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Research Article

Solving Fuzzy Volterra Integrodifferential Equations of Fractional Order by Bernoulli Wavelet Method

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A matrix method called the Bernoulli wavelet method is presented for numerically solving the fuzzy fractional integrodifferential equations. Using the collocation points, this method transforms the fuzzy fractional integrodifferential equation to a matrix equation which corresponds to a system of nonlinear algebraic equations with unknown coefficients. To illustrate the method, it is applied to certain fuzzy fractional integrodifferential equations, and the results are compared.

1. Introduction

Dynamical systems with fractional order derivatives have found many applications in various problems in science and engineering like viscoelasticity, heat conduction, electrode-electrolyte polarization, electromagnetic waves, diffusion wave, control theory, and so on. In fractional equations, the vagueness may be appearing in each part of the equation like initial condition, boundary condition, and so on. So solving fractional equations in the sense of real conditions leads to the use of interval or fuzzy calculations.

The concept of the fuzzy derivative was first introduced by Chang and Zadeh [1], followed by many authors. The starting point of the topic in the set valued differential equation and also fuzzy differential equation is Hukuhara's paper [2]. The Hukuhara derivative was the starting point for the topic of set differential equations and later also for fuzzy fractional differential equations. By the concept of Hukuhara differentiability, the fuzzy Riemann-Liouville fractional differential equation is introduced by Agarwal et al. in [3], which was the starting point of the topic in fuzzy fractional derivative. They have considered the Riemann-Liouville differentiability concept based on the Hukuhara differentiability to solve uncertain fractional differential equations. The existence and uniqueness of solutions of Riemann-Liouville fuzzy fractional differential equations is proved in [4, 5]. Allahviranloo et al. in [6] presented the explicit solutions of uncertain fractional differential equations under Riemann-Liouville H-differentiability using Mittag-Leffler functions and in [7] introduced the fuzzy fractional differential equations under Riemann-Liouville H-differentiability and obtained the solution of this equation by fuzzy Laplace transforms. They showed two new uniqueness results for fuzzy fractional differential equations involving Riemann-Liouville generalized H-differentiability with fuzzy version of Nagumo and Krasnoselskii-Krein conditions [8]. Consequently, the Caputo generalized Hukuhara derivative is introduced in [9]; the authors introduced an ordinary fractional differential equation under the generalized Hukuhara differentiability and studied the existence and uniqueness of the solution. The nonlinear fuzzy fractional integrodifferential equation under generalized fuzzy Caputo derivative is introduced in [10, 11] and proved the existence and uniqueness of the solutions of this set of equations by considering the type of differentiability. Recently, Sahu and Saha Ray [12] applied the two-dimensional Bernoulli wavelet method to solve the fuzzy integrodifferential equations and they developed the Bernoulli wavelet method to solve the nonlinear fuzzy Hammerstein Volterra integral equations with constant delay [13].

The main aim of the presented paper is concerned with the application of the proposed approach to obtain the numerical solution of fuzzy fractional integrodifferential equations of the form

$$\left({}_{\mathrm{gH}}D_{*}^{q} y\right)(t) = f\left(t, y(t), (\mathcal{S} y)(t)\right), \quad t \in J = [0, T], \quad q \in (0, 1], \qquad y(0) = y_0 \in \mathbb{R}_{\mathcal{F}}, \tag{1}$$

where ${}_{\mathrm{gH}}D_{*}^{q}$ is the fuzzy Caputo fractional derivative of order q, $f : J \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is a given function satisfying some assumptions that will be specified later, y_0 is an element of $\mathbb{R}_{\mathcal{F}}$, and $\mathcal{S}$ is a nonlinear integral operator given by

$$(\mathcal{S} y)(t) = \int_0^t k(x, t)\, y(x)\, \mathrm{d}x, \tag{2}$$

where $k : J \times J \to \mathbb{R}^{+}$, with $\gamma_0 = \max\{\int_0^t k(x,t)\mathrm{d}x\colon (x,t) \in J \times J\}$.

The paper is organized as follows. Section 2 collects some definitions of basic notions and notations concerning fuzzy calculus. In Section 3 we discuss the properties of Bernoulli wavelets. To determine the approximate solution for the fuzzy fractional integrodifferential equation, two-dimensional Bernoulli wavelet method has been applied in Section 4. Moreover according to the type of differentiability, solutions of a fuzzy fractional integrodifferential equations are investigated in different scenarios. In Section 6, numerical examples are given to solve the fuzzy fractional integrodifferential equation and show the accuracy of the method. Finally, in Section 7, the report ends with a brief conclusion and some remarks.

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary results, which will be used throughout this paper. Let $\mathbb{R}_{\mathcal{F}}$ denote the set of fuzzy subsets of the real axis, if $u : \mathbb{R} \to [0, 1]$, satisfying the following properties:

(i) u is upper semicontinuous on $\mathbb{R}$.

(ii) u is fuzzy convex.

(iii) u is normal.

(iv) closure of $\{x \in \mathbb{R} \mid u(x) > 0\}$ is compact.

Then $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers.

For $0 < r \le 1$, set $[u]_r = \{t \in \mathbb{R}^n \mid u(t) \ge r\}$, and $[u]_0 = \mathrm{cl}\{t \in \mathbb{R}^n \mid u(t) > 0\}$. We represent $[u]_r = [\underline{u}(r), \overline{u}(r)]$, so if $u \in \mathbb{R}_{\mathcal{F}}$, the r-level set $[u]_r$ is a closed interval for all $r \in [0, 1]$. For arbitrary $u, v \in \mathbb{R}_{\mathcal{F}}$ and $k \in \mathbb{R}$, the addition and scalar multiplication are defined by $[u+v]_r = [u]_r + [v]_r$, $[ku]^r = k[u]_r$, respectively.

A triangular fuzzy number is defined as a fuzzy set in $\mathbb{R}_{\mathcal{F}}$, which is specified by an ordered triple $u = (a, b, c) \in \mathbb{R}^3$ with $a \le b \le c$ such that $\underline{u}(r) = a + (b - a)r$ and $\overline{u} = c - (c - b)r$ are the endpoints of r-level sets for all $r \in [0, 1]$.

