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Research Article

Fuzzy Constrained Probabilistic Inventory Models Depending on Trapezoidal Fuzzy Numbers

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We discussed two different cases of the probabilistic continuous review mixture shortage inventory model with varying and constrained expected order cost, when the lead time demand follows some different continuous distributions. The first case is when the total cost components are considered to be crisp values, and the other case is when the costs are considered as trapezoidal fuzzy number. Also, some special cases are deduced. To investigate the proposed model in the crisp case and the fuzzy case, illustrative numerical example is added. From the numerical results we will conclude that Uniform distribution is the best distribution to get the exact solutions, and the exact solutions for fuzzy models are considered more practical and close to the reality of life and get minimum expected total cost less than the crisp models.

1. Introduction

Inventory system is one of the most diversified fields of applied sciences that are widely used in a variety of areas including operations research, applied probability, computer sciences, management sciences, production system, and telecommunications. More than fifty years ago, the analysis of inventory system has appeared in the reference books and survey papers. Hadley and Whitin [1] are considered one of the first researchers who have discussed the analysis of inventory systems, where they displayed a method for the analysis of the mathematical model for inventory systems. Also, Balkhi and Benkherouf [2] have introduced production lot size inventory model in which products deteriorate at a constant rate and in which demand and production rates are allowed to vary with time. Inventory models may be either deterministic or probabilistic, since the demand of commodity may be deterministic or probabilistic, respectively. These cases were dealt with by Hadley and Whitin [1], Abuo-El-Ata et al. [3], and Vijayan and Kumaran [4].

Some managers allow the shortage in inventory systems; this shortage may be backorder case, lost sales case, and mixture shortage case. Many authors are dealing with inventory problems with various shortage cases where the cost components are considered as crisp values which does not depict the real inventory system fully. For example, constrained probabilistic inventory model with varying order and shortage costs using Lagrangian method has been investigated by Fergany [5]. In addition, constrained probabilistic inventory model with continuous distributions and varying holding cost was discussed by Fergany and El-Saadani [6]. In 2006, several models of continuous distributions for constrained probabilistic lost sales inventory models with varying order cost under holding cost constraint using Lagrangian method by Fergany and El-Wakeel [7, 8] were discussed. Recently, El-Wakeel [9] deduced constrained backorders inventory system with varying order cost under holding cost constraint: lead time demand uniformly distributed using Lagrangian method. Also, El-Wakeel and Fergany [10] deduced constrained probabilistic continuous review inventory system with mixture shortage and stochastic lead time demand.

Sometimes, the cost components are considered as fuzzy values, because, in real life, the various physical or chemical characteristics may cause an effect on the cost components and then precise values of cost characteristics become difficult to measure as the exact amount of order, holding, and especially shortage cost. Thus, in controlling the inventory system it may allow some flexibility in the cost parameter values in order to treat the uncertainties which always fit the real situations. Since we want to satisfy our requirements for such contradictions, the fuzzy set theory meets these requirements to some extent. In 1965, Zadeh [11] first introduced the fuzzy set theory which studied the intention to accommodate uncertainty in the nonstochastic sense rather than the presence of random variables. Syed and Aziz [12] have examined the fuzzy inventory model without shortages using signed distance method. Kazemi et al. [13] have treated the inventory model with backorders with fuzzy parameters and decision variables. Gawdt [14] presented a mixture continuous review inventory model under varying holding cost constraint when the lead time demand follows Gamma distribution, where the costs were fuzzified as the trapezoidal fuzzy numbers. The continuous review inventory model with mixture shortage under constraint involving crashing cost based on probabilistic triangular fuzzy numbers by Fergany and Gawdt [15] was discussed. A probabilistic periodic review inventory model using Lagrange technique and fuzzy adaptive particle swarm optimization was presented by Fergany et al. [16]. Fuzzy inventory model for deteriorating items with time dependent demand and partial backlogging is established by Kumar and Rajput [17]. Indrajitsingha et al. [18] give fuzzy inventory model with shortages under fully backlogged using signed distance method. Recently, Patel et al. [19] introduced the continuous review inventory model under fuzzy environment without backorder for deteriorating items.

As we found earlier, many authors have studied the inventory models with different assumptions and conditions. These assumptions and conditions are represented in constraints and costs (constant or varying). Therefore, due to the importance of the inventory models we shall propose and study, in this paper, the mixture shortage inventory model with varying order cost under expected order cost constraint and the lead time demand follow Exponential, Laplace, and Uniform distributions. Our goal of studying the inventory models is to minimize the total cost. The order quantity and the reorder point are the policy variables for this model, which minimize the expected annual total cost. We evaluated the optimal order quantity and the reorder point in two cases: first case is when the cost components are considered as crisp values, and the second case is when the cost components are fuzzified as a trapezoidal fuzzy numbers, which is called the fuzzy case. Finally this work is illustrated by numerical example and we will make comparisons of all results and obtain conclusions.

2. Model Development

To develop any model of inventory models we need to put some notations and assumptions represented in Notations section.

2.1. Assumptions

(1) Consider that continuous review inventory model under order cost constraint and shortages are allowed.

(2) Demand is a continuous random variable with known probability.

(3) The lead time is constant and follows the known distributions.

(4) γ is a fraction of unsatisfied demand that will be backordered while the remaining fraction $(1-\gamma)$ is completely lost, where $0 \leq \gamma \leq 1$.

