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Research Article

Gaussian Qualitative Trigonometric Functions in a Fuzzy Circle

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We build a bridge between qualitative representation and quantitative representation using fuzzy qualitative trigonometry. A unit circle obtained from fuzzy qualitative representation replaces the quantitative unit circle. Namely, we have developed the concept of a qualitative unit circle from the view of fuzzy theory using Gaussian membership functions, which play a key role in shaping the fuzzy circle and help in obtaining sharper boundaries. We have also developed the trigonometric identities based on qualitative representation by defining trigonometric functions qualitatively and applied the concept to fuzzy particle swarm optimization using α-cuts.

1. Introduction

The term "Trigonometry" was first coined as the title of a book *(Trigonometria)*, which translates to "triangle's measurement." Although trigonometry is now taught with an emphasis on right triangles, its origin goes back to an era where it was used to determine the positions of celestial bodies and the distances between them and to understand the concept of chords in circles. Euclidean geometry, that is, planar geometry, deals with two-dimensional figures. The study of planar geometry provides constructions for planar figures, their properties, and the relationships between points, lines, and figures. The planar figure formed when a point traces a path at a fixed distance with respect to another fixed point called a circle, where the circle divides a plane into interior and exterior regions. Various theorems and properties on circles have been developed since time immemorial.

Gradually, researchers have developed theories on intersecting circles, which led to divergent properties between circular triangles and spherical triangles. The introduction of fuzzy sets and systems by Zadeh [1] changed the face of research in trigonometry. Fuzzy trigonometry was introduced by Buckley and Eslami [2], wherein continuous fuzzy numbers and sets were defined using the principle of extension. This method formed the basis of fuzzy trigonometry but failed to satisfy many criteria and identities. Furthermore, Ress [3] developed an approach for mapping standard trigonometric functions into the fuzzy realm. Using these modified fuzzy trigonometric functions, the proofs of a few inverse trigonometric identities, in addition to the standard identities, were given. A breakthrough in the study of fuzzy trigonometry was achieved by Liu et al. [4], with an aim of connecting symbolic cognitive functions to qualitative functions. The basic identities were satisfied, but a few properties could not be achieved. Ghosh and Chakraborty [5] proposed two methodologies for describing a fuzzy circle. The first methodology defines a circle as a set of points which are equidistant from a fixed point. The second methodology describes a circle using three fuzzy points. The definitions using both methods are as follows.

(1) Considering fuzzy numbers plotted along various line segments which pass through a specific point $\widetilde{P}(a,b)$ which is fuzzy and located at a distance of $\widetilde{r}$ from the fixed fuzzy point, then the fuzzy circle is defined as

$$\widetilde{C}_1 = \bigvee_{\theta} \left\{ \widetilde{B}_{\theta} \right\}, \tag{1}$$

where the distance between $\widetilde{B}_{\theta}$ and a random point on the support is always fixed.

(2) Considering three fuzzy points, the fuzzy circle is drawn by passing the circle through these three points, and the supremum of the membership values defines the fuzzy circle which was proposed.

Technology in today's world needs advanced level procedures or heuristics which involve few or zero assumptions about the problem being optimized. The need for metaheuristics has many advantages over other algorithms, with one advantage being its ability to search very large candidate solutions spaces. Particle swarm optimization (PSO) is one such metaheuristic method which allows optimization based on iterations, which in turn helps in improving the candidate solution. The solution is improved by creating a population of candidate solutions (i.e., particles) and making them move around in the search-space, taking into consideration their positions and velocities. The movement of the particle is influenced by its local best known position, simultaneously being guided towards the best known position in the search area. These are updated time-to-time for all other particles to form a swarm. PSO was first introduced by Kennedy et al. with an intention of simulating the social behaviour in a flock of birds or a school of fish. Shi and Eberhart [6] developed an optimization technique for fuzzy systems by dynamically adjusting the inertia weight, improving PSO's performance. Pang et al. [7] utilized fuzzy discrete PSO to solve the Travelling salesman problem., Abdelbar et al. [8] compared the behaviour of the Gaussian and Cauchy membership functions in fuzzy PSO and concluded that the Cauchy membership functions are best suited for fuzzy PSO. It was noted that traditional PSO converges prematurely when applied as a global optimization technique. To prevent this downfall, Anantathanavit and Munlin [9] proposed radius (R-PSO) as an extension of standard PSO. R-PSO regroups the particles into a circle of the given radius and determines the best agent particle of the group. The experiment results proved that R-PSO performed better than traditional PSO when solving multimodal complex problems.

To produce wholesome proofs for identities based on fuzzy trigonometric functions, we introduce the Gaussian qualitative trigonometric functions (GQTFs) on a fuzzy circle. The unit fuzzy circle is defined using Gaussian membership function (GMF) partitions. Using the GMF circumference, fuzzy centre, and fuzzy point on the circle, trigonometric functions and their identities are successfully developed. In this paper, we use the advantages of R-PSO and GMFs to define a fuzzy R-PSO based on a fuzzy qualitative circle. Section 2 gives the preliminaries and introduces the concept of GQTFs on the fuzzy circle. Section 3 provides insight on the formation of the fuzzy centre and fuzzy point on the circle and provides a broad perspective of the redefined trigonometric functions, their ratio identities, their Pythagorean identities, their Taylor's series expansions, their differentials, and laws for obtaining the solutions of triangles. Section 4 describes the properties of intersecting fuzzy circles, and Section 5 describes the PSO with fuzzy matrices. Section 6 describes fuzzy R-PSO with GMF, which makes an efficient model for finding the best agent. Finally, Section 7 concludes the paper.