The Hausdorff distance between fuzzy numbers is given by $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}^{+} \cup \{0\}$ as in [15]

$$\begin{aligned} D(u, v) &= \sup_{t \in [0,1]} d_H\left([u]_r, [v]_r\right) \\ &= \sup_{r \in [0,1]} \max\left\{\left|\underline{u}(r) - \underline{v}(r)\right|, \left|\overline{u}(r) - \overline{u}(r)\right|\right\}, \end{aligned} \tag{3}$$

where d_H is the Hausdorff metric. The metric space $(\mathbb{R}_{\mathcal{F}}, D)$ is complete, separable, and locally compact and the following properties from [15] for metric D are valid:

(1) $D(u \oplus w, v \oplus w) = D(u, v)$, $\forall u, v, w \in \mathbb{R}_{\mathcal{F}}$;

(2) $D(\lambda u, \lambda v) = |\lambda| D(u, v)$, $\forall \lambda \in \mathbb{R}$, $u, v \in \mathbb{R}_{\mathcal{F}}$;

(3) $D(u \oplus v, w \oplus z) \le D(u, w) + D(v, z)$, $\forall u, v, w, z \in \mathbb{R}_{\mathcal{F}}$;

(4) $D(u \ominus v, w \ominus z) \le D(u, w) + D(v, z)$, as long as $u \ominus v$ and $w \ominus z$ exist, where $u, v, w, z \in \mathbb{R}_{\mathcal{F}}$.

Here, $\ominus$ is the Hukuhara difference (H-difference); it means that $w \ominus v = u$ if and only if $u \oplus v = w$.

Definition 1 (see [16]). The generalized Hukuhara difference of two fuzzy numbers $u, v \in \mathbb{R}_{\mathcal{F}}$ is defined as follows:

$$\begin{gathered} u \ominus_{\mathrm{gH}} v = w \\ \Updownarrow \\ \text{(i) } u = v + w; \\ \text{or (ii) } v = u + (-1) w. \end{gathered} \tag{4}$$

In terms of r-levels one has $[u \ominus_{\mathrm{gH}} v]_r = [\min\{\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{u}(r)\}, \max\{\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r)\}]$ and if the H-difference exists, then $u \ominus v = u \ominus_{\mathrm{gH}} v$; the conditions for the existence of $w = u \ominus_{\mathrm{gH}} v \in \mathbb{R}_{\mathcal{F}}$ are as follows.

Case (i)

$$\begin{aligned} &\underline{w}(r) = \underline{u}(r) - \underline{v}(r), \\ &\overline{w}(r) = \overline{u}(r) - \overline{v}(r), \\ &\qquad\qquad \forall r \in [0, 1], \\ &\text{with } \underline{w}(r) \text{ increasing, } \overline{w}(r) \text{ decreasing, } \underline{w}(r) \le \overline{w}(r). \end{aligned} \tag{5}$$

Case (ii)

$$\begin{aligned} &\underline{w}(r) = \overline{u}(r) - \overline{v}(r), \\ &\overline{w}(r) = \underline{u}(r) - \underline{v}(r), \\ &\qquad\qquad \forall r \in [0, 1], \\ &\text{with } \underline{w}(r) \text{ increasing, } \overline{w}(r) \text{ decreasing, } \underline{w}(r) \le \overline{w}(r). \end{aligned} \tag{6}$$

It is easy to show that (i) and (ii) are both valid if and only if w is a crisp number.

Remark 2. Throughout the rest of this paper, we assume that $u \ominus_{\mathrm{gH}} v \in \mathbb{R}_{\mathcal{F}}$.

Definition 3 (see [17]). A fuzzy-valued function $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is said to be continuous at $t_0 \in [a, b]$ if for each $\epsilon > 0$ there is $\delta > 0$ such that $D(f(t), f(t_0)) < \epsilon$, whenever $t \in [a, b]$ and $|t - t_0| < \delta$. One says that f is fuzzy continuous on $[a, b]$ if f is continuous at each $t_0 \in [a, b]$.

Definition 4 (see [16]). The generalized Hukuhara derivative of a fuzzy-valued function $f : (a,b) \to \mathbb{R}_{\mathcal{F}}$ at $x_0 \in (a,b)$ is defined as

$$f'_{\text{gH}}(x_0) = \lim_{h\to 0} \frac{f(x_0+h) \ominus_{\text{gH}} f(x_0)}{h}. \quad (7)$$

If $f'_{\text{gH}}(x_0) \in \mathbb{R}_{\mathcal{F}}$ satisfying (7) exists, one says that f is generalized Hukuhara differentiable (gH-differentiable for short) at x_0.

Definition 5 (see [18]). Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$. One says that $f(t)$ is fuzzy Riemann integrable in $\mathbb{I} \in \mathbb{R}_F$ if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u,v];\xi\}$ with the norms $\Delta(P) < \delta$ one has

$$D\left(\sum_{p}^{*} (v-u) \odot f(\xi), I\right) < \varepsilon, \quad (8)$$

where $\sum_p^*$ denotes the fuzzy summation. One chooses to write

$$\mathbb{I} := \int_a^b f(t)\, dt. \quad (9)$$

Note that if the fuzzy-valued function $f(t,r) = [\underline{f}(t,r), \overline{f}(t,r)]$ is continuous in the metric D, the Lebesgue integral and the Riemann integral yield the same value, and also

$$\int_a^b f(t,r)\, dt = \left[\int_a^b \underline{f}(t,r)\, dt, \int_a^b \overline{f}(t,r)\, dt\right], \quad 0 \le r \le 1. \quad (10)$$

Throughout this paper, we consider the notations $A^{\mathbb{F}}[a,b]$ for the space of the fuzzy-valued functions from $[a,b]$ into $\mathbb{R}_{\mathcal{F}}$ that are absolutely continuous on $[a,b]$. Also, $C^{\mathbb{F}}[a,b]$ denote the set of the fuzzy-valued functions which are fuzzy continuous on all of $[a,b]$ such that the continuity is one-sided at endpoints a,b. Also, we denote the space of all Lebesgue integrable fuzzy-valued functions on the bounded interval $[a,b] \subset \mathbb{R}$ by $L^{\mathbb{F}}[a,b]$.

Definition 6 (see [7]). Let $f \in L^{\mathbb{F}}[a,b]$. The fuzzy Riemann-Liouville integral of a fuzzy-valued function f is defined as follows:

$$\left(\mathcal{I}_a^q f\right)(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s)\, ds \quad (11)$$

for $a \le t$, and $0 < q \le 1$.

