proposed model, which is obvious, with ϑ increasing from 0 to one, the solution is stable and coincides with the exact solution obtained in the case $\vartheta = 1$.

In Figure 2 below, we sketch the graph of $\varphi_{20}(\tau), \psi_{20}(\tau)$, and $\phi_{20}(\tau)$.

The following tables, Tables 1–3 present the LRPSM solution of systems (33) and (34) at various values of $\vartheta(\vartheta = 1, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3)$. In Table 1, we propose different values of $\varphi_{20}(\tau)$ with different values of ϑ, in Table 2, we propose different values of $\psi_{20}(\tau)$ with different values of ϑ, and in Table 3, we propose different values of $\phi_{20}(\tau)$ with different values of ϑ. We notice the efficiency and strength of the method from the agreement of the values, and they all coincide and converge to the exact solution in the integer order when $\vartheta = 1$.

In the following, we introduce Figure 3, which illustrates the absolute error between the Runge–Kutta method solution and the LRPSM solution at $\vartheta = 1$.

The following tables (Tables 4–6), present comparisons between the obtained results $\varphi_{20}(\tau)$, $\psi_{20}(\tau)$, and $\phi_{20}(\tau)$ from the LRPSM and the fourth-order Runge–Kutta method (RK4). These comparisons prove the strength and convergence of the presented method. We notice from Tables 4–6 the efficiency of the proposed method, since the difference between the methods is too small.

5. Conclusions

In this study, we introduce the fractional SIR epidemic model in the sense of Caputo's fractional derivative. The LRPSM is used to solve the proposed system. We present a series of solutions of the model and compare our results with those obtained by the fourth-order Runge–Kutta method. In addition, we sketch the graphs of the solutions with different values of ϑ and analyze the results. Finally, we conclude that the LRPSM is efficient in solving the SIR epidemic system and similar models. Moreover, as the method is applicable and easy in treating similar problems, it could provide many terms of the series solution with less calculations and effort compared to other numerical methods. In the future, we intend to solve new models by LRPSM and make comparisons to other analytical methods. We conclude the following points from our study:

(i) LRPSM is a powerful technique for solving fractional models
(ii) LRPSM is simple and could provide many terms of the series solution without requiring discretization, linearization, or special assumptions on the conditions
(iii) The method could give exact solutions when the exact one is a polynomial

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] G. P. Wormser and B. Pourbohloul, *Modeling Infectious Diseases in Humans and Animals*, Princeton University Press, Princeton, NJ, USA, 2008.
[2] R. M. Anderson May, *Infectious Diseases of Humans: Dynamics and Control*Oxford University Press, Oxford, UK, 1992.
[3] W. O. Kermack and A. G. McKendrick, "A contribution to the mathematical theory of epidemics, Proceedings of the royal society of London. Series A," *Containing papers of a mathematical and physical character*, vol. 115, no. 772, pp. 700–721, 1927.
[4] R. M. Merrill, *Introduction to Epidemiology*Jones & Bartlett Learning, Burlington, MA, USA, 2019.
[5] M. Alqhtani, K. M. Owolabi, K. M. Saad, and E. Pindza, "Efficient numerical techniques for computing the Riesz fractional-order reaction-diffusion models arising in biology," *Chaos, Solitons & Fractals*, vol. 161, Article ID 112394, 2022.
[6] I. Podlubny, *Fractional Differential Equations*, Elsevier, Netherlands, Europe, 1999.
[7] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London, UK, 2010.
[8] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," *Journal of Computational and Applied Mathematics*, vol. 264, pp. 65–70, 2014.
[9] R. Almeida and D. F. Torres, "Necessary and sufficient conditions for the fractional calculus of variations with Caputo derivatives," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 3, pp. 1490–1500, 2011.
[10] H. M. Srivastava, K. M. Saad, and W. M. Hamanah, "Certain new models of the multi-space fractal-fractional kuramoto-sivashinsky and korteweg-de Vries equations," *Mathematics*, vol. 10, no. 7, p. 1089, 2022.
[11] A. A. Kilbas, H. H. Srivasfava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equation*North-Holland Mathematics studies, Elsevier Science BV, Amsterdam, Netherlands, 2006.
[12] R. L. Magin, "Fractional calculus in bioengineering, Part 2," *Critical Reviews in Biomedical Engineering*, vol. 32, no. 2, pp. 105–194, 2004.
[13] J. Danane, K. Allali, and Z. Hammouch, "Mathematical analysis of a fractional differential model of HBV infection with antibody immune response," *Chaos, Solitons and Fractals*, vol. 136, Article ID 109787, 2020.
[14] C. J. Tomlin and J. D. Axelrod, "Biology by numbers: mathematical modelling in developmental biology," *Nature Reviews Genetics*, vol. 8, no. 5, pp. 331–340, 2007.
[15] M. Alqhtani, K. M. Saad, R. Shah, W. Weera, and W. M. Hamanah, "Analysis of the fractional-order local Poisson equation in fractal porous media," *Symmetry*, vol. 14, no. 7, p. 1323, 2022.
[16] S. T. Vittadello and M. P. Stumpf, "Open Problems in Mathematical Biology," *Mathematical Biosciences*, vol. 354, Article ID 108926, 2022.
[17] F. A. Rihan, *Delay Differential Equations and Applications to Biology*, Springer, Singapore, 2021.
[18] R. Saadeh, R. Saadeh, A. Qazza, and K. Amawi, "A new approach using integral transform to solve cancer models," *Fractal and Fractional*, vol. 6, no. 9, p. 490, 2022.
[19] Y. Ahmet and Y. Cherruault, "Analytical approximate solution of a SIR epidemic model with constant vaccination strategy by homotopy perturbation method," *Kybernetes*, vol. 38, no. 9, pp. 1566–1575, 2009.
[20] S. Hasan, A. Al-Zoubi, A. Freihet, M. Al-Smad, and S. Momani, "Solution of fractional SIR epidemic model using residual power series method," *Applied Mathematics & Information Sciences*, vol. 13, no. 2, pp. 153–161, 2019.
[21] A. Qazza, R. Saadeh, and E. Salah, "Solving fractional partial differential equations via a new scheme," *AIMS Mathematics*, vol. 8, no. 3, pp. 5318–5337, 2022.
[22] H. Khan, R. N. Mohapatra, K. Vajravelu, and S. J. Liao, "The explicit series solution of SIR and SIS epidemic models," *Applied Mathematics and Computation*, vol. 215, no. 2, pp. 653–669, 2009.
[23] N. I. Akinwande and S. Abubakar, "Approximate solution of SIR infectious disease model using homotopy perturbation method," *Pacific Journal Science and Technology*, vol. 14, no. 2, pp. 163–169, 2013.

