Research Article

Fuzzy Hypervector Spaces

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The aim of this paper is the generalization of the notion of fuzzy vector spaces to fuzzy hypervector spaces. In this regard, by considering the notion of fuzzy hypervector spaces, we characterized a fuzzy hypervector space based on its level sub-hyperspace. The algebraic nature of fuzzy hypervector space under transformations is studied. Certain conditions are obtained under which a given fuzzy hypervector space can or cannot be realized as a union of two fuzzy hypervector spaces such that none is contained in the other. The construction of a fuzzy hypervector space generated by a given fuzzy subset of a hypervector space is given. The set of all fuzzy cosets of a fuzzy hypervector space is shown to be a hypervector space. Finally, a fuzzy quotient hypervector space is defined and an analogue of a consequence of the “fundamental theorem of homomorphisms” is obtained.

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1. INTRODUCTION

The notion of a hypergroup was introduced by Marty in 1934 [1]. Since then many researchers have worked on hyperalgebraic structures and developed this theory (for more see [2–4]). In 1990, Tallini introduced the notion of hypervector spaces (see [5, 6]) and studied basic properties of them.

The concept of a fuzzy subset of a nonempty set was introduced by Zadeh in 1965 [7] as a function from a nonempty set X into the unit real interval $I = [0, 1]$. Rosenfeld [8] applied this to the theory of groupoids and groups and then many researchers developed it in all the fields of algebra. The concepts of a fuzzy field and a fuzzy linear space over a fuzzy field were introduced and discussed by Nanda [9]. In 1977, Katsaras and Liu [10] formulated and studied the notion of fuzzy vector subspaces over the field of real or complex numbers.

Recently fuzzy set theory has been well developed in the context of hyperalgebraic structure theory (see, e.g., [11–18]). In [11], the first author introduced and studied the notion of fuzzy hypervector space over valued fields. In this paper, we follow [11, 12] and study more properties of fuzzy hypervector spaces. The paper provides the suitable tools to define and study the properties of fuzzy hypervector spaces,

as a generalization of fuzzy vector spaces, and hence can be considered as an application of fuzzy sets to hyperstructure theory. In this regard, we study the algebraic nature of fuzzy hypervector spaces under transformation. Also we introduce and study fuzzy quotient of fuzzy hypervector spaces. In Section 2, some basic definitions and results of hypervector spaces and fuzzy sets which will be used in next sections are given. In Section 3, the union of fuzzy sub-hyperspaces are investigated. It is shown that every fuzzy sub-hyperspace can be written as union of two distinct fuzzy sub-hyperspaces.

In Section 4, the notion of a fuzzy sub-hyperspaces generated by a fuzzy subset is studied. Finally, in Section 5, a fuzzy coset of a fuzzy hypervector space is defined. For a fuzzy sub-hyperspace of a hypervector space, the following results are established:

- (i) the set V_μ of all fuzzy cosets of μ in V is a hypervector space;
- (ii) $\dim V_\mu = \dim V - \dim \mu_t$, where

$$\mu(t) = \{x \in V : \mu(x) = \mu(0)\}. \quad (1)$$

Finally, a fuzzy quotient hyperspace is defined and it is shown that each fuzzy sub-hyperspace of V_μ has a correspondence in a natural way to a fuzzy sub-hyperspaces of V .

2. PRELIMINARIES

In this section, we present some definitions and simple properties of hypervector spaces and fuzzy subsets, that we will use later.

A map $\circ : H \times H \rightarrow P_*(H)$ is called a *hyperoperation* or join operation, where $P_*(H)$ is the set of all nonempty subsets of H . The join operation is extended to subsets of H in natural way, so that $A \circ B$ is given by

$$A \circ B = \bigcup \{a \circ b : a \in A, b \in B\}. \quad (2)$$

The notations $a \circ A$ and $A \circ a$ are used for $\{a\} \circ A$ and $A \circ \{a\}$, respectively. Generally, the singleton $\{a\}$ is identified by its element a .

Definition 1 (see [5]). Let K be a field and $(V, +)$ an Abelian group. A hypervector space over K is a quadruple $(V, +, \circ, K)$, where “ $\circ$ ” is a mapping:

$$\circ : K \times V \longrightarrow P_*(V), \quad (3)$$

such that for all $a, b \in K$ and $x, y \in V$ the following conditions hold:

- (H₁) $a \circ (x + y) \subseteq a \circ x + a \circ y$,
- (H₂) $(a + b) \circ x \subseteq a \circ x + b \circ x$,
- (H₃) $a \circ (b \circ x) = (ab) \circ x$,
- (H₄) $a \circ (-x) = (-a) \circ x = -(a \circ x)$,
- (H₅) $x \in 1 \circ x$.

Remark 1. (i) In the right-hand side of the right distributive law (H₁), the sum is meant in the sense of Frobenius, that is we consider the set of all sums of an element of $a \circ x$ with an element of $a \circ y$. Similarly, it is in the left distributive law (H₂).

(ii) We say that $(V, +, \circ, K)$ is antileft distributive if

$$\forall a, b \in K, \quad \forall x \in V, \quad (a + b) \circ x \supseteq a \circ x + b \circ x \quad (4)$$

and strongly left distributive if

$$\forall a, b \in K, \quad \forall x \in V, \quad (a + b) \circ x = a \circ x + b \circ x \quad (5)$$

In a similar way, we define the antiright distributive and strongly right distributive hypervector spaces, respectively. The hypervector space $(V, +, \circ, K)$ is called strongly distributive if it is both strongly left and strongly right distributive.

(iii) The left-hand side of the associative law (H₃) means the set-theoretical union of all the sets $a \circ y$, where y runs over the set $b \circ x$, that is,

$$a \circ (b \circ x) = \bigcup_{y \in b \circ x} a \circ y. \quad (6)$$

(iv) Let $\Omega_V = 0 \circ \underline{0}$, where $\underline{0}$ is the zero of $(V, +)$. In [5], it is shown that if V is either strongly right or strongly left distributive, then Ω is a subgroup of $(V, +)$.

Example 1. In $(\mathbb{R}^2, +)$, we define the scalar product in $\mathbb{R}$ by setting

$$\forall a \in \mathbb{R}, \quad \forall x \in \mathbb{R}^2 : a \circ x = \begin{cases} \overline{ax} & \text{if } x \neq \underline{0}, \\ \{\underline{0}\} & \text{if } x = \underline{0}, \end{cases} \quad (7)$$

where $\overline{ax}$ is the line through the point x and $\underline{0} = (0, 0)$. Then $(\mathbb{R}^2, +, \circ, \mathbb{R})$ is a strongly left distributive hypervector space.

In the sequel of this article, unless otherwise specified, we assume that V is a hypervector space over the field K .

Definition 2. A nonempty subset W of V is a sub-hyperspace if W is itself a hypervector space with the hyperoperation on V , that is,

$$W \neq \emptyset,$$

$$\forall x, y \in W \implies x - y \in W, \quad (8)$$

$$\forall a \in K, \quad \forall x \in W \implies a \circ x \subseteq W.$$

In this case, one writes $W \leq V$.