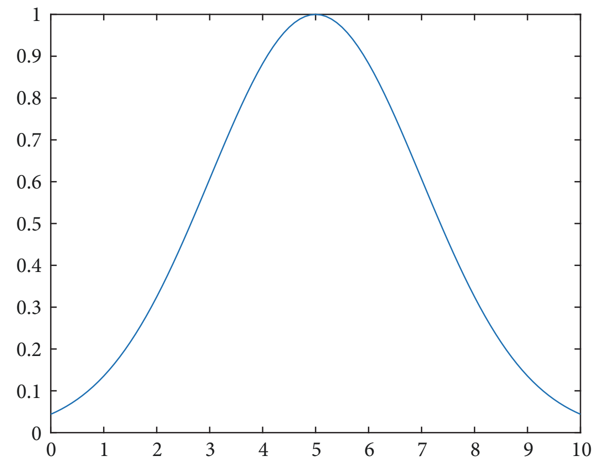

Figure 1: GMF with $c = 5$ and $\sigma = 2$.

2. Prerequisites

The prerequisites required for the developed concept have been taken from various research articles which are cited in the references. Slightly modified versions of the concepts have been given in this section.

2.1. Fuzzy Subset. Considering a set E which is either finite or infinite, for every element x in E, the set consisting of all ordered pairs of the form $\{(x, \mu_{\widetilde{A}}(x))\}$ is called a fuzzy subset, denoted by $\widetilde{A}$, in E. Here, $\mu_{\widetilde{A}}(x)$ is called the membership function. The membership function gives a mapping of the elements in set E to the membership set M.

2.2. Zadeh's Extension Principle. Considering a fuzzy subset of a universal set E, if $f : E \to Z$ is a function, then the extension principle generates a function $\widehat{f}$ whose membership function is defined over the supremum of all $f^{-1}(z) \neq \emptyset$, that is,

$$\mu_{\widehat{f}}(z) = \sup_{f^{-1}(z)} \mu_A(x), \quad x \in E. \tag{2}$$

Here, f^{-1} is the preimage of z in E.

2.3. Gaussian Membership Function (GMF). The GMF is given by

$$G_{mf}(x : c, \sigma) = \exp\left[-\frac{(c-x)^2}{2\sigma^2}\right], \tag{3}$$

where c and σ are the centre and width of the fuzzy set, respectively. An example of this function is shown in Figure 1. This membership function is also defined in terms of the interval $[m, j]$ as follows:

$$G_{mf}(x : m, j) = \exp\left[-\frac{(j-x)^2}{2m^2}\right]. \tag{4}$$

Table 1: Arithmetic operations.

Sl.no.	Operation	Notation	Result
1.	Sum	$[m,c]+[j,a]$	$[m+j,c+a]$
2	Difference	$[m,c]-[j,a]$	$[m-a,c-j]$
3	Product	$[m,c]\times[j,a]$	$[m\times j,c\times a]$
4	Quotient	$[m,c]\div[j,a]$	$[m\div a,c\div j]$

The GMF with fuzzification factor m is given by

$$G_{mf}(x:c,\sigma,m)=\exp\left[-\frac{(c-x)^m}{2\sigma^m}\right]. \tag{5}$$

2.4. Arithmetic Operations on GMFs. The Arithmetic Operations on GMFs can be summarised as follows.

Letting $G_{mf}(x:m,c)$ and $G_{mf}(x:j,a)$ be two GMFs, their arithmetic operations are shown in Table 1.

3. Proposed Method

Fuzzy Gaussian qualitative coordinates are proposed to facilitate the geometrical interpretation of GQTFs. To achieve this, we first define a fuzzy circle obtained using GMFs.

3.1. Gaussian Qualitative Coordinates. The x- and y-axes are defined for a unit circle. For the sake of simplicity and ease of use, the abscissa and ordinate are first considered as real lines. Figure 2 describes the circumference of the fuzzy unit circle which is obtained by partitioning the circumference into GMFs. The cloud formation, as described by Ghosh and Chakraborty [10], is now refined to obtain sharp boundaries for the circle. The crisp centre of the circle has a neighbourhood which is formed by extending and converging the core area of every GMF towards the centre of the circle. Thus the centre of the derived fuzzy circle will have infinitely many sides depending on the number of partitions. We can henceforth generalise the centre to be an infinitesimal circle with an infinitesimal radius, that is, l.

3.2. Fuzzy Centre and Position of a Point. The values of a and b obtained on the circumference of the fuzzy circle are now extended towards the core of the circle. Without loss of generality, a point on the proposed fuzzy circle can be assumed to be the centre of some GMF. The position of a point can be mapped as shown in Figure 3.

3.3. Trigonometric Functions. The graphs of sine, cosine, and tangent functions under the proposed concept have been obtained and verified to be in coherence with the crisp trigonometric functions. Various identities and laws for triangles are presented as follows.

3.3.1. Ratio Identities

(i) $\sin(G_{mf}[-\delta\ \ \delta])=G_{mf}[a_y\ \ b_y]/G_{mf}[\sqrt{(a_x-a_0)^2+a_y^2}\ \ \sqrt{(b_x-b_0)^2+b_y^2}]$

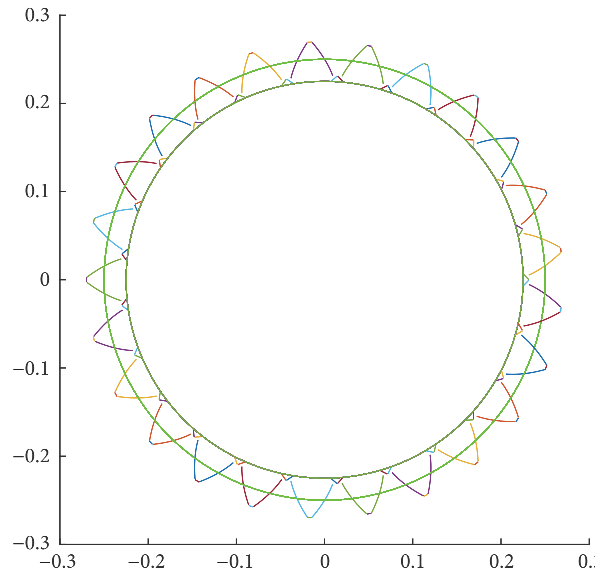

Figure 2: Fuzzy circle with Gaussian membership function.