[24] F. Brauer, C. Castillo-Chavez, and Z. Feng, *Mathematical models in epidemiology*, vol. 32, Springer, New York, NY, USA, 2019.
[25] Z. Ma, *Dynamical Modeling and Analysis of Epidemics*, World Scientific, Singapore, 2009.
[26] T. Eriqat, A. El-Ajou, M. N. Oqielat, Z. Al-Zhour, and S. Momani, "A new attractive analytic approach for solutions of linear and nonlinear neutral fractional pantograph equations," *Chaos, Solitons & Fractals*, vol. 138, Article ID 109957, 2020.
[27] A. El-Ajou, "Adapting the Laplace transform to create solitary solutions for the nonlinear time-fractional dispersive PDEs via a new approach," *The European Physical Journal Plus*, vol. 136, no. 2, p. 229, 2021.
[28] M. Alquran, M. Alsukhour, M. Ali, and I. Jaradat, "Combination of Laplace transform and residual power series techniques to solve autonomous n-dimensional fractional nonlinear systems," *Nonlinear Engineering*, vol. 10, no. 1, pp. 282–292, 2021.
[29] R. Saadeh, A. Burqan, and A. El-Ajou, "Reliable solutions to fractional Lane-Emden equations via Laplace transform and residual error function," *Alexandria Engineering Journal*, vol. 61, no. 12, pp. 10551–10562, 2022.
[30] A. Burqan, A. El-Ajou, R. Saadeh, and M. Al-Smadi, "A new efficient technique using Laplace transforms and smooth expansions to construct a series solution to the time-fractional Navier-Stokes equations," *Alexandria Engineering Journal*, vol. 61, no. 2, pp. 1069–1077, 2022.
[31] E. Salah, A. Qazza, R. Saadeh, and A. El-Ajou, "A hybrid analytical technique for solving multi-dimensional time-fractional Navier-Stokes system," *AIMS Mathematics*, vol. 8, no. 1, pp. 1713–1736, 2023.
[32] A. Burqan, R. Saadeh, A. Qazza, and S. Momani, "ARA-residual power series method for solving partial fractional differential equations," *Alexandria Engineering Journal*, vol. 62, pp. 47–62, 2023 Jan 1.
[33] A. Qazza, A. Burqan, and R. Saadeh, "Application of ARA-residual power series method in solving systems of fractional differential equations," *Mathematical Problems in Engineering*, vol. 2022, Article ID 6939045, 17 pages, 2022.
[34] A. Burqan, R. Saadeh, and A. Qazza, "A novel numerical approach in solving fractional neutral pantograph equations via the ARA integral transform," *Symmetry*, vol. 14, no. 1, p. 50, 2021.
[35] R. Saadeh, A. Qazza, and A. Burqan, "On the double ARA-sumudu transform and its applications," *Mathematics*, vol. 10, no. 15, p. 2581, 2022.