Theorem 7 (see [7]). *Let $f \in L^{\mathbb{F}}[a,b]$ be a fuzzy-valued function. The fuzzy Riemann-Liouville integral of a fuzzy-valued function f can be expressed as follows:*

$$\mathcal{I}_a^q f(t,r) = \left[I_a^q \underline{f}(t,r), I_a^q \overline{f}(t,r)\right], \quad (12)$$

where

$$\begin{aligned} I_a^q \underline{f}(t,r) &= \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} \underline{f}(s,r)\, \mathrm{d}s, \\ I_a^q \overline{f}(t,r) &= \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} \overline{f}(s,r)\, \mathrm{d}s. \end{aligned} \quad (13)$$

Definition 8 (see [9]). Let $f \in A^{\mathbb{F}}[a,b]$. The fuzzy gH-fractional Caputo differentiability of the fuzzy-valued function f is defined as follows:

$$\begin{aligned} \left({}_{\text{gH}}D_*^q f\right)(t) &= \mathcal{I}_a^{1-q}\left(f'_{\text{gH}}\right)(t) \\ &= \frac{1}{\Gamma(1-q)} \int_a^t \frac{\left(f'_{\text{gH}}\right)(s)\, ds}{(t-s)^q}, \end{aligned} \quad (14)$$

where $a < s < t$, $q \in (0,1]$.

Definition 9 (see [9]). Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be ${}^{cf}[\text{gH}]$-differentiable at $t_0 \in (a,b)$. One says that f is ${}^{cf}[(\text{i})\text{-gH}]$-differentiable at t_0 if

$$\text{(i)} \quad \left({}_{\text{gH}}D_*^\alpha f\right)(t_0,r) = \left[\left(D_*^\alpha \underline{f}\right)(t_0,r), \left(D_*^\alpha \overline{f}\right)(t_0,r)\right], \quad 0 \le r \le 1, \quad (15)$$

and that f is ${}^{cf}[(\text{ii})\text{-gH}]$-differentiable at t_0 if

$$\text{(ii)} \quad \left({}_{\text{gH}}D_*^\alpha f\right)(t_0,r) = \left[\left(D_*^\alpha \overline{f}\right)(t_0,r), \left(D_*^\alpha \underline{f}\right)(t_0,r)\right], \quad 0 \le r \le 1. \quad (16)$$

Definition 10 (see [9]). Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy function. A point $t_0 \in (a,b)$ is said to be a switching point for the ${}^{cf}[\text{gH}]$-differentiability of f, if in any neighborhood V of t_0 there exist points $t_1 < t_0 < t_2$ such that one has the following.

`Type(I)`. At t_1 (15) holds while (16) does not hold and at t_2 (16) holds and (15) does not hold.

`Type(II)`. At t_1 (16) holds while (15) does not hold and at t_2 (15) holds and (16) does not hold.

Lemma 11 (see [9]). *Let $f : [0,T] \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy-valued function such that $f \in A^{\mathbb{F}}[0,T]$; then*

$$\mathcal{I}_0^q \left({}_{gH}D_*^q f\right)(t) = f(t) \ominus_{gH} f(0). \quad (17)$$

Lemma 12 (see [14]). *Fuzzy initial value problem (1) is equivalent to one of the following integral equations.*

Case 1. If $y(t)$ is ${}^{cf}[(i)\text{-}gH]$-differentiable, then

$$y(t) = y_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s), (\mathcal{S}y)(s))\, \mathrm{d}s. \quad (18)$$

Case 2. If $y(t)$ is ${}^{cf}[(ii)\text{-}gH]$-differentiable, hence

$$y(t) = y_0 \ominus \frac{-1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s), (\mathcal{S}y)(s))\, \mathrm{d}s. \quad (19)$$

Theorem 13 (see [14]). *Assume that the following conditions hold:*

($\mathbf{H}_1$) $f : J \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ *is fuzzy continuous.*

($\mathbf{H}_2$) *There exists a constant* $q_1 \in (0, q)$ *and real-valued positive functions* $m_1(t), m_2(t) \in L^{1/q_1}(J, \mathbb{R}^+)$ *such that*

$$D\left(f\left(t, x(t), (\mathcal{S}x)(t)\right), f\left(t, y(t), (\mathcal{S}y)(t)\right)\right) \le m_1(t) D\left(x(t), y(t)\right) + m_2(t) D\left((\mathcal{S}x)(t), (\mathcal{S}) y(t)\right), \quad (20)$$

for each $t \in J$, *and all* $x(t), y(t), (\mathcal{S}x)(t), (\mathcal{S}y)(t) \in C^{\mathbb{F}}[0, T]$.

If

$$\Omega_{q,q_1,T} = \frac{MT^{q-q_1}}{\Gamma(q)\left((q-q_1)/(1-q_1)\right)^{1-q_1}} < 1, \quad (21)$$

then (1) has a unique solution on J.

3. Bernoulli Wavelets

In this section, first we recall the definitions of wavelets and Bernoulli wavelets. Our aim is to approximate the solution $y(t)$ by the truncated Bernoulli series.

3.1. Wavelets and Bernoulli Wavelets. The Bernoulli polynomials play an important role in different areas of mathematics, including number theory and the theory of finite differences. The classical Bernoulli polynomials $\beta_m(x)$ are usually defined by means of following relations:

$$\frac{d\beta_m(x)}{dx} = m\beta_{m-1}(x), \quad m \ge 1, \qquad \beta_0(x) = 1. \quad (22)$$

Also the Bernoulli polynomials can be represented in the form

$$\beta_m(x) = \sum_{i=0}^{m} \binom{i}{m} \alpha_{m-i} t^i, \quad (23)$$

where $\alpha_i,\ i = 0, 1, \ldots, m$ are the Bernoulli numbers. Thus, the first four such polynomials, respectively, are

$$\begin{aligned} \beta_0(t) &= 1, \\ \beta_1(t) &= t - \frac{1}{2}, \\ \beta_2(t) &= t^2 - t + \frac{1}{6}, \\ \beta_3(t) &= t^3 - \frac{3}{2}t^2 + \frac{1}{2}t. \end{aligned} \quad (24)$$

Also, these polynomials satisfy the following formula:

$$\int_0^1 \beta_n(t)\beta_m(t)\,\mathrm{d}t = (-1)^{n-1}\frac{m!n!}{(m+n)!}\beta_{m+n}, \quad m, n \ge 1. \quad (25)$$

The properties of Bernoulli polynomials ($\beta_m(t)$) and the sequence of Bernoulli numbers (α_m) are

(1) $\beta_m(1-t) = (-1)^m \beta_m(t),\ m \in \mathbb{Z}^+$.

(2) $\int_0^1 \beta_m(t)\beta_n(t)dt = (-1)^{m-1}(m!n!/(m+n)!),\ m, n \ge 1$.

