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Research Article

Ordered Semigroups Based on $(\in, \in \vee q_k^{\delta})$-Fuzzy Ideals

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A new trend of using fuzzy algebraic structures in various applied sciences is becoming a central focus due to the accuracy and nondecoding nature. The aim of the present paper is to develop a new type of fuzzy subsystem of an ordered semigroup S. This new type of fuzzy subsystem will overcome the difficulties faced in fuzzy ideal theory of an ordered semigroup up to some extent. More precisely, we introduce $(\in, \in \vee q_k^{\delta})$-fuzzy left (resp., right, quasi-) ideals of S. These concepts are elaborated through appropriate examples. Further, we are bridging ordinary ideals and $(\in, \in \vee q_k^{\delta})$-fuzzy ideals of an ordered semigroup S through level subset and characteristic function. Finally, we characterize regular ordered semigroups in terms of $(\in, \in \vee q_k^{\delta})$-fuzzy left (resp., right, quasi-) ideals.

1. Introduction

The breakthrough of Zadeh's pioneering idea [1] of fuzzy sets changed our traditional tools for formal modeling, reasoning, and computing. These traditional tools are crisp in nature. Crisp means dichotomous, i.e., yes or no type rather than more or less type. In set theory, an element can either belong to a set or not. Zadeh's seminal paper on fuzzy set opens a new direction for researchers to use fuzzy sets in applied sciences like automata theory, computer science, artificial intelligence, coding theory, robotics, and much more. Keeping this inspiration in view, Rosenfeld [2] introduced fuzzy subgroups. The inception of fuzzy subgroups provides a platform for other researchers to use this pioneering idea in other algebraic structures along with several diverse applications. Among other algebraic structures, semigroups (especially ordered semigroups) are having a lot of applications in error correcting codes, control engineering, performance of super computer, and information sciences. Mordeson et al.'s idea in [3] gives birth to an up to date account of fuzzy subsemigroups and fuzzy ideals of semigroups while Kehayopulu and Tsingelis [4–6] used fuzzy sets in ordered semigroups to develop fuzzy ideal theory. Shabir and Khan [7] gave a characterization of ordered semigroups by the properties of fuzzy ideals and fuzzy generalized bi-ideals.

In 1996, the idea of a quasi-coincidence of a fuzzy point with a fuzzy set [8, 9] is presented which played a vital role to generate different types of fuzzy subgroups. Bhakat and Das [8] gave the concept of (α, β)-fuzzy subgroups and introduced $(\in, \in \vee q)$-fuzzy subgroups by using the "belongs to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup. In fact this is an important and useful generalization of Rosenfeld's idea of fuzzy subgroup [2]. These new generalizations provide a motivation to other researchers for further development in other algebraic structures. Since then a variety of research has been carried out using this icebreaking idea. More precisely, the aforementioned notion is used by Ma et al. [10, 11] in R_0-algebras and Davvaz and Mozafar [12] in Lie algebra, and Davvaz and coauthors used the idea of generalized fuzzy sets

in rings [13–15], Jun et al. [16], Khan and Shabir [17], and Khan et al. [18] in ordered semigroups, and Shabir et al. [19, 20] in semigroups. For further details, the readers are referred to [21–26].

In 2009, Jun [27] initiated a more general form of quasi-coincident with relation (q) and provided (q_k), where $k \in [0, 1)$. The notion is further strengthened by other researchers to use in various algebraic structures [28–31].

Recently, Jun et al. [32] presented another comprehensive generalization of $(\in, \in \vee q)$-fuzzy subgroups and discussed fuzzy subgroups in light of generalized quasi-coincident with relation. Further, Khan et al. [33] elaborated ordered semigroups in terms of fuzzy generalized bi-ideals using this idea [32]. Also, Khan et al. [34] determined fuzzy filters of ordered semigroups for the said notion.

In this paper, we apply Jun et al.'s idea [32] to ordered semigroups and discuss ordered semigroups through fuzzy left (resp., right) ideals of type $(\in, \in \vee q_k^{\delta})$, where $k \in [0, 1)$ and $\delta \in (0, 1]$, and obtained several important results. More precisely, $(\in, \in \vee q_k^{\delta})$-fuzzy left (resp., right) ideals are introduced. Ordinary fuzzy ideals and $(\in, \in \vee q_k^{\delta})$-fuzzy ideals of an ordered semigroup S are linked. Finally, regular ordered semigroups by the properties of these newly developed fuzzy left (resp., right) ideals are characterized.

2. Preliminaries

Algebraic structures like semigroups, ordered semigroups, and monids are extensively used in applied fields such as automata theory, coding theory, and theoretical computer science. The present section discusses various fundamental definitions and results essential for the proceeding article.

A system $(S, \cdot, \leq)$, in which $(S, \cdot)$ is a semigroup, $(S, \leq)$ is a poset with both left and right compatibility, i.e., $a \leq b \to ax \leq bx$, $xa \leq xb$ for all $a, b, x \in S$, is known as ordered semigroup. For $A \subseteq S$, we denote $(A] := \{a \in S \mid a \leq x \text{ for some } x \in A\}$. If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. For $A, B \subseteq S$, we denote $AB := \{ab \mid a \in A,\ b \in B\}$. If $A \subseteq B$, then $(A] \subseteq (B]$, $(A](B] \subseteq (AB]$ and $((A]]$ [18]. Later, S will denote an ordered semigroup, unless otherwise stated. For a nonempty subset A of S, $a \in A$ and $S \ni b \leq a$, then $b \in A$ is called a right (resp., left) ideal [18] of S if $AS \subseteq A$ (resp., $SA \subseteq A$). If A is both a left and a right ideal of S then A is called a two-sided ideal or simply an ideal of S. An ordered semigroup S is regular [22] if for every $a \in S$ there exists $x \in S$ such that $a \leq axa$. Let S be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then the characteristic function χ_A of A is defined by

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \tag{1}$$

By a fuzzy subset ξ of an ordered semigroup S, we mean a function $\xi : X \to [0, 1]$.

A fuzzy subset ξ of S is called a fuzzy left (resp., right) ideal of S if, for all $x, y \in S$,

(i) $x \leq y \to \xi(x) \geq \xi(y)$,

(ii) $\xi(xy) \geq \xi(y)$ (resp. $\xi(xy) \geq \xi(x)$).

A fuzzy subset ξ of S is called a fuzzy ideal of S if it is both a fuzzy left and a fuzzy right ideal of S.

For a nonempty subset X of S, define

$X_a = \{(y, z) \in S \times S \mid a \leq yz\}$ (see [18]).

Transfer principle in fuzzy set theory [35] plays a remarkable role by providing the connection between classical and fuzzy subsets through level subset. A subset of the form

$$U(\xi; t) := \{x \in S \mid \xi(x) \geq t\}, \tag{2}$$

where $t \in (0, 1]$, is known as level subset of S. For the said purpose, the following lemma is used.

Lemma 1 (see [35]). *A fuzzy subset ξ defined on S has the property P if and only if all nonempty level subsets $U(\xi; t)$ have the property P.*

As a consequence of the above property, the following useful theorem is given.