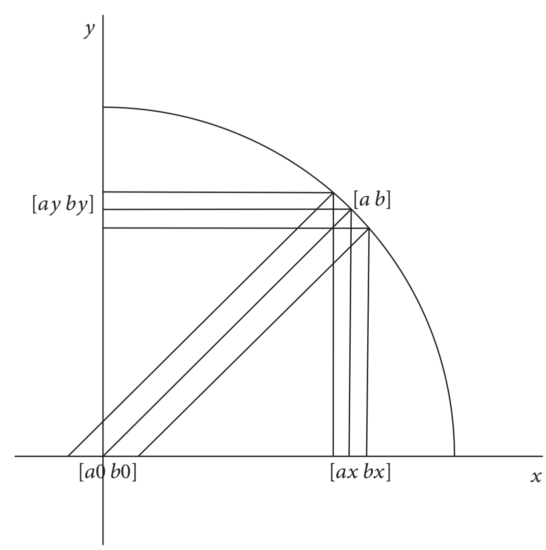

Figure 3: Coordinates of a point on the fuzzy circle.

(ii) $\cos(G_{mf}[-\delta\ \ \delta])=G_{mf}[a_0-b_x\ \ b_0-a_x]/G_{mf}[\sqrt{(a_x-a_0)^2+a_y^2}\ \ \sqrt{(b_x-b_0)^2+b_y^2}]$,

(iii) $\tan(G_{mf}[-\delta\ \ \delta])=G_{mf}[a_y\ \ b_y]/G_{mf}[a_0-b_x\ \ b_0-a_x]$.

3.3.2. Pythagorean Identities

(i) $\sin^2(G_{mf}[-\delta\ \ \delta])+\cos^2(G_{mf}[-\delta\ \ \delta])=G_{mf}[1-\delta\ \ 1+\delta]$,

(ii) $\sec^2(G_{mf}[-\delta\ \ \delta])-\tan^2(G_{mf}[-\delta\ \ \delta])=G_{mf}[1-\delta\ \ 1+\delta]$,

(iii) $\csc^2(G_{mf}[-\delta\ \ \delta])-\cot^2(G_{mf}[-\delta\ \ \delta])=G_{mf}[1-\delta\ \ 1+\delta]$.

3.3.3. Laws for Obtaining Solutions to Triangles

(i) The law of sines gives the relationship between the sines of the angles of a triangle and the lengths of the sides of the triangle:

$$\begin{aligned}\frac{\sin\left(G_{mf}\left[\theta_y-\delta_f \quad \theta_y+\delta_f\right]\right)}{G_{mf}\left[a_y \quad b_y\right]} &= \frac{\sin\left(G_{mf}\left[\theta_x-\delta_f \quad \theta_x+\delta_f\right]\right)}{G_{mf}\left[a_x \quad b_x\right]} \\ &= \frac{\sin\left(G_{mf}\left[\theta_z-\delta_f \quad \theta_z+\delta_f\right]\right)}{G_{mf}\left[a_z \quad b_z\right]}.\end{aligned} \tag{6}$$

(ii) The law of cosines relates the sides of a triangle with the cosine of an angle in the triangle:

(1) $G_{mf}[a_x-b_y \quad a_y-b_x]^2 = G_{mf}[c_x-b_y \quad c_y-b_x]^2 + G_{mf}[a_x-c_y \quad a_y-c_x]^2 - 2G_{mf}[c_x-b_y \quad c_y-b_x] \times G_{mf}[a_x-c_y \quad a_y-c_x] \times \cos(G_{mf}[C-\delta \quad C+\delta])$

(2) $G_{mf}[c_x-b_y \quad c_y-b_x]^2 = G_{mf}[a_x-c_y \quad a_y-c_x]^2 + G_{mf}[a_x-b_y \quad a_y-b_x]^2 - 2\times G_{mf}[a_x-c_y \quad a_y-c_x] \times G_{mf}[a_x-b_y \quad a_y-b_x] \times \cos(G_{mf}[A-\delta \quad A+\delta])$,

(3) $G_{mf}[a_x-c_y \quad a_y-c_x]^2 = G_{mf}[b_x-c_y \quad b_y-c_x]^2 + G_{mf}[a_x-b_y \quad a_y-b_x]^2 - 2\times G_{mf}[b_x-c_y \quad b_y-c_x] \times G_{mf}[a_x-b_y \quad a_y-b_x] \times \cos(G_{mf}[B-\delta \quad B+\delta])$.

(iii) The law of tangents gives the relationship between the sides and tangents of the angles of a triangle:

$$\begin{aligned}&\frac{G_{mf}\left[b_x+c_x-\left(a_y+c_y\right) \quad b_y+c_y-\left(a_x+c_x\right)\right]}{G_{mf}\left[b_x+a_x-2c_y \quad b_y+a_y-2c_x\right]} \\ &\quad= \frac{\tan\left((1/2)\,G_{mf}\left[A-B-2\delta \quad A-B+2\delta\right]\right)}{\tan\left((1/2)\,G_{mf}\left[A+B-2\delta \quad A+B+2\delta\right]\right)}.\end{aligned} \tag{7}$$

3.3.4. Mollweide's Identity. This identity is a tool for checking the solutions of triangles. It uses all sides and angles of the triangle.

$$\begin{aligned}&\frac{G_{mf}\left[a_x+b_x-2c_y \quad a_y+b_y-2c_x\right]}{G_{mf}\left[a_x-b_y \quad a_y-b_x\right]} \\ &\quad= \frac{\cos\left((1/2)\,G_{mf}\left[A-B-2\delta \quad A-B+2\delta\right]\right)}{\sin\left((1/2)\,G_{mf}\left[C-\delta \quad C+\delta\right]\right)}.\end{aligned} \tag{8}$$