(5) New order with size (Q) is placed when the inventory level drops to a certain level, called the reorder point (r); assume that the system repeats itself in the sense that the inventory position varies between r and $r+Q$ during each cycle.

3. Model (I): The Mixture Shortage Model Where the Cost Components Are Considered as Crisp Values

In this section, we consider that the continuous review inventory model with shortage is allowed. Some customers are willing to wait for the new replenishment and the others have no patience; this case is called mixture shortage or partial backorders.

The expected annual total cost consisted of the sum of three components:

$$\begin{aligned} E(\text{Total Cost}) &= E(\text{order Cost}) + E(\text{Holding Cost}) \\ &\quad + E(\text{Shortage Cost}), \\ E(\text{TC}(Q,r)) &= E(\text{OC}) + E(\text{HC}) + E(\text{SC}), \end{aligned} \tag{1}$$

where

$$E(\text{SC}) = E(\text{BC}) + E(\text{LC}) \tag{2}$$

and we assume the varying order cost function, where the order cost is a decreasing function of the order quantity Q. Then, the expected order cost is given by

$$\begin{aligned} E(\text{OC}) &= c_o(Q)\frac{\overline{D}}{Q} = c_o Q^{-\beta}\frac{\overline{D}}{Q} = c_o\overline{D}Q^{-\beta-1}, \\ E(\text{HC}) &= c_h\overline{H} = c_h\left[\frac{Q}{2} + r - E(x) + (1-\gamma)\overline{S}(r)\right], \\ E(\text{BC}) &= \frac{c_b\gamma\overline{D}}{Q}\overline{S}(r), \\ E(\text{LC}) &= \frac{c_l(1-\gamma)\overline{D}}{Q}\overline{S}(r). \end{aligned} \tag{3}$$

Our objective is to minimize the expected total costs $[\min E(\text{TC}(Q,r))]$ with varying order cost under the expected order cost constraint which needs to find the optimal values of order quantity Q and reorder point r. To solve this primal function, let us write it as follows:

$$\begin{aligned} E(\mathrm{TC}(Q,r)) &= c_o\overline{D}Q^{-\beta-1} \\ &\quad + c_h\left[\frac{Q}{2}+r-E(x)+(1-\gamma)\overline{S}(r)\right] \\ &\quad + \frac{c_b\gamma\overline{D}}{Q}\overline{S}(r)+\frac{c_l(1-\gamma)\overline{D}}{Q}\overline{S}(r) \\ &= c_o\overline{D}Q^{-\beta-1}+c_h\left(\frac{Q}{2}+r-E(x)\right) \\ &\quad + \frac{c_b\gamma\overline{D}}{Q}\overline{S}(r) \\ &\quad + \left(c_h+\frac{c_l\overline{D}}{Q}\right)(1-\gamma)\overline{S}(r) \end{aligned} \tag{4}$$

$$\text{Subject to:}\quad c_o\overline{D}Q^{-\beta-1}\le K. \tag{5}$$

We use the Lagrange multiplier technique to get the optimal values Q^* and r^* which minimize (4) under constraint (5) as follows:

$$\begin{aligned} G(Q,r,\lambda) &= c_o\overline{D}Q^{-\beta-1}+c_h\left(\frac{Q}{2}+r-E(x)\right) \\ &\quad + \frac{c_b\gamma\overline{D}}{Q}\overline{S}(r)+\left(c_h+\frac{c_l\overline{D}}{Q}\right)(1-\gamma)\overline{S}(r) \\ &\quad + \lambda\left(c_o\overline{D}Q^{-\beta-1}-K\right). \end{aligned} \tag{6}$$

Putting each of the corresponding first partial derivatives of (6) equal to zero at $Q=Q^*$ and $r=r^*$, respectively, we get

$$\begin{aligned} &c_hQ^{*2}+B\overline{D}Q^{*-\beta}-2A\overline{S}(r^*)=0, \\ &R(r^*)=\frac{c_hQ^*}{c_h(1-\gamma)Q^*+A}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} A &= \overline{D}\left[c_b\gamma+c_l(1-\gamma)\right], \\ B &= 2c_o(-\beta-1)[1+\lambda]. \end{aligned} \tag{8}$$

Clearly, it is difficult to find an exact solution of Q^* and r^* of (7), so we can suppose that the lead time demand follows some distributions.

3.1. Lead Time Demand Follows Exponential Distribution. Supposing that the lead time demand follows the Exponential distribution with parameters ν, then its probability density function is given by

$$\begin{aligned} &f(x)=\nu e^{-\nu x};\quad x\ge 0,\ \nu>0 \\ &\text{with } E(x)=\frac{1}{\nu}, \\ &R(r)=e^{-\nu r}, \\ &\overline{S}(r)=\frac{1}{\nu}e^{-\nu r}. \end{aligned} \tag{9}$$

The optimal order quantity and the optimal reorder level which minimize the expected relevant annual total cost can be obtained by substituting (9) into (7). Solving them simultaneously we get

$$\begin{aligned} &\nu c_h^2(1-\gamma)Q^{*3}+\nu c_hAQ^{*2}-2c_hAQ^* \\ &\quad + \nu c_h(1-\gamma)B\overline{D}Q^{*1-\beta}+\nu AB\overline{D}Q^{*-\beta}=0, \\ &r^*=-\frac{1}{\nu}\ln\left[\frac{c_hQ^*}{c_h(1-\gamma)Q^*+A}\right], \end{aligned} \tag{10}$$

which give exact solutions for model (I).

