# Research Article **Rough Fuzzy Hyperideals in Ternary Semihypergroups**

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The relations between rough sets and algebraic systems have been already considered by many mathematicians, and rough sets have been studied in various kinds of algebraic systems. This paper concerns a relationship between rough sets and ternary semihypergroups. We introduce the notion of rough hyperideals and rough bi-hyperideals in ternary semihypergroups. We also study fuzzy, rough, and rough fuzzy ternary subsemihypergroups (left hyperideals, right hyperideals, lateral hyperideals, hyperideals, and bi-hyperideals) of ternary semihypergroups.

#### 1. Introduction

The notion of a rough set was proposed by Pawlak [1] as a formal tool for modeling and processing incomplete information in information systems. Since then the subject has been investigated in many papers. The theory of rough sets is an extension of set theory, in which a subset of an universe is described by a pair of ordinary sets called the lower and upper approximations. A key notion in the Pawlak rough set model is the equivalence relation. The equivalence classes are the building blocks for the construction of the lower and upper approximations. The lower approximation of a given set is the union of all equivalence classes which are subsets of the set, and the upper approximation is the union of all equivalence classes which have a nonempty intersection with the set. It is a natural question to ask what happens if we substitute the universe set with an algebraic system. Some authors have studied the algebraic properties of rough sets. Aslam et al. [2] introduced the notion of roughness in left almost semigroups. Chinram [3], introduced rough prime ideals and rough fuzzy prime ideals in Γ-semigroups. Petchkhaew and Chinram [4], introduced the notion of rough fuzzy ideals in ternary semigroups. In [5], Davvaz considered the relationship between rough sets and ring theory, considered a ring as a universal set, and introduced the notion of rough ideals and rough subrings with respect to the ideal of a ring. Also, rough modules have been investigated by Davvaz and Mahdavipour [6]. Davvaz et al. applied rough theory to  $\Gamma$ -semihypergroups [7], hyperrings [8], and  $\Gamma$ -semihyperrings [9]. Yaqoob [10] introduced the notion of rough  $\Gamma$ -hyperideals in left almost  $\Gamma$ -semihypergroups, also see [11, 12]. Kuroki, in [13], introduced the notion of a rough ideal in a semigroup. Jun applied the rough set theory to BCK-algebras [14].

Hyperstructure theory was introduced in 1934, when Marty [15] defined hypergroups, began to analyze their properties and applied them to groups. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many mathematicians. Nowadays, hyperstructures have a lot of applications to several domains of mathematics and computer science and they are studied in many countries of the world. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. A lot of papers and several books have been written on hyperstructure theory, see [16–19]. A recent book on hyperstructures [16] points out on their applications in rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs, and hypergraphs. Another book [18] is devoted especially to the study of hyperring theory. Several kinds of hyperrings are introduced and analyzed. The volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: *e*-hyperstructures and transposition hypergroups.

Hila and Naka [20–22] worked out on ternary semihypergroups and introduced some properties of hyperideals in ternary semihypergroups, also see [23].

The concept of a fuzzy set, introduced by Zadeh in his classic paper [24], provides a natural framework for generalizing some of the notions of classical algebraic structures. Fuzzy semigroups have been first considered by Kuroki [25]. After the introduction of the concept of fuzzy sets by Zadeh, several researches conducted the researches on the generalizations of the notions of fuzzy sets with huge applications in computer, logics, and many branches of pure and applied mathematics. Fuzzy set theory has been shown to be an useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situations by attributing a degree to which a certain object belongs to a set. In 1971, Rosenfeld [26] defined the concept of fuzzy group. Since then many papers have been published in the field of fuzzy algebra. Recently fuzzy set theory has been well developed in the context of hyperalgebraic structure theory. A recent book [16] contains an wealth of applications. In [27], Davvaz introduced the concept of fuzzy hyperideals in a semihypergroup, also see [28, 29]. A several papers are written on fuzzy sets in several algebraic hyperstructures. The relationships between the fuzzy sets and algebraic hyperstructures have been considered by Corsini, Davvaz, Leoreanu, Zhan, Zahedi, Ameri, Cristea, and many other researchers. The notion of a rough set has often been compared to that of a fuzzy set, sometimes with a view to prove that one is more general, or, more useful than the other. Several researchers were conducted on the generalizations of the notion of fuzzy sets and rough sets.

In this paper, the notion of rough subsemihypergroup (rough hyperideal, rough bi-hyperideal resp.) in ternary semihypergroups has been introduced which is a generalization of subsemihypergroup (hyperideal, bi-hyperideal resp.). We also study fuzzy, rough and rough fuzzy ternary subsemihypergroups (left hyperideal, right hyperideals, lateral hyperideals, hyperideals, and bi-hyperideals) of ternary semihypergroups.

#### 2. Ternary Semihypergroups

In this section we will present some basic definitions of ternary semihypergroups.

A map  $\circ$  :  $H \times H \rightarrow \mathcal{P}^*(H)$  is called hyperoperation or join operation on the set H, where H is a nonempty set and  $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$  denotes the set of all nonempty subsets of H.

A hypergroupoid is a set H with together a (binary) hyperoperation.

*Definition 1.* A hypergroupoid  $(H, \circ)$ , which is associative, that is,  $x \circ (y \circ z) = (x \circ y) \circ z$ , for all  $x, y, z \in S$ , is called a semihypergroup.

Let A and B be two nonempty subsets of H. Then, we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \quad a \circ A = \{a\} \circ A, \quad a \circ B = \{a\} \circ B.$$
(1)

Definition 2. A map  $f : H \times H \times H \rightarrow \mathcal{P}^*(H)$  is called ternary hyperoperation on the set H, where H is a nonempty set and  $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$  denotes the set of all nonempty subsets of H.

*Definition 3.* A ternary hypergroupoid is called the pair (H, f), where f is a ternary hyperoperation on the set H.

*Definition 4.* A ternary hypergroupoid (S, f) is called a ternary semihypergroup if for all  $a_1, a_2, \ldots, a_5 \in S$ , we have

$$f(f(a_1, a_2, a_3), a_4, a_5) = f(a_1, f(a_2, a_3, a_4), a_5)$$
  
=  $f(a_1, a_2, f(a_3, a_4, a_5)).$  (2)

*Definition 5.* Let (S, f) be a ternary semihypergroup. Then S is called a ternary hypergroup if for all  $a, b, c \in S$ , there exist  $x, y, z \in S$  such that

$$c \in f(x, a, b) \cap f(a, y, b) \cap f(a, b, z).$$
(3)

Definition 6. Let (S, f) be a ternary semihypergroup and T a nonempty subset of H. Then T is called a subsemihypergroup of S if and only if  $f(T, T, T) \subseteq T$ .

*Definition 7.* A nonempty subset I of a ternary semihypergroup S is called a left (right, lateral) hyperideal of S if

$$f(S, S, I) \subseteq I(f(I, S, S) \subseteq I, f(S, I, S) \subseteq I).$$

$$(4)$$

*Definition 8.* A subsemihypergroup *B* of a ternary semihypergroup *S* is called a bi-hyperideal of *S* if

$$f(B, S, B, S, B) \subseteq B. \tag{5}$$

Definition 9. Let (S, f) be a ternary semihypergroup and Q a subset of S. Then Q is called a quasi-hyperideal of S if and only if

$$f(Q, S, S) \cap f(S, Q, S) \cap f(S, S, Q) \subseteq Q,$$
  
$$f(Q, S, S) \cap f(S, S, Q, S, S) \cap f(S, S, Q) \subseteq Q.$$
  
(6)

# 3. Rough Hyperideals in Ternary Semihypergroups

In what follows, let *S* denote a ternary semihypergroup unless otherwise specified. In this section, for simplicity we write f(a, b, c) as  $a \cdot b \cdot c$  and consider the ternary hyperoperation *f* as "·". Suppose that *S* is a ternary semihypergroup. A partition or classification of *S* is a family  $\mathcal{P}$  of nonempty subsets of *S* such that each element of *S* is contained in exactly one element of  $\mathcal{P}$ .