3.4. Taylor's Series Expansions of Trigonometric Functions. The Taylor's series expansions of the trigonometric functions are as follows:

(i) $\cos(G_{mf}[x_{mf}-\delta_{mf} \quad x_{mf}+\delta_{mf}]) = [1-(G_{mf}[x_{mf}-\delta_{mf} \quad x_{mf}+\delta_{mf}])^2/2! + (G_{mf}[x_{mf}-\delta_{mf} \quad x_{mf}+\delta_{mf}])^4/4! \mp \cdots]$

Thus, $\cos(G_{mf}[x_{mf}-\delta_{mf} \quad x_{mf}+\delta_{mf}]) = G_{mf}[\cos(x_{mf}-\delta_{mf})-\Delta_{mf} \quad \cos(x_{mf}-\delta_{mf})+\Delta_{mf}]$, where $\Delta_{mf} = 2\sum_{n=2,4,6,\ldots}^{\infty}(((x_{mf}+\delta_{mf})^n+(x_{mf}-\delta_{mf})^n)/n!)$

(ii) $[\sin(G_{mf}[x_{mf}-\delta_{mf} \quad x_{mf}+\delta_{mf}]) = G_{mf}[x_{mf}-\delta_{mf} \quad x_{mf}+\delta_{mf}]-(G_{mf}[x_{mf}-\delta_{mf} \quad x_{mf}+\delta_{mf}])^3/3! + (G_{mf}[x_{mf}-\delta_{mf} \quad x_{mf}+\delta_{mf}])^5/5! \mp \cdots]$

Thus, $[\sin(G_{mf}[x_{mf}-\delta_{mf} \quad x_{mf}+\delta_{mf}]) = G_{mf}[\sin(x_{mf}-\delta_{mf})-\Delta_1 \quad \sin(x_{mf}-\delta_{mf})+\Delta_1]]$, where $\Delta_1 = 2\sum_{n=1,3,5,\ldots}^{\infty}(((x_{mf}+\delta_{mf})^n+(x_{mf}-\delta_{mf})^n)/n!)$

(iii) $\tan(G_{mf}[x_{mf}-\delta_{mf} \quad x_{mf}+\delta_{mf}]) = G_{mf}[x_{mf}-\delta_{mf} \quad x_{mf}+\delta_{mf}] + (1/3)G_{mf}[(x_{mf}-\delta_{mf})^3 \quad (x_{mf}+\delta_{mf})^3] + (2/5)G_{mf}[(x_{mf}-\delta_{mf})^5 \quad (x_{mf}+\delta_{mf})^5] + \cdots$

Thus, $[\tan(G_{mf}[x_{mf}-\delta_{mf} \quad x_{mf}+\delta_{mf}]) = G_{mf}[\tan(x_{mf}-\delta_{mf}) \quad \tan(x_{mf}+\delta_{mf})]]$.

3.5. Differentiation of the Gaussian Trigonometric Functions. The trigonometric functions thus obtained are also fuzzy differentiable. Supposing that $F(x) = f(G_{mf}[c \quad \sigma])$ is a Gaussian fuzzy trigonometric function, then the derivatives of the function are as follows:

First Derivative: $F'(x) = f'(G_{mf}[c \quad \sigma]) \times \exp(-x^2/2\sigma^2) \times (-x/\sigma^2)$.

The second and higher order derivatives are obtained by using the chain rule for the above functions.

4. Intersection of Fuzzy Circles

We now introduce the concept of intersecting fuzzy circles and study their structure and properties while comparing them with regular intersecting circles.

4.1. Circles Touching at Exactly One Point

Case 1. Two circles have an equal number of partitions, implying that the radii of both circles are equal. Figure 4 shows the fuzzified point of intersection.

In Figure 4, we observe that the centres of both Gaussian curves intersect at a crisp point. The fuzzified areas of the intersection are congruent triangles. The area of each triangle is given by $F_{ta} = 1/2 \times (2r) \times \sigma = r \times \sigma$. Thus, the total fuzzy area is $2r\sigma$.

Case 2. The number of partitions on the intersecting circles is unequal, as shown in Figure 5.

We observe that the crisp point of intersection is obtained where the centres of both Gaussian curves meet. The point of intersection is bounded by triangles whose areas are given by

$$\text{Area of } \Delta_1 = \frac{1}{2} \times \sigma_1 \times \left(r_1+\sigma_1\right) = \frac{\sigma_1\left(r_1+\sigma_1\right)}{2}$$

$$\text{Area of } \Delta_2 = \frac{1}{2} \times \sigma_1 \times \left(r_1-\sigma_1\right) = \frac{\sigma_1\left(r_1-\sigma_1\right)}{2}$$

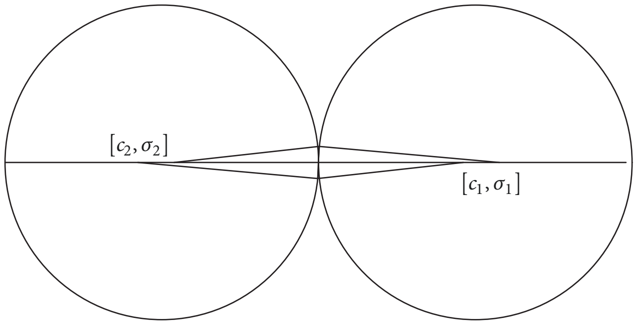

Figure 4: Circles of equal radii touching at a single point.