3.2. Lead Time Demand Follows Laplace Distribution. If the lead time demand follows the Laplace distribution with parameters μ,θ, the probability density function will be

$$\begin{aligned} &f(x)=\frac{1}{2\theta}e^{-|x-\mu|/\theta};\quad -\infty<x<\infty,\ \theta>0 \\ &\text{with } E(x)=\mu, \\ &R(r)=\frac{1}{2}e^{-((r-\mu)/\theta)}, \\ &\overline{S}(r)=\frac{\theta}{2}e^{-((r-\mu)/\theta)}. \end{aligned} \tag{11}$$

The optimal order quantity and the optimal reorder level which minimize the expected relevant annual total cost can be obtained by substituting (11) into (7), and, solving them simultaneously, we obtain

$$\begin{aligned} &c_h^2(1-\gamma)Q^{*3}+c_hAQ^{*2}-2c_h\theta AQ^* \\ &\quad + c_h(1-\gamma)B\overline{D}Q^{*1-\beta}+AB\overline{D}Q^{*-\beta}=0, \\ &r^*=\mu-\theta\ln\left[\frac{2c_hQ^*}{c_h(1-\gamma)Q^*+A}\right], \end{aligned} \tag{12}$$

which give exact solutions for model (I).

3.3. Lead Time Demand Follows Uniform Distribution. Similarly, suppose that the lead time demand follows the Uniform distribution over the range from 0 to b, that is, $x \sim \mathrm{Uniform}(0,b)$; then its probability density function is given by

$$\begin{aligned} &f(x)=\frac{1}{b};\quad 0\le x\le b \\ &\text{with } E(x)=\frac{b}{2}, \\ &R(r)=1-\frac{r}{b}, \\ &\overline{S}(r)=\frac{1}{2b}(r-b)^2. \end{aligned} \tag{13}$$

The optimal order quantity and the optimal reorder level which minimize the expected relevant annual total cost can be obtained by substituting (13) into (7). Solving them simultaneously, we find

$$c_h^3(1-\gamma)^2 Q^{*4} + 2c_h^2(1-\gamma)AQ^{*3} + c_h A[A - bc_h]Q^{*2} + c_h^2(1-\gamma)^2 B\overline{D}Q^{*2-\beta} + 2c_h(1-\gamma)AB\overline{D}Q^{*1-\beta} + A^2 B\overline{D}Q^{*-\beta} = 0,$$
$$r^* = b\left[1 - \frac{c_h Q^*}{c_h(1-\gamma)Q^* + A}\right], \tag{14}$$

which give exact solutions for model (I).

Thus, the exact solution for constrained continuous review inventory model with mixture shortage and varying order cost can obtained by solving previous equations for each distribution separately at different values of β and varying λ until the smallest positive value is found such that the constraint holds.

4. Model ($\mathrm{I_f}$): The Mixture Shortage Model Where the Cost Components Are Considered as Fuzzy Numbers

Consider continuous review inventory model similar to model (I), but assuming that the cost components c_o, c_h, c_b, and c_l are all fuzzy numbers, to control various uncertainties from various physical or chemical characteristics where there may be an effect on the cost components.

We represent these costs by trapezoidal fuzzy numbers as given below:

$$\begin{aligned} \tilde{c}_o &= (c_o - \delta_1, c_o - \delta_2, c_o + \delta_3, c_o + \delta_4), \\ \tilde{c}_h &= (c_h - \delta_5, c_h - \delta_6, c_h + \delta_7, c_h + \delta_8), \\ \tilde{c}_b &= (c_b - \theta_1, c_b - \theta_2, c_b + \theta_3, c_b + \theta_4), \\ \tilde{c}_l &= (c_l - \theta_5, c_l - \theta_6, c_l + \theta_7, c_l + \theta_8), \end{aligned} \tag{15}$$

where δ_i and θ_i, $i = 1, 2, \ldots, 8$ are arbitrary positive numbers and should satisfy the following constraints:

$$\begin{aligned} &c_o > \delta_1 > \delta_2, \\ &\delta_3 < \delta_4, \\ &c_h > \delta_5 > \delta_6, \\ &\delta_7 < \delta_8. \end{aligned} \tag{16}$$

Similarly,

$$\begin{aligned} &C_b > \theta_1 > \theta_2, \\ &\theta_3 < \theta_4, \\ &c_l > \theta_5 > \theta_6, \\ &\theta_7 < \theta_8. \end{aligned} \tag{17}$$

We can represent the order cost as a trapezoidal fuzzy number as shown in Figure 1 and similarly for the remaining costs.

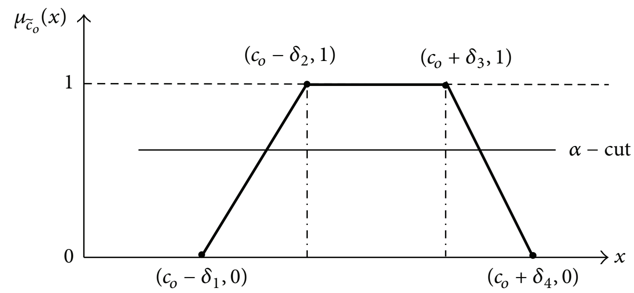

Figure 1: Order cost as a trapezoidal fuzzy number.

