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Research Article

Cardinal Basis Piecewise Hermite Interpolation on Fuzzy Data

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A numerical method along with explicit construction to interpolation of fuzzy data through the extension principle results by widely used fuzzy-valued piecewise Hermite polynomial in general case based on the cardinal basis functions, which satisfy a vanishing property on the successive intervals, has been introduced here. We have provided a numerical method in full detail using the linear space notions for calculating the presented method. In order to illustrate the method in computational examples, we take recourse to three prime cases: linear, cubic, and quintic.

1. Introduction

Fuzzy interpolation problem was posed by Zadeh [1]. Lowen presented a solution to this problem, based on the fundamental polynomial interpolation theorem of Lagrange (see, e.g., [2]). Computational and numerical methods for calculating the fuzzy Lagrange interpolate were proposed by Kaleva [3]. He introduced an interpolating fuzzy spline of order l. Important special cases were $l = 2$, the piecewise linear interpolant, and $l = 4$, a fuzzy cubic spline. Moreover, Kaleva obtained an interpolating fuzzy cubic spline with the not-a-knot condition. Interpolating of fuzzy data was developed to simple Hermite or osculatory interpolation, $E(3)$ cubic splines, fuzzy splines, complete splines, and natural splines, respectively, in [4–8] by Abbasbandy et al. Later, Lodwick and Santos presented the Lagrange fuzzy interpolating function that loses smoothness at the knots at every α-cut; also every α-cut ($\alpha \neq 1$) of fuzzy spline with the not-α-knot boundary conditions of order k has discontinuous first derivatives on the knots and based on these interpolants some fuzzy surfaces were constructed [9]. Zeinali et al. [10] presented a method of interpolation of fuzzy data by Hermite and piecewise cubic Hermite that was simpler and consistent and also inherited smoothness properties of the generator interpolation. However, probably due to the switching points difficulties, the method was expressed in a very special case and none of three remaining important cases was not investigated and this is a fundamental reason for the method weakness.

In total, low order versions of piecewise Hermite interpolation are widely used and when we take more knots, the error breaks down uniformly to zero. Using piecewise-polynomial interpolants instead of high order polynomial interpolants on the same material and spaced knots is a useful way to diminish the wiggling and to improve the interpolation. These facts, as well as cardinal basis functions perspective, motivated us in [11] to patch cubic Hermite polynomials together to construct piecewise cubic fuzzy Hermite polynomial and provide an explicit formula in a succinct algorithm to calculate the fuzzy interpolant in cubic case as a new replacement method for [4, 10].

Now, in this paper, in light of our previous work, we want to introduce a wide general class of fuzzy-valued interpolation polynomials by extending the same approach in [11] applying a very special case of which general class of fuzzy polynomials could be an alternative to fuzzy osculatory interpolation in [4] and so its lowest order case ($m = 1$), namely, the piecewise linear polynomial, is an analogy of fuzzy linear spline in [3]. Meanwhile, when $m = 2$ with exactly the same data, we will simply produce the second lower order form of mentioned general class that was introduced in [11] and the interpolation of fuzzy data in [10].

The paper is organized in five sections. In Section 2, we have reviewed definitions and preliminary results of several basic concepts and findings; next, we construct piecewise fuzzy Hermite polynomial in detail based on cardinal basis functions and prove some new properties of the introduced general interpolant (Section 3). In Section 4, we have produced three initial, linear, cubic [11], and quintic cases and shown the relationship between some of the mentioned cases and the newly presented interpolants in [3, 4, 10]. Furthermore, to illustrate the method, some computational examples are provided. Finally, the conclusions of this interpolation are in Section 5.

2. Preliminaries

To begin, let us introduce some brief account of notions used throughout the paper. We shall denote the set of fuzzy numbers by $\mathbb{R}_{\mathcal{F}}$ the family of all nonempty convex, normal, upper semicontinuous, and compactly supported fuzzy subsets defined on the real axis $\mathbb{R}$. Obviously, $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$. If $u \in \mathbb{R}_{\mathcal{F}}$ is a fuzzy number, then $u^{\alpha} = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}$, $0 < \alpha \leq 1$, shows the α-cut of u. For $\alpha = 0$ by the closure of the support, $u^0 = \mathrm{cl}\{x \mid x \in \mathbb{R},\ u(x) > 0\}$. It is well known the α-cuts of $u \in \mathbb{R}_{\mathcal{F}}$ are closed bounded intervals in $\mathbb{R}$ and we will denote them by $u^{\alpha} = [\underline{u}^{\alpha}, \overline{u}^{\alpha}]$; functions $\underline{u}^{(\cdot)}, \overline{u}^{(\cdot)}$ are the lower and upper branches of u. The core of u is $u^1 = \{x \mid x \in \mathbb{R},\ u(x) = 1\}$. In terms of α-cuts, we have the addition and the scalar multiplication:

$$\begin{aligned}(u+v)^{\alpha} &= u^{\alpha} + v^{\alpha} = \{x + y \mid x \in u^{\alpha},\ y \in v^{\alpha}\}\\ (\lambda u)^{\alpha} &= \lambda u^{\alpha} = \{\lambda x \mid x \in u^{\alpha}\}\\ (0)^{\alpha} &= \{0\}\end{aligned} \tag{1}$$

for all $0 \leq \alpha \leq 1$, $u, v \in \mathbb{R}_{\mathcal{F}}$, and $\lambda \in \mathbb{R}$.