Given a ternary semihypergroup S, by  $\mathcal{P}(S)$  we will denote the power-set of S. Let A and B be two nonempty

subsets of *S*. We define  $(A, B) \in \overline{\rho}$  if for every  $a \in A$  there exists  $b \in B$  such that  $(a, b) \in \rho$  and for every  $d \in B$  there exists  $c \in A$  such that  $(c, d) \in \rho$ . If  $\rho$  is an equivalence relation on *S*, then, for every  $x \in S$ ,  $[x]_{\rho}$  stands for the equivalence class of *x* with the represent  $\rho$ .

*Definition 10.* Let *S* be a ternary semihypergroup. An equivalence relation  $\rho$  on *S* is called regular on *S* if

$$(a,b) \in \rho \text{ implies } (x \cdot y \cdot a, x \cdot y \cdot b) \in \overline{\rho},$$
  
$$(x \cdot a \cdot y, x \cdot b \cdot y) \in \overline{\rho}, \qquad (a \cdot x \cdot y, b \cdot x \cdot y) \in \overline{\rho},$$
  
(7)

for all  $a, b, x, y \in S$ .

A regular relation  $\rho$  on *S* is called complete if  $[a]_{\rho} \cdot [b]_{\rho} \cdot [c]_{\rho} = [a \cdot b \cdot c]_{\rho}$  for all  $a, b \in S$ .

**Lemma 11.** Let *S* be a ternary semihypergroup and  $\rho$  be a regular relation on *S*. If  $a, b \in S$ , then  $[a]_{\rho} \cdot [b]_{\rho} \cdot [c]_{\rho} \subseteq [a \cdot b \cdot c]_{\rho}$ .

*Proof.* Let  $x \in [a]_{\rho} \cdot [b]_{\rho} \cdot [c]_{\rho}$ . Then there exist  $p \in [a]_{\rho}$ ,  $q \in [b]_{\rho}$  and  $r \in [c]_{\rho}$  such that  $x \in p \cdot q \cdot r$ . Since  $(a, p) \in \rho$ ,  $(b, q) \in \rho$  and  $(c, r) \in \rho$  then by regularity of  $\rho$ , we have

$$(a \cdot b \cdot c, p \cdot q \cdot r) \in \overline{\rho}.$$
 (8)

So  $x \in p \cdot q \cdot r$  implies that there exists  $y \in a \cdot b \cdot c$  such that  $(x, y) \in \rho$ , and therefore  $x \in [a \cdot b \cdot c]_{\rho}$ .

Let *A* be a nonempty subset of a ternary semihypergroup *S* and  $\rho$  be a regular relation on *S*. Then, the sets

$$\overline{Apr}_{\rho}(A) = \left\{ x \in S : [x]_{\rho} \cap A \neq \emptyset \right\},$$

$$\underline{Apr}_{\rho}(A) = \left\{ x \in S : [x]_{\rho} \subseteq A \right\}$$
(9)

are called  $\rho$ -upper and  $\rho$ -lower approximations of A, respectively. For a nonempty subset A of S,  $Apr_{\rho}(A) = (\underline{Apr}_{\rho}(A), \overline{Apr}_{\rho}(A))$  is called a rough set with respect to  $\rho$ if  $\underline{Apr}_{\rho}(A) \neq \overline{Apr}_{\rho}(A)$ .

**Theorem 12.** Let  $\rho$  and  $\varphi$  be regular relations on a ternary semihypergroup S. If A and B are nonempty subsets of S, then the following hold:

$$(1) \underline{Apr}_{\rho}(A) \subseteq A \subseteq \overline{Apr}_{\rho}(A);$$

$$(2) \overline{Apr}_{\rho}(A \cup B) = \overline{Apr}_{\rho}(A) \cup \overline{Apr}_{\rho}(B);$$

$$(3) \underline{Apr}_{\rho}(A \cap B) = \underline{Apr}_{\rho}(A) \cap \underline{Apr}_{\rho}(B);$$

$$(4) A \subseteq B \text{ implies } \underline{Apr}_{\rho}(A) \subseteq \underline{Apr}_{\rho}(B);$$

$$(5) A \subseteq B \text{ implies } \overline{Apr}_{\rho}(A) \subseteq \overline{Apr}_{\rho}(B);$$

$$(6) \underline{Apr}_{\rho}(A \cup B) \supseteq \underline{Apr}_{\rho}(A) \cup \underline{Apr}_{\rho}(B);$$

$$(7) \overline{Apr}_{\rho}(A \cap B) \subseteq \overline{Apr}_{\rho}(A) \cap \overline{Apr}_{\rho}(B);$$

$$(8) \rho \subseteq \varphi \text{ implies } \underline{Apr}_{\rho}(A) \supseteq \underline{Apr}_{\varphi}(A);$$

$$(9) \rho \subseteq \varphi \text{ implies } \overline{Apr}_{\rho}(A) \subseteq \overline{Apr}_{\varphi}(A).$$

*Proof.* The proof of this theorem is similar to [13, Theorem 2.1].  $\Box$ 

**Theorem 13.** Let  $\rho$  be a regular relation on a ternary semihypergroup S and let A, B and C be nonempty subsets of S. Then

(1) 
$$Apr_{\rho}(A) \cdot Apr_{\rho}(B) \cdot Apr_{\rho}(C) \subseteq Apr_{\rho}(A \cdot B \cdot C),$$
  
(2) If  $\rho$  is complete, then  $\underline{Apr}_{\rho}(A) \cdot \underline{Apr}_{\rho}(B) \cdot \underline{Apr}_{\rho}(C) \subseteq Apr_{\rho}(A \cdot B \cdot C).$ 

*Proof.* (1) Let  $x \in \overline{Apr}_{\rho}(A) \cdot \overline{Apr}_{\rho}(B) \cdot \overline{Apr}_{\rho}(C)$ . Then  $x \in a \cdot b \cdot c$  for  $a \in \overline{Apr}_{\rho}(A)$ ,  $b \in \overline{Apr}_{\rho}(B)$  and  $c \in \overline{Apr}_{\rho}(C)$ . There exist  $r, s, t \in S$  such that  $r \in [a]_{\rho} \cap A, s \in [b]_{\rho} \cap B$  and  $t \in [c]_{\rho} \cap C$ . Since  $\rho$  is regular, it follows that

$$r \cdot s \cdot t \subseteq [a]_{\rho} \cdot [b]_{\rho} \cdot [c]_{\rho} \subseteq [a \cdot b \cdot c]_{\rho}.$$
(10)

On the other hand, since  $r \cdot s \cdot t \subseteq A \cdot B \cdot C$ , we have

$$r \cdot s \cdot t \subseteq [a \cdot b \cdot c]_{\rho} \cap A \cdot B \cdot C, \tag{11}$$

and so  $x \in a \cdot b \cdot c \subseteq \overline{Apr}_{\rho}(A \cdot B \cdot C)$ . This shows that  $\overline{Apr}_{\rho}(A) \cdot \overline{Apr}_{\rho}(B) \cdot \overline{Apr}_{\rho}(C) \subseteq \overline{Apr}_{\rho}(A \cdot B \cdot C)$ .

(2) Let  $x \in \underline{Apr}_{\rho}(A) \cdot \underline{Apr}_{\rho}(B) \cdot \underline{Apr}_{\rho}(C)$ . Then  $x \in a \cdot b \cdot c$ for  $a \in \underline{Apr}_{\rho}(A)$ ,  $b \in \underline{Apr}_{\rho}(B)$  and  $c \in \underline{Apr}_{\rho}(C)$ .