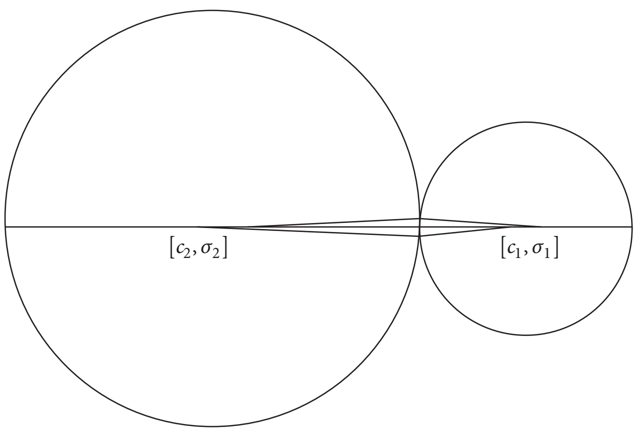

Figure 5: Circles of unequal radii touching at a single point.

$$\text{Area of } \Delta_3 = \frac{1}{2} \times \sigma_2 \times (r_2 + \sigma_2) = \frac{\sigma_2 (r_2 + \sigma_2)}{2}$$
$$\text{Area of } \Delta_4 = \frac{1}{2} \times \sigma_2 \times (r_2 - \sigma_2) = \frac{\sigma_2 (r_2 - \sigma_2)}{2}. \quad (9)$$

Hence, the total area is $= r_1\sigma_1 + r_2\sigma_2$. The area obtained is in coherence with the traditional way of determining the area of intersection.

The intersection of circles becomes interesting when we consider circles intersecting at two points. In such a case, the area of intersection can be either symmetric or dominate a single circle with a small portion of the second circle as a common area.

4.2. Fuzzy Circles Intersecting at Two Points

Case 1. Two circles have an equal number of partitions. Figure 6 shows the intersecting fuzzy circles. From the following diagram, we observe that the crisp points of intersection and the common chord are obtained during the fuzzification process. We obtain congruent triangles in the fuzzy area of intersection. The centres of the Gaussian curves meet to produce the crisp point of intersection. The two points, when joined, produce the crisp common chord for the intersecting circles.

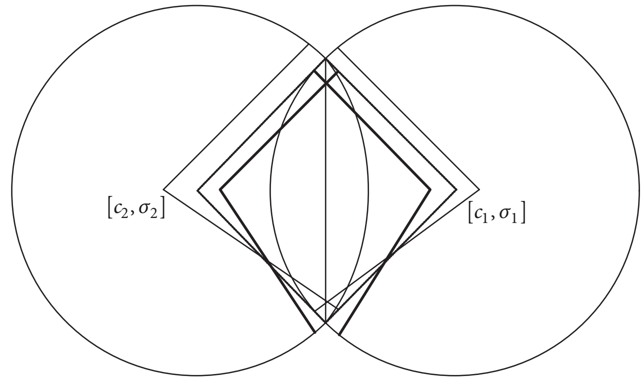

Figure 6: Intersecting circles of equal radii.

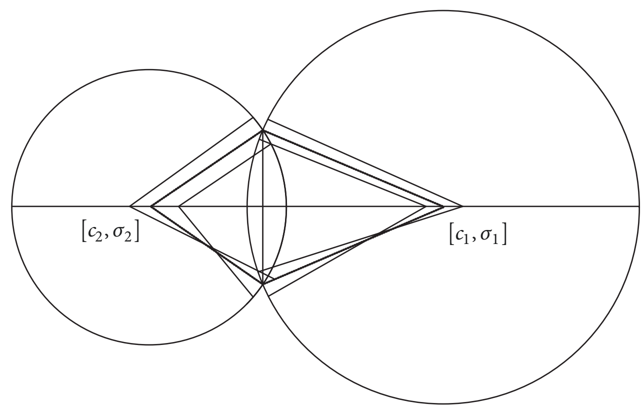

Figure 7: Intersecting circles of unequal radii.

The area of the overlapping region is calculated as follows:

(i) The area of each sector is $(r_- \times \theta)/2$.

(ii) The length of the arc of the sector is given by $r \times \theta$.

(iii) Thus, the area of the fuzzified sector is $(r_+^2 \times \theta)/2$.

(iv) The area of the smaller sector is $(r_-^2 \times \theta)/2$. Here, $r_+ = r + \sigma$ and $r_- = r - \sigma$.

(v) Supposing the line r_+ cuts the smaller radius in the ratio $m : n$, then $m = r_+/2$. Thus, the area is $(r_+^2 \times \theta_1)/4$, and the total area is $r_+^2/2 \times (2\theta + \theta_1)$.

Case 2. Two circles have an unequal number of partitions, which implies that the radii of both circles are different (Figure 7).

The area of the bigger fuzzy sector is $(r^2 \times \theta)/2 + (r^2 \times \theta_1)/4 = (r^2/4)(2\theta + \theta_1)$.

Similarly, the area of the smaller fuzzy sector is $(r_1^2/4)(2\delta + \delta_1)$.

Thus, in both cases, we obtain a fuzzy diagram bounded by ten points. The area of the overlapping section is obtained using the following formulas:

$$\begin{aligned} \text{Area}_{\text{fuzzy}} &= \frac{1}{\text{Dis}}\sqrt{C \times J \times A \times M}, \\ C &= \left(-\text{Dis} + \text{rad}_1 + \text{rad}_2\right), \\ J &= \left(\text{Dis} - \text{rad}_1 + \text{rad}_2\right), \\ A &= \left(\text{Dis} + \text{rad}_1 - \text{rad}_2\right), \\ M &= \left(\text{Dis} + \text{rad}_1 + \text{rad}_2\right), \end{aligned} \tag{10}$$

where $\text{Dis} = [O_1 - O_2 - (\sigma_1 + \sigma_2) \;\; O_1 - O_2 + (\sigma_1 + \sigma_2)]$ is the distance between the centres and $\text{rad}_1 = [r_1 - \sigma_1 \;\; r_1 + \sigma_1]$ and $\text{rad}_2 = [r_2 - \sigma_2 \;\; r_2 + \sigma_2]$ are the Gaussian radii of the fuzzy circles.

4.3. Intersection of Three Fuzzy Circles. Three circles can intersect in many ways, but we consider two special cases wherein we obtain circular triangles and irregular convex quadrilaterals. However, we must check whether an overlapping area exists between the considered circles. To verify, we check if either of the following conditions are satisfied.