Theorem 2 (see [14]). *A fuzzy subset ξ of S is a fuzzy left (resp., right) ideal of S if and only if each nonempty level subset $U(\xi; t)$, for all $0 < t \leq 1$, is a left (resp., right) ideal of S.*

We denote by $\xi(S)$ the set of all fuzzy subsets of S and define an order relation "$\subseteq$" on the fuzzy subsets of $\xi(S)$ as follows:

$$\xi \subseteq G \quad \text{iff } \xi(x) \leq G(x) \ \ \forall x \in S,\ \ \xi, G \in \xi(S). \tag{3}$$

If $\xi, G \in \xi(S)$, the intersection $\xi \cap G$, union $\xi \cup G$, and product $\xi \circ G$ can be defined as follows.

For all $x \in S$,

$$\begin{aligned} (\xi \cap G)(x) &= \xi(x) \wedge G(x), \\ (\xi \cup G)(x) &= \xi(x) \vee G(x), \\ (\xi \circ G)(a) &= \begin{cases} \bigvee_{(y,z) \in X_a} (\xi(y) \wedge G(z)) & \text{if } X_a \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4}$$

Theorem 3 (see [4]). *A nonempty subset A of an ordered semigroup S is a left (resp., right) ideal of S if and only if the characteristic function χ_A of A is a fuzzy left (resp., right) ideal of S.*

3. Fuzzy Ideals Based on δ-Quasi-Coincident with Relation

A verity of latest research, conducting multiple conferences and workshops on fuzzy mathematics, shows the importance of this particular field of study. It is worth mentioning that the researchers are more interested to use algebraic structures in upcoming completed problems not even in mathematics but in an engineering, computer science, robotics coding theory, and others. The idea of quasi-coincident with relation was based on the aforementioned concept, where Jun's [27] generalization of quasi-coincident with relation played a vital role in generating several types of fuzzy subsystems which have already been used in various applied fields which leads

to a verity of productive research [20, 28–31, 33, 34]. Jun et al. [32] further generalized his idea [27] and initiated an essential generalization of $(\in, \in \vee q)$-fuzzy subgroups. Keeping in view Jun et al.'s idea [32], we introduce a new generalization called δ-quasi-coincident with relation. In this section, new classification of ordered semigroups based on $(\in, \in \vee q_k^{\delta})$ -fuzzy ideals, where k is an arbitrary element of $[0, 1)$ and $\delta \in (0, 1]$, is determined. A fuzzy subset ξ of S of the form

$$\xi : S \longrightarrow [0, 1],$$
$$y \longmapsto \begin{cases} a & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases} \tag{5}$$

is fuzzy point [36] with support x and value $a \in (0, 1]$ and is denoted by x_a. A fuzzy point x_a is said to belong to (resp., quasi-coincident with) a fuzzy set ξ, written as $x_a \in \xi$ (resp., $x_a q \xi$) if $\xi(x) \geq a$ (resp., $\xi(x) + a > 1$). If $x_a \in \xi$ or $x_a q \xi$, then we write $x_a \in \vee q \xi$. The symbol $\overline{\in \vee q}$ means $\in \vee q$ does not hold. Generalizing the concept of $x_a q \xi$, in ordered semigroups, we define $x_a q_k^{\delta} \xi$, as $\xi(x) + a + k > \delta$, where $k \in [0, 1)$ and $\delta \in (0, 1]$. Note that $x_a q_k \xi$ implies $x_a q_k^{\delta} \xi$, but the converse of the statement is not always true which will later be elaborated through examples. Particularly, if $\delta = 1$, then every δ-quasi-coincident with relation will lead to quasi-coincident with relation, symbolically $x_a q_k^1 \xi = x_a q_k \xi$. Also, $x_a \in \vee q_k^{\delta} \xi$ (resp. $x_a \in \wedge q_k^{\delta} \xi$) means that $x_a \in \xi$ or $x_a q_k^{\delta} \xi$ (resp. $x_a \in \xi$ and $x_a q_k^{\delta} \xi$).

Definition 4. A fuzzy subset ξ of S is called an $(\in, \in \vee q_k^{\delta})$-fuzzy left (resp., right) ideal of S if it satisfies the conditions:

(1) $(\forall x, y \in S)(\forall a \in (0, 1])(x \leq y \rightarrow y_a \in \xi \rightarrow x_a \in \vee q_k^{\delta} \xi)$.

(2) $(\forall x, y \in S)(\forall a \in (0, 1])(y_a \in \xi \rightarrow (xy)_a \in \vee q_k^{\delta} \xi$ (resp. $(yx)_a \in \vee q_k^{\delta} \xi))$.

Note that every $(\in, \in \vee q_k^{\delta})$-fuzzy left (resp., right) ideal with $\delta = 1$ is an $(\in, \in \vee q_k)$-fuzzy left (resp., right) ideal of S [30].

Example 5. Consider an ordered semigroup $S = \{a_1, a_2, a_3, a_4, a_5\}$ with multiplication Table 1 and order relation:

$$\leq := \{(a_1, a_1), (a_1, a_3), (a_1, a_4), (a_1, a_5), (a_2, a_2), (a_3, a_3), (a_3, a_5), (a_4, a_4), (a_5, a_5)\}. \tag{6}$$

This order relation is represented by Figure 1.

Let $\xi : S \rightarrow [0, 1]$ be a fuzzy subset of S, for $x \in S$, and define a fuzzy subset ξ as follows:

S	a_1	a_2	a_3	a_4	a_5
$\xi(x)$	0.60	0.20	0.50	0.40	0.45

(7)

Then ξ is an $(\in, \in \vee q_k^{\delta})$-fuzzy ideal of S with $\delta = 0.8$.

Table 1: Multiplication table of ordered semigroup S.

$\cdot$	a_1	a_2	a_3	a_4	a_5
a_1	a_1	a_4	a_1	a_4	a_4
a_2	a_1	a_2	a_1	a_4	a_4
a_3	a_1	a_4	a_3	a_4	a_5
a_4	a_1	a_4	a_1	a_4	a_4
a_5	a_1	a_4	a_3	a_4	a_5

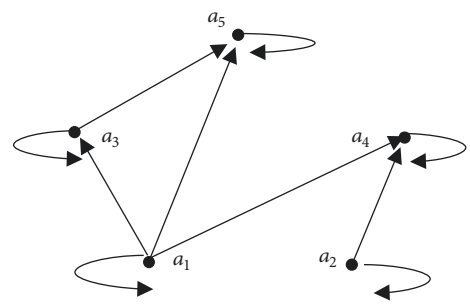

Figure 1: Order relation of S.

Obviously, every $(\in, \in \vee q_k)$-fuzzy left (resp., right) ideal is an $(\in, \in \vee q_k^{\delta})$-fuzzy left (resp., right) ideal of S, but the converse in not true in general. Because in the aforementioned example, if $a = 0.42$ and $k \in [0, 0.18]$ then ξ is an $(\in, \in \vee q_k^{\delta})$-fuzzy ideal of S with $\delta = 0.8$ but not $(\in, \in \vee q_k)$-fuzzy ideal. $(a_5)_{0.42} \in \xi$ but $(a_1 a_5)_{0.42} \overline{\in \vee q_k^{\delta}} \xi$. If $\delta \in (0.8, 1]$, then ξ is not an $(\in, \in \vee q_k^{\delta})$-fuzzy ideal of S.