It follows that  $[a]_{\rho} \subseteq A$ ,  $[b]_{\rho} \subseteq B$  and  $[c]_{\rho} \subseteq C$ . Since  $\rho$  is complete, we have

$$[a]_{\rho} \cdot [b]_{\rho} \cdot [c]_{\rho} = [a \cdot b \cdot c]_{\rho} \subseteq A \cdot B \cdot C, \qquad (12)$$

and so  $x \in a \cdot b \cdot c \subseteq \underline{Apr}_{\rho}(A \cdot B \cdot C)$ . Hence  $\underline{Apr}_{\rho}(A) \cdot \underline{Apr}_{\rho}(B) \cdot \underline{Apr}_{\rho}(C) \subseteq \underline{Apr}_{\rho}(A \cdot B \cdot C)$ .

Definition 14. Let S be a ternary semihypergroup. A nonempty subset A of S is called a subsemihypergroup of S if  $A \cdot A \cdot A \subseteq A$ .

Let  $\rho$  be a regular relation on a ternary semihypergroup S. Then a nonempty subset A of S is called a  $\rho$ -upper ( $\rho$ -lower) rough subsemihypergroup of S if  $\overline{Apr}_{\rho}(A)(\underline{Apr}_{\rho}(A))$  is a subsemihypergroup of S.

**Theorem 15.** Let  $\rho$  be a regular relation on ternary semihypergroup S and let A be a subsemihypergroup of S. Then

- (1)  $\overline{Apr}_{o}(A)$  is a subsemilypergroup of S,
- (2) if  $\rho$  is complete, then  $Apr_{\rho}(A)$  is, if it is nonempty, a subsemihypergroup of S.

*Proof.* (1) Let A be a subsemihypergroup of S. Now by Theorem 13(1),

$$\overline{Apr}_{\rho}(A) \cdot \overline{Apr}_{\rho}(A) \cdot \overline{Apr}_{\rho}(A) 
\subseteq \overline{Apr}_{\rho}(A \cdot A \cdot A) \subseteq \overline{Apr}_{\rho}(A).$$
(13)

This shows that  $Apr_{\rho}(A)$  is a subsemilypergroup of *S*, that is, *A* is a  $\rho$ -upper rough subsemilypergroup of *S*.

(2) Let A be a subsemilypergroup of S. Now by Theorem 13(2)

$$\underline{Apr}_{\rho}(A) \cdot \underline{Apr}_{\rho}(A) \cdot \underline{Apr}_{\rho}(A) \\
\subseteq \underline{Apr}_{\rho}(A \cdot A \cdot A) \subseteq \underline{Apr}_{\rho}(A).$$
(14)

This shows that  $\underline{Apr}_{\rho}(A)$  is a subsemilypergroup of *S*, that is, *A* is a  $\rho$ -lower rough subsemilypergroup of *S*.

Definition 16. A nonempty subset A of a ternary semihypergroup S is called left (right, lateral) hyperideal of S if  $S \cdot S \cdot A \subseteq A(A \cdot S \cdot S \subseteq A, S \cdot A \cdot S \subseteq A)$ .

A nonempty subset *A* of a ternary semihypergroup *S* is called a hyperideal of *S* if it is a left, right and lateral hyperideal of *S*. A nonempty subset *A* of a ternary semihypergroup *S* is called two-sided hyperideal of *S* if it is a left and right hyperideal of *S*. A lateral hyperideal *A* of a ternary semihypergroup *S* is called a proper lateral hyperideal of *S* if  $A \neq S$ .

Let  $\rho$  be a regular relation on a ternary semihypergroup S. Then a nonempty subset A of S is called a  $\rho$ -upper ( $\rho$ -lower) rough left hyperideal of S if  $\overline{Apr}_{\rho}(A)$  ( $\underline{Apr}_{\rho}(A)$ ) is a left hyperideal of S. Similarly  $\rho$ -upper ( $\rho$ -lower) rough right and rough lateral hyperideals of S can be defined.

**Theorem 17.** Let  $\rho$  be a regular relation on a ternary semihypergroup S and let A be a left (right, lateral) hyperideal of S. Then

- (1)  $\overline{Apr}_{o}(A)$  is a left (right, lateral) hyperideal of S,
- (2) if  $\rho$  is complete, then  $\underline{Apr}_{\rho}(A)$  is, if it is nonempty, a left (right, lateral) hyperideal of S.

*Proof.* (1) Let A be a right hyperideal of S. Now by Theorem 13(1),

$$\overline{Apr}_{\rho}(A) \cdot S \cdot S = \overline{Apr}_{\rho}(A) \cdot \overline{Apr}_{\rho}(S) \cdot \overline{Apr}_{\rho}(S)$$

$$\subseteq \overline{Apr}_{\rho}(A \cdot S \cdot S) \subseteq \overline{Apr}_{\rho}(A).$$
(15)

This shows that  $\overline{Apr}_{\rho}(A)$  is a right hyperideal of *S*, that is, *A* is a  $\rho$ -upper rough right hyperideal of *S*.

(2) Let A be a right hyperideal of S. Now by Theorem 13(2),

$$\underline{Apr}_{\rho}(A) \cdot S \cdot S = \underline{Apr}_{\rho}(A) \cdot \underline{Apr}_{\rho}(S) \cdot \underline{Apr}_{\rho}(S)$$

$$\subseteq \underline{Apr}_{\rho}(A \cdot S \cdot S) \subseteq \underline{Apr}_{\rho}(A).$$
(16)

This shows that  $\underline{Apr}_{\rho}(A)$  is a right hyperideal of *S*, that is, *A* is a  $\rho$ -lower rough right hyperideal of *S*. The case for left (lateral) hyperideal can be seen in a similar way.

**Theorem 18.** Let  $\rho$  be a regular relation on a ternary semihypergroup S. If A, B, and C are a right hyperideal, a lateral hyperideal, and a left hyperideal of S, respectively. Then

(1) 
$$\overline{Apr}_{\rho}(A \cdot B \cdot C) \subseteq \overline{Apr}_{\rho}(A) \cap \overline{Apr}_{\rho}(B) \cap \overline{Apr}_{\rho}(C),$$
  
(2)  $\underline{Apr}_{\rho}(A \cdot B \cdot C) \subseteq \underline{Apr}_{\rho}(A) \cap \underline{Apr}_{\rho}(B) \cap \underline{Apr}_{\rho}(C).$ 

*Proof.* Since *A* is a right hyperideal of *S*, so  $A \cdot B \cdot C \subseteq A \cdot S \cdot S \subseteq A$ . Since *B* is a lateral hyperideal of *S*, so  $A \cdot B \cdot C \subseteq S \cdot B \cdot S \subseteq B$ , also *C* is a left hyperideal of *S*, so  $A \cdot B \cdot C \subseteq S \cdot S \cdot C \subseteq C$ , thus  $A \cdot B \cdot C \subseteq A \cap B \cap C$ . Then by Theorem 12(7), we have

$$Apr_{\rho}(A \cdot B \cdot C) \subseteq Apr_{\rho}(A \cap B \cap C)$$
$$\subseteq \overline{Apr}_{\rho}(A) \cap \overline{Apr}_{\rho}(B) \cap \overline{Apr}_{\rho}(C).$$
(17)

Also by Theorem 12(3), we have

$$\underline{Apr}_{\rho}(A \cdot B \cdot C) \subseteq \underline{Apr}_{\rho}(A \cap B \cap C)$$

$$= \underline{Apr}_{\rho}(A) \cap \underline{Apr}_{\rho}(B) \cap \underline{Apr}_{\rho}(C).$$
(18)

This completes the proof.

*Definition 19.* A subsemihypergroup *B* of a ternary semihypergroup *S* is called a bi-hyperideal of *S* if  $B \cdot S \cdot B \cdot S \cdot B \subseteq B$ .