$u = \langle a, b, c, d\rangle$ specifies a trapezoidal fuzzy number, where $a \leq b \leq c \leq d$ and if $b = c$ we obtain a triangular fuzzy number. For $\alpha \in [0,1]$, we have $u^{\alpha} = [a + \alpha(b-a), d - \alpha(d-c)]$. In the rest of this paper, we will assume that u is a triangular fuzzy number.

Definition 1 (see, e.g., [5]). An L-R fuzzy number $u = (m, l, r)_{LR}$ is a function from the real numbers into the interval $[0,1]$ satisfying

$$u(x) = \begin{cases} R\left(\dfrac{x-m}{r}\right) & \text{for } m \leq x \leq m + r,\\ L\left(\dfrac{m-x}{l}\right) & \text{for } m - l \leq x \leq m,\\ 0 & \text{otherwise,}\end{cases} \tag{2}$$

where R and L are continuous and decreasing functions from $[0,1]$ to $[0,1]$ fulfilling the conditions $R(0) = L(0) = 1$ and $R(1) = L(1) = 0$. When $R(x) = L(x) = 1 - x$, we will have $L-L$ fuzzy numbers that involve the triangular fuzzy numbers. For an $L-L$ fuzzy number $u = (m, l, r)$, the support is the closed interval $[m-l, m+r]$ (see, e.g., [6]).

The linear space of all polynomials of degree at most N will be designated by P_N. Full Hermite interpolation problem defines a unique polynomial, called $p_N(x)$, which solves the following problem.

Theorem 2 (see [12] (existence and uniqueness)). *Let $x_0, x_1, \ldots, x_n$ be $n+1$ distinct points, $\alpha_0, \alpha_1, \ldots, \alpha_n$ be positive integers, $k = 0, 1, \ldots, \alpha_i$, and $N = \alpha_0 + \alpha_1 + \cdots + \alpha_n + n$. Set $w(x) = \prod_{i=0}^{n}(x - x_i)^{\alpha_i + 1}$ and*

$$\begin{aligned} l_{ik}(x) &= w(x)\frac{(x-x_i)^{k-\alpha_i}}{k!\,(x-x_i)^{\alpha_i+1-k}}\frac{d^{(\alpha_i-k)}}{dx^{(\alpha_i-k)}}\left[\frac{(x-x_i)^{\alpha_i+1}}{w(x)}\right]_{x=x_i}\\ p_N(x) &= \sum_{i=0}^{n} r_i l_{i0}(x) + \sum_{i=0}^{n} r_i' l_{i1}(x) + \cdots + \sum_{i=0}^{n} r_i^{(\alpha_i)} l_{i\alpha_i}(x)\end{aligned} \tag{3}$$

is a unique member of P_N for which

$$\begin{gathered} p_N(x_0) = r_0,\ p_N'(x) = r_0', \ldots,\ p_N^{(\alpha_0-1)}(x_0) = r_0^{(\alpha_0)}\\ \vdots\\ p_N(x_n) = r_n,\ p_N'(x_n) = r_n', \ldots,\ p_N^{(\alpha_n-1)}(x_n) = r_n^{(\alpha_n)}.\end{gathered} \tag{4}$$

When $\alpha_0 = \alpha_1 = \cdots = \alpha_n = 1$, the full Hermite interpolation simplifies into simple Hermite or osculatory interpolation.

Definition 3. Given distinct knots $x_0, x_1, \ldots, x_n$, associated function values $f_0, f_1, \ldots, f_n$, and a linear space Φ of specific real functions generated by continuous cardinal basis functions $\phi_j : \mathbb{R} \to \mathbb{R}$, $(j = 0, 1, \ldots, n)$, $\phi_j(x_i) = \delta_{ij}$, $(i = 0, 1, \ldots, n)$, we say that the function F organized in the shape $F(x) = \sum_{j=0}^{n} f_j\phi_j(x)$ is an interpolant based on cardinal basis and such a procedure is the cardinal basis functions method.

3. Piecewise Fuzzy Hermite Interpolation Polynomial

A special case of full Hermite interpolation is piecewise Hermite interpolation (see, e.g., [13, 14]). Let us assume throughout the paper that $\Delta : a = x_0 < x_1 < \cdots < x_n = b$ is a grid of $I = [a, b]$ with knots x_i and m is a positive integer. All piecewise Hermite polynomials form a certain finite dimensional smooth linear space which we name $H_{2m-1}(\Delta; I)$.

