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Research Article

Two New Types of Rings Defined by Using a Translational Invariant Fuzzy Subset

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We use a translational invariant fuzzy subset p of a ring R to define two new types of commutative rings namely, p-presimplifiable and p-associate rings. We present some results of these rings. The interest of these results is that most of them are mirrors of corresponding results of presimplifiable and associate rings in classical ring theory.

1. Introduction

Throughout this paper, all rings are assumed to be commutative with unity. If R is a ring, then $J(R)$, $\mathrm{nil}(R)$, $Z(R)$ and $U(R)$ denote the Jacobson radical of R, the nilradical of R, the set of zero divisors of R, and the set of units of R, respectively.

Recall that a ring R is presimplifiable if whenever $a, b \in R$ with property $a = ab$ we have $a = 0$ or $b \in U(R)$. And, a ring R is associate if whenever $a, b \in R$ satisfy $a = tb$ and $b = sa$ for some $t, s \in R$, there is a unit u such that $a = ub$. The class of associate rings contains a large class of rings such as presimplifiable rings, Principal ideal rings, Artinian rings, regular rings, pp-rings, and stable range one rings. This class of rings was originally studied in Kaplansky [1]. Then Bouvier studied presimplifiable rings in a series of papers [2–5]. Recently, the class of associate rings was studied more extensively by Anderson and Valdes-Leon [6, 7], Spellman et al. [8], and Anderson et al. [9].

Some well-known properties of presimplifiable and associate rings will be given in the following.

Remark 1. (1) A ring R is presimplifiable if and only if $Z(R) \subseteq J(R)$; see Anderson et al. [9].

(2) A homomorphic image of a presimplifiable ring need not be associate; see Spellman et al. [8].

(3) Integral domains, domain-like rings (i.e., $Z(R) \subseteq \mathrm{nil}(R)$) and local rings are examples of presimplifiable rings; see Anderson et al. [9].

(4) If R_1 and R_2 are integral domains and R_3 is a ring with epimorphisms $f_i : R_i \to R_3$, $i = 1, 2$, which are not isomorphisms, then the pullback $\mathbf{P}$ of $R_1 \to R_3 \leftarrow R_2$ is presimplifiable if and only if $f_i^{-1}(1) \subseteq U(R_i)$, $i = 1, 2$; see Anderson et al. [9].

(5) presimplifiable, regular rings are examples of associate rings; see Anderson et al. [9].

Algebraic structures play a prominent role in mathematics with wide ranging applications in many areas such as topological spaces, theoretical physics, coding theory, and computer sciences. This provides sufficient motivation to researchers to extend various concepts and results from the realm of abstract algebra to the broader framework of fuzzy setting, although not all results in algebra can be fuzzified. The concept of a fuzzy subset of a set was introduced by Zadeh [10]. Fuzzy subgroup and their important properties were defined and established by Rosenfeld [11]. Fuzzy ideal of rings was introduced by Liu [12]. The notion of translational invariant fuzzy subset was introduced by Ray in [13]. This notion gave immense scope for extending the classical results of different algebraic structures. Our aim of this paper is to extend the classical results of presimplifiable and associate rings to the fuzzy setting.

Recall that a fuzzy subset p of a set A is a mapping from A into the closed unit interval $[0, 1]$. A fuzzy subset p is called translational invariant with respect to a binary operation "$*$"

if it satisfies the following condition: $p(x) = p(y)$ implies that $p(x * a) = p(y * a)$ for every $x, y, a \in A$. An element $u \in R$ with $p(u) \neq p(0)$ is called a p-unit of a ring R if there exists $v \in R$ with $p(v) \neq p(0)$ such that $p(uv) = p(1)$. If p is a translational invariant fuzzy subset of R with respect to both "+" and "$\cdot$" and $p(1) \neq p(0)$, then it is easy to prove that every unit is a p-unit. And uv is p-unit if and only if u and v are p-units.

Now, we define two new types of commutative rings by using a translational invariant fuzzy subset of a ring.

Definition 2. Let R be a ring and p a translational invariant fuzzy subset of R with respect to both "+" and "$\cdot$" satisfying $p(0) \neq p(1)$ and $p(x) = p(-x)$ for every $x \in R$.