Let  $\rho$  be a regular relation on a ternary semihypergroup *S*. Then a subsemihypergroup *B* of *S* is called a  $\rho$ -upper ( $\rho$ -lower) rough bi-hyperideal of *S* if  $\overline{Apr}_{\rho}(A)(\underline{Apr}_{\rho}(A))$  is a bi-hyperideal of *S*.

**Theorem 20.** Let  $\rho$  be a regular relation on a ternary semihypergroup S and let B be a bi-hyperideal of S. Then

- (1)  $\overline{Apr}_{o}(B)$  is a bi-hyperideal of S,
- (2) if  $\rho$  is complete, then  $\underline{Apr}_{\rho}(B)$  is, if it is nonempty, a bi-hyperideal of S.

*Proof.* (1) Let *B* be a bi-hyperideal of *S*. Now by Theorem 13(1)

$$\overline{Apr}_{\rho}(B) \cdot S \cdot \overline{Apr}_{\rho}(B) \cdot S \cdot \overline{Apr}_{\rho}(B)$$

$$= \overline{Apr}_{\rho}(B) \cdot \overline{Apr}_{\rho}(S) \cdot \overline{Apr}_{\rho}(B)$$

$$\cdot \overline{Apr}_{\rho}(S) \cdot \overline{Apr}_{\rho}(B) \qquad (19)$$

$$\subseteq \overline{Apr}_{\rho}(B \cdot S \cdot B \cdot S \cdot B)$$

$$\subseteq \overline{Apr}_{\rho}(B).$$

From this and Theorem 15(1),  $\overline{Apr}_{\rho}(B)$  is a bi-hyperideal of *S*, that is, *A* is a  $\rho$ -upper rough bi-hyperideal of *S*.

(2) Let *B* be a bi-hyperideal of *S*. Now by Theorem 13(2)

$$\underline{Apr}_{\rho}(B) \cdot S \cdot \underline{Apr}_{\rho}(B) \cdot S \cdot \underline{Apr}_{\rho}(B)$$

$$= \underline{Apr}_{\rho}(B) \cdot \underline{Apr}_{\rho}(S) \cdot \underline{Apr}_{\rho}(B)$$

$$\cdot \underline{Apr}_{\rho}(S) \cdot \underline{Apr}_{\rho}(B)$$

$$\subseteq \underline{Apr}_{\rho}(B \cdot S \cdot B \cdot S \cdot B)$$

$$\subseteq \underline{Apr}_{\rho}(B).$$
(20)

Table 1							
*	0	а	b	С	d	е	g
0	0	0	0	0	0	0	0
а	0	а	$\{a,b\}$	С	$\{c,d\}$	е	$\{e,g\}$
b	0	b	b	d	d	g	g
С	0	С	$\{c,d\}$	С	$\{c, d\}$	С	$\{c,d\}$
d	0	d	d	d	d	d	d
е	0	е	$\{e,g\}$	С	$\{c, d\}$	е	$\{e,g\}$
g	0	g	g	d	d	g	g

From this and Theorem 15(2),  $\underline{Apr}_{\rho}(B)$  is a bi-hyperideal of *S*, that is, *A* is a  $\rho$ -lower rough bi-hyperideal of *S*.

The following example shows that the converse of above theorem does not hold.

*Example 21.* Let  $H = \{0, a, b, c, d, e, g\}$  and f(x, y, z) = (x \* y) \* z for all  $x, y, z \in H$ , where \* is defined by Table 1.

Then (H, f) is a ternary semihypergroup. Let  $\rho$  be a complete regular relation on *S* such that  $\rho$ -regular classes are the subsets  $\{0\}$ ,  $\{a, b, c, d, e, g\}$ . Now for  $A = \{0, e, g\} \subseteq S$ ,  $\overline{Apr}_{\rho}(A) = \{0, a, b, c, d, e, g\}$  and  $\underline{Apr}_{\rho}(A) = \{0\}$ . It is clear that  $\overline{Apr}_{\rho}(A)$  and  $\underline{Apr}_{\rho}(A)$  are bi-hyperideals of *S*, but the subsemihypergroup  $\{0, e, g\}$  of *S* is not a bi-hyperideal of *S*.

*Definition 22.* A subset Q of a ternary semihypergroup S is called a quasi-hyperideal of S if

$$Q \cdot S \cdot S \cap S \cdot Q \cdot S \cap S \cdot S \cdot Q \subseteq Q,$$

$$Q \cdot S \cdot S \cap S \cdot S \cdot Q \cdot S \cdot S \cap S \cdot S \cdot Q \subseteq Q.$$
(21)

Let  $\rho$  be a regular relation on a ternary semihypergroup S. Then a subset Q of S is called a  $\rho$ -upper ( $\rho$ -lower) rough quasi-hyperideal of S if  $\overline{Apr}_{\rho}(A)(\underline{Apr}_{\rho}(A))$  is a quasi-hyperideal of S.

**Theorem 23.** Let  $\rho$  be a complete regular relation on a ternary semihypergroup S and let Q be a bi-hyperideal of S. Then  $Apr_{o}(Q)$  is, if it is nonempty, a quasi-hyperideal of S.

*Proof.* Let *Q* be a quasi-hyperideal of *S*. Now by Theorems 13(2) and 12(3)

$$\underline{Apr}_{\rho}(Q) \cdot S \cdot S \cap S \cdot \underline{Apr}_{\rho}(Q) \cdot S \cap S \cdot S \cdot \underline{Apr}_{\rho}(Q)$$

$$= \underline{Apr}_{\rho}(Q) \cdot \underline{Apr}_{\rho}(S) \cdot \underline{Apr}_{\rho}(S) \cap \underline{Apr}_{\rho}(S) \cdot \underline{Apr}_{\rho}(Q)$$

$$\cdot \underline{Apr}_{\rho}(S) \cap \underline{Apr}_{\rho}(S) \cdot \underline{Apr}_{\rho}(S) \cdot \underline{Apr}_{\rho}(Q)$$

$$\subseteq \underline{Apr}_{\rho}(Q \cdot S \cdot S) \cap \underline{Apr}_{\rho}(S \cdot Q \cdot S) \cap \underline{Apr}_{\rho}(S \cdot S \cdot Q)$$

$$= \underline{Apr}_{\rho}(Q \cdot S \cdot S \cap S \cdot Q \cdot S \cap S \cdot S \cdot Q)$$

$$\subseteq \underline{Apr}_{\rho}(Q).$$

Also we can show that

$$\underline{Apr}_{\rho}(Q) \cdot S \cdot S \cap S \cdot S \cdot \underline{Apr}_{\rho}(Q) \cdot S \cdot S \cap S \cdot S \cdot \underline{Apr}_{\rho}(Q)$$
$$\subseteq \underline{Apr}_{\rho}(Q). \tag{23}$$

Hence  $\underline{Apr}_{\rho}(Q)$  is a quasi-hyperideal of *S*, that is, *A* is a  $\rho$ -lower rough quasi-hyperideal of *S*.

# 4. Rough Hyperideals in the Quotient Ternary Semihypergroups

Let  $\rho$  be a regular relation on a ternary semihypergroup *S*. The  $\rho$ -upper approximation and  $\rho$ -lower approximation of a nonempty subset *A* of *S* can be presented in an equivalent form as shown below:

$$\overline{\overline{Apr}}_{\rho}(A) = \left\{ [x]_{\rho} \in \frac{S}{\rho} : [x]_{\rho} \cap A \neq \emptyset \right\},$$

$$\underline{Apr}_{\rho}(A) = \left\{ [x]_{\rho} \in \frac{S}{\rho} : [x]_{\rho} \subseteq A \right\},$$
(24)

respectively.

**Theorem 24.** Let  $\rho$  be a regular relation on a ternary semihypergroup S and let A be a subsemihypergroup of S. Then

- (1)  $\overline{Apr}_{\rho}(A)$  is a subsemihypergroup of S/ $\rho$ ,
- (2) if  $\rho$  is complete, then Apr(A) is, if it is nonempty, a subsemihypergroup of  $\overline{S/\rho}$ .