Theorem 6. *Let I be a left (resp., right) ideal of S and ξ a fuzzy subset in S defined by*

$$\xi(x) = \begin{cases} \geq \dfrac{\delta - k}{2} & \text{if } x \in I, \\ 0 & \text{otherwise}. \end{cases} \tag{8}$$

Then

(1) *ξ is a $(q, \in \vee q_k^{\delta})$-fuzzy ideal of S.*

(2) *ξ is an $(\in, \in \vee q_k^{\delta})$-fuzzy ideal of S.*

Proof. (1) We discuss only the case of left ideals since the case for right ideals can be proved similarly. Let $x, y \in S$, $x \leq y$, and $a \in (0, 1]$ be such that $y_a q \xi$. Then $\xi(y) + a > 1$ and so $y \in I$. Since I is a left ideal of S and $x \leq y \in I$, we have $x \in I$. Thus $\xi(x) \geq (\delta - k)/2$. If $a \leq (\delta - k)/2$, then $\xi(x) \geq a$ and so $x_a \in \xi$. If $a > (\delta - k)/2$, then $\xi(x) + a + k > (\delta - k)/2 + (\delta - k)/2 + k = (\delta - k + \delta - k + 2k)/2 = \delta$ and so $x_a q_k^{\delta} \xi$. Therefore $x_a \in \vee q_k^{\delta} \xi$.

Let $x, y \in S$ and $a \in (0, 1]$ be such that $y_a q \xi$. Then $\xi(y) + a > 1$ so $y \in I$. Since I is a left ideal of S, we have $xy \in I$. Thus $\xi(xy) \geq (\delta - k)/2$. Now, if $a > (\delta - k)/2$, then $\xi(xy) + a + k > (\delta - k)/2 + (\delta - k)/2 + k = \delta$ and so $(xy)_a q_k^{\delta} \xi$. If $a \leq (1 - k)/2$, then $\xi(xy) \geq a$ and so $(xy)_a \in \xi$. Therefore, $(xy)_a \in \vee q_k^{\delta} \xi$.

(2) Let $x, y \in S$, $x \leq y$, and $a \in (0, 1]$ be such that $y_a \in \xi$. Then $\xi(y) \geq a$ and $y \in I$. Since I is a left ideal of S and

$x \leq y \in I$, we have $x \in I$. Thus $\xi(x) \geq (\delta - k)/2$. If $a \leq (\delta - k)/2$, then $\xi(x) \geq a$ and so $x_a \in \xi$. If $a > (\delta - k)/2$, then $\xi(x) + a + k > (\delta - k)/2 + (\delta - k)/2 + k = \delta$ and so $x_a q_k^\delta \xi$. Therefore $x_a \in \vee q_k^\delta \xi$.

If $x, y \in S$ and $a \in (0, 1]$ be such that $y_a \in \xi$. Then $x, y \in I$ and so $xy \in I$. Thus $\xi(xy) \geq (\delta - k)/2$. If $a > (\delta - k)/2$, then $\xi(xy) + a + k > (\delta - k)/2 + (\delta - k)/2 + k = \delta$ and so $(xy)_a q_k^\delta \xi$. If $a \leq (\delta - k)/2$, then $\xi(xy) \geq a$ and we have $(xy)_a \in \xi$. Therefore $(xy)_a \in \vee q_k^\delta \xi$. Consequently, ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S. □

If we take $\delta = 1$ in Theorem 6, then we get the following corollary.

Corollary 7 (see [30]). *Let I be a left (resp., right) ideal of S and ξ, a fuzzy subset in S defined by*

$$\xi(x) = \begin{cases} \geq \dfrac{1-k}{2} & \text{if } x \in I, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Then

(1) *ξ is a $(q, \in \vee q_k)$-fuzzy ideal of S.*

(2) *ξ is an $(\in, \in \vee q_k)$-fuzzy ideal of S.*

Theorem 8. *A fuzzy subset ξ of S is an $(\in, \in \vee q_k^\delta)$-fuzzy left (resp., right) ideal of S if and only if*

(1) $(\forall x, y \in S)(x \leq y \to \xi(x) \geq \xi(y) \wedge (\delta - k)/2)$,

(2) $(\forall x, y \in S)(\xi(xy) \geq \xi(y) \wedge (\delta - k)/2$ *(resp.* $\xi(xy) \geq \xi(x) \wedge (\delta - k)/2)$).

Proof. Suppose that ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy ideal of S. On the contrary, assume that there exist $x, y \in S$, $x \leq y$ such that $\xi(x) < \xi(y) \wedge (\delta - k)/2$. Choose $a \in (0, 1]$ such that $\xi(x) < a \leq \xi(y) \wedge (\delta - k)/2$. Then $y_a \in \xi$, but $\xi(x) < a$ and $\xi(x) + a + k < (\delta - k)/2 + (\delta - k)/2 + k = \delta$, so $x_a \overline{\in \vee q_k^\delta} \xi$, a contradiction. Hence $\xi(x) \geq \xi(y) \wedge (\delta - k)/2$ for all $x, y \in S$ with $x \leq y$.

If there exist $x, y \in S$ such that $\xi(xy) < \xi(y) \wedge (\delta - k)/2$, choose $a \in (0, 1]$ such that $\xi(xy) < a \leq \xi(y) \wedge (\delta - k)/2$. Then $y_a \in \xi$ but $\xi(xy) < a$ and $\xi(xy) + a + k < (\delta - k)/2 + (\delta - k)/2 + k = \delta$, so $(xy)_a \overline{q_k^\delta} \xi$. Thus, $(xy)_a \overline{\in \vee q_k^\delta} \xi$, a contradiction. Therefore, $\xi(xy) \geq \xi(y) \wedge (\delta - k)/2$ for all $x, y \in S$.

Conversely, let $y_a \in \xi$ for some $a \in (0, 1]$. Then $\xi(y) \geq a$. Now, $\xi(x) \geq \xi(y) \wedge (\delta - k)/2 \geq a \wedge (\delta - k)/2$. If $a > (\delta - k)/2$, then $\xi(x) \geq (\delta - k)/2$ and $\xi(x) + a + k > (\delta - k)/2 + (\delta - k)/2 + k = \delta$, and $x_a q_k^\delta \xi$ follows. If $a \leq (\delta - k)/2$, then $\xi(x) \geq a$ and so $x_a \in \xi$. Thus, $x_a \in \vee q_k^\delta \xi$.