(3) $\int_0^1 |\beta_m(t)|dt < 16m!/(2\pi)^{m+1},\ m \ge 0$.

(4) $\int_a^x \beta_m(t)dt = |\beta_{m+1}(x) - \beta_{m+1}(a)|/(m+1)$.

(5) $\sup_{t\in[0,1]}|\beta_{2m}(t)| = |\alpha_{2m}|$.

(6) $\sup_{t\in[0,1]}|\beta_{2m+1}(t)| \le ((2m+1)/4)|\alpha_{2m}|$.

(7) $\alpha_{2m+1} = 0,\ \alpha_{2m} = \beta_{2m}(1)$.

(8) $\beta_m(1/2) = (2^{1-m} - 1)\alpha_m$.

(9) $\alpha_m = -(1/(m+1))\sum_{k=0}^{m-1}\binom{m+1}{k}\alpha_k$.

Wavelets constitute a family of functions constructed from dilation and translation of single function called the mother wavelet $\psi(t)$. They are defined by

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R}, \quad (26)$$

where a is dilation parameter and b is a translation parameter. Bernoulli wavelets $\mathfrak{B}_{n,m}(t) = \mathfrak{B}(k, n, m, t)$ have four arguments, defined on interval $[0, 1)$ by

$$\mathfrak{B}_{n,m}(t) = \begin{cases} 2^{(k-1)/2}\widehat{\beta}_m\left(2^{k-1}t - n + 1\right), & \dfrac{n-1}{2^{k-1}} \le t < \dfrac{n}{2^{k-1}}, \\ 0, & \text{elsewhere}, \end{cases} \quad (27)$$

with

$$\widehat{\beta}_m(t) = \begin{cases} 1, & m = 0, \\ \dfrac{1}{\sqrt{\left((-1)^{m-1}(m!)^2/(2m)!\right)\alpha_{2m}}}\beta_m(t) & m > 0, \end{cases} \quad (28)$$

where $k \in \mathbb{Z}^+,\ n = 1, 2, 3, \ldots, 2^{k-1}$ and $m = 0, 1, \ldots, M-1$ is the order of the Bernoulli polynomials and M is a fixed positive integer. The coefficient $1/\sqrt{((-1)^{m-1}(m!)^2/(2m)!)\alpha_{2m}}$ is for orthonormality, the dilation parameter is $a = 2^{-(k-1)}$, and translation parameter is $b = (n-1)2^{-(k-1)}$.

The two-dimensional Bernoulli wavelets are defined as

$$\mathfrak{B}_{n,i,l,j}(x,t)=\begin{cases}2^{(k_1-1)/2}2^{(k_2-1)/2}\widehat{\beta}_i\left(2^{k_1-1}x-n+1\right)\widehat{\beta}_j\left(2^{k_2-1}t-l+1\right), & \dfrac{n-1}{2^{k_1-1}}\le x<\dfrac{n}{2^{k_1-1}},\ \dfrac{l-1}{2^{k_2-1}}\le t<\dfrac{l}{2^{k_2-1}}\\ 0, & \text{elsewhere,}\end{cases} \tag{29}$$

where $n=1,2,\ldots,2^{k_1-1}$, $l=1,2,\ldots,2^{k_2-1}$, $i=0,1,\ldots,M_1-1$ and $j=0,1,\ldots,M_2-1$ and k_1 and k_2 are any positive integers.

3.2. Function Approximation. A function $y(x,t)$ defined over $[0,1)\times[0,1)$ can be expanded in terms of Bernoulli wavelets as

$$y(x,t)=\sum_{n=1}^{\infty}\sum_{i=0}^{\infty}\sum_{l=1}^{\infty}\sum_{j=0}^{\infty}c_{n,i,l,j}\mathfrak{B}_{n,i,l,j}(x,t). \tag{30}$$

If the infinite series in (30) is truncated, then it can be written as

$$\begin{aligned}y(x,t)&=\sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1}c_{n,i,l,j}\mathfrak{B}_{n,i,l,j}(x,t)\\&=C^T\mathbf{B}(x,t),\end{aligned} \tag{31}$$

where $\mathbf{B}(x,t)$ is $(2^{k_1-1}2^{k_2-1}M_1M_2\times 1)$ matrix, given by

$$\begin{aligned}\mathbf{B}(x,t)=\big[&\mathfrak{B}_{1,0,1,0}(x,t),\mathfrak{B}_{1,0,1,1}(x,t),\ldots,\\&\mathfrak{B}_{1,0,1,M_2-1}(x,t),\mathfrak{B}_{1,0,2,M_2-1}(x,t),\ldots,\\&\mathfrak{B}_{1,0,2^{k_2-1},M_2-1}(x,t),\mathfrak{B}_{1,1,2^{k_2-1},M_2-1}(x,t),\ldots,\\&\mathfrak{B}_{1,M_1-1,2^{k_2-1},M_2-1}(x,t),\mathfrak{B}_{2,M_1-1,2^{k_2-1},M_2-1}(x,t),\ldots,\\&\mathfrak{B}_{2^{k_1-1},M_1-1,2^{k_2-1},M_2-1}(x,t)\big].\end{aligned} \tag{32}$$

Also, C is $(2^{k_1-1}2^{k_2-1}M_1M_2\times 1)$ matrix and

$$\begin{aligned}C=\big[&c_{1,0,1,0},c_{1,0,1,1},\ldots,c_{1,0,1,M_2-1},c_{1,0,2,M_2-1},\ldots,\\&c_{1,0,2^{k_2-1},M_2-1},\ldots,c_{1,M_1-1,2^{k_2-1},M_2-1},\ldots,\\&c_{2^{k_1-1},M_1-1,2^{k_2-1},M_2-1}\big]^T.\end{aligned} \tag{33}$$

3.3. The Fractional Order Integration of the Bernoulli Wavelet. The fractional order integration of the Bernoulli wavelets is as follows:

$$\begin{aligned}I_a^q\Psi(t)=\big[&I_a^q\psi_{1,0}(t),\ldots,I_a^q\psi_{1,M-1}(t),I_a^q\psi_{2,0}(t),\ldots,\\&I_a^q\psi_{2,M-1}(t),\ldots,I_a^q\psi_{2^{k-1},0}(t),\ldots,I_a^q\psi_{2^{k-1},M-1}(t)\big]^T,\end{aligned} \tag{34}$$

where

$$\begin{aligned}&I_a^q\psi_{n,m}(t)\\&\quad=\begin{cases}2^{(k-1)/2}I_a^q\left(\widehat{\beta}_m\left(2^{k-1}t-n+1\right)\right), & \dfrac{n-1}{2^{k-1}}\le t<\dfrac{n}{2^{k-1}}\\ 0, & \text{elsewhere,}\end{cases}\\&I_a^q\left(\widehat{\beta}_m\left(2^{k-1}t-n+1\right)\right)\\&\quad=\frac{1}{\Gamma(q)\sqrt{\left((-1)^{m-1}(m!)^2/(2m)!\right)\alpha_{2m}}}\left(\sum_{m=0}^{r}\binom{i}{m}\alpha_{r-m}\right.\\&\quad\left.\cdot\int_a^t(t-s)^{q-1}\left(2^{k-1}s-n+1\right)\mathrm{d}s\right)\end{aligned} \tag{35}$$

for $k\in\mathbb{Z}^+$, $n=1,2,\ldots,2^{k-1}$, and $m=0,1,2,\ldots,M-1$ is the order of the Bernoulli polynomial and M is a fixed positive integer.