Definition 4. $H_{2m-1}(\Delta; I)$ is a collection of all real-valued piecewise-polynomial functions $s(x)$ of degree at most $2m - 1$, defined on I, such that $s(x) \in C^{m-1}(I)$. The associated function to $s(x)$ on successive intervals $[x_{i-1}, x_i]$, $1 \leq i \leq n$, with knots from Δ, is defined by $s_i(x)$, that is, a $(m-1)$-times continuously differentiable piecewise Hermite polynomial of degree $2m-1$, on I.

Definition 5 (see [15]). Given any real-valued function, $f(x) \in C^{m-1}(I)$. Let its unique $H_{2m-1}(\Delta; I)$-interpolate, for

m and grid Δ of I, be the element $s(x)$ of degree $2m-1$ on each interval $[x_i-1, x_i]$, $1 \le i \le n$, such that

$$D^k s(x_i) = D^k f(x_i) \quad \forall 0 \le k \le m-1,\ 0 \le i \le n,\ D^k = \frac{d^k}{dx^k}. \tag{5}$$

Existence and uniqueness of full Hermite interpolation is provided in [12]. Because of this, presentation (5) is actually a special case of such interpolation on a gridded interval and it follows that each function belonging to $C^{m-1}(I)$ has a unique interpolate in $H_{2m-1}(\Delta; I)$.

A particular cardinal basis for linear space $H_{2m-1}(\Delta; I)$ of dimension $m(n+1)$ is $\mathscr{B} = \{\phi_{ik}(x)\}_{i=0,k=0}^{n,m-1}$, (see, e.g., [16]), where the basis function $\phi_{ik}(x)$ is defined by

$$D^l \phi_{ik}(x_j) = \delta_{kl}\delta_{ij}, \quad 0 \le k, l \le m-1,\ 0 \le i, j \le n,\ D^l = \frac{d^l}{dx^l}. \tag{6}$$

Some important results based on (6) are simple to see in the sequel, as $\phi_{i0}(x_i) = 1$ and $\phi_{i0}(x_j) = 0$ at all knots x_j and since $s(x)$ outside $[x_{i-1}, x_i]$, $1 \le i \le n$, satisfies zero data, then $\phi_{i0} \equiv 0$ for all $x_0 \le x \le x_{i-1}$ and $x_{i+1} \le x \le x_n$. $\phi_{i1}(x)$ is of degree $2m-1$, and $\phi_{i1}^{l}(x_i) = 1$ but it is zero at all other knots. Moreover, because outside the interval $[x_{i-1}, x_{i+1}]\phi_{i1}(x)$ interpolates zero data, then $\phi_{i1}(x)$ must be vanished identically for all $x \ge x_{i+1}$ and $x \le x_{i-1}$ (see, e.g., [13, 14, 17]). Analogous reasoning applies to

$$\phi_{i1}(x) = \begin{cases} (x - x_{i-1})^m (x - x_i) \sum_{j=0}^{m-2} a_j x^j, & x_{i-1} \le x \le x_i, \\ (x - x_{i+1})^m (x - x_i) \sum_{j=0}^{m-2} b_j x^j, & x_i \le x \le x_{i+1}, \\ 0, & \text{otherwise}. \end{cases} \tag{7}$$

In the following theorem, we will use the recent features.

Theorem 6. *Assume that $\phi_{ik} \in \mathscr{B}$ and satisfies the piecewise Hermite polynomial cardinal basis function constraints (6). Then,*

(i) $\phi_{i\,0}(x) + \phi_{i+1\,0}(x) \ge 1$, *for all $x \in (x_i, x_{i+1})$, $i = 0, 1, \ldots, n-1$.*

(ii) *For all $i = 1, 2, \ldots, n-1$, ϕ_{i1} changes the sign at x_i. The sign of ϕ_{i1} is not positive on any subinterval $[x_{i-1}, x_i]$, and that is not negative on $[x_i, x_{i+1}]$.*

(iii) *The sign of all other elements of $\mathscr{B}$ is not negative on I.*

Proof. With the assumption of (6), let $\phi_{i\,0}(x) + \phi_{i+1\,0}(x)$ be polynomial of degree $2m-1$ on the interval $[x_{i-1}, x_{i+2}]$ and interpolate the data (x_j, f_j), where $f_j = 1$ for $j = i, i+1$ and zero on the other knots of partition Δ. Suppose that $0 < i < n-1$ and $\phi_{i\,0}(x) + \phi_{i+1\,0}(x) < 1$ for some $x \in (x_i, x_{i+1})$. By the mean value theorem, its derivative has a zero on (x_i, x_{i+1}). The derivative has two $(m-2)$th order zeros at x_i and x_{i+1} and its two other zeros are x_{i-1}, x_{i+2}. Then, it has at least $2m-1$ zeros on the interval $[x_{i-1}, x_{i+2}]$, which is a contradiction. The cases $i = 0$ and $i = n-1$ are treated similarly.