Let $y_a \in \xi$, then $\xi(y) \geq a$ and so $\xi(xy) \geq \xi(y) \wedge (\delta - k)/2 \geq a \wedge (\delta - k)/2$. If $a > (\delta - k)/2$, then $\xi(xy) \geq (\delta - k)/2$ and $\xi(xy) + a + k > (\delta - k)/2 + (\delta - k)/2 + k = \delta$ and so $(xy)_a q_k^\delta \xi$. If $a \leq (\delta - k)/2$, then $\xi(xy) \geq a$ implies that $(xy)_a \in \xi$. Thus $(xy)_a \in \vee q_k^\delta \xi$ and consequently, ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S. Similarly we can prove that ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy right ideal of S. □

If we take $\delta = 1$ in Theorem 8, then we have the following corollary [30].

Corollary 9. *A fuzzy subset ξ of S is an $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideal of S if and only if*

(1) $(\forall x, y \in S)(x \leq y \to \xi(x) \geq \xi(y) \wedge (1 - k)/2)$,

(2) $(\forall x, y \in S)(\xi(xy) \geq \xi(y) \wedge (1 - k)/2$ *(resp* $\xi(xy) \geq \xi(x) \wedge (1 - k)/2)$).

Using Theorem 8, we have the following characterization of fuzzy left (resp., right) ideals of ordered semigroups.

Proposition 10. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $\emptyset \neq I \subseteq S$. Then I is a left (resp., right) ideal of S if and only if the characteristic function χ_I of I is an $(\in, \in \vee q_k^\delta)$-fuzzy left (resp., right) ideal of S.*

Corollary 11. *A fuzzy subset ξ of an ordered semigroup S is an $(\in, \in \vee q_k^\delta)$-fuzzy left (resp., right) ideal of S if and only if it satisfies conditions (1) and (2) of Theorem 8.*

Remark 12. Every fuzzy left (resp., right) ideal of an ordered semigroup is an $(\in, \in \vee q_k^\delta)$-fuzzy left (resp., right) ideal of S. But the converse is not true.

Example 13. Consider the ordered semigroup as given in Example 5, and define a fuzzy subset $\xi : S \to [0, 1]$ as follows:

S	a_1	a_2	a_3	a_4	a_5
$\xi(x)$	0.60	0.20	0.50	0.40	0.45

(10)

Then ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy ideal with $\delta = 0.8$ but $U(\xi; a) = \{a_1, a_3\}$ for all $a \in (0.45, 0.50]$ is not an ideal of S. Hence by Theorem 2, ξ is not a fuzzy ideal of S for all $a \in (0.45, 0.50]$.

Theorem 14. *Every $(\in, \in)$-fuzzy left (resp., right) ideal is an $(\in, \in \vee q_k^\delta)$-fuzzy left (resp., right) ideal of S.*

Proof. The proof is straightforward and is omitted here. □

The converse of the above theorem is not true in general as shown in Example 15.

Example 15. Consider the ordered semigroup as given in Example 5, and define a fuzzy subset $\xi : S \to [0, 1]$ as follows:

S	a_1	a_2	a_3	a_4	a_5
$\xi(x)$	0.80	0.20	0.70	0.50	0.60

(11)

Then ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy ideal but not an $(\in, \in)$-fuzzy ideal of S. This is because $(a_1)_{0.50} \in \xi$ but $(a_1 a_2)_{0.50} = (a_4)_{0.50} \overline{\in} \xi$.

Theorem 16. *Every $(\in \vee q_k^\delta, \in \vee q_k^\delta)$-fuzzy left (resp., right) ideal is an $(\in, \in \vee q_k^\delta)$-fuzzy left (resp., right) ideal of S.*

Proof. Suppose that ξ is an $(\in \vee q_k^\delta, \in \vee q_k^\delta)$-fuzzy left ideal of S. Let $x, y \in S$, and $a \in (0, 1]$ be such that $x \leq y$, $y_a \in \xi$, then

$y_a \in \vee q_k^\delta \xi$. So by hypothesis, we have $x_a \in \vee q_k^\delta \xi$. Now let $x, y \in S$, and $a \in (0,1]$ be such that $y_a \in \xi$. Then $y_a \in \vee q_k^\delta \xi$. Hence $(xy)_a \in \vee q_k^\delta \xi$. Similarly we can prove that $(yx)_a \in \vee q_k^\delta \xi$. □

If $\delta = 1$ in Theorem 16, then the following corollary [30] can be obtained.

Corollary 17. *Every $(\in \vee q_k, \in \vee q_k)$-fuzzy left (resp., right) ideal is an $(\in, \in \vee q_k)$-fuzzy left (resp., right) ideal of S.*

The importance of the following theorem is to provide the bridge between ordinary ideals and fuzzy ideals of type $(\in, \in \vee q_k^\delta)$. This linkage is very much essential for the characterization of ordered semigroups. Some time it is quite difficult to understand weather a particular ideal is fuzzy (resp., ordinary) ideal or not. In such a situation, the following result plays its role by providing the required information.

Theorem 18. *A fuzzy subset ξ of S is an $(\in, \in \vee q_k^\delta)$-fuzzy left (resp., right) ideal of S if and only if $U(\xi; a)(\neq \emptyset)$ is left (resp., right) ideal of S for all $a \in (0, (\delta - k)/2]$.*

Proof. Assume that ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S. Let $x, y \in S$, $x \leq y$ be such that $y \in U(\xi; a)$ for some $a \in (0, (\delta - k)/2]$. Then $\xi(y) \geq a$ and by hypothesis

$$\xi(x) \geq \xi(y) \wedge \frac{\delta - k}{2} \geq a \wedge \frac{\delta - k}{2} = a. \tag{12}$$

Hence $x \in U(\xi; a)$. Let $x, y \in S$ be such that $y \in U(\xi; a)$ for some $a \in (0, (\delta - k)/2]$. Then $\xi(y) \geq a$ and we have

$$\xi(xy) \geq \xi(y) \wedge \frac{\delta - k}{2} \geq a \wedge \frac{\delta - k}{2} = a. \tag{13}$$

Thus $xy \in U(\xi; a)$.

Conversely, suppose that $U(\xi; a)(\neq \emptyset)$ is a left ideal of S for all $a \in (0, (\delta - k)/2]$. If there exist $x, y \in S$, $x \leq y$ such that $\xi(x) < \xi(y) \wedge (\delta - k)/2$, choose $a \in (0, (\delta - k)/2]$ such that $\xi(x) < a \leq \xi(y) \wedge (\delta - k)/2$, then, $\xi(y) \geq a$ implies that $y \in U(\xi; a)$ but $x \notin U(\xi; a)$, a contradiction. Hence, $\xi(x) \geq \xi(y) \wedge (\delta - k)/2$. If $x, y \in S$ such that $\xi(xy) < \xi(y) \wedge (\delta - k)/2$, then choose $a \in (0, (\delta - k)/2]$, and $\xi(xy) < a \leq \xi(y) \wedge (\delta - k)/2$. Thus, $y \in U(\xi; a)$ but $xy \notin U(\xi; a)$, a contradiction. Hence, $\xi(xy) \geq \xi(y) \wedge (\delta - k)/2$ for all $x, y \in S$, $k \in [0, 1)$ and $\delta \in (0, 1]$. Therefore, ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S. By a similar way, we can prove that ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy right ideal of S. □

Note that $q_k^1 = q_k$. Therefore, if $\delta = 1$ in Theorem 18, then this new result can produce the following corollary of the literature [30].