4. The Numerical Method

In this paper, we focus on the fuzzy fractional integrodifferential equation:

$$\begin{aligned}\left({}_{\mathrm{gH}}D_*^q y\right)(t)&=y(t)+\int_0^t k(x,t)\,y(x)\,\mathrm{d}x+g(t),\\&\qquad t\in\mathbb{J}=[0,1),\end{aligned} \tag{36}$$

with initial condition

$$y(0)=y_0\in\mathbb{R}_{\mathscr{F}}, \tag{37}$$

where $y(t)$ and $g(t)$ are fuzzy functions and $k(x,t):\mathbb{J}\times\mathbb{J}\to\mathbb{R}^+$. Applying $\mathscr{I}_0^q$ on both sides of (36), using Lemmas 11 and 12, if $y(t)$ is ^{cf}[(i)-gH]-differentiable, then

$$y(t)=y_0+\mathscr{I}_0^q\left(y(t)+\int_0^t k(x,t)\,y(x)\,\mathrm{d}x+g(t)\right) \tag{38}$$

and $y(t)$ is ^{cf}[(ii)-gH]-differentiable:

$$\begin{aligned}&y(t)\\&\quad=y_0\\&\qquad\ominus(-1)\,\mathscr{I}_0^q\left(y(t)+\int_0^t k(x,t)\,y(x)\,\mathrm{d}x+g(t)\right).\end{aligned} \tag{39}$$

Consider $y(t,r)=[\underline{y}(t,r),\overline{y}(t,r)]$ is the solution of (36) and we approximate the unknown function $y(t,r)$ as given by (30). Assume that $y(t)$ is ^{cf}[(i)-gH]-differentiable, so by Theorem 7 and (38) we have the following fractional integrodifferential equations system:

$$\begin{aligned} y(x,r) &= y(0,r) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} y(s,r)\,\mathrm{d}s \\ &+ \frac{1}{\Gamma(q)}\int_0^t\int_0^s (t-s)^{q-1} k(x,s)\, y(x,r)\,\mathrm{d}x\,\mathrm{d}s \\ &+ \int_0^t (t-s)^{q-1} g(s,r)\,\mathrm{d}s. \end{aligned} \tag{40}$$

We see that

$$\begin{aligned} \begin{pmatrix} \underline{y}(t,r) \\ \overline{y}(t,r) \end{pmatrix} &= \begin{pmatrix} \underline{y}(0,r) \\ \overline{y}(0,r) \end{pmatrix} + \begin{pmatrix} \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\underline{y}(s,r)\,\mathrm{d}s \\ \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\overline{y}(s,r)\,\mathrm{d}s \end{pmatrix} \\ &+ \begin{pmatrix} \frac{1}{\Gamma(q)}\int_0^t\int_0^s (t-s)^{q-1} k(x,s)\,\underline{y}(x,r)\,\mathrm{d}x\,\mathrm{d}s \\ \frac{1}{\Gamma(q)}\int_0^t\int_0^s (t-s)^{q-1} k(x,s)\,\overline{y}(x,r)\,\mathrm{d}x\,\mathrm{d}s \end{pmatrix} \\ &+ \begin{pmatrix} \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\underline{g}(s,r)\,\mathrm{d}s \\ \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\overline{g}(s,r)\,\mathrm{d}s \end{pmatrix}. \end{aligned} \tag{41}$$

Hence we obtain

$$\begin{aligned} \underline{y}(t,r) = \underline{y}(0,r) + \frac{1}{\Gamma(q)}\Bigg(&\int_0^t (t-s)^{q-1}\underline{y}(s,r)\,\mathrm{d}s \\ &+ \int_0^t\int_0^s (t-s)^{q-1} k(x,s)\,\underline{y}(x,r)\,\mathrm{d}x\,\mathrm{d}s \\ &+ \int_0^t (t-s)^{q-1}\underline{g}(s,r)\,\mathrm{d}s\Bigg), \end{aligned} \tag{42}$$

$$\begin{aligned} \overline{y}(t,r) = \overline{y}(0,r) + \frac{1}{\Gamma(q)}\Bigg(&\int_0^t (t-s)^{q-1}\overline{y}(s,r)\,\mathrm{d}s \\ &+ \int_0^t\int_0^s (t-s)^{q-1} k(x,s)\,\overline{y}(x,r)\,\mathrm{d}x\,\mathrm{d}s \\ &+ \int_0^t (t-s)^{q-1}\overline{g}(s,r)\,\mathrm{d}s\Bigg). \end{aligned} \tag{43}$$

In order to apply the Bernoulli wavelets in (42), we first approximate the unknown function $\underline{y}(t,r)$ as

$$\begin{aligned} \underline{y}(t,r) &= \sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1} c_{n,i,l,j}\mathfrak{B}_{n,i,l,j}(t,r) \\ &= C^T\mathbf{B}(t,r). \end{aligned} \tag{44}$$

Putting (44) in (42) we obtain

$$\begin{aligned} \sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1} c_{n,i,l,j}\mathfrak{B}_{n,i,l,j}(t,r) &= \underline{y}(0,r) + \frac{1}{\Gamma(q)}\Bigg(\sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1} c_{n,i,l,j}\int_0^t (t-s)^{q-1}\mathfrak{B}_{n,i,l,j}(s,r)\,\mathrm{d}s \\ &+ \sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1} c_{n,i,l,j}\int_0^t\int_0^s (t-s)^{q-1} k(x,s)\,\mathfrak{B}_{n,i,l,j}(x,r)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t (t-s)^{q-1}\underline{g}(s,r)\,\mathrm{d}s\Bigg). \end{aligned} \tag{45}$$