(1) R is said to be a p-presimplifiable if whenever $a, b \in R$ with property $p(a) = p(ab)$ we have $p(a) = p(0)$ or b is a p-unit.

(2) R is said to be a p-associate if whenever $a, b \in R$ satisfy $p(a) = p(sb)$ and $p(b) = p(ta)$ for some $s, t \in R$ implies that there is a p-unit u such that $p(a) = p(ub)$.

It is clear that the concepts of p-presimplifiable and p-associate rings are generalization of the presimplifiable and the associate rings in classical ring theory.

In this paper, in section two, we will study some properties of p-presimplifiable and p-associate rings and we will show that most of them are very close to that of presimplifiable and associate rings in classical ring theory. In section three, unlike the classical ring case, we will show the structure preserving nature of p-presimplifiable and p-associate rings under ring epimorphism.

Remark 3. In the next two sections, p is assumed to be a translational invariant fuzzy subset of R with respect to both "+" and "$\cdot$" satisfying $p(0) \neq p(1)$ and $p(x) = p(-x)$ for every $x \in R$.

2. p-Presimplifiable and p-Associate Rings

Our aim of this section is to study some properties of p-presimplifiable and p-associate rings.

Next, we give a characterization of p-presimplifiable rings but first we need to state the following.

Definition 4. An element $a \in R$ is said to be a p-zero if $p(a) = p(0)$.

Definition 5 (Ray and Ali [14]). An element $a \in R$ with $p(a) \neq p(0)$ is said to be a p-divisor of zero if there exists $b \in R$ with $p(b) \neq p(0)$ such that $p(ab) = p(0)$.

The proof of the following lemma is straightforward.

Lemma 6. (1) *The set $Z_p(R) = \{a \in R : a$ is a p-divisor of zero or a is a p-zero $\}$ is containing the set $Z(R)$.*

(2) *The set $N_p(R) = \{a \in R : p(a^n) = p(0)$ for some $n \in \mathbb{N}\}$ is an ideal containing $N(R)$.*

(3) *The set $J_p(R) = \{a \in R : 1 + at$ is a p-unit for every $t \in R\}$ is an ideal of R containing $J(R)$.*

Theorem 7. *The following statements are equivalent.*

(1) *R is a p-presimplifiable ring.*

(2) *$Z_p(R) \subseteq J_p(R)$.*

Proof. (1)$\Rightarrow$(2) Suppose that R is p-presimplifiable ring. Let $a \in Z_p(R)$, then there exists $b \in R$ such that $p(b) \neq p(0)$ and $p(ab) = p(0)$. But $p((at + 1)b) = p(b)$ for every $t \in R$ because p is a translational invariant. So, $at + 1$ is a p-unit and hence $Z_p(R) \subseteq J_p(R)$.

(2)$\Rightarrow$(1) Suppose that $a, b \in R$ such that $p(a) = p(ba)$ and $p(a) \neq p(0)$. Then $p(a(1 - b)) = p(0)$. Hence $p(b) = p(1)$ or $1 - b \in Z_p(R) \subseteq J_p(R)$. Therefore, b is a p-unit of R. Thus R is p-presimplifiable. □

Definition 8. (1) A ring R is said to be a p-integral domain if R has no p-divisor of zero.

(2) A ring R is said to be a p-domain-like if $Z_p(R) \subseteq N_p(R)$.

Clearly, every p-integral domain is p-domain-like. However, the converse need not be true for example, $\mathbb{Z}_4[i]$ with a fuzzy subset p defined by $p(0) = p(2) = p(2i) = p(2 + 2i) = 0$, $p(\pm 1) = p(\pm 1 + 2i) = 0.3, p(\pm i) = p(2 \pm i) = 0.5$, and $p(\pm 1 \pm i) = 1$ is p-domain-like but not p-integral domain.

Theorem 9. *Every p-domain-like ring is p-presimplifiable.*

Proof. It is enough to show that $N_p(R) \subseteq J_p(R)$. So, let $a \in N_p(R)$; then $p(a^n) = p(0)$ for some $n \in \mathbb{N}$. Hence $p(1 - (-at)^n) = p(1)$ for every $t \in R$ because p is translational invariant. Thus $1 + at$ is a p-unit. So $a \in J_p(R)$. □

Definition 10. A ring R is said to be a p-local if for every $r, s \in R$ with $p(r + s) = p(1)$, it follows that r or s is a p-unit.