In light of representation (7) and condition (6), the polynomial $\phi_{i1}(x)$ is of degree $2m-1$. It has only one minimum point on $[x_{i-1}, x_i]$ and a single maximum on the subinterval $[x_i, x_{i+1}]$. Suppose that each of the above points are one more. Then, by the mean value theorem, first derivative of $\phi_{i1}(x)$ has at least three zeros on (x_{i-1}, x_i) and three zeros on (x_i, x_{i+1}). Also, the derivative has two $(m-2)$th order zeros at x_{i-1}, x_{i+1}. Then, it has at least $2m+2$ zeros, which is a contradiction. Hence, $\phi_{i1}'(x)$ has only one zero on each of the intervals (x_{i-1}, x_i) and (x_i, x_{i+1}). Now, recall $\phi_{i1}'(x_i) = 1$; it follows that $\phi_{i1}(x) \le 0$, on $[x_{i-1}, x_i]$ and $\phi_{i1}(x) \ge 0$, on $[x_i, x_{i+1}]$. This gives (ii).

A similar proof via definition of basis functions and (6) follows the claim (iii). □

For a given $f(x) \in C^{m-1}(I)$ and its piecewise Hermite interpolate $s(x) \in H_{2m-1}(\Delta; I)$, an equivalent explicit representation of $s(x)$ in (5) can be uniquely expressed (see, e.g., [13, 17]); namely,

$$s(x) = \sum_{k=0}^{m-1} \sum_{i=0}^{n} f^{(k)}(x_i)\, \phi_{ik}(x). \tag{8}$$

Now, we want to construct a fuzzy-valued function as $s : I \to \mathbb{R}_{\mathscr{F}}$ such that $s^{(k)}(x_i) = f^{(k)}(x_i) = u_{ki} \in \mathbb{R}_{\mathscr{F}}$, $0 \le k \le m-1$, $0 \le i \le n$. Also, if for all $0 \le k \le m-1$, $0 \le i \le n$, $u_{ki} = y_i^{(k)}$ are crisp numbers in $\mathbb{R}$ and $f^{(k)}(x_i) = \chi_{y_i^{(k)}}$ (see, e.g., [2]), then there is a polynomial of degree $2m-1$ on successive intervals $[x_{i-1}, x_i]$, $0 \le i \le n$, with $s^{(k)}(x_i) = y_i^{(k)}$, $0 \le k \le m-1$, $0 \le i \le n$ such that $s(x) = \chi_{f(x)}$ for all $x \in \mathbb{R}$, where $\{(x_i, f_i, f_i', \ldots, f_i^{(m-1)}) \mid f_i^{(k)} \in \mathbb{R}_{\mathscr{F}},\ 0 \le k \le m-1,\ 0 \le i \le n\}$ is given.

We suppose that such a fuzzy function exists and we attempt to find and compute it with respect to interpolation polynomial presented by Lowen [2]. Let, for each $x \in [x_0, x_n]$, $s(x)$ be a fuzzy piecewise Hermite polynomial and $\Lambda = \{y_i^{(k)}\}_{i=0,k=0}^{n,m-1}$; then, from Kaleva [3] and Nguyen [18], we obtain the α-cuts of $s(x)$ in a succinctly algorithm as follows:

$$\begin{aligned} s^{\alpha}(x) &= \left\{t \in \mathbb{R} \mid \mu_{s(x)}^{(t)} \ge \alpha\right\} = \Big\{t \in \mathbb{R} \mid \exists y_i^{(k)} : \mu_{u_{ki}}^{(y_i^{(k)})} \\ &\ge \alpha,\ 0 \le k \le m-1,\ 0 \le i \le n,\ s_{\Lambda}(x) = t\Big\} = \Big\{t \\ &\in \mathbb{R} \mid t = s_{\Lambda}(x),\ y_i^{(k)} \in u_{ki}^{\alpha},\ 0 \le k \le m-1,\ 0 \le i \\ &\le n\Big\} = \sum_{k=0}^{m-1} \sum_{i=0}^{n} u_{ki}^{\alpha} \phi_{ik}(x), \end{aligned} \tag{9}$$

where

$$s_{\Lambda}(x) = \sum_{k=0}^{m-1} \sum_{i=0}^{n} y_i^{(k)} \phi_{ik}(x) \tag{10}$$

is a piecewise Hermite polynomial in crisp case and by definition

$$s^{\alpha}(x)=\sum_{k=0}^{m-1}\sum_{i=0}^{n}u_{ki}^{\alpha}\phi_{ik}(x) \tag{11}$$

we obtain a formula that comprises a simple practical way for calculating $s(x)$:

$$s(x)=\sum_{k=0}^{m-1}\sum_{i=0}^{n}u_{ki}\phi_{ik}(x). \tag{12}$$

Since, for each $0\le k\le m-1$, $0\le i\le n$, $u_{ki}^{\alpha}=[\underline{u}_{ki}^{\alpha},\overline{u}_{ki}^{\alpha}]$, then we will have $s^{\alpha}(x)$ by solving the following optimization problems:

$$\begin{aligned}\max \,\&\, \min \quad & \sum_{k=0}^{m-1}\sum_{i=0}^{n}y_i^{(k)}\phi_{ik}(x)\\ \text{subject to} \quad & \underline{u}_{ki}^{\alpha}\le y_i^{(k)}\le\overline{u}_{ki}^{\alpha},\\ & 0\le k\le m-1,\ 0\le i\le n.\end{aligned} \tag{13}$$