(i) A point of intersection of two circles should be a point inside the third circle;

(ii) One of the circles should be engulfed completely inside another circle.

Case 1. Three fuzzy circles with equal numbers of partitions are considered, implying that the radii of all three circles are equal. The overlapping area formed by the three segments and an inner triangle whose sides are the arcs of the three circles is called a circular triangle (Figure 8).

The area of the triangle is calculated using Heron's formula:

$$\text{Area}_{abc} = \sqrt{S(S-A)(S-B)(S-C)}, \tag{11}$$

where $S = (1/2)(A + B + C)$, $A = G_{mf}[a_x \;\; a_y]$, $B = G_{mf}[b_x \;\; b_y]$, and $C = G_{mf}[c_x \;\; c_y]$. The area of the combined overlap is given by

$$\begin{aligned} \text{Area}_{\text{seg}} = {} & \sum_{n=1}^{3} R_n^2 \sin^{-1}\left(\frac{A}{2R_n}\right) - \sum_{n=1}^{3} \frac{A}{4}\sqrt{4 \times R_n^2 - A^2} \\ & + \sqrt{S(S-A)(S-B)(S-C)}. \end{aligned} \tag{12}$$

Case 2. Three fuzzy circles with unequal numbers of partitions will have different radii. In this case, we must recalculate the overlapping segment area using the area of the bigger circle (Figure 9).

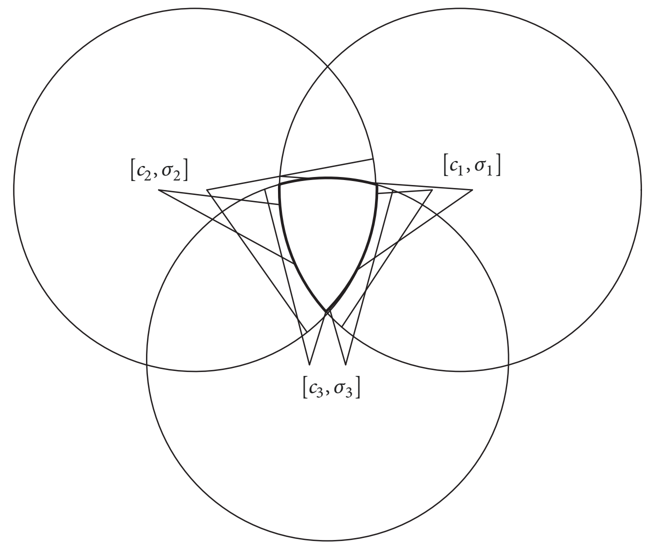

Figure 8: Circles of equal radii intersecting and forming a circular triangle.

Step 1. We find the midpoint of the chord in the segment, which is given by

$$\begin{aligned} x_{\text{mid}} &= \frac{x_{i12} + x_{i13}}{2}, \\ y_{\text{mid}} &= \frac{y_{i12} + y_{i13}}{2}. \end{aligned} \tag{13}$$

Step 2. We then find the equation of the line joining the midpoint and the centre of the circle, which is given by

$$y = \frac{B_1 - y_{\text{mid}}}{A_1 - x_{\text{mid}}} \times x - \frac{x_{\text{mid}}\left(B_1 - y_{\text{mid}}\right)}{A_1 - x_{\text{mid}}} + y_{\text{mid}}. \tag{14}$$

Step 3. Next, we determine the points of intersection of the above line and circle, which are given by $X_{ic1,ic2} = (a_1 + b_1 \times m - m \times n \pm \sqrt{\delta})/(1 + m^2)$, $Y_{ic1,ic2} = (n + a_1 \times m + b_1 \times m^2 \pm m \times \sqrt{\delta})/(1 + m^2)$, where $m = (B_1 - y_{\text{mid}})/(A_1 - y_{\text{mid}})$, $n = (-x_{\text{mid}} \times (B_1 - y_{\text{mid}}))/(A_1 - x_{\text{mid}}) + y_{\text{mid}}$.

Step 4. Finally, we check whether the other two radii of the circles are greater than the distance between the centre and the midpoint.

Thus, the area of the bigger segment completely encompassed by the bigger circle is $(R_1^2/2)(\theta_1 - \sin\theta_1)$.

Results

(i) The regions of intersection of three equal fuzzy circles have been considered. Hence the common value of radii has been denoted by R. Here R denotes the common radii.

(ii) Given three intersecting fuzzy circles, the lines joining the points of intersection outside the circular triangle satisfy $ACE = BDF$ which defines Haruki's theorem.

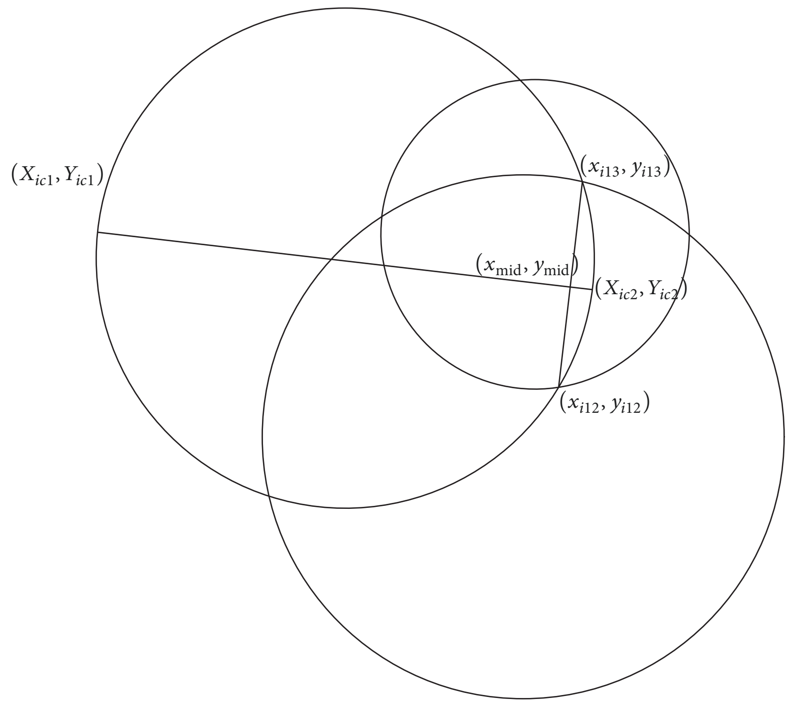

FIGURE 9: Three intersecting circles of unequal radii.