*Proof.* (1) Let  $[a]_{\rho}, [b]_{\rho}, [c]_{\rho} \in \overline{Apr}_{\rho}(A)$ . Then  $[a]_{\rho} \cap A \neq \emptyset$ ,  $[b]_{\rho} \cap A \neq \emptyset$  and  $[c]_{\rho} \cap A \neq \emptyset$ . So there exist  $x \in [a]_{\rho} \cap A$ ,  $y \in [b]_{\rho} \cap A$  and  $z \in [c]_{\rho} \cap A$ . Since A is a subsemilypergroup of S, we have  $x \cdot y \cdot z \subseteq A$ . By Lemma 11, we have

$$x \cdot y \cdot z \subseteq [a]_{\rho} \cdot [b]_{\rho} \cdot [c]_{\rho} \subseteq [a \cdot b \cdot c]_{\rho}.$$
(25)

Thus  $[a \cdot b \cdot c]_{\rho} \cap A \neq \emptyset$ , which implies that  $[a]_{\rho} \cdot [b]_{\rho} \cdot [c]_{\rho} \subseteq \overline{\overline{Apr}}_{\rho}(A)$ . Hence  $\overline{\overline{Apr}}_{\rho}(A)$  is a subsemilypergroup of  $S/\rho$ .

 $\overline{Apr}_{\rho}(A). \text{ Hence } \overline{Apr}_{\rho}(A) \text{ is a subsemihypergroup of } S/\rho.$ (2) Let  $[a]_{\rho}, [b]_{\rho}, [c]_{\rho} \in \underline{Apr}_{\rho}(A). \text{ Then } [a]_{\rho} \subseteq A, [b]_{\rho} \subseteq A \text{ and } [c]_{\rho} \subseteq A. \text{ Since } A \text{ is a subsemihypergroup of } S, we have$ 

$$[a]_{\rho} \cdot [b]_{\rho} \cdot [c]_{\rho} = [a \cdot b \cdot c]_{\rho} \subseteq A \cdot A \cdot A \subseteq A.$$
(26)

This means that  $\underbrace{Apr}_{\rho}(A)$  is a subsemihypergroup of  $S/\rho$ .  $\Box$ 

**Theorem 25.** Let  $\rho$  be a regular relation on a ternary semihypergroup S and let A be a left (right, lateral) hyperideal of S. Then

- (1)  $\overline{Apr}_{\rho}(A)$  is a left (right, lateral) hyperideal of S/ $\rho$ ,
- (2) if  $\rho$  is complete, then  $\underline{Apr}(A)$  is, if it is nonempty, a left (right, lateral) hyperideal of  $S/\rho$ .

(22)

*Proof.* (1) Let *A* be a left hyperideal of *S*. Let  $[x]_{\rho} \in \overline{Apr_{\rho}}(A)$ and  $[y]_{\rho}, [z]_{\rho} \in S/\rho$ . Then  $[x]_{\rho} \cap A \neq \emptyset$ , hence  $x \in \overline{Apr_{\rho}}(A)$ . Since *A* is a left hyperideal of *S*, by Theorem 17(1),  $\overline{Apr_{\rho}}(A)$ is a left hyperideal of *S*. So, we have

$$y \cdot z \cdot x \subseteq \overline{Apr}_{\rho}(A). \tag{27}$$

Now, for every  $m \in y \cdot z \cdot x$ , we have  $[m]_{\rho} \cap A \neq \emptyset$ . On the other hand, from  $m \in y \cdot z \cdot x$ , we obtain  $[m]_{\rho} \subseteq [y]_{\rho} \cdot [z]_{\rho} \cdot [x]_{\rho}$ . Therefore  $[y]_{\rho} \cdot [z]_{\rho} \cdot [x]_{\rho} \subseteq \overline{Apr}_{\rho}(A)$ . This means that  $\overline{Apr}_{\rho}(A)$  is a left hyperideal of  $S/\rho$ .

(2) Let *A* be a left hyperideal of *S*. Let  $[x]_{\rho} \in \overline{Apr}_{\rho}(A)$  and  $[y]_{\rho}, [z]_{\rho} \in S/\rho$ . Then,  $[x]_{\rho} \subseteq A$ , which implies  $x \in \underline{Apr}_{\rho}(A)$ . Since *A* is a left hyperideal of *S*, by Theorem 17(2),  $\underline{Apr}_{\rho}(A)$  is a left hyperideal of *S*. Thus, we have

$$y \cdot z \cdot x \subseteq \underline{Apr}_{o}(A).$$
<sup>(28)</sup>

Now, for every  $m \in y \cdot z \cdot x$ , we have  $m \in \underline{Apr}_{\rho}(A)$ , which implies that  $[m]_{\rho} \subseteq A$ . Hence,  $[m]_{\rho} \in \underline{Apr}_{\rho}(A)$ . On the other hand, from  $m \in y \cdot z \cdot x$ , we have  $[m]_{\rho} \subseteq [y]_{\rho} \cdot [z]_{\rho} \cdot [x]_{\rho}$ . Therefore  $[y]_{\rho} \cdot [z]_{\rho} \cdot [x]_{\rho} \subseteq \underline{Apr}(A)$ . This means that  $\underline{Apr}(A)$  is, if it is nonempty, a left hyperideal of  $S/\rho$ . The other cases can be seen in a similar way.

**Theorem 26.** Let  $\rho$  be a regular relation on a ternary semihypergroup S and let A be a bi-hyperideal of S. Then

- (1)  $\overline{\overline{Apr}}_{\rho}(A)$  is a bi-hyperideal of S/ $\rho$ ,
- (2) if  $\rho$  is complete, then  $\underline{Apr}(A)$  is, if it is nonempty, a *bi-hyperideal of S/p*.

*Proof.* (1) Let *A* be a bi-hyperideal of *S*. Let  $[x]_{\rho}, [y]_{\rho}, [z]_{\rho} \in \overline{\overline{Apr}}_{\rho}(A)$  and  $[s]_{\rho}, [t]_{\rho} \in S/\rho$ . Then,

$$[x]_{\rho} \cap A \neq \emptyset, \qquad [y]_{\rho} \cap A \neq \emptyset, \qquad [z]_{\rho} \cap A \neq \emptyset.$$
(29)

Hence,  $x \in \overline{Apr}_{\rho}(A)$ ,  $y \in \overline{Apr}_{\rho}(A)$  and  $z \in \overline{Apr}_{\rho}(A)$ . By Theorem 20(1),  $\overline{Apr}_{\rho}(A)$  is a bi-hyperideal of *S*. So, we have

$$x \cdot s \cdot y \cdot t \cdot z \subseteq \overline{Apr}_{\rho}(A). \tag{30}$$

Now, for every  $m \in x \cdot s \cdot y \cdot t \cdot z$ , we obtain

$$[m]_{\rho} \subseteq [x]_{\rho} \cdot [s]_{\rho} \cdot [y]_{\rho} \cdot [t]_{\rho} \cdot [z]_{\rho}.$$
(31)

On the other hand, since  $m \in \overline{Apr}_{\rho}(A)$ , we have  $[m]_{\rho} \cap A \neq \emptyset$ . Thus,

$$[x]_{\rho} \cdot [s]_{\rho} \cdot [y]_{\rho} \cdot [t]_{\rho} \cdot [z]_{\rho} \subseteq \overline{Apr}_{\rho}(A).$$
(32)

Therefore, from this and Theorem 24(1),  $\overline{Apr}_{\rho}(A)$  is a bihyperideal of  $S/\rho$ . (2) Let *A* be a bi-hyperideal of *S*. Let  $[x]_{\rho}, [y]_{\rho}, [z]_{\rho} \in \overline{Apr}_{\rho}(A)$  and  $[s]_{\rho}, [t]_{\rho} \in S/\rho$ . Then,

$$[x]_{\rho} \subseteq A, \qquad [y]_{\rho} \subseteq A, \qquad [z]_{\rho} \subseteq A.$$
 (33)