Corollary 19. *A fuzzy subset ξ of S is an $(\in, \in \vee q_k)$-fuzzy left (resp., right) ideal of S if and only if $U(\xi; a)(\neq \emptyset)$ is left (resp., right) ideal of S for all $a \in (0, (1 - k)/2]$.*

Proposition 20. *Let ξ be a nonzero $(\in, \in \vee q_k^\delta)$-fuzzy left (resp., right) ideal of S. Then the set $\xi_0 = \{x \in S \mid \xi(x) > 0\}$ is a left (resp., right) ideal of S.*

Proof. Let ξ be an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S. If , $y \in S$, $x \leq y$, and $y \in \xi_0$, then, $\xi(y) > 0$. Since ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S, we have

$$\xi(x) \geq \xi(y) \wedge \frac{\delta - k}{2} > 0, \quad \text{because } \xi(y) > 0. \tag{14}$$

Thus $\xi(x) > 0$ and so $x \in \xi_0$. Let $y \in \xi_0$, then, $\xi(y) > 0$ and by hypothesis,

$$\xi(xy) \geq \xi(y) \wedge \frac{\delta - k}{2} > 0. \tag{15}$$

Thus $xy \in \xi_0$. Consequently, ξ_0 is a left ideal of S. Similarly, we can prove that ξ_0 is a right ideal of S. □

If $\delta = 1$, then Proposition 20 leads to the following result [30].

Result 1. Let ξ be a nonzero $(\in, \in \vee q_k)$-fuzzy left (resp., right) ideal of S. Then the set ξ_0 is a left (resp., right) ideal of S.

In the following lemma, we establish a relationship between $(\in, \in \vee q_k^\delta)$-fuzzy left (resp., right) ideals and $(\in, \in)$-fuzzy left (resp., right) ideals of S.

Lemma 21. *Suppose that ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy left (resp., right) ideal of S such that $\xi(x) < (\delta - k)/2$ for all $x \in S$. Then ξ is an $(\in, \in)$-fuzzy left (resp., right) ideal of S.*

Proof. Let $x, y \in S$ and $a \in (0, 1]$ be such that $x \leq y$, $y_a \in \xi$. Then $\xi(y) \geq a$. Since ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal and $\xi(x) < (\delta - k)/2$, we have

$$\xi(x) \geq \xi(y) \wedge \frac{\delta - k}{2} \geq a \wedge \frac{\delta - k}{2} = a. \tag{16}$$

Hence, $x_a \in \xi$. If $x, y \in S$ and $a \in (0, 1]$ be such that $y_a \in \xi$. Then $\xi(y) \geq a$ and we have

$$\xi(xy) \geq \xi(y) \wedge \frac{\delta - k}{2} \geq a \wedge \frac{\delta - k}{2} = a. \tag{17}$$

Thus, $(xy)_a \in \xi$. By a similar way, we can prove that $(yx)_a \in \xi$. □

If $\delta = 1$, then Lemma 21 leads to the following result [30].

Result 2. If ξ is an $(\in, \in \vee q_k)$-fuzzy left (resp., right) ideal of S such that $\xi(x) < (1-k)/2$ for all $x \in S$, then ξ is an $(\in, \in)$-fuzzy left (resp., right) ideal of S.

For any fuzzy subset ξ of S and $a \in (0, 1]$, we denote $\Xi_k^\delta(\xi; a)$ as follows:

$$\begin{aligned} \Xi_k^\delta(\xi; a) &:= \left\{x \in S \mid x_a q_k^\delta \xi\right\}, \\ \Omega_k^\delta(\xi; a) &:= \left\{x \in S \mid x_a \in \vee q_k^\delta \xi\right\}. \end{aligned} \tag{18}$$

Clearly $\Omega_k^\delta(\xi; a) = U(\xi; a) \cup \Xi_k^\delta(\xi; a)$.

We call $\Omega_k^\delta(\xi; a)$ an $(\in \vee q_k^\delta)$-fuzzy level left (resp., right) ideal I and $\Xi_k^\delta(\xi; a)$ a q_k^δ-level left (resp., right) ideal of S.

We give another characterization of $(\in, \in \vee q_k^\delta)$-fuzzy left (resp., right) ideals by using $\Omega_k^\delta(\xi; a)$.

Theorem 22. *A fuzzy subset ξ of S is an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S if and only if $\Omega_k^\delta(\xi; a)$ is a left (resp., right) ideal of S for all $a \in (0, 1]$.*

Proof. Assume ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy left (resp., right) ideal of S. Let $x, y \in S$, $x \leq y$, and $a \in (0, 1]$ be such that $y \in \Omega_k^\delta(\xi; a)$. Then $y_a \in \vee q_k^\delta \xi$, that is, $\xi(y) \geq a$ or $\xi(y) + a + k > \delta$. Since ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S and $x \leq y$, we have $\xi(x) \geq \xi(y) \wedge (\delta - k)/2$. We discuss the following cases.

Case 1 ($\xi(y) \geq a$). If $a > (\delta - k)/2$, then $\xi(x) \geq \xi(y) \wedge (\delta - k)/2 = (\delta - k)/2$ and so

$$\xi(x) + a + k > \frac{\delta - k}{2} + \frac{\delta - k}{2} + k = \delta, \tag{19}$$

so $x_a q_k^\delta \xi$. If $a \leq (\delta - k)/2$, then $\xi(x) \geq \xi(y) \wedge (\delta - k)/2 \geq a$. Hence $x_a \in \xi$.

Case 2 ($\xi(y) + a + k > \delta$). If $a > (\delta - k)/2$, then

$$\begin{aligned} \xi(x) &\geq \xi(y) \wedge \frac{\delta - k}{2} > \delta - a - k \wedge \frac{\delta - k}{2} \\ &= \delta - a - k, \end{aligned} \tag{20}$$

that is, $\xi(x) + a + k > \delta$. Thus $x_a q_k^\delta \xi$. If $a \leq (\delta - k)/2$, then

$$\begin{aligned} \xi(x) &\geq \xi(y) \wedge \frac{\delta - k}{2} > \delta - a - k \wedge \frac{\delta - k}{2} = \frac{\delta - k}{2} \\ &\geq a. \end{aligned} \tag{21}$$

Therefore $x_a \in \xi$. Hence $x_a \in \vee q_k^\delta \xi$. Let $x \in S$ and $y \in \Omega_k^\delta(\xi; a)$ for $a \in (0, 1]$. Then $y_a \in \vee q_k^\delta \xi$, that is, $\xi(y) \geq a$ or $\xi(y) + a + k > \delta$. Since ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S, we have $\xi(xy) \geq \xi(y) \wedge (\delta - k)/2$.