Note that the membership function of $\tilde{c}_o$ is 1 at points $c_o - \delta_2$ and $c_o + \delta_3$, decreases as the point deviates from $c_o - \delta_2$ and $c_o + \delta_3$, and reaches zero at the endpoints $c_o - \delta_1$ and $c_o + \delta_4$.

The left and right limits of α – cut of c_o, c_h, c_b, and c_l are given as follows:

$$\begin{aligned} \tilde{c}_{ov}(\alpha) &= c_o - \delta_1 + (\delta_1 - \delta_2)\alpha, \\ \tilde{c}_{ou}(\alpha) &= c_o + \delta_4 - (\delta_4 - \delta_3)\alpha, \\ \tilde{c}_{hv}(\alpha) &= c_h - \delta_5 + (\delta_5 - \delta_6)\alpha, \\ \tilde{c}_{hu}(\alpha) &= c_h + \delta_8 - (\delta_8 - \delta_7)\alpha, \\ \tilde{c}_{bv}(\alpha) &= c_b - \theta_1 + (\theta_1 - \theta_2)\alpha, \\ \tilde{c}_{bu}(\alpha) &= c_b + \theta_4 - (\theta_4 - \theta_3)\alpha, \\ \tilde{c}_{lv}(\alpha) &= c_l - \theta_5 + (\theta_5 - \theta_6)\alpha, \\ \tilde{c}_{lu}(\alpha) &= c_l + \theta_8 - (\theta_8 - \theta_7)\alpha. \end{aligned} \tag{18}$$

The expected annual total cost for this case under the expected order cost constraint and when all cost components are fuzzy can be expressed as follows:

$$\begin{aligned} \tilde{E}(\tilde{c}_o, \tilde{c}_h, \tilde{c}_b, \tilde{c}_l) &= \tilde{c}_o\overline{D}Q^{-\beta-1} + \tilde{c}_h\left[\frac{Q}{2} + r - E(x) + (1-\gamma)\overline{S}(r)\right] + \frac{\tilde{c}_b\gamma\overline{D}}{Q}\overline{S}(r) + \frac{\tilde{c}_l(1-\gamma)\overline{D}}{Q}\overline{S}(r) \\ &= \tilde{c}_o\overline{D}Q^{-\beta-1} + \tilde{c}_h\left(\frac{Q}{2} + r - E(x)\right) + \frac{\tilde{c}_b\gamma\overline{D}}{Q}\overline{S}(r) + \left(\tilde{c}_h + \frac{\tilde{c}_l\overline{D}}{Q}\right)(1-\gamma)\overline{S}(r) \end{aligned} \tag{19}$$

Subject to: $\tilde{c}_o\overline{D}Q^{-\beta-1} \le K.$ (20)

We use the Lagrange multiplier technique to find the optimal values Q^* and r^* which minimize (19) under constraint (20) as follows:

$$\begin{aligned}\widetilde{G}(\widetilde{c}_o,\widetilde{c}_h,\widetilde{c}_b,\widetilde{c}_l) &= \widetilde{c}_o\overline{D}Q^{-\beta-1}+\widetilde{c}_h\left(\frac{Q}{2}+r-E(x)\right)\\ &+\frac{\widetilde{c}_b\gamma\overline{D}}{Q}\overline{S}(r)\\ &+\left(\widetilde{c}_h+\frac{\widetilde{c}_l\overline{D}}{Q}\right)(1-\gamma)\overline{S}(r)\\ &+\lambda\left(\widetilde{c}_o\overline{D}Q^{-\beta-1}-K\right).\end{aligned} \tag{21}$$

We can obtain the form of left and right α – cut of the fuzzified cost function (21), respectively, as follows:

$$\begin{aligned}\widetilde{G}(\widetilde{c}_o,\widetilde{c}_h,\widetilde{c}_b,\widetilde{c}_l)_v(\alpha) &= \widetilde{c}_{ov}\overline{D}Q^{-\beta-1}\\ &+\widetilde{c}_{hv}\left(\frac{Q}{2}+r-E(x)\right)\\ &+\frac{\widetilde{c}_{bv}\gamma\overline{D}}{Q}\overline{S}(r)\\ &+\left(\widetilde{c}_{hv}+\frac{\widetilde{c}_{lv}\overline{D}}{Q}\right)(1-\gamma)\overline{S}(r)\\ &+\lambda\left(\widetilde{c}_{ov}\overline{D}Q^{-\beta-1}-K\right),\\ \widetilde{G}(\widetilde{c}_o,\widetilde{c}_h,\widetilde{c}_b,\widetilde{c}_l)_u(\alpha) &= \widetilde{c}_{ou}\overline{D}Q^{-\beta-1}\\ &+\widetilde{c}_{hu}\left(\frac{Q}{2}+r-E(x)\right)\\ &+\frac{\widetilde{c}_{bu}\gamma\overline{D}}{Q}\overline{S}(r)\\ &+\left(\widetilde{c}_{hu}+\frac{\widetilde{c}_{lu}\overline{D}}{Q}\right)(1-\gamma)\overline{S}(r)\\ &+\lambda\left(\widetilde{c}_{ou}\overline{D}Q^{-\beta-1}-K\right).\end{aligned} \tag{22}$$