Example 11. (1) Let $R = \mathbb{Z}$ and p a fuzzy subset of R defined by $p(x) = 1$ if x is even and $p(x) = 0$ if x is odd; then R is p-local ring.

(2) Let $R = \mathbb{Z}_4[i]$ and p a fuzzy subset of R defined by $p(0) = p(2) = p(2i) = p(2+2i) = 0.1, p(\pm 1) = p(\pm 1+2i) = p(\pm 1 \pm i) = 0.5, p(\pm i) = p(2 \pm i) = 1$. Then R is p-local ring.

Theorem 12. *Every p-local ring is p-presimplifiable.*

Proof. Note that, $J_p(R)$ is the set of all non-p-units of R. □

Now, we consider the p-pullback.

Definition 13. Let R_1, R_2, and R_3 be any three rings with homomorphisms $f_i : R_i \to R_3$, $i = 1, 2$, which preserves the unity. Let $p_i : R_i \to [0, 1]$, $i = 1, 2$, be fuzzy subsets of R_i. If for every $x \in R_1$ and $y \in R_2$ satisfying $f_1(x) = f_2(y)$ we have that $p_1(x) = p_2(y)$, then the set $\mathbf{P} = \{(x, y) \in R_1 \times R_2 : f_1(x) = f_2(y)\}$ with the fuzzy subset p of $\mathbf{P}$ defined by $p((x, y)) = p_1(x)$ for every $(x, y) \in \mathbf{P}$ is a subring of $R_1 \times R_2$ called a p-pullback of $R_1 \to R_3 \leftarrow R_2$ with the set of p-units $U_p(\mathbf{P}) = \{(u_1, u_2) \in \mathbf{P} : u_1$ and u_2 are p-units$\}$.

Theorem 14. *Let R_1 be a p_1-presimplifiable, R_2 a p_2-presimplifiable, and R_3 a ring with epimorphisms $f_i : R_i \to R_3$, $i = 1, 2$. If for every $x \in R_1$ and $y \in R_2$ satisfying $f_1(x) = f_2(y)$ we have that $p_1(x) = p_2(y)$, then the p-pullback of $R_1 \to R_3 \leftarrow R_2$ is p-presimplifiable.*

Proof. Let $(a,b) \in \mathbf{P}$ such that $p((a,b)) \neq p((0,0))$ and $p((a,b)) = p((m,n)(a,b))$ for some $(m,n) \in \mathbf{P}$. Then $p_1(a) \neq p_1(0)$, $p_1(ma) = p_1(a)$, and $p_2(b) \neq p_2(0)$, $p_2(nb) = p_2(b)$. But R_1 is p_1-presimplifiable and R_2 is p_2-presimplifiable. Thus $(m,n) \in U_p(\mathbf{P})$. □

Example 15. (1) Let p_1 be a fuzzy subset defined on $R_1 = \mathbb{Z}_4$ by $p_1(0) = p_1(2) = 0$ and $p_1(1) = p_1(3) = 0.5$. And p_2 be a fuzzy subset defined on $R_2 = \mathbb{Z}_6$ by $p_2(0) = p_2(2) = p_2(4) = 0$ and $p_2(1) = p_2(3) = p_2(5) = 0.5$. Let $f_i : R_i \to \mathbb{Z}_2, i = 1, 2$, be the epimorphisms map x to $\bar{x}$. Then the p-pullback $\mathbf{P}$ of $R_1 \to \mathbb{Z}_2 \leftarrow R_2$ is p-presimplifiable.

(2) Let $R_m = \mathbb{Z}_4[i]$ for $m = 1, 2, 3$. Let p_1 be a fuzzy subset defined on R_1 by $p_1(0) = p_1(2) = p_1(2i) = p_1(2 + 2i) = 0$, $p_1(\pm 1) = p_1(\pm 1 + 2i) = 0.2$, $p_1(\pm i) = p_1(2 \pm i) = 0.4$ and $p_1(\pm 1 \pm i) = 0.7$. If $p_1 = p_2$ and $f_1 = f_2 = I_{\mathbb{Z}_4[i]}$, then the p-pullback $\mathbf{P}$ of $R_1 \to R_3 \leftarrow R_2$ is p-presimplifiable.