Proposition 1. The intersection of a family of sub-hyperspaces is a sub-hyperspace.

Proof. Straightforward. □

Definition 3. If S is a nonempty subset of V , then the linear span of S is the smallest sub-hyperspace of V containing S , that is,

$$L(S) = \langle S \rangle = \bigcap_{S \subseteq W \leq V} W. \quad (9)$$

Lemma 1 (see [12]). If S is a nonempty subset of V , then

$$L(S) = \left\{ t \in \sum_{i=1}^n a_i \circ s_i, \quad a_i \in K, \quad s_i \in S, \quad n \in \mathbb{N} \right\}. \quad (10)$$

Definition 4. A subset S of V is called linearly independent if for every vector $v_1, v_2, \dots, v_n$ in S , and $c_1, \dots, c_n \in K$, $\underline{0} \in c_1 \circ v_1 + \dots + c_n \circ v_n$, implies that $c_1 = c_2 = \dots = c_n = 0$. A subset S of V is called linearly dependent if it is not linearly independent.

Definition 5. A basis for V is a linearly independent subset β of V which linearly spans V , that is, $L(\beta) = V$. One says that V is finite dimensional if it has a finite basis.

Remark 2. Note that some hypervector spaces V (some set W of vectors) may not have any collection of linearly independent vectors. Such hypervector space (set) is called independentless. Clearly if V is independentless, then V has not any basis; and for such hypervector spaces, dimension is not defined. In this case we say that V is dimensionless. The hypervector space $(\mathbb{R}^2, +, \circ, \mathbb{R})$ in Example 1 is a nontrivial example of an independentless hypervector space, since $\underline{0}$ belongs to every line through the $\underline{0}$.

Definition 6. A hypervector space V over K is said to be K -invertible or shortly invertible if and only if $u \in a \circ v$ implies that $v \in a^{-1} \circ u$, for $u, v \in V, a \in K \setminus \{0\}$.

Remark 3. Let $(V, +, \circ, K)$ be a hypervector space and W a sub-hyperspace of V . Consider the quotient Abelian group $(V/W, +)$. Define the rule

$$\begin{aligned} * : K \times V/W &\longrightarrow P_*(V/W) \\ (a, v + W) &\longmapsto a \circ v + W. \end{aligned} \quad (11)$$

Then it is easy to verify that $(V/W, +, *, K)$ is a hypervector space over K and it is called the quotient hypervector space of V over W .

Theorem 1 (see [19]). *Let V be strongly left distributive and invertible. If V is finite dimensional and W is sub-hyperspace of V , then the following hold:*

- (i) W is finite dimensional and $\dim W \leq \dim V$,
- (ii) $\dim V/W = \dim V - \dim W$.

Definition 7. Let V and W be hypervector spaces over K . A mapping $T : V \rightarrow W$ is called

- (i) weak linear transformation if and only if

$$T(x + y) = T(x) + T(y), \quad T(a \circ x) \cap a \circ T(x) \neq \emptyset, \quad (12)$$

- (ii) (inclusion) linear transformation if and only if

$$T(x + y) = T(x) + T(y), \quad T(a \circ x) \subseteq a \circ T(x), \quad (13)$$

- (iii) good transformation if and only if

$$T(x + y) = T(x) + T(y), \quad T(a \circ x) = a \circ T(x). \quad (14)$$

Definition 8. Let $T : V \rightarrow W$ be a linear transformation. Then the kernel of T is denoted by $\ker T$ and defined by

$$\ker T = \{x \in V : T(x) \in \Omega_W\}. \quad (15)$$

Definition 9. (i) For a fuzzy subset μ of X , the level subset μ_t is defined by

$$\mu_t = \{x \in X : \mu(x) \geq t\}, \quad t \in [0, 1]. \quad (16)$$

- (ii) The image of μ is denoted by $\text{Im}(\mu)$ and is defined by

$$\text{Im}(\mu) = \{\mu(x) : x \in X\}. \quad (17)$$

- (iii) If $\mu \in \text{FS}(X)$ and $A \subseteq X$, then by $\mu(A)$, one means

$$\mu(A) = \bigwedge_{a \in A} \mu(a). \quad (18)$$

Definition 10. (i) (Extension principle) let $f : X \rightarrow Y$ be a mapping, $\mu \in \text{FS}(X)$, and $\nu \in \text{FS}(Y)$. Then, $f(\mu) \in \text{FS}(Y)$ and $f^{-1}(\nu) \in \text{FS}(X)$, respectively, are defined:

$$f(\mu)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} \quad (19)$$

- (ii)

$$f^{-1}(\nu)(x) = \nu(f(x)), \quad \forall x \in X. \quad (20)$$

Definition 11. Let $f : X \rightarrow Y$ be a mapping and $\mu \in \text{FS}(X)$. Then μ is called f -invariant if

$$\forall x, y \in X, \quad f(x) = f(y) \implies \mu(x) = \mu(y). \quad (21)$$

Clearly if μ is f -invariant, then $f^{-1}(f(\mu)) = \mu$.

Definition 12. Let K be a field and $\nu \in \text{FS}(K)$. Suppose the following conditions hold:

- (i) $\nu(a + b) \geq \nu(a) \wedge \nu(b)$, for all $a, b \in K$,
- (ii) $\nu(-a) \geq \nu(a)$, for all $a \in K$,
- (iii) $\nu(ab) \geq \nu(a) \wedge \nu(b)$, for all $a, b \in K$,
- (iv) $\nu(a^{-1}) \geq \nu(a)$, $\forall a \in K \setminus \{0\}$,

Then, call ν a fuzzy field in K and denote it by ν_K .

Obviously, Definition 12 is a generalization of the classical field notion.

Definition 13 (see [11]). Let V be a hypervector space over a field K and ν a fuzzy field of K . A fuzzy set μ of V is said to be a fuzzy hypervector space of V over fuzzy field ν_K , if for all $x, y \in V$ and all $a \in K$, the following conditions are satisfied:

- (i) $\mu(x + y) \geq \mu(x) \wedge \mu(y)$,
- (ii) $\mu(-x) \geq \mu(x)$,
- (iii) $\bigwedge_{y \in a \circ x} \mu(y) \geq \nu(a) \wedge \mu(x)$,
- (iv) $\nu(1) \geq \mu(0)$.

Obviously, Definition 13 is a generalization of the concept of a fuzzy vector space and also of the classical notion of a hypervector space (in sense of Tallini [5]). If we consider $\nu = \chi_K$, the characteristic function of K , then μ is called a fuzzy sub-hyperspace of V .

Definition 14. Let $\{\mu_i\}_{i \in I}$ be a nonempty collection of fuzzy sub-hyperspaces of V . Then the fuzzy subset $\bigcap_{i \in I} \mu_i$ of V is defined by the following:

$$\left(\bigcap_{i \in I} \mu_i \right)(x) = \bigwedge_{i \in I} \mu_i(x). \quad (22)$$

Proposition 2. *The intersection of a family of fuzzy sub-hyperspaces is a fuzzy sub-hyperspace.*

Proposition 3 (see [11]). *Let V be a strongly left distributive hypervector space over the field K and ν_K a fuzzy field. Let $\mu \in \{\text{FS}\}(V)$. Then μ is a fuzzy hypervector space over ν_K if and only if*

$$\bigwedge_{z \in a \circ x + b \circ y} \mu(z) \geq (\nu(a) \wedge \mu(x)) \wedge (\nu(b) \wedge \mu(y)), \quad (23)$$

and $\nu(1) \geq \mu(x)$, for all $a, b \in K$ and $x, y \in V$.