From the ϕ_{ij}'s sign that we represented in Theorem 6, these problems have the following optimal solutions:

Maximization is as follows:

$$y_i^{(k)}=\begin{cases}\overline{u}_{ki}^{\alpha} & \text{if } \phi_{ik}(x)\ge 0\\ \underline{u}_{ki}^{\alpha} & \text{if } \phi_{ik}(x)<0,\end{cases}\qquad 0\le k\le m-1,\ 0\le i\le n. \tag{14}$$

Minimization is as follows:

$$y_i^{(k)}=\begin{cases}\underline{u}_{ki}^{\alpha} & \text{if } \phi_{ik}(x)\ge 0\\ \overline{u}_{ki}^{\alpha} & \text{if } \phi_{ik}(x)<0,\end{cases}\qquad 0\le k\le m-1,\ 0\le i\le n. \tag{15}$$

Theorem 7. *If $s(x)=\sum_{k=0}^{m-1}\sum_{i=0}^{n}u_{ki}\phi_{ik}(x)$ is an interpolating piecewise fuzzy Hermite polynomial, then for all $\alpha\in[0,1]$, $i\in\{0,1,\ldots,n-1\}$, $x\in[x_i,x_{i+1}]$,*

$$\mathit{len}\, s^{\alpha}(x)\ge\min\{\mathit{len}\, s^{\alpha}(x_i),\mathit{len}\, s^{\alpha}(x_{i+1})\}, \tag{16}$$

where

$$\begin{aligned}s^{\alpha}(x)&=[\underline{s}^{\alpha}(x),\overline{s}^{\alpha}(x)],\\ \mathit{len}\, s^{\alpha}(x)&=\overline{s}^{\alpha}(x)-\underline{s}^{\alpha}(x).\end{aligned} \tag{17}$$

Proof. By using Theorem 6 and (11), we have $s^{\alpha}(x_i)=u_{0\,i}^{\alpha}$, $s^{\alpha}(x_{i+1})=u_{0\,i+1}^{\alpha}$ and $\operatorname{len} s^{\alpha}(x_i)=\operatorname{len} u_{0i}^{\alpha}$, $\operatorname{len} s^{\alpha}(x_{i+1})=\operatorname{len} u_{0\,i+1}^{\alpha}$. Since the addition does not decrease the length of an interval from (11), we can write $s^{\alpha}(x)=\sum_{k=0}^{m-1}\sum_{j=0}^{n}u_{k_j}^{\alpha}\phi_{jk}(x)$; then,

$$\begin{aligned}\operatorname{len} s^{\alpha}(x)&\ge\sum_{k=0}^{m-1}\sum_{j=0}^{n}\left|\phi_{jk}(x)\right|\operatorname{len} u_{kj}^{\alpha}\\ &\ge\sum_{j=0}^{n}\left|\phi_{j0}(x)\right|\operatorname{len} u_{0j}\\ &\ge\left|\phi_{i0}(x)\right|\operatorname{len} u_{0i}+\left|\phi_{i+1\,0}(x)\right|\operatorname{len} u_{0\,i+1}\\ &\ge\min\{\operatorname{len} u_{0\,i},\operatorname{len} u_{0\,i+1}\}\left(\left|\phi_{i0}(x)\right|+\left|\phi_{i+1\,0}(x)\right|\right)\\ &\ge\min\{\operatorname{len} u_{0\,i},\operatorname{len} u_{0\,i+1}\}\\ &=\min\{\operatorname{len} s^{\alpha}(x_i),\operatorname{len} s^{\alpha}(x_{i+1})\}.\end{aligned} \tag{18}$$

□

Theorem 8. *Let $u_{ki}=(m_{ki},l_{ki},r_{ki})$, $0\le k\le m-1$, $0\le i\le n$, be a triangular $L-L$ fuzzy number; then, also $s(x)$, the piecewise fuzzy Hermite polynomial interpolation, is such a fuzzy number for each x.*

Proof. The closed interval $[m-l,m+r]$ is the support of $u=(m,l,r)$, a triangular $L-L$ fuzzy number; then for each x and u_{ki}, we have