We end this section by studying some properties of p-associate rings. Recall that, a ring R is p-associate if $a, b \in R$ which satisfy $p(a) = p(sb)$ and $p(b) = p(ta)$ for some $s, t \in R$ implies that there is a p-unit u such that $p(a) = p(ub)$. It is clear that every p-presimplifiable ring is p-associate while the converse is not necessarily true as we shall see (Example 18).

Definition 16. A ring R is said to be a p-Boolean if $p(x) = p(x^2)$ for all $x \in R$.

Clearly, every *Boolean* ring is *p-Boolean* but the converse need not be true for example, if p is a fuzzy subset of $\mathbb{Z}$ defined by $p(x) = 1$ if x is even and $p(x) = 0$ if x is odd, then the ring $\mathbb{Z}$ is *p-Boolean* but it is not Boolean.

Theorem 17. *Every p-Boolean ring is p-associate.*

Proof. Let $x, y \in R$ such that $p(x) = p(ny)$ and $p(y) = p(mx)$ for some $m, n \in R$. Then $p(xy) = p(ny) = p(x)$ and $p(xy) = p(mx) = p(y)$ since p is a translational invariant fuzzy subset of R. So, $p(x) = p(y)$. □

Example 18. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ and p a fuzzy subset of R defined as follows $p((0,0)) = 0$, $p((1,0)) = 0.2$, $p((0,1)) = 0.3$, and $p((1,1)) = 1$. Then R is p-Boolean and hence it is p-associate. However, R is not p-presimplifiable since $p((1,0)(1,0)) = p((1,0))$ and $(1,0)$ is not a p-unit of R.

Definition 19. A ring R is said to be a p-stable range one if for every $x, y, a, b \in R$ satisfying $p(ax + by) = p(y)$, implies the existence of an element z and a p-unit u such that $p(a + bz) = p(u)$.

Theorem 20. *Every p-stable range one ring is p-associate.*

Proof. Suppose that R is p-stable range one and $a, b \in R$ such that $p(a) = p(tb)$ and $p(b) = p(sa)$ for some $s, t \in R$. Then $p((1 - ts) + ts) = p(1)$. So, there exist an element z and a p-unit u of R such that $p((1 - ts)z + t) = p(u)$. So, $p(((1 - ts)z + t)b) = p(a)$ and hence R is p-associate. □

Definition 21. A ring R is said to be a p-regular if for any $a \in R$ there exist a p-idempotent e (i.e., $p(e) = p(e^2)$) and a p-unit u such that $p(a) = p(ue)$.

Theorem 22. *Every p-regular ring is p-associate.*

Proof. Let e, n be two p-idempotents and u, v two p-units such that $p(ue) = p(tvn)$ and $p(vn) = p(sue)$ for some $s, t \in R$. Then $p(uen) = p(ue)$ and $p(vne) = p(vn)$. Hence $p(u(en - e)) = p(0)$ and $p(v(en - n)) = p(0)$. But u and v are p-units so, $p(en - e) = p(0)$ and $p(en - n) = p(0)$. Therefore, $p(e) = p(n)$. Then $p(ue) = p(tvn)$ implies that $p(ue) = p(tve)$ in R. Hence $p(ue) = p(teve)$ in Re. But ue is a p-unit in Re so, te is a p-unit in Re. Thus $p(teke) = p(e)$ for some $k \in R$. So, $p((te + 1 - e)(ke + (1 - e))) = p(1 - e + teke) = p(1)$. Hence, $te + 1 - e$ is a p-unit of R and $p(ue) = p((te + (1 - e)ve)$. So the result holds. □

3. Images and Inverse Image under Homomorphism

In this section, we study p-presimplifiable and p-associate properties under ring homomorphism.