Hence,  $x \in \underline{Apr}_{\rho}(A)$ ,  $y \in \underline{Apr}_{\rho}(A)$ , and  $z \in \underline{Apr}_{\rho}(A)$ . By Theorem 20(2),  $\underline{Apr}_{\rho}(A)$  is a bi-hyperideal of *S*. So, we have

$$x \cdot s \cdot y \cdot t \cdot z \subseteq \underline{Apr}_{o}(A).$$
(34)

Then, for every  $m \in x \cdot s \cdot y \cdot t \cdot z$ , we obtain

$$[m]_{\rho} \in [x]_{\rho} \cdot [s]_{\rho} \cdot [y]_{\rho} \cdot [t]_{\rho} \cdot [z]_{\rho}.$$
(35)

On the other hand, since  $m \in \underline{Apr}_{\rho}(A)$ , we have  $[m]_{\rho} \subseteq A$ . So,

$$[x]_{\rho} \cdot [s]_{\rho} \cdot [y]_{\rho} \cdot [t]_{\rho} \cdot [z]_{\rho} \subseteq \underline{Apr}_{\rho}(A).$$
(36)

Therefore from this and Theorem 24(2),  $\underline{Apr}(A)$  is, if it is nonempty, a bi-hyperideal of  $S/\rho$ .

# 5. Fuzzy Hyperideals of Ternary Semihypergroups

In this section we introduce and study fuzzy ternary subsemihypergroups, fuzzy left hyperideals, fuzzy right hyperideals, fuzzy lateral hyperideals, and fuzzy hyperideals of ternary semihypergroups.

*Definition 27.* Let S be a ternary semihypergroup. A fuzzy subset f of S is called

- (1) a fuzzy ternary subsemilypergroup of S if  $\inf_{t \in xyz} f(t) \ge \min\{f(x), f(y), f(z)\}$  for all  $x, y, z \in S$ ,
- (2) a fuzzy left hyperideal of *S* if  $\inf_{t \in xyz} f(t) \ge f(z)$  for all  $x, y, z \in S$ ,
- (3) a fuzzy right hyperideal of *S* if  $\inf_{t \in xyz} f(t) \ge f(x)$  for all  $x, y, z \in S$ ,
- (4) a fuzzy lateral hyperideal of *S* if  $\inf_{t \in xyz} f(t) \ge f(y)$  for all *x*, *y*, *z*  $\in$  *S*,
- (5) a fuzzy hyperideal of *S* if  $\inf_{t \in xyz} f(t) \ge \max\{f(x), f(y), f(z)\}$  for all  $x, y, z \in S$ .

**Theorem 28.** Let *S* be a ternary semihypergroup and *A* a nonempty subset of *S*. The following statements hold true.

- (1) A is a ternary subsemihypergroup of S if and only if  $f_A$  is a fuzzy ternary subsemihypergroup of S.
- (2) A is a left hyperideal (right hyperideal, lateral hyperideal, hyperideal) of S if and only if  $f_A$  is a fuzzy left hyperideal (fuzzy right hyperideal, fuzzy lateral hyperideal, fuzzy hyperideal) of S.

*Proof.* (1) Let us assume that A is a ternary subsemilypergroup of S. Let  $x, y, z \in S$ . *Case* 1. *x*, *y*, *z*  $\in$  *A*. Since *A* is a ternary subsemihypergroup of *S*, we have  $xyz \subseteq A$ . Then  $\inf_{t \in xyz} f_A(t) = 1 \ge \min\{f(x), f(y), f(z)\}$ .

*Case* 2.  $x \notin A$  or  $y \notin A$  or  $z \notin A$ . Thus  $f_A(x) = 0$  or  $f_A(y) = 0$  or  $f_A(z) = 0$ . Therefore min $\{f(x), f(y), f(z)\} = 0 \le \inf_{t \in xyz} f_A(t)$ .

Conversely, let  $x, y, z \in A$ . We have  $f_A(x) = f_A(y) = f_A(z) = 1$ . Since  $f_A$  is a fuzzy ternary subsemihypergroup of S,  $\inf_{t \in xyz} f_A(t) \ge \min\{f(x), f(y), f(z)\} = 1$ . Hence  $xyz \subseteq A$ .

(2) Let us assume that A is a left hyperideal of S. Let  $x, y, z \in S$ .

*Case* 1.  $z \in A$ . Since *A* is a left hyperideal of *S*, then  $xyz \subseteq A$ . Then  $\inf_{t \in xyz} f_A(t) = 1$ . Therefore  $\inf_{t \in xyz} f_A(t) \ge f_A(z)$ . *Case* 2.  $z \notin A$ . We have  $f_A(z) = 0$ . Hence  $\inf_{t \in xyz} f_A(t) \ge f_A(z)$ .

Conversely, let  $x, y \in S$  and  $z \in A$ . Since  $f_A$  is a fuzzy left hyperideal of S and  $z \in A$ ,  $\inf_{t \in xyz} f_A(t) \ge f_A(z) = 1$ . Thus  $xyz \subseteq A$ .

The remaing parts can be seen in similarly way.  $\Box$ 

Let S be a ternary semihypergroup. A nonempty subset T of S is called *prime subset* of S if for all  $x, y, z \in S$ ,  $xyz \subseteq T$  implies  $x \in T$  or  $y \in T$  or  $z \in T$ . A ternary subsemihypergroup T of S is called *prime ternary subsemihypergroup* of S if T is a prime subset of S. *Prime left hyperideals, prime right hyperideals, prime lateral hyperideals,* and *prime hyperideals* of S are defined analogously. A fuzzy subset f of S is called a *prime fuzzy subset* of S if f(x), f(y), f(z) for all  $x, y, z \in S$ . A fuzzy ternary subsemihypergroup of S if f is a prime fuzzy subset of S. *Prime fuzzy ternary subsemihypergroup* of S if f is a prime fuzzy subset of S. *Prime fuzzy ternary subsemihypergroup* of S if f is a prime fuzzy subset of S. *Prime fuzzy left hyperideals, prime fuzzy right hyperideals, prime fuzzy lateral hyperideals, and prime fuzzy hyperideals* of S are defined analogously.

**Theorem 29.** Let *S* be a ternary semihypergroup and *A* a nonempty subset of *S*. The following statements hold true.

- (1) *A* is a prime subset of *S* if and only if *f*<sub>A</sub> is a prime fuzzy subset of *S*.
- (2) A is a prime ternary subsemihypergroup (prime left hyperideal, prime right hyperideal, prime lateral hyperideal, prime hyperideal) of S if and only if  $f_A$  is a prime fuzzy ternary subsemihypergroup (prime fuzzy left hyperideal, prime fuzzy right hyperideal, prime fuzzy lateral hyperideal, prime fuzzy hyperideal) of S.

*Proof.* (1) Let us assume that A is a prime subset of S. Let  $x, y, z \in S$ .

*Case* 1.  $xyz \subseteq A$ . Since *A* is prime,  $x \in A$  or  $y \in A$  or  $z \in A$ . Thus  $\max\{f_A(x), f_A(y), f_A(z)\} = 1 \ge \inf_{t \in xyz} f_A(t)$ . *Case* 2.  $xyz \notin A$ . Thus  $\inf_{t \in xyz} f_A(t) = 0 \le \max\{f_A(x), f_A(y), f_A(z)\}$ .

Conversely, let  $x, y, z \in S$  such that  $xyz \subseteq A$ . Thus  $f_A(t) = 1$  for all  $t \in xyz$ . Since  $f_A$  is prime,  $\max\{f_A(x), f_A(y), f_A(z)\} = 1$ . This implies  $f_A(x) = 1$  or  $f_A(y) = 1$  or  $f_A(z) = 1$ . Hence  $x \in A$  or  $y \in A$  or  $z \in A$ .