Since $\widetilde{G}_v(\alpha)$ and $\widetilde{G}_u(\alpha)$ exist and are integrable for $\alpha \in [0,1]$, as in Yao and Wu [20], we have

$$d\left(\widetilde{G},\widetilde{0}\right)=\frac{1}{2}\int_0^1\left(\widetilde{G}_v(\alpha)+\widetilde{G}_u(\alpha)\right)d\alpha. \tag{23}$$

We get the defuzzified value of $\widetilde{G}(\widetilde{c}_o,\widetilde{c}_h,\widetilde{c}_b,\widetilde{c}_l)(\alpha)$ by using (23) for (22) as follows:

$$\begin{aligned}d\left(\widetilde{G},\widetilde{0}\right) &= A_1\overline{D}Q^{-\beta-1}+A_2\left(\frac{Q}{2}+r-E(x)\right)\\ &+\frac{A_3\gamma\overline{D}}{Q}\overline{S}(r)\\ &+\left(A_2+\frac{A_4\overline{D}}{Q}\right)(1-\gamma)\overline{S}(r)\\ &+\lambda\left(A_1\overline{D}Q^{-\beta-1}-K\right),\end{aligned} \tag{24}$$

where

$$\begin{aligned}A_1 &= \frac{(4c_o-\delta_1-\delta_2+\delta_3+\delta_4)}{4},\\ A_2 &= \frac{(4c_h-\delta_5-\delta_6+\delta_7+\delta_8)}{4},\\ A_3 &= \frac{(4c_b-\theta_1-\theta_2+\theta_3+\theta_4)}{4},\\ A_4 &= \frac{(4c_l-\theta_5-\theta_6+\theta_7+\theta_8)}{4}.\end{aligned} \tag{25}$$

Similarly, as in model (I), to get the optimal values Q^* and r^* put each of the corresponding first partial derivatives of (24) equal to zero at $Q = Q^*$ and $r = r^*$, respectively; we obtain

$$\begin{aligned}&\left(2A_1(-\beta-1)\overline{D}Q^{*-\beta}\right)(1+\lambda)+A_2Q^{*2}\\ &\quad-2A_3\gamma\overline{DS}(r)-2A_4(1-\gamma)\overline{DS}(r)=0\end{aligned} \tag{26}$$

and the probability of the shortage is

$$R(r^*)=\frac{A_2Q^*}{A_2(1-\gamma)Q^*+A_3\gamma\overline{D}+A_4(1-\gamma)\overline{D}}. \tag{27}$$

Clearly, there is no closed form solution of (26) and (27). We can solve these equations by using the same manner as in model (I).

5. Special Cases

(1) Letting $\gamma = 0$, $\beta = 0$ and $K \to \infty \Rightarrow C_o(Q) = c_o$ and $\lambda = 0$, thus $A = c_l\overline{D}$, $B = -2c_o$ and hence (7) reduces to

$$\begin{aligned}Q^* &= \sqrt{\frac{2\overline{D}(c_o+c_l\bar{s}(r^*))}{c_h}},\\ R(r^*) &= \frac{c_hQ^*}{c_hQ^*+c_l\overline{D}}.\end{aligned} \tag{28}$$

This is an unconstrained lost sales continuous review inventory model with constant units of costs, which are the same results as in Hadley and Whitin [1].

(2) Letting $\gamma = 1$, $\beta = 0$ and $K \to \infty \Rightarrow C_o(Q) = c_o$ and $\lambda = 0$, thus $A = c_b\overline{D}$, $B = -2c_o$; thus (7) reduces to

$$\begin{aligned}Q^* &= \sqrt{\frac{2\overline{D}(c_o+c_b\bar{s}(r^*))}{c_h}},\\ R(r^*) &= \frac{c_hQ^*}{c_b\overline{D}}.\end{aligned} \tag{29}$$

This is an unconstrained backorders continuous review inventory model with constant units of costs, which are the same results as in Hadley and Whitin [1].

(i) Equations (10) give unconstrained backorders continuous review of inventory model with constant units of cost and the lead time demand follows the Exponential distribution, which are the same results as in Hillier and Lieberman [21].

(ii) Equations (12) give unconstrained backorders continuous review inventory model with constant units of cost and the lead time demand follows the Laplace distribution, which agree with results of Nahmias [22].

(iii) Equations (14) give unconstrained backorders continuous review inventory model with constant units of cost and the lead time demand follows the Uniform distribution, which are the same results as in Fabrycky and Banks [23].

6. Numerical Example

Consider an inventory system with the following data:

$\overline{D} = 1050$ units per year,

$c_o = 70$ SR per unit ordered,

$c_h = 25$ SR per unit per year,

$c_b = 7$ SR per unit backorder,

$c_l = 15$ SR per unit lost,

the backorder fraction has the values $\gamma = 0.1, \gamma = 0.3$, and $\gamma = 0.7$,

let $K = 140$ SR,

and take

$$\delta_1 = 60, \quad \delta_2 = 48, \quad \delta_3 = 10, \quad \delta_4 = 50, \quad \delta_5 = 19, \quad \delta_6 = 10, \quad \delta_7 = 1, \quad \delta_8 = 2,$$
$$\theta_1 = 6, \quad \theta_2 = 4, \quad \theta_3 = 2, \quad \theta_4 = 4, \quad \theta_5 = 12, \quad \theta_6 = 7, \quad \theta_7 = 1, \quad \theta_8 = 2. \tag{30}$$