Recall that if f is a function from a ring R into a ring S, then a fuzzy subset p of R is f-invariant if $f(x) = f(y)$ implies that $p(x) = p(y)$ for every $x, y \in R$. Ray and Ali [14] proved that if $f : R \to S$ is an epimorphism then a translational invariant of a fuzzy subset p of R implies a translational invariant of a fuzzy subset $f(p)$ of S, where $f(p)(x) = \sup\{p(a) : a \in R, f(a) = x\}$ for every $x \in S$. Also, the inverse image of a translational invariant fuzzy subset Q of S is a translational invariant fuzzy subset of R, where $f^{-1}(Q)(x) = Q(f(x))$ for every $x \in R$.

It is easy to prove the following lemma.

Lemma 23. *Let $f : R \to S$ be a ring epimorphism, p an f-invariant fuzzy subset of R, and Q be a fuzzy subset of S.*

(1) *$p(0_R) \neq p(1_R)$ if and only if $f(p)(0_S) \neq f(p)(1_S)$.*

(2) *u is a p-unit of R if and only if $f(u)$ is an $f(p)$-unit of S.*

(3) *For every $x \in R$, $p(x) = p(-x)$ if and only if $f(p)(x) = f(p)(-x)$.*

(4) *$f^{-1}(Q)(0_R) \neq f^{-1}(Q)(1_R)$ if and only if $Q(0_S) \neq Q(1_S)$.*

(5) *u is an $f^{-1}(Q)$-unit of R if and only if $f(u)$ is a Q-unit of S.*

(6) *$f^{-1}(Q)(x) = f^{-1}(Q)(-x)$ if and only if $Q(f(x)) = Q(f(-x))$.*

Theorem 24. *Let $f : R \to S$ be an epimorphism and p an f-invariant fuzzy subset of R.*

(1) *R is p-presimplifiable if and only if S is $f(p)$-presimplifiable.*

(2) *R is p-associate if and only if S is $f(p)$-associate.*

Proof. (1) Let $f(a)$ and $f(p)$ be two non-$f(p)$-zero elements of S such that $f(p)(f(a)f(b)) = f(p)(f(a))$. Then $p(ab) = p(a)$ and $p(a) \neq p(0)$ because p is an f-invariant. But R is p-presimplifiable, so b is a p-unit and hence $f(b)$ is an $f(p)$-unit of S.

Conversely, let a and b be two non-p-zero elements of R such that $p(ab) = p(a)$. Then $f(p)(f(a)f(b)) = f(p)(f(a))$ and $f(p)(f(a)) \neq f(p)(0_S)$. But $f(b)$ is $f(p)$-unit of S because S is $f(p)$-presimplifiable. Hence b is a p-unit of R. □

Corollary 25. *Let $f : R \to R/I$ be the natural homomorphism and p a fuzzy subset of R satisfying $p(x) = p(0)$ for every $x \in I$.*

(1) *R is p-presimplifiable if and only if R/I is $f(p)$-presimplifiable.*

(2) *R is p-associate if and only if R/I is $f(p)$-associate.*

Proof. Note that, if $p(x) = p(0)$ for every $x \in I$, then p is an f-invariant fuzzy subset of R. □

Corollary 26. *Let R be a p-Boolean ring or p-integral domain ring and $f : R \to R/N(R)$ the natural homomorphism.*

(1) *R is p-presimplifiable if and only if $R/N(R)$ is $f(p)$-presimplifiable.*

(2) *R is p-associate if and only if $R/N(R)$ is $f(p)$-associate.*

Theorem 27. *Let $f : R \to S$ be an epimorphism and Q a translational invariant fuzzy subset of S.*

(1) *R is $f^{-1}(Q)$-presimplifiable if and only if S is Q-presimplifiable.*

(2) *R is $f^{-1}(Q)$-associate if and only if S is Q-associate.*

Proof. (2) Suppose that $f(a), f(b) \in S$ satisfy $Q(f(a)f(t)) = Q(f(b))$ and $Q(f(b)f(s)) = Q(f(a))$ for some $t, s \in R$. Then $f^{-1}(Q)(at) = f^{-1}(Q)(b)$ and $f^{-1}(Q)(bs) = f^{-1}(Q)(a)$. But R is $f^{-1}(Q)$-associate, so there exists an $f^{-1}(Q)$-unit u satisfying $f^{-1}(Q)(au) = f^{-1}(Q)(b)$. Hence $f(u)$ is a Q-unit of S satisfying $Q(f(a)f(u)) = Q(f(b))$. Thus S is Q-associate. □