Proposition 4 (see [11]). Let $\mu \in \{\text{FS}\}(V)$ and $\nu \in \{\text{FS}\}(K)$. Then μ is a fuzzy hypervector space over ν_K if and only if μ_α is a hypervector space over the field ν_α , for all $\alpha \in \{\text{Im}\}(\nu)$ and $\nu(1) \geq \mu(\underline{0})$. μ_α is called a level sub-hyperspace of V .

Proposition 5 (see [11]). Let V and W be strongly left distributive hypervector spaces, and $T : V \rightarrow W$ a good transformation. Let μ_V and η_W be fuzzy sub-hyperspaces over ν_K . Then $T(\mu)_W$ and $T^{-1}(\eta)_V$ are fuzzy sub-hyperspaces over ν_K .

3. UNION OF FUZZY SUB-HYPERSPACES

We start this section with some basic properties of fuzzy hypervector spaces.

Proposition 6. If μ is a fuzzy hypervector space over the fuzzy field ν_K , then

$$\bigwedge_{y \in 1 \circ x} \mu(y) = \mu(x). \quad (24)$$

Proof. By (H₅) $x \in 1 \circ x$, so $\bigwedge_{y \in 1 \circ x} \mu(y) \leq \mu(x)$. On the other hand, using Definition 13, we have

$$\bigwedge_{y \in 1 \circ x} \mu(y) \geq \nu_K(1) \wedge \mu(x) = 1 \wedge \mu(x) = \mu(x). \quad (25)$$

Thus $\bigwedge_{y \in 1 \circ x} \mu(y) = \mu(x)$. $\square$

Theorem 2. Let μ be a fuzzy sub-hyperspace of V and let μ_{t_1}, μ_{t_2} (with $t_1 < t_2$) any two level sub-hyperspaces of μ . Then $\mu_{t_1} = \mu_{t_2}$ if and only if there is no x in V such that $t_1 \leq \mu(x) < t_2$.

Proof. Straightforward. $\square$

From Theorem 2 it follows that the level sub-hyperspaces of a fuzzy sub-hyperspace μ of V need not be distinct. If $\{\text{Im}\}(\mu) = \{t_0, \dots, t_n\}$ such that $t_0 > t_1 > \dots > t_n$, then the family of level sub-hyperspaces of μ consists of $\{\mu_{t_1}, \dots, \mu_{t_n}\}$ and we have the chain $\mu_{t_1} \subseteq \dots \subseteq \mu_{t_n} = V$.

Proposition 7. Let W be a proper sub-hyperspace of V . Then the fuzzy subset μ of V defined by

$$\mu(x) = \begin{cases} 1 & \text{if } x \in W, \\ t & \text{otherwise,} \end{cases} \quad (26)$$

where $t \in [0, 1)$ is a fuzzy sub-hyperspace of V .

Proof. Obvious. $\square$

Theorem 3. Let V and W be hypervector spaces over the field K , and let $f : V \rightarrow W$ be a mapping. Let μ_V and η_W be fuzzy sub-hyperspaces over ν_K such that

$$\begin{aligned} \{\text{Im}\}(\mu) &= \{t_0, \dots, t_n\}, & \text{with } t_0 > t_1 > \dots > t_n, \\ \{\text{Im}\}(\eta) &= \{s_0, \dots, s_m\}, & \text{with } s_0 > s_1 > \dots > s_m. \end{aligned} \quad (27)$$

Then

(i) $\{\text{Im}\}f(\mu) \subseteq \{\text{Im}\}(\mu)$ and the chain of level sub-hyperspaces of $f(\mu)$ is

$$f(\mu_{t_1}) \subseteq f(\mu_{t_2}) \subseteq \dots \subseteq f(\mu_{t_n}) = W. \quad (28)$$

(ii) $\{\text{Im}\}f^{-1}(\eta) = \{\text{Im}\}(\eta)$ and the chain of level sub-hyperspaces of $f^{-1}(\eta)$ is

$$f^{-1}(\eta_{s_1}) \subset f^{-1}(\eta_{s_2}) \subset \dots \subset f^{-1}(\eta_{s_m}) = V. \quad (29)$$

Proof. (i) Clearly $\text{Im}f(\mu) \subseteq \text{Im}(\mu)$, since

$$f(\mu)(y) = \bigvee_{x \in f^{-1}(y)} \mu(x), \quad \forall y \in W. \quad (30)$$

Also $f(\mu_{t_i}) = (f(\mu))_{t_i}$, for all $i = 0, \dots, n$, because

$$\begin{aligned} y \in f(\mu_{t_i}) &\iff \exists x \in f^{-1}(y), x \in \mu_{t_i} \\ &\iff \exists x \in f^{-1}(y), \mu(x) \geq t_i \\ &\iff \bigvee_{z \in f^{-1}(y)} \mu(z) \geq t_i \\ &\iff f(\mu)(y) \geq t_i \\ &\iff y \in (f(\mu))_{t_i}. \end{aligned} \quad (31)$$

Hence the chain of level sub-hyperspaces of $f(\mu)$ is $f(\mu_{t_1}) \subseteq f(\mu_{t_2}) \subseteq \dots \subseteq f(\mu_{t_n}) = W$.

(ii) Clearly $\text{Im}f^{-1}(\eta) = \text{Im}(\eta)$, since

$$(f^{-1}(\eta))(x) = \eta(f(x)), \quad \forall x \in V. \quad (32)$$

Also $f^{-1}(\eta_{s_i}) = (f^{-1}(\eta))_{s_i}$, for all $i = 0, \dots, m$, because

$$\begin{aligned} y \in f^{-1}(\eta_{s_i}) &\iff f(y) \in \eta_{s_i} \\ &\iff \eta(f(y)) \geq s_i \\ &\iff (f^{-1}(\eta))(y) \geq s_i \\ &\iff y \in (f^{-1}(\eta))_{s_i}. \end{aligned} \quad (33)$$

Hence the chain of level sub-hyperspaces of $f^{-1}(\eta)$ is

$$f^{-1}(\eta_{s_1}) \subset f^{-1}(\eta_{s_2}) \subset \dots \subset f^{-1}(\eta_{s_m}) = V. \quad (34)$$

$\square$

Theorem 4. Let V and W be strongly left-distributive hypervector spaces over the field K and $T : V \rightarrow W$ a good transformation. Then the mapping $\mu \mapsto f(\mu)$ defines a one-to-one correspondence between the set of all f -invariant fuzzy sub-hyperspaces of V and the set of all fuzzy sub-hyperspaces of W .