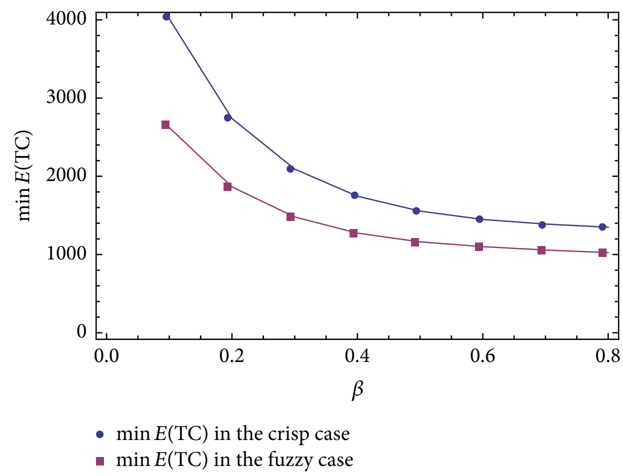

Figure 2: The comparison between the crisp and fuzzy cases for Exponential at $\gamma = 0.7$.

Determine Q^* and r^* for both cases of the previous model, when the lead time demand has the following distributions:

(i) Exponential distribution with $\nu = 0.077$ units.

(ii) Laplace distribution with $\mu = 13$ and $\theta = 10$ units.

(iii) Uniform distribution with $b = 26$ units.

Depending on the above data, we can obtain all results by solving the previous deduced equations at different values of β, λ, and γ as shown in the Tables 1, 2, and 3 which give the optimal values of Q^* and r^* that minimize the expected total cost, when the lead time demand follows Exponential, Laplace, and Uniform distribution, respectively, for model (I) and model ($\mathrm{I_f}$).

From Table 1 we have that

at $\gamma = 0.1$, we will make backorders by 10% of new orders quantity;

at $\gamma = 0.3$, we will make backorders by 30% of new orders quantity;

at $\gamma = 0.7$, we will make backorders by 70% of new orders quantity.

After comparison of the crisp case and fuzzy case for Exponential distribution, we can deduce that the least min $E(\mathrm{TC})$ was obtained at $\gamma = 0.7$. We can draw the minimum expected total cost for model (I) and model ($\mathrm{I_f}$) against β for the Exponential distribution at $\gamma = 0.7$ as shown in Figure 2.

From Table 2 we have that

at $\gamma = 0.1$, we will make backorders by 10% of new orders quantity;

at $\gamma = 0.3$, we will make backorders by 30% of new orders quantity;

Table 1: The exact solutions and min $E(\mathrm{TC})$ for model (I) and model ($\mathrm{I_f}$) at Exponential distribution.

γ	β	Crisp case Q^*	Crisp case r^*	Crisp case min $E(\mathrm{TC})$	Fuzzy case Q^*	Fuzzy case r^*	Fuzzy case min $E(\mathrm{TC})$
0.1	0.1	297.092	13.8608	4199.84	250.412	15.426	2741.44
	0.2	184.893	18.4055	2910.93	158.061	20.016	1972.09
	0.3	123.76	22.6467	2252.78	107.107	24.237	1578.8
	0.4	87.6955	26.5107	1898.63	76.6735	28.054	1367.98
	0.5	65.1132	29.9807	1703	57.4583	31.4585	1253.06
	0.6	50.1308	33.1065	1593.97	46.7207	33.9499	1189.84
	0.7	41.8943	35.2866	1533.94	43.0554	34.9436	1146.05
	0.8	39.0054	36.1611	1491.79	39.9907	35.846	1112.46
0.3	0.1	297.079	11.8026	4148.23	250.456	13.6144	2708.31
	0.2	184.905	16.4977	2863.37	158.08	18.3589	1941.59
	0.3	123.759	20.8394	2207.59	107.132	22.6783	1550.15
	0.4	87.7063	24.768	1855.18	76.7397	26.5535	1340.67
	0.5	65.1619	28.2749	1660.8	57.4816	30.0074	1226.34
	0.6	50.1482	31.4365	1552.36	46.8468	32.4964	1163.56
	0.7	41.9983	33.6078	1492.73	43.1748	33.498	1119.93
	0.8	39.1053	34.4876	1450.74	40.1052	34.4067	1086.48
0.7	0.1	297.118	6.33535	4012.01	250.446	8.99977	2622.85
	0.2	184.851	11.5619	2739.35	158.081	14.2364	1865.33
	0.3	123.742	16.2344	2092.27	107.118	18.864	1479.48
	0.4	87.6963	20.3768	1745.29	76.7206	22.9372	1273.64
	0.5	65.1605	24.0241	1554.52	57.4204	26.5318	1161.7
	0.6	50.2168	27.266	1448.65	47.2184	28.9822	1100.36
	0.7	42.327	29.4105	1390.22	43.5262	30.0069	1057.21
	0.8	39.4206	30.3066	1348.72	40.4411	30.9342	1024.17

at $\gamma = 0.7$, we will make backorders by 70% of new orders quantity.

After comparison of the crisp case and fuzzy case for Laplace distribution, we can deduce that the least min $E(\mathrm{TC})$ was obtained at $\gamma = 0.7$. We can draw the minimum expected total cost for model (I) and model ($\mathrm{I_f}$) against β for the Laplace distribution at $\gamma = 0.7$ as shown in Figure 3.