Now we collocate (45) at $(2^{k_1-1}2^{k_2-1}M_1M_2)$ collocation points by $t_i = (2i-1)/2^{k_1}M_1$, $r_j = (2j-2)/2^{k_2}M_2$ for $i = 1,2,\ldots,2^{k_1-1}M_1$, $j = 1,2,\ldots,2^{k_2-1}M_2$ yielding

$$\begin{aligned} \sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1} c_{n,i,l,j}\mathfrak{B}_{n,i,l,j}(t_i,r_j) &= \underline{y}(0,r_j) \\ &+ \frac{1}{\Gamma(q)}\Bigg(\sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1} c_{n,i,l,j} \\ &\cdot\int_0^{t_i}(t_i-s)^{q-1}\mathfrak{B}_{n,i,l,j}(s,r_j)\,\mathrm{d}s \\ &+ \sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1} c_{n,i,l,j} \\ &\cdot\int_0^{t_i}\int_0^s (t_i-s)^{q-1} k(x,s)\,\mathfrak{B}_{n,i,l,j}(x,r_j)\,\mathrm{d}x\,\mathrm{d}s \\ &+ \int_0^{t_i}(t_i-s)^{q-1}\underline{g}(s,r_j)\,\mathrm{d}s\Bigg). \end{aligned} \tag{46}$$

Now consider (43); we approximate the unknown function $\overline{y}(t,r)$ by Bernoulli wavelet as

$$\overline{y}(t,r) = \sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1} c'_{n,i,l,j}\mathfrak{B}_{n,i,l,j}(t,r). \tag{47}$$

Working the same way we find

$$\begin{aligned}
&\sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1} c'_{n,i,l,j}\mathfrak{B}_{n,i,l,j}\left(t_i,r_j\right)=\overline{y}\left(0,r_j\right)\\
&\quad+\frac{1}{\Gamma(q)}\left(\sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1} c'_{n,i,l,j}\right.\\
&\quad\cdot\int_0^{t_i}\left(t_i-s\right)^{q-1}\mathfrak{B}_{n,i,l,j}\left(s,r_j\right)\mathrm{d}s\\
&\quad+\sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1} c'_{n,i,l,j}\\
&\quad\cdot\int_0^{t_i}\int_0^{s}\left(t_i-s\right)^{q-1}k(x,s)\,\mathfrak{B}_{n,i,l,j}\left(x,r_j\right)\mathrm{d}x\,\mathrm{d}s\\
&\quad\left.+\int_0^{t_i}\left(t_i-s\right)^{q-1}\overline{g}\left(s,r_j\right)\mathrm{d}s\right).
\end{aligned}\tag{48}$$

Equations (46) and (48) yield $2(2^{k_1-1}M_1)(2^{k_2-1}M_2)$ equations in $2(2^{k_1-1}M_1)(2^{k_2-1}M_2)$ unknowns in $c_{n,i,l,j}$ and $c'_{n,i,l,j}$. By solving this system of equations using mathematical software, the Bernoulli wavelet coefficients $c_{n,i,l,j}$ and $c'_{n,i,l,j}$ can be obtained and, hence, substituting them in (44) and (47), the approximate solutions can be obtained.

Now, consider $y(t)$ is ${}^{cf}[(\text{ii})\text{-gH}]$-differentiable; we have the following fractional integrodifferential equations system:

$$\begin{aligned}
&y(x,r)\\
&\quad=y(0,r)+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}y(s,r)\,\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(q)}\int_0^t\int_0^s(t-s)^{q-1}k(x,s)\,y(x,r)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\int_0^t(t-s)^{q-1}g(s,r)\,\mathrm{d}s.
\end{aligned}\tag{49}$$

Using (39) and definition of Hukuhara difference, this system can be written in the form

$$\begin{aligned}
&\begin{pmatrix}\underline{y}(t,r)\\ \overline{y}(t,r)\end{pmatrix}\\
&=\begin{pmatrix}\underline{y}(0,r)\\ \overline{y}(0,r)\end{pmatrix}+\begin{pmatrix}\dfrac{1}{\Gamma(q)}\displaystyle\int_0^t(t-s)^{q-1}\overline{y}(s,r)\,\mathrm{d}s\\ \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t(t-s)^{q-1}\underline{y}(s,r)\,\mathrm{d}s\end{pmatrix}\\
&\quad+\begin{pmatrix}\dfrac{1}{\Gamma(q)}\displaystyle\int_0^t\int_0^s(t-s)^{q-1}k(x,s)\,\overline{y}(x,r)\,\mathrm{d}x\,\mathrm{d}s\\ \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t\int_0^s(t-s)^{q-1}k(x,s)\,\underline{y}(x,r)\,\mathrm{d}x\,\mathrm{d}s\end{pmatrix}\\
&\quad+\begin{pmatrix}\dfrac{1}{\Gamma(q)}\displaystyle\int_0^t(t-s)^{q-1}\overline{g}(s,r)\,\mathrm{d}s\\ \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t(t-s)^{q-1}\underline{g}(s,r)\,\mathrm{d}s\end{pmatrix}.
\end{aligned}\tag{50}$$

Then in similar way to previous case, we get the values of unknown vectors $c_{n,i,l,j}$ and $c'_{n,i,l,j}$ and then obtain the solutions $\underline{y}(t,r)$ and $\overline{y}(t,r)$ from (44) and (47), respectively.

5. Error Estimation Algorithm and Convergence Analysis

In this section, we will show an efficient estimation for the Bernoulli wavelets approximation and also a technique to obtain the corrected solution of problem (40) by using the residual method and to describe the convergence behavior of the proposed numerical method.

Let $y^*(t,r)$ be the truncated series which approximate the unknown function of (40), so we observe that

$$\begin{aligned}
&y^*(t,r)\\
&\quad=y(0,r)+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}y^*(s,r)\,\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(q)}\int_0^t\int_0^s(t-s)^{q-1}k(x,s)\,y^*(x,r)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\int_0^t(t-s)^{q-1}g(s,r)\,\mathrm{d}s+R^*(t,r),
\end{aligned}\tag{51}$$

where $R^*(t,r)$ is the residual function. Now, let us consider

$$\begin{aligned}
&L\left[y^*(t,r)\right]\\
&\quad=y^*(t,r)-\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}y^*(s,r)\,\mathrm{d}s\\
&\quad-\frac{1}{\Gamma(q)}\int_0^t\int_0^s(t-s)^{q-1}k(x,s)\,y^*(x,r)\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}\tag{52}$$

Hence, we have

$$\begin{aligned}
L\left[y^*(t,r)\right]&=y(0,r)+\int_0^t(t-s)^{q-1}g(s,r)\,\mathrm{d}s\\
&\quad+R^*(t,r)
\end{aligned}\tag{53}$$

with initial condition $y^*(0,r)=y(0,r)$ for all $0\le r\le 1$.