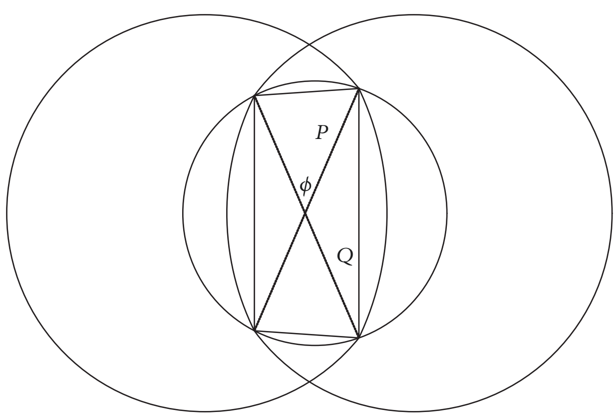

FIGURE 10: Formation of irregular convex quadrilateral.

(iii) The crisp circle which touches the inner circles of the circular triangle and its neighbouring triangles is either tangent to the innermost circle and internally tangent to the inner circles of the neighbouring triangles or vice versa. Such a circle is called the "Hart's circle." We can construct 8 such Hart circles for a given circular triangle.

Case 3. The most complex case is that of an overlapping area which is obtained when the circular segments form the boundary shown in Figure 10. In the process, we obtain an irregular convex quadrilateral.

The area of the irregular convex quadrilateral is given by $\text{Area}_{\text{quad}} = (1/2)P \times Q \times \sin\phi$, where ϕ is the angle between the diagonals P and Q.

Hence the combined area of the segments is given by

$$\text{Area} = \sum_{n=1} 4R_n^2 \sin^{-1}\left(\frac{A_n}{2 \times R_n}\right) - \sum_{n=1}^{4} \frac{A_n}{4}\sqrt{4 \times R_n^2 - A_n^2}. \quad (15)$$

Here A_n is the length of the chord joining the points of intersection of the circles. There are four chords which form the edges of the convex quadrilateral. And, R_n is the radii of the circles.

5. Fuzzy Particle Swarm Optimization (PSO) with Fuzzy Matrices and GMFs

5.1. Fuzzy Matrices. A matrix of the form $P = [p_{ij\mu}]_{m\times n}$ where $p_{ij\mu}$ is the membership degree of the element $p_{ij} \in \widetilde{P}$ is called a fuzzy matrix.

In PSO, a finite number of particles define a swarm. The particles move in the n-dimensional space with certain velocity. The particle's velocity and position are updated using the following formula:

$$V_i^{k+1} = w \otimes V_i^k \oplus (c_1 \otimes r(\cdot)) \otimes \left(P_i^k \ominus X_i^k\right) \oplus \sum_{h\in\eta(i,k)} c_2 \otimes r(\cdot) \otimes \psi(p_h)\left(P_g^k \ominus X_i^k\right), \quad (16)$$

where $\eta(i,k)$ denotes the k best particles in the neighbourhood of the particle and $\psi(p_h)$ denotes the matrix containing the corresponding membership degrees of the particles defined using the GMF.

5.2. Particle Position Initialization. The positions of the particles are initialized and represented in the form of a fuzzy matrix. The position matrix is given by

$$P_{rij} = \begin{bmatrix} p_{r11} & p_{r12} & \cdots & p_{r1n} \\ \vdots & \ddots & & \vdots \\ p_{rn1} & p_{rn2} & \cdots & p_{rnn} \end{bmatrix}. \quad (17)$$

The positions of the particles for the fuzzy matrix are generated randomly and satisfy the following conditions:

$$\sum_{j=1}^{n} p_{rij} = 1, \quad P_{rij} \in [0,1], \; i = 1,2,\ldots,n. \quad (18)$$

5.3. Particle Velocity Initialization. The velocity matrix is given by

$$V_{rij} = \begin{bmatrix} v_{r11} & v_{r12} & \cdots & v_{r1n} \\ \vdots & \ddots & & \vdots \\ v_{rn1} & v_{rn2} & \cdots & v_{rnn} \end{bmatrix}, \quad (19)$$

where the velocities of the particles are generated according to the randomly generated position of the particles with conditions

$$\sum_{j=1}^{n} v_{rij} = 0, \quad i = 1,2,\ldots,n. \quad (20)$$

5.4. Normalization of the Particle Position Matrix. The particle position matrix generated using random positions may sometimes not adhere to the constraint $P_{rij} \in [0,1]$. Hence, we need to normalize the particle position matrix. The first step in normalization involves replacing all negative values in the matrix with zeros. Then, the matrix is transformed into

$$P_{rij} = \begin{bmatrix} \frac{p_{r11}}{\sum_{i=1}^{n} p_{r1i}} & \frac{p_{r12}}{\sum_{i=1}^{n} p_{r1i}} & \cdots & \frac{p_{r1n}}{\sum_{i=1}^{n} p_{r1i}} \\ \vdots & \ddots & & \vdots \\ \frac{p_{rn1}}{\sum_{i=1}^{n} p_{rni}} & \frac{p_{rn2}}{\sum_{i=1}^{n} p_{rni}} & \cdots & \frac{p_{rnn}}{\sum_{i=1}^{n} p_{rni}} \end{bmatrix}. \quad (21)$$

5.5. Choosing the Best Agents with α-levels. The agent particles are now selected based on the value of α, where α is a real number in the interval $[0,1]$. Once the agent particles are selected, they are placed on the circumference of the circle of radius α. Furthermore, the global best position of the particle is calculated using the distance formula

$$d\left(p_{rij}, p_{rwj}\right) = \frac{1}{8(\sigma)^2} \sum_{i=1}^{n} \left[\left(\mu_{p_{rij}}(x_i) - \mu_{p_{rwj}}(x_i) \right)^2 + (c_1 - c_2)^2 \right]; \quad (22)$$

$$d \le 2\alpha.$$

This process is illustrated in Figure 11.