From Table 3 we have that

at $\gamma = 0.1$, we will make backorders by 10% of new orders quantity;

at $\gamma = 0.3$, we will make backorders by 30% of new orders quantity;

at $\gamma = 0.7$, we will make backorders by 70% of new orders quantity.

After comparison of the crisp case and fuzzy case for Uniform distribution, we can deduce that the least min $E(\mathrm{TC})$ was obtained at $\gamma = 0.7$. We can draw the minimum expected total cost for model (I) and model ($\mathrm{I_f}$) against β for the Uniform distribution at $\gamma = 0.7$ as shown in Figure 4.

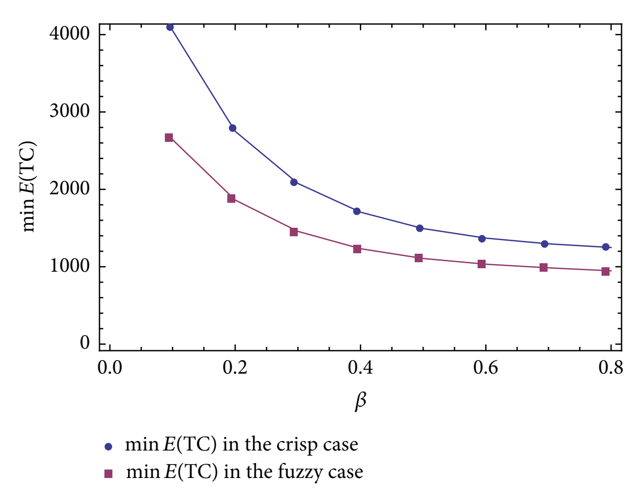

Figure 3: The comparison between the crisp and fuzzy cases for Laplace at $\gamma = 0.7$.

7. Conclusion

In this study we discussed two cases for mixture shortage inventory model under varying order cost constraint when lead time demand follows Exponential, Laplace, and Uniform distributions. We have evaluated the exact solutions of Q^* and r^* for each value of β and λ^* which yields our expected order cost constraint and then obtain the minimum expected total cost by using Lagrangian multiplier technique.

By comparing between the minimum expected total cost for model (I) and model ($\mathrm{I_f}$) at each distribution, we can

Table 2: The exact solutions and min $E(\mathrm{TC})$ for model (I) and model ($\mathrm{I_f}$) at Laplace distribution.

γ	β	Crisp case			Fuzzy case		
		Q^*	r^*	min $E(\mathrm{TC})$	Q^*	r^*	min $E(\mathrm{TC})$
0.1	0.1	297.123	16.7406	4197.53	250.409	17.9466	2732.79
	0.2	184.844	20.2428	2881.62	158.099	21.4789	1944.2
	0.3	123.72	23.5092	2199.23	107.093	24.7322	1532.59
	0.4	87.7213	26.4792	1823.44	76.7474	27.6615	1305.96
	0.5	65.081	29.1581	1607.46	57.4314	30.2959	1176.14
	0.6	50.1314	31.5604	1480.65	44.6116	32.6421	1100.83
	0.7	39.9135	33.6954	1405.74	40.0084	33.6658	1052.46
	0.8	35.8352	34.715	1357.91	36.8761	34.4363	1014.82
0.3	0.1	297.09	15.1563	4157.53	250.449	16.5519	2707.33
	0.2	184.858	18.7738	2845.06	158.053	20.2063	1920.28
	0.3	123.74	22.1162	2164.62	107.069	23.536	1510.28
	0.4	87.6956	25.141	1789.72	76.6717	26.5228	1284.38
	0.5	65.0979	27.8493	1574.89	57.4223	29.1839	1155.52
	0.6	50.2117	30.2628	1448.85	44.5733	31.5604	1080.66
	0.7	39.8244	32.4508	1374.04	40.0876	32.5659	1032.46
	0.8	35.9	33.4388	1326.43	36.9506	33.3419	994.931
0.7	0.1	297.084	10.9477	4052.24	250.405	12.9997	2641.24
	0.2	184.853	14.9711	2749.93	158.116	17.0285	1862.01
	0.3	123.776	18.5665	2076.28	107.095	20.5958	1456.08
	0.4	87.7171	21.7564	1705.32	76.744	23.7273	1233.15
	0.5	65.084	24.5783	1492.99	57.4997	26.4847	1106.02
	0.6	50.1989	27.0667	1368.85	44.5736	28.9432	1032.24
	0.7	39.9021	29.2867	1295.45	40.3191	29.9172	984.487
	0.8	36.1029	30.2592	1248.3	37.1679	30.7094	947.275

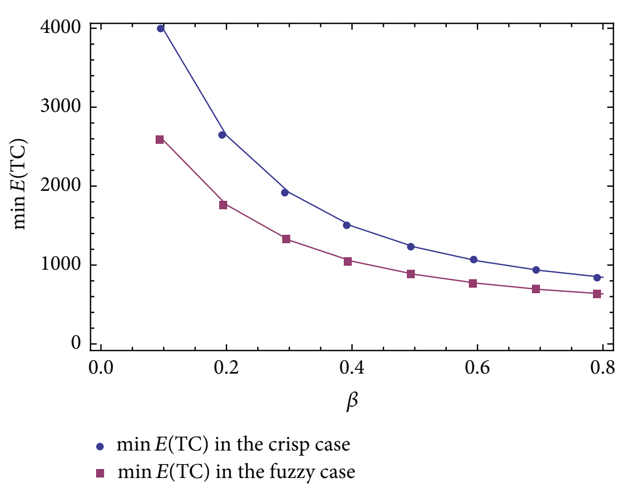

Figure 4: The comparison between the crisp and fuzzy cases for Uniform at $\gamma = 0.7$.