$$\begin{aligned}s(x)&=\left(\sum_{k=0}^{m-1}\sum_{i=0}^{n}u_{ki}\phi_{ik}(x)\right)\\ &=\Bigg[\sum\sum_{\phi_{ik}\ge0}(m_{ki}-l_{ki})\phi_{ik}(x)\\ &\quad+\sum\sum_{\phi_{ik}<0}(m_{ki}+r_{ki})\phi_{ik}(x),\\ &\quad\sum\sum_{\phi_{ik}\ge0}(m_{ki}+r_{ki})\phi_{ik}(x)\\ &\quad+\sum\sum_{\phi_{ik}<0}(m_{ki}-l_{ki})\phi_{ik}(x)\Bigg]=\Bigg[\sum_{k=0}^{m-1}\sum_{i=0}^{n}m_{ki}\phi_{ik}(x)\\ &\quad-\left(\sum\sum_{\phi_{ik}\ge0}l_{ki}\phi_{ik}(x)-\sum\sum_{\phi_{ik}<0}r_{ki}\phi_{ik}(x)\right)\\ &\quad+\sum_{k=0}^{m-1}\sum_{i=0}^{n}m_{ki}\phi_{ik}(x)\\ &\quad+\left(\sum\sum_{\phi_{ik}\ge0}r_{ki}(x)\phi_{ik}(x)-\sum\sum_{\phi_{ik}<0}l_{ik}\phi_{ik}(x)\right)\Bigg]\\ &=(m(x)-l(x),m(x)+r(x)).\end{aligned} \tag{19}$$

□

It follows that if $s(x) = (m(x), l(x), r(x))$, is a triangular $L-L$ fuzzy number for each x, then

$$
\begin{aligned}
m(x) &= \sum_{k=0}^{m-1}\sum_{i=0}^{n} m_{ki}\phi_{ki}, \\
l(x) &= \sum\sum_{\phi_{ik}\geq 0} l_{ki}\phi_{ik}(x) - \sum\sum_{\phi_{ik}<0} r_{ki}\phi_{ik}(x), \\
r(x) &= \sum\sum_{\phi_{ik}\geq 0} r_{ki}\phi_{ik} - \sum\sum_{\phi_{ik}<0} l_{ik}\phi_{ik}(x).
\end{aligned} \tag{20}
$$

4. Piecewise-Polynomial Linear, Cubic, and Quintic Fuzzy Hermite Interpolation

We consider $m = 1$ and compute the piecewise fuzzy linear interpolant as the initial case of the presented method based on (12) and for a given set of fuzzy data $\{(x_i, f_i) \mid f_i \in \mathbb{R}_{\mathcal{F}},\ 0 \leq i \leq n\}$, as follows:

$$
s(x) = \sum_{i=0}^{n} u_{0i}\phi_{i0}(x), \tag{21}
$$

where $u_{0i} = f_i$, $0 \leq i \leq n$, and subject to conditions (6),

$$
\begin{aligned}
\phi_{00}(x) &= \begin{cases} 0, & x \geq x_1 \\ \left(\dfrac{x_1 - x}{x_1 - x_0}\right), & x_0 \leq x \leq x_1 \end{cases} \\
\phi_{i0}(x) &= \begin{cases} 0, & x \leq x_{i-1},\ x \geq x_{i+1} \\ \left(\dfrac{x - x_{i-1}}{x_i - x_{i-1}}\right), & x_{i-1} \leq x \leq x_i \\ \left(\dfrac{x_{i+1} - x}{x_{i+1} - x_i}\right), & x_i \leq x \leq x_{i+1} \end{cases} \\
&\qquad\qquad 1 \leq i \leq n-1 \\
\phi_{n0}(x) &= \begin{cases} 0, & x \leq x_{n-1} \\ \left(\dfrac{x - x_{n-1}}{x_n - x_{n-1}}\right), & x_{n-1} \leq x \leq x_n. \end{cases}
\end{aligned} \tag{22}
$$

The obtained $s(x)$ is the same as fuzzy spline of order $l = 2$ that had been introduced in [3] because the basic splines and the cardinal basis functions in two interpolants are equal.

Example 9 (see [4]). Suppose the data (1, (0, 2, 1), (1, 0, 3)), (1.3, (5, 1, 2), (0, 2, 1)), (2.2, (1, 0, 3), (4, 4, 3)), (3, (4, 4, 3), (5, 1, 2)), (3.5, (0, 3, 2), (1, 1, 1)), (4, (1, 1, 1), and (0, 3, 2)). In Figure 1, the dashed line is the 0.5-cut set of piecewise cubic fuzzy interpolation $s(x)$, $x \in [1, 4]$ and the solid lines represent the support and the core of $s(x)$.

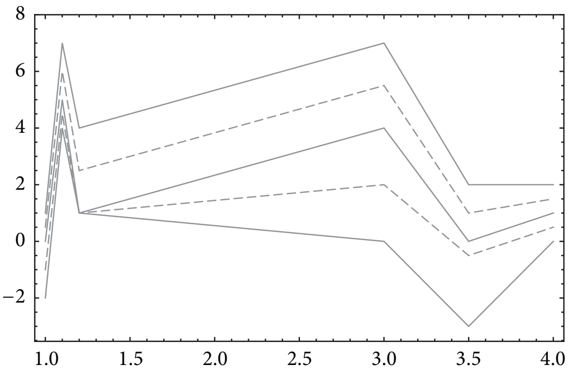

Figure 1: Graph of Example 9.