Case 1 ($\xi(y) \geq a$). If $a > (\delta - k)/2$, then $\xi(xy) \geq \xi(y) \wedge (\delta - k)/2 = (\delta - k)/2$ and so

$$\xi(xy) + a + k > \frac{\delta - k}{2} + \frac{\delta - k}{2} + k = \delta, \tag{22}$$

so $(xy)_a q_k^\delta \xi$. If $a \leq (\delta - k)/2$, then $\xi(xy) \geq \xi(y) \wedge (\delta - k)/2 \geq a$. Hence $(xy)_a \in \xi$.

Case 2 ($\xi(y) + a + k > \delta$). If $a > (\delta - k)/2$, then

$$\begin{aligned} \xi(xy) &\geq \xi(y) \wedge \frac{\delta - k}{2} > \delta - a - k \wedge \frac{\delta - k}{2} \\ &= \delta - a - k, \end{aligned} \tag{23}$$

that is, $\xi(xy) + a + k > \delta$. Thus $(xy)_a q_k^\delta \xi$. If $a \leq (\delta - k)/2$, then

$$\begin{aligned} \xi(xy) &\geq \xi(y) \wedge \frac{\delta - k}{2} > \delta - a - k \wedge \frac{\delta - k}{2} = \frac{\delta - k}{2} \\ &\geq a. \end{aligned} \tag{24}$$

Therefore, $(xy)_a \in \xi$. Hence $(xy)_a \in \vee q_k^\delta \xi$. Thus $\Omega_k^\delta(\xi; a)$ is left ideal. Similarly, we can prove that $(yx)_a \in \vee q_k^\delta \xi$. Consequently, $\Omega_k^\delta(\xi; a)$ is left (resp., right) ideal of S.

Conversely, assume that ξ is a fuzzy subset of S and let $a \in (0, 1]$ be such that $\Omega_k^\delta(\xi; a)$ is left ideal of S. If there exist $x, y \in S$, $x \leq y$, and $a \in (0, (\delta - k)/2]$ such that

$$\xi(x) < a \leq \xi(y) \wedge \frac{\delta - k}{2}, \tag{25}$$

then $y \in U(\xi; a) \subseteq \Omega_k^\delta(\xi; a)$. Hence $y \in \Omega_k^\delta(\xi; a)$, since $\Omega_k^\delta(\xi; a)$ is left ideal and $x \leq y$. Thus $x \in \Omega_k^\delta(\xi; a)$ and we have $\xi(x) \geq a$ or $\xi(x) + a + k > \delta$, a contradiction. Hence $\xi(x) \geq \xi(y) \wedge (\delta - k)/2$ for all $x, y \in S$ with $x \leq y$. Let $x, y \in S$ be such that

$$\xi(xy) < a \leq \xi(y) \wedge \frac{\delta - k}{2} \tag{26}$$

for some $a \in (0, (\delta - k)/2]$. Then $y \in U(\xi; a) \subseteq \Omega_k^\delta(\xi; a)$. Hence $y \in \Omega_k^\delta(\xi; a)$, since $\Omega_k^\delta(\xi; a)$ is left ideal. Thus $xy \in \Omega_k^\delta(\xi; a)$ and we have $\xi(xy) \geq a$ or $\xi(xy) + a + k > \delta$, a contradiction. Hence $\xi(xy) \geq \xi(y) \wedge (\delta - k)/2$ for all $x, y \in S$. Thus ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S. Similarly we can prove that ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy right ideal of S. □

If $\delta = 1$, then, a fuzzy subset ξ of S is an $(\in, \in \vee q_k)$-fuzzy left ideal of S if and only if $\Omega_k(\xi; a)$ is a left (resp., right) ideal of S for all $a \in (0, 1]$ [30].

From Theorems 2 and 22, $U(\xi; a)$ and $\Omega_k^\delta(\xi; a)$ are left ideals of S for all $a \in (0, (\delta - k)/2]$, but $\Xi_k^\delta(\xi; a)$ is not left ideal of S for $a \in (0, (\delta - k)/2]$ in general, as shown in the next example.

Example 23. Consider an ordered semigroup as given in Example 5, and define a fuzzy subset $\xi : S \to [0, 1]$ as follows:

S	a_1	a_2	a_3	a_4	a_5
$\xi(x)$	0.60	0.40	0.50	0.30	0.10

(27)

Then $\Xi_k^\delta(\xi; a) = \{a_1, a_2, a_3, a_4\}$ is left ideal for $a \in (0.1, (1 - k)/2]$, $\delta = 0.8$, and $k = 0.3$ but neither $U(\xi; a)$ or $\Omega_k^\delta(\xi; a)$ is left ideal because of $(a_2)_{0.20} q_k^\delta \xi$ but $(a_1 a_2)_{0.20} = (a_4)_{0.20} \overline{q_k^\delta} \xi$.

4. Characterization of Regular Ordered Semigroups by δ-Quasi-Coincident with Relation

Regular ordered semigroups are known to be the most important classes of semigroup theory. In this section, fuzzy quasi-ideals of type $(\in, \in \vee q_k^\delta)$ are introduced. Further, the connection between $(\in, \in \vee q_k^\delta)$-fuzzy left (resp., right) ideals and $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideals is provided. The classification of regular ordered semigroups by the properties of $(\in, \in \vee q_k^\delta)$-fuzzy left (resp., right) ideals is also the milestone of the present section.

Definition 24. A fuzzy subset ξ of S is called an $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal of S if it satisfies the conditions:

(1) $(\forall x, y \in S)(x \leq y \rightarrow \xi(x) \geq \xi(y) \wedge (\delta - k)/2)$,

(2) $(\forall x \in S)(\xi(x) \geq ((\xi \circ 1) \cap (1 \circ \xi))(x) \wedge (\delta - k)/2)$.

Theorem 25. *Let ξ be a nonzero $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal of S, then the set $\xi_0 = \{x \in S \mid \xi(x) > 0\}$ is a quasi-ideal of S.*

Proof. Let $x, y \in S$ be such that $x \leq y$. If $y \in \xi_0$, then $\xi(y) > 0$. Since ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal of S, therefore

$$\xi(x) \geq \xi(y) \wedge \frac{\delta - k}{2} > 0, \quad \text{because } \xi(y) > 0. \tag{28}$$

Thus $\xi(x) > 0$ and so $x \in \xi_0$. Let $a \in (\xi_0 S] \cap (S\xi_0]$, then $a \in (\xi_0 S]$ and $a \in (S\xi_0]$. Hence there exist $r, s \in S$ and $x, y \in \xi_0$ such that $a \leq xs$, and $a \leq ry$. Then $(x, s), (r, y) \in X_a$. Since $X_a \neq \phi$, we have

$$\begin{aligned}
\xi(x) &\geq ((\xi \circ 1) \cap (1 \circ \xi))(x) \wedge \frac{\delta - k}{2} \\
&= \left\{ \bigvee_{(p,q) \in X_a} (\xi(p) \wedge 1(q)) \right\} \\
&\quad \wedge \left\{ \bigvee_{(q,p) \in X_a} (1(q) \wedge \xi(p)) \right\} \wedge \frac{\delta - k}{2} \\
&\geq \{\xi(x) \wedge 1(r)\} \wedge \{1(r) \wedge \xi(y)\} \wedge \frac{\delta - k}{2} \\
&= \xi(x) \wedge \xi(y) \wedge \frac{\delta - k}{2} > 0, \\
&\qquad \text{because } x, y \in \xi_0.
\end{aligned} \tag{29}$$

Hence $a \in \xi_0$ implies that $(\xi_0 S] \cap (S\xi_0] \subseteq \xi_0$. □

The proof of the following proposition is straightforward and is omitted.