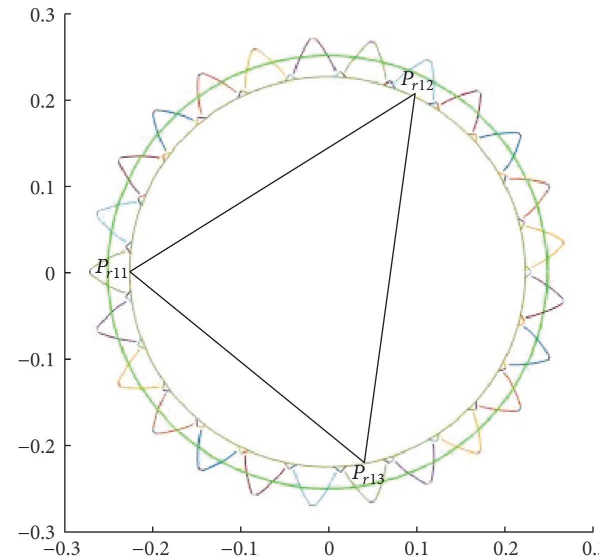

FIGURE 11: Neighbourhood for obtaining the global best agent.

6. Algorithm for Fuzzy R-PSO with GMFs

The proposed algorithm provides the local best position of particles based on the threshold, as given in Figure 12.

(1) Start.

(2) Initialize: maximum number of particles in the target neighbourhood: $\max_p$; maximum number of iterations: $\max_n$.

(3) Assign random position and velocity matrices for the particles.

(4) For $i = 0$ to $\max_p$

  (a) Calculate positions and velocities of the particles as given in the equation.

  (b) Normalize the particle position matrix using the equation.

(5) Fix the α-level for the normalized particle position matrix.

(6) If a particle's position in the normalized matrix is $\le\alpha$, then update the first neighbourhood circle.

(7) Calculate the distance between the particles and locate the global best position.

(8) Stop.

7. Discussion and Conclusion

The proposed methodology of constructing a fuzzy qualitative circle with GMF provides a smoother graphs of fuzzy trigonometric functions when compared to previously described methods. The graphs of the curves as obtained by Kaufmann and Gupta [11] flatten either the min- or max-curve depending upon the point of inflection, which led to a restriction on the extent of the spread in terms of angular

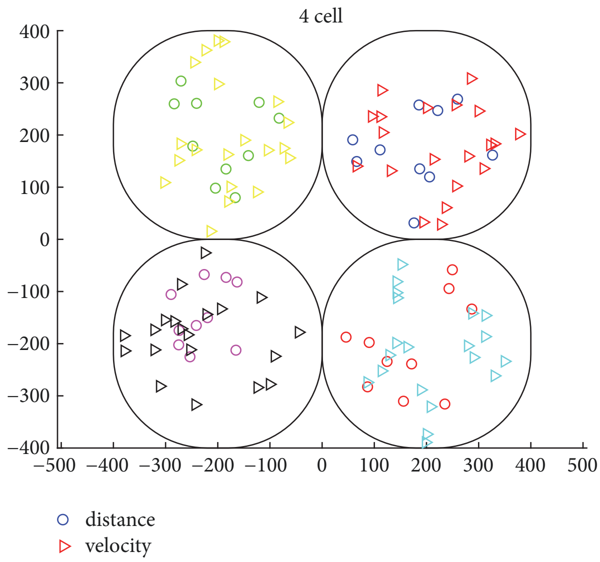

Figure 12: Neighbourhood obtained using α-cuts.

measurements. This angular spread has now been defined in terms of σ which is the width of the GMF. When $\sigma = 0$, the Gaussian Fuzzy Circle reduces to the conventional crisp circle. The centre of the fuzzy circle is also fuzzy and the radius is a linear combination of fuzzy numbers. Also, we have demonstrated geometrical interpretation of the GQTFs and the representation of a fuzzy circle with $n = 25$ GMFs. The position of a point on the fuzzy circle has been explained. Trigonometrical identities, Pythagorean identities, and laws for obtaining solutions to triangles (i.e., law of sines, law of cosines, and law of tangents) have been derived. The GMFs on a fuzzy circle and Taylor's series expansions have been verified. The relationship between qualitative and quantitative trigonometric states is very difficult to describe; however, Gaussian fuzzy qualitative trigonometry helps in describing their relationship. In this paper, we have successfully described the intersection of fuzzy circles when the circle is divided into either equal or unequal numbers of partitions, and a study of circular triangles has been given. The results on circular triangles have also been verified. This research article gives a theoretical interpretation of applying R-PSO inside a fuzzy circle. The R-PSO with fuzzy circle of radius α (developed using α-cuts) reduced the premature convergence of the standard PSO algorithm. By fixing various levels of α, global best position of the particles can be obtained. Furthermore, the development of fuzzy circles with partitions provides a sharper boundary when compared to the existing ideas of fuzzy circles and fuzzy qualitative circles.

7.1. Future Direction. The authors intend to implement this idea in designing a navigation stick using the concept of radar. We propose to utilize this concept in designing navigation systems for people with special needs. Further, this research, being an extension to fuzzy qualitative trigonometry, will be implemented in extensive research towards robot kinematics and human movement analysis.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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