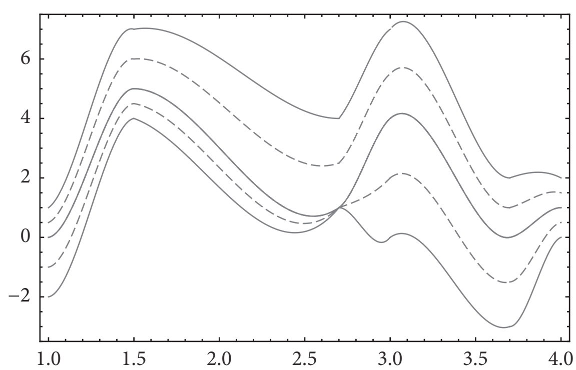

Figure 2: Graph of Example 10.

When $m = 2$, we get the piecewise cubic fuzzy Hermite polynomial interpolant in [11] for a given set of data $\{(x_i, f_i, f_i') \mid f_i, f_i' \in \mathbb{R}_{\mathcal{F}},\ 0 \leq i \leq n\}$,

$$
s(x) = \sum_{i=0}^{n} u_{0i}\phi_{i0}(x) + \sum_{i=0}^{n} u_{1i}\phi_{i1}(x), \tag{23}
$$

where $u_{ki} = f_i^{(k)}$, $k = 0, 1$, $0 \leq i \leq n$.

An outstanding feature of this study is that, by simply applying the second case of the presented general method and exactly the same data, we have produced an alternative to simple fuzzy Hermite polynomial interpolation in [4]. Heretofore, the mentioned cubic case (23) was independently introduced in [10] but only in very weak conditions and without using the extension principle.

The cardinal basis functions $\phi_{ik}(x)$, $k = 0, 1$, $0 \leq i \leq n$, were computed in [17].

Example 10. Suppose the data (1, (0, 2, 1), (1, 0, 3)), (1.5, (5, 1, 2), (0, 2, 1)), (2.7, (1, 0, 3), (4, 4, 3)), (3, (4, 4, 3), (5, 1, 2)), (3.7, (0, 3, 2), (1, 1, 1)), and (4, (1, 1, 1), (0, 3, 2)). In Figure 2, the dashed line is the 0.5-cut set of piecewise cubic fuzzy interpolations $s(x)$, $x \in [1, 4]$ and the solid lines represent the support and the core of $s(x)$.

Let $m = 3$; from (6), we shall construct the cardinal basis for $H_5(\Delta; I)$. The quintic Hermite polynomials

$\phi_{i0}(x)$, $\phi_{i1}(x)$, and $\phi_{i2}(x)$ are solving the interpolation problem

$$D^l \phi_{ik}\left(x_j\right) = \delta_{kl}\delta_{ij}, \quad 0 \le k, l \le 2,\ 0 \le i, j \le n. \tag{24}$$

To this end, we determine uniquely all the pervious ϕ_{ij}'s by the (24).

For $1 \le i \le n-1$, let

$$\phi_{i0}(x) = \begin{cases} \dfrac{\left(x_{i-1}-x\right)^3}{\left(x_{i-1}-x_i\right)^5}\left[\left(x_{i-1}+3x\right)\left(x_{i-1}-5x_i\right)+6x^2+10x_i^2\right], & x_{i-1} \le x \le x_i, \\ \dfrac{\left(x_{i+1}-x\right)^3}{\left(x_{i+1}-x_i\right)^5}\left[\left(x_{i+1}+3x\right)\left(x_{i+1}-5x_i\right)+6x^2+10x_i^2\right], & x_i \le x \le x_{i+1}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\phi_{i1}(x) = \begin{cases} \left(\dfrac{x_{i-1}-x}{x_{i-1}-x_i}\right)^3 \left(x-x_i\right)\left[1+3\dfrac{x-x_i}{x_{i-1}-x_i}\right], & x_{i-1} \le x \le x_i, \\ \left(\dfrac{x_{i+1}-x}{x_i-x_{i+1}}\right)^3 \left(x_i-x\right)\left[1+3\dfrac{x_i-x}{x_i-x_{i+1}}\right], & x_i \le x \le x_{i+1}, \\ 0, & \text{otherwise}, \end{cases} \tag{25}$$

$$\phi_{i2}(x) = \begin{cases} \left(\dfrac{x_{i-1}-x}{x_{i-1}-x_i}\right)^3 \dfrac{\left(x_i-x\right)^2}{2}, & x_{i-1} \le x \le x_i, \\ \left(\dfrac{x-x_{i+1}}{x_i-x_{i+1}}\right)^3 \dfrac{\left(x_i-x\right)^2}{2}, & x_i \le x \le x_{i+1}, \\ 0, & \text{otherwise}. \end{cases}$$