Proof. It immediately follows from Proposition 5 and the following results:

(i) $f^{-1}(f(\mu)) = \mu$, where μ is a f -invariant fuzzy sub-hyperspace of V ,

(ii) $f(f^{-1}(\eta)) = \eta$, where η is any fuzzy sub-hyperspace of W . $\square$

For sub-hyperspaces W_1 and W_2 of V , it is easy to verify that $W_1 \cup W_2$ is a sub-hyperspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. In the sequel, an attempt is made to study the following problem. Is it possible to realize a fuzzy sub-hyperspace as a union of two fuzzy sub-hyperspaces such that none of them is contained in the other?

It is shown that there exist fuzzy sub-hyperspaces such that their union is a fuzzy sub-hyperspace, but none of them is contained in the other. Furthermore, it turns out that an answer to the above problem depends on the image set of the underlying fuzzy sub-hyperspace. The complete story is explained in Theorems 5, 6, 7, and 8.

We begin by offering an example showing that the union of two fuzzy sub-hyperspaces need not be so.

Example 2. In Example 1, set

$$W_1 = \{(b, 0) : b \in \mathbb{R}\}, \quad W_2 = \{(0, c) : c \in \mathbb{R}\}. \quad (35)$$

Obviously, $(W_1, +, \circ, \mathbb{R})$ and $(W_2, +, \circ, \mathbb{R})$ are sub-hyperspaces of $V = (\mathbb{R}^2, +, \circ, \mathbb{R})$, such that for all $a, b, c \in \mathbb{R}$,

$$\begin{aligned} a \circ (b, 0) &= \begin{cases} W_1 & \text{if } b \neq 0, \\ \{0\} & \text{if } b = 0, \end{cases} \\ a \circ (0, c) &= \begin{cases} W_2 & \text{if } c \neq 0, \\ \{0\} & \text{if } c = 0. \end{cases} \end{aligned} \quad (36)$$

Choose numbers $t_i \in [0, 1]$, $0 \leq i \leq 3$, such that $t_0 > t_1 > t_2 > t_3$. Define fuzzy subsets μ_1 and μ_2 by

$$\begin{aligned} \mu_1(x) &= \begin{cases} t_0 & \text{if } x \in W_1, \\ t_3 & \text{otherwise,} \end{cases} \\ \mu_2(x) &= \begin{cases} t_1 & \text{if } x \in W_2, \\ t_2 & \text{otherwise.} \end{cases} \end{aligned} \quad (37)$$

Then $\mu_{1t_0} = W_1$, $\mu_{1t_3} = \mathbb{R}^2$, $\mu_{2t_1} = W_2$ and $\mu_{2t_2} = \mathbb{R}^2$. Thus from Proposition 4, it follows that μ_1 and μ_2 are fuzzy sub-hyperspaces of V . Clearly $\mu = \mu_1 \cup \mu_2$, given by

$$(\mu_1 \cup \mu_2)(x) = \begin{cases} t_0 & \text{if } x \in W_1, \\ t_1 & \text{if } x \in W_2 \setminus \{0\}, \\ t_2 & \text{if } x \in V \setminus W_1 \cup W_2, \end{cases} \quad (38)$$

is not a fuzzy sub-hyperspace of V , since $\mu_{t_1} = W_1 \cup W_2$ is not a sub-hyperspace of V .

Theorem 5. Let μ be a fuzzy sub-hyperspace of V such that $\{\text{Im}\}(\mu) = \{\alpha\}$, where $\alpha \in [0, 1]$. If $\mu = \mu_1 \cup \mu_2$, where μ_1 and μ_2 are fuzzy sub-hyperspaces of V , then either $\mu_1 \subseteq \mu_2$ or $\mu_2 \subseteq \mu_1$.

Proof. Since $\mu = \mu_1 \cup \mu_2$ is a fuzzy sub-hyperspace of V , the level fuzzy sub-hyperspace $(\mu_1 \cup \mu_2)_\alpha$ is a sub-hyperspace of V such that

$$\begin{aligned} (\mu_1 \cup \mu_2)_\alpha &= \{x \in V : (\mu_1 \cup \mu_2)(x) \geq \alpha\} \\ &= \{x \in V : \mu_1(x) \vee \mu_2(x) \geq \alpha\} \\ &= \{x \in V : \mu_1(x) \geq \alpha\} \cup \{x \in V : \mu_2(x) \geq \alpha\} \\ &= \mu_{1\alpha} \cup \mu_{2\alpha}. \end{aligned} \quad (39)$$

So either $\mu_{1\alpha} \subseteq \mu_{2\alpha}$ or $\mu_{2\alpha} \subseteq \mu_{1\alpha}$. Thus either $\mu_1 \subseteq \mu_2$ or $\mu_2 \subseteq \mu_1$. $\square$

Next we give an example showing that the union of two fuzzy sub-hyperspaces, such that none is contained in the other, may not be a fuzzy sub-hyperspace.

Example 3. Let V and t_i be the same as in Example 2. Define fuzzy subsets μ_1 and μ_2 of V by

$$\begin{aligned} \mu_1(x) &= \begin{cases} t_0 & \text{if } x = 0, \\ t_3 & \text{otherwise,} \end{cases} \\ \mu_2(x) &= \begin{cases} t_1 & \text{if } x = 0, \\ t_2 & \text{otherwise.} \end{cases} \end{aligned} \quad (40)$$

Then

$$(\mu_1 \cup \mu_2)(x) = \begin{cases} t_0 & \text{if } x = 0, \\ t_2 & \text{otherwise,} \end{cases} \quad (41)$$

such that $\mu_{1t_0} = \{0\}$, $\mu_{1t_3} = \mathbb{R}^2$, $\mu_{2t_1} = \{0\}$, $(\mu_1 \cup \mu_2)_{t_0} = \{0\}$ and $(\mu_1 \cup \mu_2)_{t_2} = \mathbb{R}^2$. Thus from Proposition 4, it follows that μ_1, μ_2 , and $\mu_1 \cup \mu_2$ are fuzzy sub-hyperspaces of V . However, $\mu_1 \not\subseteq \mu_2$ and $\mu_2 \not\subseteq \mu_1$.

Lemma 2. Let μ be a fuzzy sub-hyperspace of V . If $\mu(x) < \mu(y)$, for some $x, y \in V$, then

$$\mu(x - y) = \mu(x) = \mu(y - x). \quad (42)$$

Theorem 6. Let μ be a fuzzy sub-hyperspace of V such that $\{\text{Im}\}(\mu) = \{0, t\}$, where $t \in (0, 1]$. If $\mu = \mu_1 \cup \mu_2$, where μ_1 and μ_2 are fuzzy sub-hyperspaces of V , then either $\mu_1 \subseteq \mu_2$ or $\mu_2 \subseteq \mu_1$.