Corollary 28. *Let $f : R \to R/I$ be the natural homomorphism and Q a translational invariant fuzzy subset of R/I.*

(1) *R is $f^{-1}(Q)$-presimplifiable if and only if R/I is Q-presimplifiable.*

(2) *R is $f^{-1}(Q)$-associate if and only if R/I is Q-associate.*

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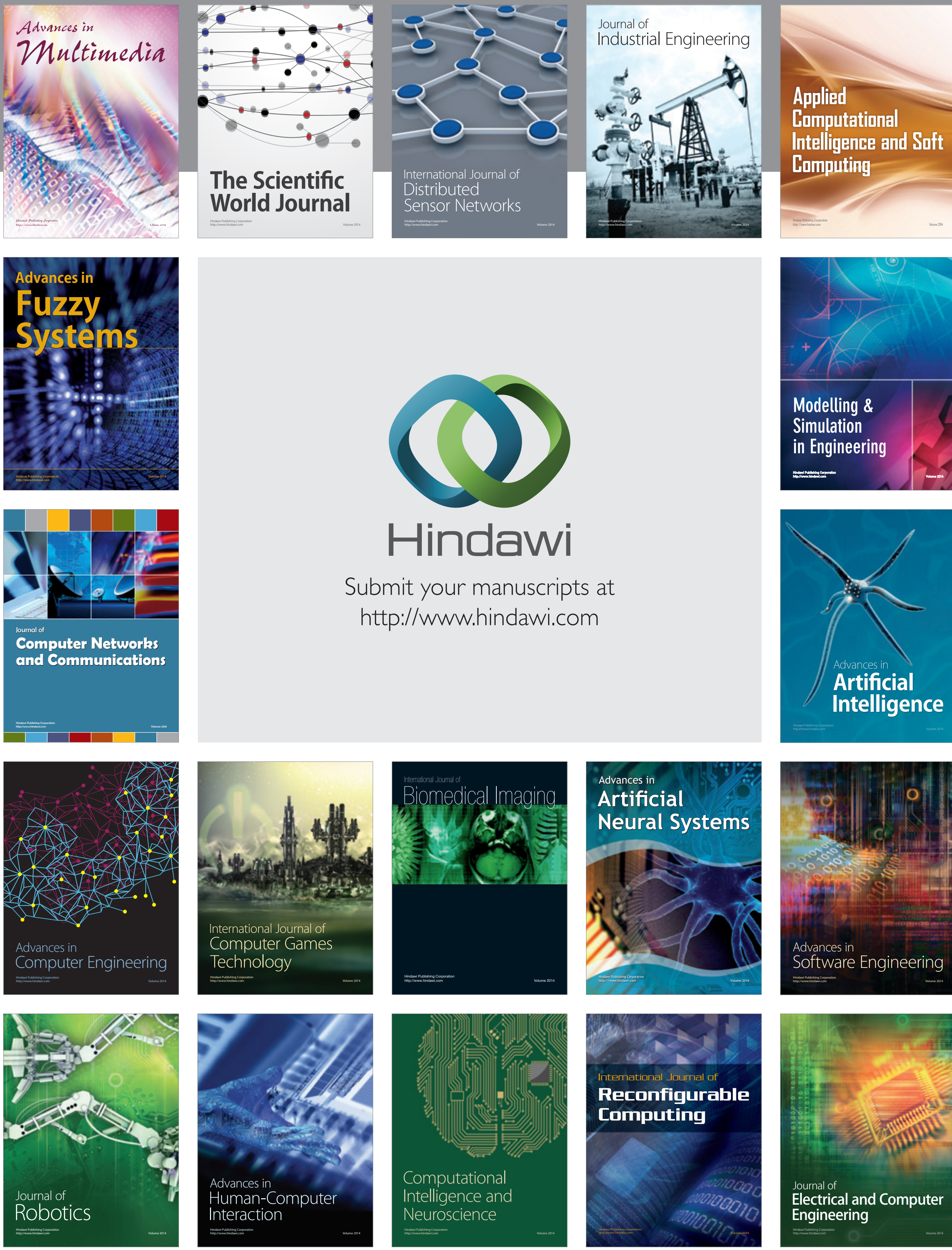
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