deduce that the least min $E(\mathrm{TC})$ was obtained when the lead time demand follows Uniform distribution and equals 844.584 SR with order quantity $Q^* = 32.4596$ and reorder point $r^* = 23.9138$ for model (I), while the minimum expected annual total cost for model ($\mathrm{I_f}$) is 634.709 SR with order quantity $Q^* = 29.3328$ and reorder point $r^* = 24.2447$ as shown in Table 3. This means that we can conclude that the minimum expected total cost in fuzzy case is less than in the crisp case, which indicates that the fuzziness is very close to the actuality of life and gets minimum expected total cost less than the crisp case.

For the results of the numerical example, we note that when β increases, r^* increases, and thus Q^* decreases which indicate that the min $E(\mathrm{TC})$ decreases.

Also, the different values of β lead to changes of Q^* in each distribution separately. But in all distributions we note that values of Q^* are almost fixed, due to the constraint on the varying order cost. Also, we note that when γ increases, min $E(\mathrm{TC})$ decreases; this indicates that 70% of the shortages can be met at the lowest possible cost.

Finally, our study in particular provides the ample scope for further research and exploration. For instance, we have considered probabilistic mixture shortage inventory model under varying order cost constraint. This work can be further developed by considering an ample range of different assumptions and conditions represented in constraints and costs (constant or varying), such as varying two costs under two constraints or varying two costs under constraint or varying one cost under two constraints. Also, we can study some of the inventory models with the system multiechelon-multisource.

Table 3: The exact solutions and min $E(\mathrm{TC})$ for model (I) and model ($\mathrm{I_f}$) at Uniform distribution.

γ	β	Crisp case			Fuzzy case		
		Q^*	r^*	min $E(\mathrm{TC})$	Q^*	r^*	min $E(\mathrm{TC})$
0.1	0.1	297.124	17.0568	4067.24	250.44	18.0722	2623.71
	0.2	184.926	19.6971	2697.71	158.084	20.4323	1791.22
	0.3	123.756	21.4539	1955.07	107.098	21.978	1333.88
	0.4	87.7442	22.6221	1519.47	76.748	22.9994	1062.47
	0.5	65.1225	23.415	1246.58	57.4541	23.6935	890.457
	0.6	50.187	23.9661	1066.66	44.6672	24.1744	776.304
	0.7	39.8908	24.3597	942.705	35.7373	24.5207	696.802
	0.8	32.4803	24.6502	853.887	29.23	24.7787	639.579
0.3	0.1	297.121	15.5207	4048	250.42	16.8872	2612.58
	0.2	184.862	18.7021	2684.53	158.097	19.6747	1784.32
	0.3	123.725	20.7762	1946.25	107.066	21.4673	1328.91
	0.4	87.7476	22.1372	1513.44	76.7256	22.6354	1058.95
	0.5	65.132	23.0538	1242.15	57.475	23.421	888.053
	0.6	50.1549	23.6892	1062.94	44.6562	23.9647	774.318
	0.7	39.8284	24.1411	939.565	35.7041	24.3546	695.179
	0.8	32.4725	24.4703	851.602	29.2817	24.6397	638.328
0.7	0.1	297.1	10.0378	3979.21	250.426	12.9987	2576.66
	0.2	184.862	15.3252	2642.32	158.11	17.311	1762.56
	0.3	123.72	18.5524	1918.4	107.112	19.9169	1314.92
	0.4	87.7011	20.5852	1493.56	76.6763	21.5568	1048.64
	0.5	65.1351	21.9128	1227.92	57.4479	22.6277	880.563
	0.6	50.1559	22.8182	1052.06	44.6218	23.3576	768.557
	0.7	39.8823	23.4508	931.288	35.6999	23.873	690.713
	0.8	**32.4596**	**23.9138**	**844.584**	**29.3328**	**24.2447**	**634.709**

Notations

$\overline{D}$: A random variable denoting the demand rate per period

Q: A decision variable representing the order quantity per cycle

r: A decision variable representing the reorder point

L: The lead time between the placement of an order and its receipt

x: The continuous random variable representing the demand during L

$f(x)$: The probability density function of the lead time demand and (x) is its distribution function

$R(r)$: The probability of the shortage $= 1 - F(r) = \int_r^\infty f(x)\,dx$

$\overline{S}(r)$: The expected value of shortages per cycle $= \int_r^\infty (x-r) f(x)\,dx$

c_o: The order cost per unit

$C_o(Q) = c_o Q^{-\beta}$: The varying order cost per cycle

β: A constant real number selected to provide the best fit of estimated expected cost function

c_h: The holding cost per unit per period

c_s: The shortage cost per unit

c_b: The backorders cost per unit

c_l: The lost sales cost per unit

K: The limitation on the expected annual order cost

λ: The Lagrangian multiplier.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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