Furthermore, the error function $E_N(t,r)$ can be defined as

$$E^{*}(t,r) = y(t,r) - y^{*}(t,r), \tag{54}$$

where $y(t,r)$ is the exact solution of problem (40).

By using (53) and (54), we have the error equation

$$L[E^{*}(t,r)] = L[y(t,r)] - L[y^{*}(t,r)] = -R^{*}(t,r) \tag{55}$$

with initial condition

$$E^{*}(0,r) = 0, \quad 0 \le r \le 1. \tag{56}$$

Subsequently, the error problem by using (52) and (55) can be written as

$$\begin{aligned} &E^{*}(t,r) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}E^{*}(s,r)\,\mathrm{d}s \\ &\quad - \frac{1}{\Gamma(q)}\int_0^t\int_0^s (t-s)^{q-1}k(x,s)E^{*}(x,r)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad = -R^{*}(t,r) \\ &E^{*}(0,r) = 0, \quad 0 \le r \le 1. \end{aligned} \tag{57}$$

Solving (57) in the way as in Section 4, we get the approximation $E^{*}(t,r)$ which is the error function based on residual function. We note that if the exact solution of problem (40) is unknown, then the error function can be estimated by $E^{*}(t,r)$ which is found without the exact solution and also clearly seen from given error estimation algorithm.

Let us consider $y(t,r)$ can be expanded in terms of Bernoulli wavelets as

$$y(t,r) = \sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=1}^{\infty}\sum_{j=0}^{\infty} c_{n,i,l,j}\mathfrak{B}_{n,i,l,j}(t,r). \tag{58}$$

And $y^{*}(t,r)$ can be the truncated series:

$$y^{*}(t,r) = \sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1} c_{n,i,l,j}\mathfrak{B}_{n,i,l,j}(t,r). \tag{59}$$

Hence, the truncated error term can be calculated as

$$E_{n,i,l,j}(t,r) = y(t,r) - y^{*}(t,r). \tag{60}$$

Now, we have the following Theorems based on [13].

Theorem 14 (see [13]). *If $y(x,t)$ is defined on $[0,1) \times [0,1)$ and $|y(x,t)| \le K$, then the Bernoulli wavelets expansion of $y(x,t)$ defined in (31) converges uniformly and also*

$$\left|c_{n,i,l,j}\right| < K\frac{A_1A_2}{2^{(k_1-1)/2}2^{(k_2-1)/2}}\frac{16_i!}{(2\pi)^{i+1}}\frac{16_j!}{(2\pi)^{j+1}}, \tag{61}$$

where $A_1 = 1/\sqrt{(-1)^{i-1}(i!)^2/(2i)!}$ and $A_1 = 1/\sqrt{(-1)^{j-1}(j!)^2/(2j)!}$.

Theorem 15 (see [13]). *If a continuous function $y(t,r) \in L^2(\mathbb{R}\times\mathbb{R})$ defined on $[0,1)\times[0,1)$ is bounded, namely, $|y(t,r)| \le K$, then*

$$\left\|E_{n,i,l,j}(t,r)\right\|^2_{L^2([0,1)\times[0,1))} \le \sum_{n=2^{k_1-1}+1}^{\infty}\sum_{i=M_1}^{\infty}\sum_{l=2^{k_2-1}+1}^{\infty}\sum_{j=M_2}^{\infty}\left(K\frac{A_1A_2}{2^{(k_1-1)/2}2^{(k_2-1)/2}}\frac{16_i!}{(2\pi)^{i+1}}\frac{16_j!}{(2\pi)^{j+1}}\right)^2, \tag{62}$$

where $A_1 = 1/\sqrt{(-1)^{i-1}(i!)^2/(2i)!}$ and $A_1 = 1/\sqrt{(-1)^{j-1}(j!)^2/(2j)!}$.

6. Numerical Examples

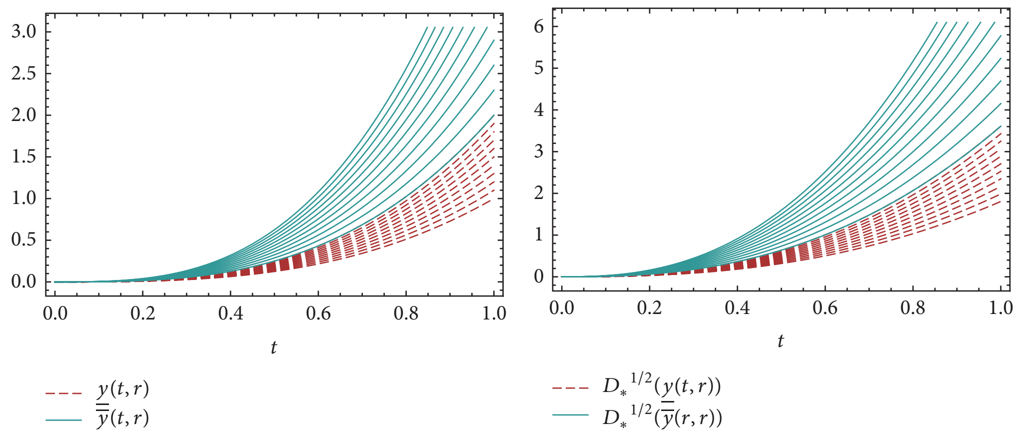
In order to illustrate the effectiveness of the proposed method, we consider numerical examples of fuzzy fractional differential equation.

Example 1. Consider the following fuzzy fractional integrodifferential equation:

$$\begin{aligned} \left({}_{\mathrm{gH}}D_{*}^{1/2}y\right)(t) &= \frac{[r+1,5-3r]}{15\sqrt{\pi}}t^{5/2}\left(48-\sqrt{\pi}t^{7/2}\right) \\ &\quad + \int_0^t \frac{xt}{3}y(x)\,\mathrm{d}x, \\ y(0) &= 0, \end{aligned} \tag{63}$$

where 0 denotes the crisp set $\{0\}$ and the exact solution of (63) is given by $y(t) = [r+1, 5-3r]t^3$. The exact solution is plotted in Figure 1(a) and its ${}^{\mathrm{gH}}D_{*}^{1/2}y(t)$ is plotted in Figure 1(b). As you see, $y(t)$ is ${}^{cf}[(\mathrm{i})\text{-gH}]$-differentiable. So, by applying the method which is discussed in detail, we presented numerical solution of this example for $M_1 = M_2 = 4$, $k_1 = k_2 = 1$. Also, we calculated the absolute errors as $|\underline{E}_r| = |\underline{y}(t,r) - \underline{y}^{*}(t,r)|$ and $|\overline{E}_r| = |\overline{y}(t,r) - \overline{y}^{*}(t,r)|$. To compare the absolute errors of presented method and Legendre method proposed in [14], see Table 1.