Proof. To obtain a proof by contradiction, assume that $\mu_1(x) > \mu_2(x)$ and $\mu_2(y) > \mu_1(y)$, for some $x, y \in V$. Then

$$\begin{aligned} \mu(x) &= (\mu_1 \cup \mu_2)(x) = \mu_1(x) > \mu_2(x) \geq 0, \\ \mu(y) &= (\mu_1 \cup \mu_2)(y) = \mu_2(y) > \mu_1(y) \geq 0. \end{aligned} \quad (43)$$

Therefore

$$\mu(x) = t = \mu_1(x) = \mu(y) = \mu_2(y), \quad (44)$$

since $\text{Im}(\mu) = \{0, t\}$. So $\mu_1(y) < \mu_1(x)$, thus by Lemma 2 we obtain that

$$\mu_1(x - y) = \mu_1(y) < \mu_2(y) = t, \quad (45)$$

and similarly

$$\mu_2(x - y) = \mu_2(x) < \mu_1(x) = t. \quad (46)$$

Hence

$$\mu(x - y) = \mu_1(x - y) \vee \mu_2(x - y) < t. \quad (47)$$

Again, $\mu(x - y) = \mu(x) \wedge \mu(y) = t$, the desired contradiction. $\square$

Theorem 7. Let μ be a fuzzy sub-hyperspace of V such that $3 \leq |\text{Im}\{\mu\}| < \infty$. Then there exist fuzzy sub-hyperspaces μ_1 and μ_2 of V such that $\mu = \mu_1 \cup \mu_2$, $\mu_1 \not\subseteq \mu_2$ and $\mu_2 \not\subseteq \mu_1$.

Proof. Let $\text{Im}(\mu) = \{t_0, t_1, \dots, t_n\}$, where $2 \leq n < \infty$, and $t_0 > t_1 > \dots > t_n$. Choose $s_1, s_2 \in [0, 1]$ such that

$$1 \geq t_0 > s_1 > t_1 > s_2 > t_2 > t_3 > \dots > t_n. \quad (48)$$

Define fuzzy subsets μ_1 and μ_2 of V by

$$\begin{aligned} \mu_1(x) &= \begin{cases} t_0 & \text{if } x \in \mu_{t_0}, \\ s_2 & \text{if } x \in \mu_{t_1} \setminus \mu_{t_0}, \\ \mu(x) & \text{otherwise,} \end{cases} \\ \mu_2(x) &= \begin{cases} s_1 & \text{if } x \in \mu_{t_0}, \\ t_1 & \text{if } x \in \mu_{t_1} \setminus \mu_{t_0}, \\ \mu(x) & \text{otherwise.} \end{cases} \end{aligned} \quad (49)$$

Clearly μ_1, μ_2 , and $\mu_1 \cup \mu_2$ are fuzzy sub-hyperspaces of V such that $\mu = \mu_1 \cup \mu_2$. However, $\mu_1 \not\subseteq \mu_2$ and $\mu_2 \not\subseteq \mu_1$. $\square$

Theorem 8. If μ is a fuzzy sub-hyperspace of V such that $|\text{Im}\{\mu\}| = \{t_0, t_1\}$, where $t_0, t_1 \in (0, 1]$ and $t_0 > t_1$, then there exist fuzzy sub-hyperspaces μ_1 and μ_2 of V such that $\mu = \mu_1 \cup \mu_2$, $\mu_1 \not\subseteq \mu_2$ and $\mu_2 \not\subseteq \mu_1$.

4. FUZZY SUB-HYPERSPACE GENERATED BY A FUZZY SUBSET

In this section, we give the construction of the fuzzy sub-hyperspace $\bar{\mu}$ generated by a fuzzy subset μ of V . It is possible that $|\text{Im}(\mu)|$ may be strictly greater than $|\text{Im}(\bar{\mu})|$. For any $W \subseteq V$, we will write $\langle W \rangle$ for the sub-hyperspace generated by W in V .

Theorem 9. Let μ be a fuzzy subset of V such that $|\text{Im}(\mu)| < \infty$. Define sub-hyperspaces $V_i, 0 \leq i \leq k$, by

$$\begin{aligned} V_0 &= \left\langle \left\{ x \in V : \mu(x) = \bigvee_{z \in V} \mu(z) \right\} \right\rangle, \\ V_i &= \left\langle V_{i-1}, \left\{ x \in V : \mu(x) = \bigvee_{z \in V \setminus V_{i-1}} \mu(z) \right\} \right\rangle, \end{aligned} \quad (50)$$

where $k \leq |\text{Im}\{\mu\}|$ and $V_k = V$. Then the fuzzy subset $\bar{\mu}$ of V defined by

$$\bar{\mu}(x) = \begin{cases} \bigvee_{z \in V} \mu(z) & \text{if } x \in \hat{V}_0 = V_0, \\ \bigvee_{z \in V \setminus V_{i-1}} \mu(z) & \text{if } x \in \hat{V}_i = V_i \setminus V_{i-1} \end{cases} \quad (51)$$

is the fuzzy sub-hyperspace generated by μ in V .

Proof. By Proposition 4, we obtain that $\bar{\mu}$ is a fuzzy sub-hyperspace of V . Now $V_0 = \langle S_0 \rangle$, where $S_0 = \{x \in V : \mu(x) = \bigvee_{z \in V} \mu(z)\}$ and for $i = 1, \dots, k$, $V_i = \langle V_{i-1}, S_i \rangle$, where

$$S_i = \left\{ x \in V : \mu(x) = \bigvee_{z \in V \setminus V_{i-1}} \mu(z) \right\}. \quad (52)$$

Let η be any fuzzy sub-hyperspace of V containing μ . It is sufficient to prove that

$$\eta(x) \geq \bar{\mu}(x) \geq \mu(x), \quad \text{for all } x \in \hat{V}_i, i = 1, \dots, k. \quad (53)$$

Let $x \in \hat{V}_0$. Then there exist $a_{01}, a_{02}, \dots, a_{0m} \in K$ and $s_{01}, s_{02}, \dots, s_{0m} \in S_0$, such that $x \in a_{01} \circ s_{01} + \dots + a_{0m} \circ s_{0m}$. Now

$$\begin{aligned} \eta(x) &\geq \bigwedge_{t \in a_{01} \circ s_{01} + \dots + a_{0m} \circ s_{0m}} \mu(t) \\ &\geq \mu(s_{01}) \wedge \mu(s_{02}) \wedge \dots \wedge \mu(s_{0m}) \\ &\quad (\text{since } \eta \text{ is a fuzzy sub-hyperspace}) \\ &= \bigvee_{z \in V} \mu(z) \\ &\geq \mu(x). \end{aligned} \quad (54)$$

Hence $\eta(x) \geq \bar{\mu}(x) \geq \mu(x)$, $\forall x \in V_0$.

Let $x \in \hat{V}_i, i = 1, \dots, k$. Then there exist $a_{i1}, a_{i2}, \dots, a_{im} \in K$ and $s_{i1}, s_{i2}, \dots, s_{im} \in S_i$ such that $x \in a_{i1} \circ s_{i1} + \dots + a_{im} \circ s_{im}$. Now

$$\begin{aligned} \eta(x) &\geq \bigwedge_{t \in a_{i1} \circ s_{i1} + \dots + a_{im} \circ s_{im}} \mu(t) \\ &\geq \mu(s_{i1}) \wedge \mu(s_{i2}) \wedge \dots \wedge \mu(s_{im}) \\ &\quad (\text{since } \eta \text{ is a fuzzy sub-hyperspace}) \\ &= \bigvee_{z \in V \setminus V_{i-1}} \mu(z) \\ &\geq \bigvee_{z \in \hat{V}_i} \mu(z) \quad (\text{since } \hat{V}_i \subseteq V \setminus V_{i-1}) \\ &\geq \mu(x). \end{aligned} \quad (55)$$

Hence $\eta(x) \geq \bar{\mu}(x) \geq \mu(x)$, for all $x \in \hat{V}_i$, as desired. $\square$

5. FUZZY COSETS

The following theorem will serve as a guiding factor in defining a fuzzy coset of a fuzzy sub-hyperspace.