Proposition 26. *Every $(\in, \in \vee q_k^\delta)$-fuzzy left (resp., right) or two-sided ideal of S is an $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal of S.*

In the following example, it is shown that the converse of Proposition 26 is not true in general.

Example 27. Consider an ordered semigroup $S = \{0, a, b, c\}$ with multiplication Table 2 and order relation:

$$\leq := \{(0,0), (a,a), (b,b), (c,c), (0,a), (0,b), (0,c)\}. \tag{30}$$

The order relation is given by Figure 2.

Then $\{0, a\}$ is a quasi-ideal of S, but not a left (resp., right) ideal of S.

Define a fuzzy subset $\xi : S \rightarrow [0, 1]$ as follows:

S	0	a	b	c
$\xi(x)$	0.25	0.25	0	0

(31)

Then ξ is fuzzy quasi-ideal of S but not an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S for $\delta = 0.8$ and $k = 0.4$ since $\xi(ca) = \xi(c) = 0 < \xi(a) \wedge (\delta - k)/2$.

Table 2: Multiplication table of ordered semigroup S.

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	0
b	0	0	0	0
c	0	c	0	0

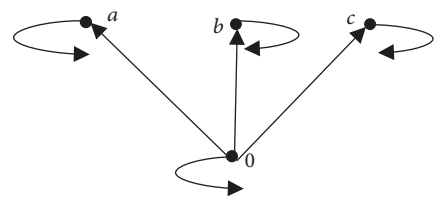

Figure 2: Order relation of S.

Definition 28. Let S be an ordered semigroup and ξ, G fuzzy subsets of S. Then the $(\delta - k)/2$-product of ξ and G is defined by

$$(\xi \circ_{(\delta-k)/2} G)(a) = \begin{cases} \displaystyle\bigvee_{(y,z) \in X_a} \left(\xi(y) \wedge G(z) \wedge \frac{\delta - k}{2} \right) & \text{if } X_a \neq \emptyset, \\ 0 & \text{if } X_a = \emptyset. \end{cases} \tag{32}$$

Proposition 29. *Let $(S, \cdot, \leq)$ be an ordered semigroup and ξ_1, ξ_2, G_1, G_2 fuzzy subsets of S such that $\xi_1 \subseteq \xi_2$ and $G_1 \subseteq G_2$. Then $\xi_1 \circ_{(\delta-k)/2} G_1 \subseteq \xi_2 \circ_{(\delta-k)/2} G_2$.*

Lemma 30. *If ξ and G are $(\in, \in \vee q_k^\delta)$-fuzzy left (resp., right) ideals of an ordered semigroup S, then $\xi \cap_{(\delta-k)/2} G$ is an $(\in, \in \vee q_k^\delta)$-fuzzy left (resp., right) ideal of S, where $(\xi \cap_{(\delta-k)/2} G)(x) := \xi(x) \wedge G(x) \wedge (\delta - k)/2$ for all $x \in S$.*

Proof. Let $x, y \in S$ such that $x \leq y$. Since ξ and G are $(\in, \in \vee q_k^\delta)$-fuzzy left ideals of S, hence $\xi(x) \geq \xi(y) \wedge (\delta - k)/2$ and $G(x) \geq G(y) \wedge (\delta - k)/2$. Therefore

$$\begin{aligned}
(\xi \cap_{(1-k)/2} G)(x) &= \xi(x) \wedge G(x) \wedge \frac{\delta - k}{2} \\
&\geq \left\{ \xi(y) \wedge \frac{\delta - k}{2} \right\} \\
&\quad \wedge \left\{ G(y) \wedge \frac{\delta - k}{2} \right\} \wedge \frac{\delta - k}{2} \\
&= \left(\xi(y) \wedge G(y) \wedge \frac{\delta - k}{2} \right) \\
&\quad \wedge \frac{\delta - k}{2} \\
&= (\xi \cap_{(\delta-k)/2} G)(y) \wedge \frac{\delta - k}{2}.
\end{aligned} \tag{33}$$

Hence $(\xi \cap_{(\delta-k)/2} G)(x) \geq (\xi \cap_{(\delta-k)/2} G)(y) \wedge (\delta - k)/2$.

Let $x, y \in S$. Since ξ and G are $(\in, \in \vee q_k^\delta)$-fuzzy left ideals of S. Then, $\xi(xy) \geq \xi(y) \wedge (\delta - k)/2$ and $G(xy) \geq G(y) \wedge (\delta - k)/2$. Thus

$$\begin{aligned}(\xi \cap_{(\delta-k)/2} G)(xy) &= \xi(xy) \wedge G(xy) \wedge \frac{\delta-k}{2} \\ &\geq \left\{\xi(y) \wedge \frac{\delta-k}{2}\right\} \\ &\quad \wedge \left\{G(y) \wedge \frac{\delta-k}{2}\right\} \wedge \frac{\delta-k}{2} \\ &= \left(\xi(y) \wedge G(y) \wedge \frac{\delta-k}{2}\right) \\ &\quad \wedge \frac{\delta-k}{2} \\ &= (\xi \cap_{(\delta-k)/2} G)(y) \wedge \frac{\delta-k}{2}.\end{aligned} \tag{34}$$

Hence $\xi \cap_{(\delta-k)/2} G$ is an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S. Similarly we can prove that $\xi \cap_{(\delta-k)/2} G$ is an $(\in, \in \vee q_k^\delta)$-fuzzy right ideal of S. □

Let ξ and G be fuzzy subsets of S; we define $\xi \cup G$ as follows:

$$(\xi \cup_{(\delta-k)/2} G)(x) = (\xi(x) \vee G(x)) \wedge \frac{\delta-k}{2} \tag{35}$$

for all $x \in S$.

Putting $\delta = 1$ in Lemma 30 leads to a result [30]: "If ξ and G are $(\in, \in \vee q_k)$-fuzzy left (resp., right) ideals of an ordered semigroup S, then $\xi \cap_{(1-k)/2} G$ is an $(\in, \in \vee q_k)$-fuzzy left (resp., right) ideal of S".