It is evident from Table 1 that the numerical solutions converge to the exact solution. It is also concluded that the proposed method is very efficient for numerical solutions of these problems.



(a) (b)

Figure 1: The level sets of $y(t)$ (a) and ${}^{\mathrm{gH}}D_*^{1/2}y(t)$ (b) of Example 1.

Table 1: Comparing the absolute errors of the presented method and the method of [14] for Example 1.

r	t	$\lvert \underline{E}_r \rvert$ Presented method	$\lvert \underline{E}_r \rvert$ Legendre method [14]	$\lvert \overline{E}_r \rvert$ Presented method	$\lvert \overline{E}_r \rvert$ Legendre method [14]
0.3	0.3	1.011×10^{-14}	3.81639×10^{-16}	3.497×10^{-15}	6.38378×10^{-16}
	0.6	5.898×10^{-16}	9.71445×10^{-17}	3.413×10^{-15}	7.63278×10^{-16}
	0.9	2.359×10^{-16}	8.18789×10^{-16}	6.453×10^{-16}	1.31839×10^{-16}
0.6	0.3	2.164×10^{-15}	1.66533×10^{-16}	2.664×10^{-15}	9.99201×10^{-16}
	0.6	4.996×10^{-16}	3.33067×10^{-16}	1.110×10^{-16}	4.44089×10^{-16}
	0.9	4.996×10^{-16}	5.55112×10^{-17}	0	4.99693×10^{-16}
0.9	0.3	3.330×10^{-16}	2.22045×10^{-16}	8.881×10^{-16}	1.77636×10^{-15}
	0.6	6.661×10^{-16}	0	1.332×10^{-15}	0
	0.9	8.881×10^{-16}	2.22045×10^{-16}	1.110×10^{-15}	2.22045×10^{-16}

Example 2. Consider the following fuzzy fractional integrodifferential equation:

$$\begin{aligned} \left({}_{\mathrm{gH}}D_*^{1/2}y\right)(t) &= \frac{3s}{140}\left(35\sqrt{\pi}-s^{5/2}\right)[1+2r,8-5r] \\ &\quad + \int_0^t (x+t)\,y(x)\,\mathrm{d}x, \qquad (64) \\ y(0) &= 0, \end{aligned}$$

where 0 denotes the crisp set $\{0\}$. The exact solution of this equation is given by $y(t) = [1+2r, 8-5r]t^{3/2}$. By applying the proposed method to obtain ${}^{cf}[(\mathrm{i})\text{-gH}]$-differentiable solution, we solve this problem numerically and calculate the absolute errors by using $|E_r| = |y(t,r) - y^*(t,r)|$. To see these absolute errors, one can refer to Table 2. Also, in Figure 2, we present the graph of the Bernoulli wavelet approximation error for $r = 1$, and $M_1 = M_2 = 8$, $M_1 = M_2 = 10$.

7. Conclusion

In the present paper, the two-dimensional Bernoulli wavelet method was applied to approximate the solution of fuzzy fractional integrodifferential equation. We transformed our problem to a system of algebraic equations so that by solving this system we obtained the solution of this kind of equation by considering the type of differentiability. Finally, the solution obtained using the suggested method shows that this approach can solve the problem effectively.

Table 2: The absolute errors for Example 2.

r	t	$\lvert\underline{E}_r\rvert$ $M_1 = M_2 = 15$	$\lvert\overline{E}_r\rvert$ $M_1 = M_2 = 15$
0.3	0.3	3.69879×10^{-6}	1.39805×10^{-7}
	0.6	3.76203×10^{-6}	7.20378×10^{-6}
	0.9	3.56742×10^{-6}	1.61695×10^{-6}
0.6	0.3	2.42246×10^{-6}	7.26592×10^{-6}
	0.6	3.76347×10^{-6}	1.35717×10^{-6}
	0.9	6.06565×10^{-6}	4.59658×10^{-6}
0.9	0.3	1.09187×10^{-6}	3.75328×10^{-5}
	0.6	8.77415×10^{-6}	1.35462×10^{-5}
	0.9	9.98774×10^{-6}	1.96437×10^{-5}

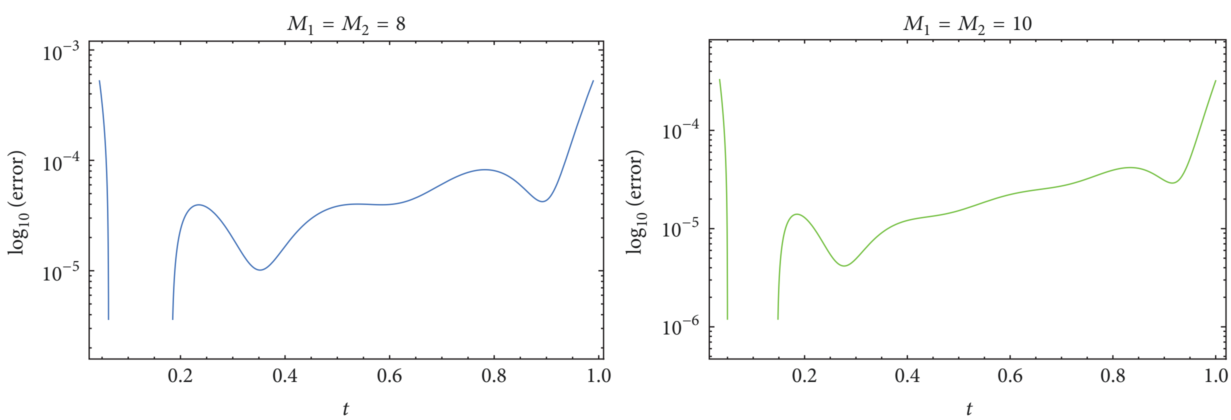


Figure 2: Graph of the Bernoulli wavelet approximation error for $r = 1$ and $t \in [0, 1)$ of Example 2.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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