Theorem 10. (i) Let μ be a fuzzy sub-hyperspace of V and $t = \mu(0)$. Then the fuzzy subset

$$\begin{aligned} \mu^* : V/\mu_t &\longrightarrow [0, 1], \\ \mu^*(x + \mu_t) &= \mu(x) \end{aligned} \quad (56)$$

is a fuzzy sub-hyperspace of V/μ_t .

(ii) If W is a sub-hyperspace of V and η is a fuzzy sub-hyperspace of V/W such that $\mu^*(x + W) = \eta(W)$ only when $x \in W$, then there exists a fuzzy sub-hyperspace μ of V such that $\mu_t = W(t = \mu(0))$ and $\eta = \mu^*$.

Proof. (i) It is easy to verify that μ^* is well defined. We now show that μ^* is a fuzzy sub-hyperspace of V/μ_t . Let $x, x_1, x_2 \in V$ and $a \in K$. Then

$$\begin{aligned}\mu^*((x_1 + \mu_t) - (x_2 + \mu_t)) &= \mu^*(x_1 - x_2 + \mu_t) \\ &= \mu(x_1 - x_2) \\ &\geq \mu(x_1) \wedge \mu(x_2) \\ &\geq \mu^*(x_1 + \mu_t) \wedge \mu^*(x_2 + \mu_t),\end{aligned}\quad (57)$$

and if $s \in a \circ x$, then

$$\begin{aligned}\mu^*(s + \mu_t) &= \mu(s) \\ &\geq \bigwedge_{t \in a \circ x} \mu(t) \\ &\geq \mu(x) \\ &= \mu^*(x + \mu_t),\end{aligned}\quad (58)$$

thus

$$\bigwedge_{s \in a \circ x} \mu^*(s + \mu_t) \geq \mu^*(x + \mu_t). \quad (59)$$

Therefore

$$\bigwedge_{t \in a \circ (x + \mu_t) = a \circ x + \mu_t} \mu(t) \geq \mu^*(x + \mu_t). \quad (60)$$

(ii) Define a fuzzy subset μ of V by

$$\mu(x) = \eta(x + W), \quad \forall x \in V. \quad (61)$$

Clearly μ is a fuzzy sub-hyperspace of V . Now $\mu_t = W$, because

$$\begin{aligned}x \in \mu_t &\iff \mu(x) = t = \mu(\underline{0}) \\ &\iff \eta(x + W) = \eta(W) \\ &\iff x \in W.\end{aligned}\quad (62)$$

Moreover $\eta = \mu^*$, since

$$\mu^*(x + W) = \mu^*(x + \mu_t) = \mu(x) = \eta(x + W). \quad (63)$$

□

Definition 15. Let μ be a fuzzy sub-hyperspace of V . For $x \in V$, the fuzzy subset μ_x^* of V defined by

$$\mu_x^*(z) = \mu(x - z), \quad \forall z \in V, \quad (64)$$

is called the fuzzy coset determined by x and μ . The set of all fuzzy cosets of μ is denoted by V_μ , that is, $V_\mu = \{\mu_x^* : x \in V\}$.

Proposition 8. Let μ be a fuzzy sub-hyperspace of $(V, +, \circ, K)$. Then the set of all fuzzy cosets of μ , $V_\mu = \{\mu_x^* : x \in V\}$, with operation

$$\begin{aligned}\oplus : V_\mu \times V_\mu &\longrightarrow V_\mu, \\ \mu_x^* \oplus \mu_y^* &= \mu_{x+y}^*\end{aligned}\quad (65)$$

and external operation

$$\begin{aligned}\odot : K \times V_\mu &\longrightarrow P_*(V_\mu), \\ a \odot \mu_x^* &= \mu_{a \circ x}^* = \{\mu_t^* : t \in a \circ x\}\end{aligned}\quad (66)$$

is a hypervector space over the field K .

Proof. It is easy to verify that “+” and “ $\circ$ ” are well defined. Let $a, b \in K$ and $x, y \in V$. Then

(H₁)

$$\begin{aligned}a \odot (\mu_x^* \oplus \mu_y^*) &= a \odot (\mu_{x+y}^*) \\ &= \{\mu_t^* : t \in a \circ (x + y)\} \\ &\subseteq \{\mu_t^* : t \in a \circ x + a \circ y\} \\ &= \{\mu_t^* : t = t_1 + t_2, t_1 \in a \circ x, t_2 \in a \circ y\} \\ &= \{\mu_{t_1+t_2}^* : t_1 \in a \circ x, t_2 \in a \circ y\} \\ &= \{\mu_{t_1}^* \oplus \mu_{t_2}^* : t_1 \in a \circ x, t_2 \in a \circ y\} \\ &= \{\mu_{t_1}^* : t_1 \in a \circ x\} \oplus \{\mu_{t_2}^* : t_2 \in a \circ y\} \\ &= (a \odot \mu_x^*) \oplus (a \odot \mu_y^*);\end{aligned}\quad (67)$$

(H₂)

$$\begin{aligned}(a + b) \odot \mu_x^* &= \{\mu_t^* : t \in (a + b) \circ x\} \\ &\subseteq \{\mu_t^* : t \in a \circ x + b \circ x\} \\ &= \{\mu_t^* : t = t_1 + t_2, t_1 \in a \circ x, t_2 \in b \circ x\} \\ &= \{\mu_{t_1+t_2}^* : t_1 \in a \circ x, t_2 \in b \circ x\} \\ &= \{\mu_{t_1}^* \oplus \mu_{t_2}^* : t_1 \in a \circ x, t_2 \in b \circ x\} \\ &= \{\mu_{t_1}^* : t_1 \in a \circ x\} \oplus \{\mu_{t_2}^* : t_2 \in b \circ x\} \\ &= (a \odot \mu_x^*) \oplus (b \odot \mu_x^*);\end{aligned}\quad (68)$$

(H₃)

$$\begin{aligned}a \odot (b \odot \mu_x^*) &= \bigcup_{\mu_t^* \in b \odot \mu_x^*} a \odot \mu_t^* \\ &= \bigcup_{t \in b \circ x} a \odot \mu_t^* \\ &= \bigcup_{t \in b \circ x} \{\mu_s^* : s \in a \circ t\} \\ &= \{\mu_s^* : s \in a \circ (b \circ x)\} \\ &= \{\mu_s^* : s \in (ab) \circ x\} \\ &= (ab) \odot \mu_x^*;\end{aligned}\quad (69)$$

(H₄)

$$\begin{aligned}a \odot (-\mu_x^*) &= a \odot (\mu_{-x}^*) \\ &= \{\mu_t^* : t \in a \circ (-x)\} \\ &= \{\mu_t^* : t \in (-a) \circ x\} \\ &= (-a) \odot \mu_x^* = -(a \odot \mu_x^*);\end{aligned}\quad (70)$$

(H₅) for all $\mu_x^* \in V_\mu$, $\mu_x^* \in 1 \odot \mu_x^* = \{\mu_t^* : t \in 1 \circ x\}$, since $x \in 1 \circ x$. □

The following lemma will serve as a powerful tool in proving Theorem 11 regarding the dim of the hypervector space V_μ .