(2) It follows from (1) and Theorem 28.  $\Box$ 

Let f be a fuzzy subset of a set (a ternary semihypergroup) S. For any  $t \in [0, 1]$ , the set

$$f_t = \{ x \in S \mid f(x) \ge t \}, \qquad f_t^s = \{ x \in S \mid f(x) > t \}$$
(37)

are called a t—level set and a t—stronglevel set of f, respectively.

**Theorem 30.** Let *f* be a fuzzy subset of a ternary semihypergroup *S*. The following statements hold true:

- (1) f is a fuzzy ternary subsemihypergroup of S if and only if for all  $t \in [0.1]$ , if  $f_t \neq \emptyset$ , then  $f_t$  is a ternary subsemihypergroup of S.
- (2) *f* is a fuzzy left hyperideal (fuzzy right hyperideal, fuzzy lateral hyperideal, fuzzy hyperideal) of S if and only if for all  $t \in [0, 1]$ , if  $f_t \neq \emptyset$ , then  $f_t$  is a left hyperideal (right hyperideal, lateral hyperideal, hyperideal) of S.

*Proof.* (1) Let us assume that f is a fuzzy ternary subsemihypergroup of S. Let  $t \in [0, 1]$  such that  $f_t \neq \emptyset$ . Let  $x, y, z \in f_t$ . So  $f(x), f(y), f(z) \ge t$ . Thus  $\min\{f(x), f(y), f(z)\} \ge t$ . Since f is a fuzzy ternary subsemihypergroup of S,  $\inf_{h \in xyz} f(h) \ge t$ . Hence,  $xyz \subseteq f_t$ . Conversely, let  $x, y, z \in S$ . Let we take  $t = \min\{f(x), f(y), f(z)\}$ . Then  $f(x), f(y), f(z) \ge t$ . Thus,  $x, y, z \in f_t$ . Since  $f_t$  is a ternary subsemihypergroup of S,  $xyz \subseteq f_t$ . Thus  $\inf_{h \in xyz} f(h) \ge t = \min\{f(x), f(y), f(z)\}$ .

(2) Let us assume that f is a fuzzy left hyperideal of S. Let  $t \in [0, 1]$ . Let us suppose that  $f_t \neq \emptyset$ . Let  $x, y \in S$  and  $z \in f_t$ . Thus  $\inf_{h \in xyz} f(h) \ge f(z) \ge t$ . Therefore,  $xyz \subseteq f_t$ .

Conversely, let  $x, y, z \in S$ . Let we take t = f(z). Thus  $z \in f_t$ , this implies  $f_t \neq \emptyset$ . By assumption, we have  $f_t$  is a left hyperideal of S. So  $xyz \subseteq f_t$ . Therefore,  $\inf_{h \in xyz} f(h) \ge t$ . Thus  $\inf_{h \in xyz} f(h) \ge f(z)$ .

The remain parts can be proved in a similar way.  $\Box$ 

**Theorem 31.** Let *S* be a ternary semihypergroup and f be a fuzzy of *S*. The following statements hold true:

- (1) *f* is prime fuzzy subset of *S* if and only if for all  $t \in [0, 1]$ , if  $f_t \neq \emptyset$ , then  $f_t$  is a prime subset of *S*.
- (2) f is a prime fuzzy ternary subsemihypergroup (prime fuzzy left hyperideal, prime fuzzy right hyperideal, prime fuzzy lateral hyperideal, prime fuzzy hyperideal) of S if and only if for all  $t \in [0,1]$ , if  $f_t \neq \emptyset$ , then  $f_t$  is a prime ternary subsemihypergroup (prime left hyperideal, prime right hyperideal, prime lateral hyperideal, prime hyperideal) of S.

*Proof.* (1) Let us assume that f is a prime fuzzy subset of S. Let  $t \in [0, 1]$ . Let us suppose that  $f_t \neq \emptyset$ . Let  $x, y, z \in S$  such that  $xyz \subseteq f_t$ . Thus  $\inf_{h \in xyz} f(h) \ge t$ . Since f is prime,  $f(x) \ge t$  or  $f(y) \ge t$  or  $f(z) \ge t$ . This implies  $x \in f_t$  or  $y \in f_t$  or  $z \in f_t$ .

Conversely, let  $x, y, z \in S$ . Let we take  $t = \inf_{h \in xyz} f(h)$ . Then  $xyz \subseteq f_t$ . Since  $f_t$  is prime,  $x \in f_t$  or  $y \in f_t$  or  $z \in f_t$ . Then  $f(x) \ge t$  or  $f(y) \ge t$  or  $f(z) \ge t$ . Hence  $\max\{f(x), f(y), f(z)\} \ge t = \inf_{h \in xyz} f(h)$ .

(2) It follows from (1) and Theorem 29.

**Theorem 32.** Let *S* be a ternary semihypergroup and *f* be a fuzzy subset of *S*. Then *f* is a prime fuzzy subset (prime fuzzy ternary subsemihypergroup, prime fuzzy left hyperideal, prime fuzzy right hyperideal, prime fuzzy lateral hyperideal, prime fuzzy hyperideal) of *S* if and only if for all  $t \in [0, 1]$ , if  $f_t^s \neq \emptyset$ , then  $f_t^s$  is a prime subset (prime ternary subsemihypergroup, prime left hyperideal, prime right hyperideal, prime lateral hyperideal, prime hyperideal, prime lateral hyperideal, prime hyperideal) of *S*.

*Proof.* The proof is similar to the proof of Theorem 30.  $\Box$ 

#### 6. Rough Fuzzy Hyperideals of Ternary Semihypergroups

In this section we study rough fuzzy ternary semihypergroups, left hyperideals, right hyperideals, lateral hyperideals and hyperideals of ternary semihypergroups.

Let S be a ternary semihypergroup and f be a fuzzy subset of S. Then the sets

$$\overline{Apr}_{\rho}(f)(x) = \sup_{a \in [x]_{\rho}} f(a), \qquad \underline{Apr}_{\rho}(f)(x) = \inf_{a \in [x]_{\rho}} f(a)$$
(38)

are called the  $\rho$ -upper and  $\rho$ -lower approximations of a fuzzy set f, respectively.

**Lemma 33.** Let *S* be a ternary semihypergroup,  $\rho$  a regular relation on *S*, *f* a fuzzy subset of *S* and  $t \in [0, 1]$ , then

(1) 
$$(\underline{Apr}_{\rho}(f))_t = \underline{Apr}_{\rho}(f_t),$$
  
(2)  $(\overline{Apr}_{\rho}(f))_t^s = \overline{Apr}_{\rho}(f_t^s).$ 

*Proof.* (1) Let  $x \in (\underline{Apr}_{\rho}(f))_t$ . Then  $\underline{Apr}_{\rho}(f)(x) \ge t$ . So  $\inf_{a \in [x]_{\rho}} f(a) \ge t$ . Therefore,  $f(a) \ge t$  for all  $a \in [x]_{\rho}$ . This implies  $[x]_{\rho} \subseteq f_t$ . Therefore,  $x \in \underline{Apr}_{\rho}(f_t)$ .

Conversely, let us assume that  $x \in \underline{Apr}_{\rho}(f_t)$ . Thus  $[x]_{\rho} \subseteq f_t$ . Then  $f(a) \ge t$  for all  $a \in [x]_{\rho}$ . This implies  $\inf_{a \in [x]_{\rho}} f(a) \ge t$ . Thus,  $\underline{Apr}_{\rho}(f)(x) \ge t$ . Hence  $x \in (\underline{Apr}_{\rho}(f))_t$ .