The six next functions are similarly defined. In particular,

$$\phi_{00}(x) = \begin{cases} \dfrac{\left(x_1-x\right)^3}{\left(x_1-x_0\right)^5}\left[\left(x_1+3x\right)\left(x_1-5x_0\right)+6x^2+10x_0^3\right], & x_0 \le x \le x_1, \\ 0, & x_1 \le x \le x_n, \end{cases}$$

$$\phi_{n0}(x) = \begin{cases} \dfrac{\left(x_{n-1}-x\right)^3}{\left(x_{n-1}-x_n\right)^5}\left[\left(x_{n-1}+3x\right)\left(x_{n-1}-5x_n\right)+6x^2+10x_n^2\right], & x_{n-1} \le x \le x_n, \\ 0, & x_0 \le x \le x_{n-1}, \end{cases}$$

$$\phi_{01}(x) = \begin{cases} \left(\dfrac{x_1-x}{x_0-x_1}\right)^3 \left(x_0-x\right)\left[1+3\dfrac{x_0-x}{x_0-x_1}\right], & x_0 \le x \le x_1, \\ 0, & x_1 \le x \le x_n, \end{cases}$$

$$\phi_{n1}(x) = \begin{cases} \left(\dfrac{x_{n-1}-x}{x_{n-1}-x_n}\right)^3 \left(x-x_n\right)\left[1+3\dfrac{x-x_n}{x_{n-1}-x_n}\right], & x_{n-1} \le x \le x_n, \\ 0, & x_0 \le x \le x_{n-1}, \end{cases}$$

$$\phi_{02}(x) = \begin{cases} \left(\dfrac{x-x_1}{x_0-x_1}\right)^3 \dfrac{\left(x_0-x\right)^2}{2}, & x_0 \le x \le x_1, \\ 0, & x_1 \le x \le x_n, \end{cases}$$

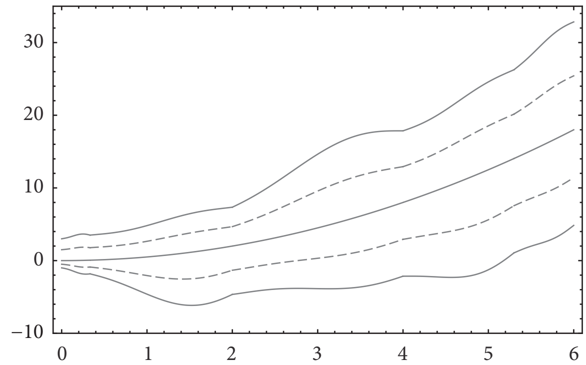

FIGURE 3: Graph of Example 11.

$$\phi_{n2}(x) = \begin{cases} \left(\dfrac{x_{n-1}-x}{x_{n-1}-x_n}\right)^3 \dfrac{(x_n - x)^2}{2}, & x_{n-1} \le x \le x_n, \\ 0, & x_0 \le x \le x_{n-1}. \end{cases} \tag{26}$$

Thus, we can immediately write down piecewise quintic fuzzy Hermite interpolation polynomial $s(x)$ using

$$s(x) = \sum_{i=0}^{n} u_{0i}\phi_{i0}(x) + \sum_{i=0}^{n} u_{1i}\phi_{i1}(x) + \sum_{i=0}^{n} u_{2i}\phi_{i2}(x), \tag{27}$$

where $\{(x_i, f_i, f_i', f_i'') \mid f_i^{(k)} \in \mathbb{B}_{\mathcal{F}},\ 0 \le k \le 2,\ 0 \le i \le n\}$, is given and $u_{ki} = f_i^{(k)}$.

Example 11. Suppose that (0, (0, 1, 3), (0, 2, 2), (1, 4, 4)), (1.3, (0.05, 1.9, 3.5), (0.3, 3.2, 0.8), (1, 3.1, 3)), (2, (2, 6.7, 5.3), (2, 0.5, 3.5), (1, 2.6, 2.4)), (4, (8, 10.1, 9.9), (4, 4, 0), (1, 0.6, 0.5)), (5.3, (14, 13, 12), (5.3, 0.2, 3.8), (1, 1.5, 1.7)), (6, (18, 13.2, 14.8), (6, 0.9, 3), and (1, 3.4, 3.2)) are the interpolation data. In Figure 3, the solid lines denote the support and the core of piecewise quintic fuzzy Hermite interpolation $s(x)$, $x \in [0, 6]$, and the dashed line is the 0.5-cut set of $s(x)$.

5. Conclusions and Further Work

Based on the cardinal basis functions for $m(n+1)$ dimension $H_{2m-1}(\Delta, I)$ linear space, interpolation of fuzzy data by the fuzzy-valued piecewise Hermite polynomial as the extension of same approach in [11] has been successfully introduced in general case and provided a succinct formula for calculating the new fuzzy interpolant. Moreover, two first cases of the presented method have been applied as an analogy to fuzzy spline of order two in [3] and an alternative to fuzzy osculatory interpolation in [4], respectively. In the guise of a remarkable achievement, the piecewise fuzzy cubic Hermite polynomial interpolation that was constructed with poor conditions and without using the extension principle in [10] has been produced in the role of a very special subdivision for the presented general method in this study. Finally, the third initial case, piecewise fuzzy quintic Hermite polynomial, has been described in detail. The next step to improve this method is interpolation of fuzzy data including switching points by a fuzzy differentiable piecewise interpolant.

Competing Interests

The authors declare that they have no competing interests.

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