Lemma 3. Let μ be a fuzzy sub-hyperspace of V . Then

$$\forall x \in V, \quad \mu(x) = \mu(\underline{0}) \iff \mu_x^* = \mu_{\underline{0}}^*. \quad (71)$$

Proof. Let $\mu(x) = \mu(\underline{0})$. Then for all $z \in V$, $\mu(z) \leq \mu(x) = \mu(\underline{0})$. If $\mu(z) < \mu(x) = \mu(\underline{0})$, then $\mu(z - x) = \mu(z)$, by Lemma 2. If $\mu(z) = \mu(x) = \mu(\underline{0})$, then $z, x \in \mu_t$, where $t = \mu(\underline{0})$. Hence $\mu(z - x) = t = \mu(\underline{0}) = \mu(z)$. Thus in either case, we have shown that for all $z \in V$, $\mu(z - x) = \mu(z)$. That is, for all $z \in V$, $\mu(x - z) = \mu(z)$. Consequently $\mu_x^* = \mu_{\underline{0}}^*$. The converse is straightforward. $\square$

Theorem 11. Let μ be a fuzzy sub-hyperspace of V and $t = \mu(\underline{0})$. Then

$$\dim V_\mu = \dim V/\mu_t. \quad (72)$$

Proof. If for all $z \in V$, $\mu(z) = \alpha$, then, for all $z \in V$, $\mu(z) = \mu(\underline{0})$. So that $V_\mu = \mu_{\underline{0}}^*$ and $\mu_t = V$. Hence

$$\dim V_\mu = 0 = \dim V/\mu_t. \quad (73)$$

So we assume that μ is not constant. Let $\dim V = n$ and $\dim \mu_t = m$. Let

$$\{v_1, v_2, \dots, v_m, x_1, x_2, \dots, x_r\}, \quad r = n - m, \quad (74)$$

be a basis of V such that $\{v_1, v_2, \dots, v_m\}$ is a basis of μ_t . Then $\{x_1 + \mu_t, x_2 + \mu_t, \dots, x_m + \mu_t\}$ is a basis of V/μ_t . We will show that the set $\beta^* = \{\mu_{x_1}^*, \mu_{x_2}^*, \dots, \mu_{x_r}^*\}$ is a basis of V_μ over the field K .

Clearly $\mu_{x_i}^* \neq \mu_{x_j}^*$ whenever $i \neq j$, because

$$\begin{aligned} \mu_{x_i}^* &= \mu_{x_j}^* \\ \implies \mu_{x_i - x_j}^* &= \mu_{\underline{0}}^*, \\ \implies \mu(x_i - x_j) &= \mu(\underline{0}), \text{ by Lemma 3,} \\ \implies x_i - x_j &\in \mu_t, \\ \implies x_i + \mu_t &= x_j + \mu_t, \text{ a contradiction.} \end{aligned} \quad (75)$$

Next we prove that the set β^* generates V_μ over K . For this, let $\mu_x^* \in V_\mu$, $\mu_x^* \neq \mu_{\underline{0}}^*$. Then $\mu(x) \neq \mu(\underline{0})$, by Lemma 3. So that $x + \mu_t$ is a nonzero element of V/μ_t . Hence there exist $a_1, \dots, a_r \in K$ such that

$$\begin{aligned} x + \mu_t &\in a_1 * (x_1 + \mu_t) + \dots + a_r * (x_r + \mu_t) \\ &= a_1 \circ x_1 + \mu_t + \dots + a_r \circ x_r + \mu_t \\ &= a_1 \circ x_1 + \dots + a_r \circ x_r + \mu_t, \\ \implies \exists t_1, \dots, t_r, \quad t_i &\in a_i \circ x_i : x + \mu_t = t_1 + \dots + t_r + \mu_t, \\ \implies x - (t_1 + \dots + t_r) &\in \mu_t, \\ \implies \mu(x - t_1 - \dots - t_r) &= \mu(\underline{0}), \\ \implies \mu_{(x - t_1 - \dots - t_r)}^* &= \mu_{\underline{0}}^*, \text{ by Lemma 3,} \end{aligned} \quad (76)$$

thus

$$\begin{aligned} \mu_x^* &= \mu_{t_1 + \dots + t_r}^* \\ &= \mu_{t_1}^* \oplus \dots \oplus \mu_{t_r}^* \\ &\in a_1 \odot \mu_{x_1}^* \oplus \dots \oplus a_r \odot \mu_{x_r}^*. \end{aligned} \quad (77)$$

Also β^* is linearly independent over K , because if $\mu_{\underline{0}}^* \in a_1 \odot \mu_{x_1}^* \oplus \dots \oplus a_r \odot \mu_{x_r}^*$, where $a_1, \dots, a_r \in K$, then $\exists \mu_{t_i}^* \in a_1 \odot \mu_{x_i}^*$, $1 \leq i \leq r$, such that

$$\begin{aligned} \mu_{\underline{0}}^* &= \mu_{t_1}^* \oplus \dots \oplus \mu_{t_r}^* = \mu_{t_1 + \dots + t_r}^*, \\ \implies \mu(t_1 + \dots + t_r) &= \mu(\underline{0}), \\ \implies t_1 + \dots + t_r &\in \mu_t, \end{aligned} \quad (78)$$

so there exist $b_1, \dots, b_m \in K$, such that

$$t_1 + \dots + t_r \in b_1 \circ v_1 + \dots + b_m \circ v_m, \quad (79)$$

thus there exist $s_j \in b_j \circ v_j$, $1 \leq j \leq m$, such that

$$\begin{aligned} t_1 + \dots + t_r &= s_1 + \dots + s_m, \\ \implies \underline{0} &= t_1 + \dots + t_r - s_1 - \dots - s_m, \\ \implies \underline{0} &\in a_1 \odot x_1 + \dots + a_r \odot x_r - b_1 \odot v_1 - \dots - b_m \odot v_m, \\ \implies a_1 = \dots = a_r &= b_1 = \dots = b_m = 0. \end{aligned} \quad (80)$$

Hence

$$\dim V_\mu = r = n - m = \dim V - \dim \mu_t = \dim V/\mu_t. \quad (81) \quad \square$$

Theorem 12. Let μ be a fuzzy sub-hyperspace of V and $t = \mu(\underline{0})$. Then

(a) the fuzzy subset μ' of V_μ defined by

$$\forall \mu_x^* \in V_\mu, \quad \mu'(\mu_x^*) = \mu(x) \quad (82)$$

is a fuzzy subhyperspace of V_μ ;

(b) the map

$$\begin{aligned} f : V &\longrightarrow V_\mu, \\ x &\longmapsto \mu_x^* \end{aligned} \quad (83)$$

is an onto good transformation with kernel μ_t , where $t = \mu(\underline{0})$.