(2) Let  $x \in (\overline{Apr}_{\rho}(f))_t^s$ . Then  $\overline{Apr}_{\rho}(f)(x) > t$ . So  $\sup_{a \in [x]_{\rho}} f(a) > t$ . Therefore, f(a) > t for some  $a \in [x]_{\rho}$ . This implies  $[x]_{\rho} \cap f_t^s \neq \emptyset$ . Therefore  $x \in \overline{Apr}_{\rho}(f_t^s)$ .

Conversely, let us assume  $x \in \overline{Apr}_{\rho}(f_t^s)$ . Thus  $[x]_{\rho} \cap f_t^s \neq \emptyset$ . Then f(a) > t for some  $a \in [x]_{\rho}$ . This implies  $\inf_{a \in [x]_{\rho}} f(a) > t$ . Thus  $\overline{Apr}_{\rho}(f)(x) > t$ . Hence,  $x \in (\overline{Apr}_{\rho}(f))_t^s$ .

**Theorem 34.** Let *S* be a ternary semihypergroup and  $\rho$  be a regular relation on *S*. If *f* is a fuzzy ternary subsemihypergroup (fuzzy left hyperideal, fuzzy right hyperideal, fuzzy lateral hyperideal, fuzzy hyperideal) of *S*, then  $\overline{Apr}_{\rho}(f)$  and  $\underline{Apr}_{\rho}(f)$  are fuzzy ternary subsemihypergroups (fuzzy left hyperideals, fuzzy right hyperideals, fuzzy lateral hyperideals, fuzzy hyperideals) of *S*.

*Proof.* It can be obtained easily by Theorems 29, 31, 15, and 17 and Lemma 33.

## 7. Fuzzy Bi-Hyperideals of Ternary Semihypergroups

 $\inf_{h \in xaybz} f_A(h).$ 

Let *S* be a ternary semihypergroup. A fuzzy subset *f* of *S* is called a *fuzzy bi-hyperideal* of *S* if  $\inf_{h \in xyz} f(h) \ge \min\{f(x), f(y), f(z)\}$  and  $\inf_{h \in xyzpq} f(h) \ge \min\{f(x), f(z), f(q)\}$  for all *x*, *y*, *z*, *p*, *q*  $\in$  *S*.

**Theorem 35.** Let *S* be a ternary semihypergroup and *A* a nonempty subset of *S*. Then *A* is a bi-hyperideal of *S* if and only if  $f_A$  is a fuzzy bi-hyperideal of *S*.

*Proof.* Let us assume that A is a bi-hyperideal of S. Let  $a, b, x, y, z \in S$ .

*Case* 1. *x*, *y*, *z*  $\in$  *A*. Since *A* is a bi-hyperideal of *S*, then *xy*, *xaybz*  $\subseteq$  *A*. Therefore  $\inf_{h \in xyz} f_A(h) = 1 \ge \min\{f_A(x), f_A(y), f_A(z)\}$  and  $\inf_{h \in xaybz} f_A(h) = 1 \ge \min\{f_A(x), f_A(y), f_A(z)\}$ . *Case* 2. *x*  $\notin$  *A* or *y*  $\notin$  *A* or *z*  $\notin$  *A*. Thus  $f_A(x) = 0$  or  $f_A(y) = 0$  or  $f_A(z) = 0$ . Hence  $\min\{f_A(x), f_A(y), f_A(z)\} = 0 \le \inf_{h \in xyz} f_A(h)$  and  $\min\{f_A(x), f_A(y), f_A(z)\} = 0 \le$ 

Let *S* be a ternary semihypergroup. A bi-hyperideal *T* of *S* is called a *prime bi-hyperideal* of *S* if *T* is a prime subset of *S*. A fuzzy bi-hyperideal f of *S* is called a *prime fuzzy bi-hyperideal* of *S* if f is a prime fuzzy subset of *S*.

**Theorem 36.** Let *S* be a ternary semihypergroup and *A* a nonempty subset of *S*. Then *A* is a prime bi-hyperideal of *S* if and only if  $f_A$  is a prime fuzzy bi-hyperideal of *S*.

*Proof.* It follows from Theorems 29 and 35.

**Theorem 37.** Let *S* be a ternary semihypergroup and *f* a fuzzy subset of *S*. Then *f* is a fuzzy bi-hyperideal of *S* if and only if for all  $t \in [0, 1]$ , if  $f_t \neq \emptyset$ , then  $f_t$  is a bi-hyperideal of *S*.

*Proof.* Let us assume that f is a fuzzy bi-hyperideal of S. Let  $t \in [0,1]$  such that  $f_t \neq \emptyset$ . Let  $x, y, z \in f_t$ . Then  $f(x) \ge t, f(y) \ge t$  and  $f(z) \ge t$ . Since f is a fuzzy bi-hyperideal of S,  $\inf_{h \in xyz} f(h) \ge \min\{f(x), f(y), f(z)\} \ge t$ , and  $\inf_{h \in xaybz} f(h) \ge \min\{f(x), f(y), f(z)\} \ge t$  for all  $a, b \in S$ . Therefore,  $xyz, xaybz \subseteq f_t$ . Hence  $f_t$  is a bi-hyperideal of S.

Conversely, let us assume for all  $t \in [0,1]$ , if  $f_t \neq \emptyset$ , then  $f_t$  is a bi-hyperideal of S. Let  $a, b, x, y, z \in S$ . Let we take  $t = \min\{f(x), f(y), f(z)\}$ . Then  $x, y, z \in$  $f_t$ . This implies that  $f_t \neq \emptyset$ . By assumption, we have  $f_t$ is a bi-hyperideal of S. So xyz,  $xaybz \subseteq f_t$ . Therefore,  $\inf_{h \in xyz} f(h) \ge t$  and  $\inf_{h \in xaybz} f(h) \ge t$ . Hence  $\inf_{h \in xyz} f(h) \ge \min\{f(x), f(y), f(z)\}$  and  $\inf_{h \in xaybz} f(h) \ge$  $\min\{f(x), f(y), f(z)\}$ .

**Theorem 38.** Let *S* be a ternary semihypergroup and f a fuzzy subset of *S*. Then f is a prime fuzzy bi-hyperideal of *S* if and

only if for all  $t \in [0,1]$ , if  $f_t \neq \emptyset$ , then  $f_t$  is a prime bihyperideal of S.

*Proof.* It follows from Theorems 31 and 37.

**Theorem 39.** Let *S* be a ternary semihypergroup and *f* a fuzzy subset of *S*. Then *f* is a fuzzy bi-hyperideal of *S* if and only if for all  $t \in [0, 1]$ , if  $f_t^s \neq \emptyset$ , then  $f_t^s$  is a bi-hyperideal of *S*.

*Proof.* The proof is similar to the proof of Theorem 37.  $\Box$ 

**Theorem 40.** Let *S* be a ternary semihypergroup and *f* a fuzzy subset of *S*. Then *f* is a prime bi-hyperideal of *S* if and only if for all  $t \in [0, 1]$ , if  $f_t^s \neq \emptyset$ , then  $f_t^s$  is a prime bi-hyperideal of *S*.

*Proof.* It follows from Theorems 31 and 39.  $\Box$ 

### 8. Rough Fuzzy Bi-Hyperideals of Ternary Semihypergroups

**Theorem 41.** Let *S* be a ternary semihypergroup and  $\rho$  be a complete regular relation on *S*. If *f* is a fuzzy bi-hyperideal of *S*, then  $\overline{Apr}_{\rho}(f)$  and  $\underline{Apr}_{\rho}(f)$  are fuzzy bi-hyperideals.

*Proof.* This proof follows from Theorems 37, 39, and 20, and Lemma 33.

Note that if  $\overline{Apr}_{\rho}(f)$  and  $\underline{Apr}_{\rho}(f)$  are fuzzy bihyperideals of a ternary semihypergroup *S*, in general, *f* need not be a fuzzy bi-hyperideal of *S*.

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