Lemma 31. *If ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy right of S and G is an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S, respectively, then $\xi \circ_{(\delta-k)/2} G \subseteq \xi \cap_{(\delta-k)/2} G$.*

Proof. Let $a \in S$. If $X_a = \phi$, then $(\xi \circ_{(\delta-k)/2} G)(a) = 0 \leq (\xi \cap_{(\delta-k)/2} G)(a)$. Assume that $X_a \neq \phi$, then

$$(\xi \circ_{(\delta-k)/2} G)(a) := \bigvee_{(y,z)\in X_a} \left(\xi(y) \wedge G(z) \wedge \frac{\delta-k}{2}\right). \tag{36}$$

Since $a \leq yz$, and ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy right and G is an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S, thus

$$\begin{aligned}\xi(a) &\geq \xi(yz) \wedge \frac{\delta-k}{2} \geq \left(\xi(y) \wedge \frac{\delta-k}{2}\right) \wedge \frac{\delta-k}{2} \\ &= \xi(y) \wedge \frac{\delta-k}{2}, \\ G(a) &\geq G(yz) \wedge \frac{\delta-k}{2} \geq \left(\xi(y) \wedge \frac{\delta-k}{2}\right) \wedge \frac{\delta-k}{2} \\ &= G(y) \wedge \frac{\delta-k}{2}.\end{aligned} \tag{37}$$

Hence, $\xi(y) \wedge G(y) \wedge (\delta - k)/2 \leq \xi(a) \wedge G(a) \wedge (\delta - k)/2 = (\xi \cap_{(\delta-k)/2} G)(a)$. □

Regular ordered semigroups are characterized by right and left ideals in [5]. In the following we give a characterization of regular ordered semigroups in terms of $(\in, \in \vee q_k^\delta)$-fuzzy right and $(\in, \in \vee q_k^\delta)$-fuzzy left ideals.

Lemma 32 (see [5]). *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S is regular.*

(2) *$R \cap L = (RL]$ for every right ideal R and every left ideal L of S.*

Lemma 33. *Let S be an ordered semigroup and $\phi \neq A, B \subseteq S$. Then, we have*

(i) *$(\chi_A \circ_{(\delta-k)/2} \chi_B)(a) = \chi_{(AB]}(a) \wedge (\delta - k)/2$ for all $a \in S$.*

(ii) *$\chi_A \cap_{(\delta-k)/2} \chi_B = \chi_{A\cap B} \wedge (\delta - k)/2$.*

(iii) *$\chi_A \cup_{(\delta-k)/2} \chi_B = \chi_{A\cup B} \wedge (\delta - k)/2$.*

Theorem 34. *An ordered semigroup S is regular if and only if, for every $(\in, \in \vee q_k^\delta)$-fuzzy right ideal ξ and every $(\in, \in \vee q_k^\delta)$-fuzzy left ideal G of S, one has $\xi \circ_{(\delta-k)/2} G = \xi \cap_{(\delta-k)/2} G$.*

Proof. Let S be regular ordered semigroup and $a \in S$. Then there exists $x \in S$ such that $a \leq axa$. Hence $(ax, a) \in X_a$ and we have

$$\begin{aligned}(\xi \circ_{(\delta-k)/2} G)(a) &= \bigvee_{(y,z)\in X_a} \left(\xi(y) \wedge G(z) \wedge \frac{\delta-k}{2}\right) \\ &\geq \xi(ax) \wedge G(a) \wedge \frac{\delta-k}{2}.\end{aligned} \tag{38}$$

Since ξ is an $(\in, \in \vee q_k^\delta)$-fuzzy right ideal of S, therefore $\xi(ax) \geq \xi(a) \wedge (\delta - k)/2$. Thus

$$\begin{aligned}&\xi(ax) \wedge G(a) \wedge \frac{\delta-k}{2} \\ &\quad \geq \left(\xi(a) \wedge \frac{\delta-k}{2}\right) \wedge G(a) \wedge \frac{\delta-k}{2} \\ &\quad = \xi(a) \wedge G(a) \wedge \frac{\delta-k}{2} = (\xi \cap_{(\delta-k)/2} G)(a).\end{aligned} \tag{39}$$

Hence $(\xi \cap_{(\delta-k)/2} G)(a) \leq (\xi \circ_{(\delta-k)/2} G)(a)$. On the other hand, by Lemma 31, we have $(\xi \circ_{(\delta-k)/2} G)(a) \leq (\xi \cap_{(\delta-k)/2} G)(a)$. Therefore $(\xi \circ_{(\delta-k)/2} G)(a) = (\xi \cap_{(\delta-k)/2} G)(a)$.

Conversely, suppose for every $(\in, \in \vee q_k^\delta)$-fuzzy right ideal ξ and every $(\in, \in \vee q_k^\delta)$-fuzzy left ideal G of S we have $\xi \circ_{(\delta-k)/2} G = \xi \cap_{(\delta-k)/2} G$. By Lemma 32, we need to show that $R \cap L = (RL]$ for every right ideal R and left ideal L of S. Let $y \in R \cap L$, then $y \in R$ and $y \in L$. Since R is right ideal and L is left ideal of S, thus by Lemma 33, χ_R is a fuzzy right and χ_L is a fuzzy left ideal S. Therefore by Proposition 10, χ_R is an $(\in, \in \vee q_k^\delta)$-fuzzy right ideal and χ_L is an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal

of S. By hypothesis, $(\chi_R \circ_{(\delta-k)/2} \chi_L)(y) = (\chi_R \cap_{(\delta-k)/2} \chi_L)(y)$. Since $y \in R$ and $y \in L$ so $\chi_R(y) = 1$ and $\chi_L(y) = 1$, then $(\chi_R \cap_{(\delta-k)/2} \chi_L)(y) = \chi_R(y) \wedge \chi_L(y) \wedge (\delta - k)/2 = 1 \wedge 1 \wedge (\delta - k)/2 = (\delta - k)/2$. It follows that $(\chi_R \circ_{(\delta-k)/2} \chi_L)(y) = (\delta - k)/2$. By Lemma 33, $(\chi_R \circ_{(\delta-k)/2} \chi_L)(y) = \chi_{(RL]}(y) \wedge (\delta - k)/2$. Thus $\chi_{(RL]}(y) = (\delta - k)/2$. It follows that $y \in (RL]$. Hence $R \cap L \subseteq (RL]$ but $(RL] \subseteq R \cap L$, always holds. Therefore $R \cap L = (RL]$. Consequently S is regular. □

5. Conclusion

Since the traditional mathematical tools are dichotomous in nature, i.e., yes or no type rather than more or less type, therefore, these tools can not be used to handle complex problems involving uncertainties of various applied fields like decision-making problems, control engineering, coding theory, theoretical physics, computer sciences, and fuzzy automata theory. For the said problems, fuzzy sets and their various generalizations are used. A quasi-coincident with relation is one of the most important relations used in algebraic structures along with a verity of applications in the aforementioned areas. The aim of the present paper was to introduce a comprehensive type of relation known as δ-quasi-coincident with relation which can lead to the existing systems and also some new types of subsystems involving quasi-coincident with relation. More precisely, $(\in, \in \vee q_k^\delta)$-fuzzy left (resp., right, quasi-) ideals of S are developed which were elaborated through examples. The bridging of ordinary ideals and $(\in, \in \vee q_k^\delta)$-fuzzy ideals of an ordered semigroup S is constructed while regular ordered semigroups were characterized through $(\in, \in \vee q_k^\delta)$-fuzzy left (resp., right, quasi-) ideals. As $\delta \in (0, 1]$, therefore for particular value $\delta = 1$, our algebraic system reduced to [30].

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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