Proof. (a) (i)

$$\begin{aligned} \mu'(\mu_x^* \oplus \mu_y^*) &= \mu'(\mu_{x+y}^*) = \mu(x + y) \\ &\geq \mu(x) \wedge \mu(y) = \mu'(\mu_x^*) \wedge \mu'(\mu_y^*), \end{aligned} \quad (84)$$

$$(ii) \mu'(-\mu_x^*) = \mu'(\mu_{-x}^*) = \mu(x) \geq \mu(-x) = \mu'(\mu_x^*),$$

$$(iii) \text{ for all } \mu_t^* \in a \odot \mu_x^*,$$

$$\mu'(\mu_t^*) = \mu(t) \geq \bigwedge_{s \in a \odot x} \mu(s) \geq \mu(x) = \mu'(\mu_x^*), \quad (85)$$

thus $\bigwedge_{\mu_t^* \in a \odot \mu_x^*} \mu'(\mu_t^*) \geq \mu'(\mu_x^*)$.

(b) For every $a \in K$ and $x, y \in V$,

$$\begin{aligned} f(x + y) &= \mu_{x+y}^* = \mu_x^* \oplus \mu_y^* = f(x) \oplus f(y), \\ f(a \circ x) &= \mu_{a \circ x}^* = a \odot f(x), \end{aligned} \quad (86)$$

thus f is an onto good transformation. Moreover, $\ker f = \{x \in V : f(x) \in \Omega_{V_\mu}\}$, where

$$\Omega_{V_\mu} = 0 \odot \mu_0^* = \{\mu_t^* : t \in 0 \circ \underline{0} = \Omega_V\}, \quad (87)$$

therefore

$$\begin{aligned} \ker f &= \{x \in V : \mu_x^* = \mu_t^*, \text{ for some } t \in 0 \circ \underline{0}\} \\ &= \{x \in V : \mu(x) = \mu(t), \text{ for some } t \in 0 \circ \underline{0}\}, \end{aligned} \quad (88)$$

but $\mu(t) \geq \bigwedge_{s \in 0 \circ \underline{0}} \mu(s) \geq \mu(\underline{0})$, so

$$\ker f = \{x \in V : \mu(x) \geq \mu(\underline{0})\}. \quad (89)$$

Thus $\ker f = \mu_t$, where $t = \mu(\underline{0})$. $\square$

Definition 16. Let μ be a fuzzy sub-hyperspace of V . The fuzzy subset μ' of V_μ is called the fuzzy quotient hypervector space determined by μ .

Finally, we establish for fuzzy sub-hyperspace an analogue of a consequence of the “fundamental theorem of homomorphisms.”

Theorem 13. Let μ be a fuzzy sub-hyperspace of V . Then each fuzzy sub-hyperspace of V_μ corresponds in a natural way to a fuzzy sub-hyperspace of V .

Proof. Let μ' be any fuzzy sub-hyperspace of V_μ . Then it is easy to see that the fuzzy subset μ of V defined by

$$\forall x \in V, \quad \mu(x) = \mu'(\mu_x^*), \quad (90)$$

is a fuzzy sub-hyperspace of V . $\square$

6. CONCLUSION

As discussed above, the notion of a fuzzy hypervector space is a generalization of fuzzy vector spaces and the paper provides the basic notions and results to study the fuzzy hypervector spaces. We hope that this paper encourages the researchers to study the more properties of fuzzy hypervector spaces and its application.

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REFERENCES

- [1] F. Marty, “Sur une generalization de la notion de groupe,” in *Proceedings of the 8th Congress des Mathematiciens Scandinaves*, pp. 45–49, Stockholm, Sweden, 1934.

- [2] P. Corsini, *Prolegomena of Hypergroup Theory*, Aviani Editor, Udine, Italy, 2nd edition, 1993.
- [3] P. Corsini and V. Leoreanu, *Applications of Hyperstructure Theory*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.
- [4] T. Vougiuklis, *Hyperstructures and Their Representations*, Hadronic Press, Palm Harbor, Fla, USA, 1994.
- [5] M. S. Tallini, “Hypervector spaces,” in *Proceeding of the 4th International Congress in Algebraic Hyperstructures and Applications*, pp. 167–174, Xanthi, Greece, 1990.
- [6] M. S. Tallini, “Weak hypervector spaces and norms in such spaces,” in *Algebraic Hyperstructures and Applications*, pp. 199–206, Hadronic Press, Palm Harbor, Fla, USA, 1994.
- [7] L. A. Zadeh, “Fuzzy sets,” *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.
- [8] A. Rosenfeld, “Fuzzy groups,” *Journal of Mathematical Analysis and Applications*, vol. 35, no. 3, pp. 512–517, 1971.
- [9] S. Nanda, “Fuzzy linear spaces over valued fields,” *Fuzzy Sets and Systems*, vol. 42, no. 3, pp. 351–354, 1991.
- [10] A. K. Katsaras and D. B. Liu, “Fuzzy vector spaces and fuzzy topological vector spaces,” *Journal of Mathematical Analysis and Applications*, vol. 58, no. 1, pp. 135–146, 1977.
- [11] R. Ameri, “Fuzzy hypervector spaces over valued fields,” *Iranian Journal of Fuzzy Systems*, vol. 2, no. 1, pp. 37–47, 2005.
- [12] R. Ameri, “Fuzzy (Co-)norm hypervector spaces,” in *Proceeding of the 8th International Congress in Algebraic Hyperstructures and Applications*, pp. 71–79, Samotraki, Greece, September 2002.
- [13] R. Ameri and M. M. Zahedi, “Hypergroup and join spaces induced by a fuzzy subset,” *Pure Mathematics and Applications*, vol. 8, pp. 155–168, 1997.
- [14] R. Ameri and M. M. Zahedi, “Fuzzy subhypermodules over fuzzy hyperrings,” in *Proceedings of the 6th International Congress in Algebraic Hyperstructures and Applications*, pp. 1–14, Democritus University, Prague, Czech Republic, September 1996.
- [15] P. Corsini and V. Leoreanu, “Fuzzy sets and join spaces associated with rough sets,” *Rendiconti del Circolo Matematico di Palermo*, vol. 51, no. 3, pp. 527–536, 2002.
- [16] P. Corsini and I. Tofan, “On fuzzy hypergroups,” *Pure Mathematics and Applications*, vol. 8, no. 1, pp. 29–37, 1997.
- [17] B. Davvaz, “Fuzzy H_μ -submodules,” *Fuzzy Sets and Systems*, vol. 117, no. 3, pp. 477–484, 2001.
- [18] B. Davvaz, “Fuzzy H_μ -groups,” *Fuzzy Sets and Systems*, vol. 101, no. 1, pp. 191–195, 1999.
- [19] R. Ameri and O. R. Dehghan, “On dimension of hypervector spaces,” to appear in *European Journal in Mathematics*.

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